Quantum noise in a transversely-pumped-cavity Bose-Hubbard model

Dávid Nagy, Gábor Kónya, Peter Domokos, and Gergely Szirmai

Institute for Solid State Physics and Optics, Wigner Research Centre for Physics, Hungarian Academy of Sciences,
P.O. Box 49, H-1525 Budapest, Hungary

We investigate the quantum measurement noise effects on the dynamics of an atomic Bose lattice gas inside an optical resonator. We describe the dynamics by means of a hybrid model consisting of a Bose-Hubbard Hamiltonian for the atoms and a Heisenberg-Langevin equation for the lossy cavity-field mode. We assume that the atoms are prepared initially in the ground state of the lattice Hamiltonian and then start to interact with the cavity mode. We show that the cavity-field fluctuations originating from the dissipative outcoupling of photons from the resonator lead to vastly different effects in the different possible ground-state phases, i.e., the superfluid, the supersolid, the Mott and charge-density-wave phases. In the former two phases with the presence of a superfluid wavefunction, the quantum measurement noise appears as a driving term leading to depletion of the ground state. The timescale for the system to leave the ground state is presented in a simple analytical form. For the latter two incompressible phases, the quantum noise results in the fluctuation of the chemical potential. We derive an analytical expression for the corresponding broadening of the quasiparticle resonances.

I. INTRODUCTION

Cavity quantum electrodynamics (CQED) is devoted to studying the interaction of the electromagnetic field with atoms under the best possible control of circumstances. The radiation field is tailored by resonators in order to select spatially and spectrally one or a few relevant field modes. The atomic positions are controlled as well; either by well-defined trajectories across the resonator or by trapping the atoms in well-defined positions [1,2]. The control over the components allows for accessing the coherent quantum dynamics of the coupled atom-field system. The first milestone has been the demonstration of the vacuum Rabi splitting [3], which is the benchmark of strong coupling between the induced electric dipole of the atom and the cavity mode. The possibility of observing coherent processes, such as the Rabi oscillation, is limited by the spontaneous photon scattering into modes other than the cavity mode. The larger the electric dipole coupling strength g is with respect to the atomic spontaneous emission rate γ and the cavity-mode linewidth κ, the shorter the dynamical timescale needed to resolve the Jaynes-Cummings spectrum is. Various nonlinear quantum effects, e.g., the photon blockade [4] and the two-photon gateway [5], have been observed in the strong-coupling regime of cavity QED.

With the use of ultracold atoms in CQED experiments, the magnitude of the collective coupling of an atomic ensemble to the cavity mode can be significantly enhanced to values higher than those characterizing the loss rates. This allows for applying large detuning between all laser excitations and the internal atomic resonances, and thereby, the atomic scattering loss processes can be significantly suppressed. As a consequence, the atom-cavity dynamics can be controlled on much longer timescales. This opportunity opened the way to a new regime of cavity QED experiments where the spatial motion of an atom cloud couples coherently to the dynamics of the cavity-field mode [6]. The effective Hamiltonian describing the system includes characteristic frequencies well below the single-atom coupling strength g, e.g., the so-called recoil frequency $\omega_R = \hbar k^2/(2m)$, where k is the cavity-mode wave number and m is the atomic mass. These experiments revealed, for example, a Dicke-type superradiant phase transition of an atomic superfluid in the cavity [7–9] and demonstrated optomechanical strong coupling between vibration and field intensity [10]. The limitation of the coherent dynamics on timescales longer than the inverse of the recoil frequency originates from the spontaneous photon scattering; again, however, (i) the spontaneous emission from atoms into free-space modes is strongly suppressed, and (ii) the photon loss from the cavity mode into external modes is weakly coupled to the dynamics in a rather indirect way.

A very new generation of CQED experiments [11,12] introduced a timescale which is even longer than the inverse recoil frequency. Ultracold atoms trapped in optical lattices sustained by the cavity can tunnel between adjacent sites. The many-body quantum state of the atoms in the lattice sites becomes relevant to the dynamics of the cavity-field amplitude [13–15]. The tunneling time depends on the depth of the trapping potential, which is a novel control parameter in the cavity QED system. The effective Hamiltonian corresponds to the family of bosonic Hubbard-type lattice models extended to include the cavity-field mode [16]. In these Hamiltonian systems exotic new phases of lattice bosons appear due to the cavity-mediated global-range interactions [17–28]. In fermionic lattices, cavity-induced topologically nontrivial [29–31] states can be generated. All these research topics are currently stimulating significant theoretical and experimental interest in atomic lattice gases integrated in CQED systems.

Despite both the pronounced theoretical and experimental interest, very little effort has been made to study the limitations of a Hamiltonian approach. In a very recent paper Chiacchio
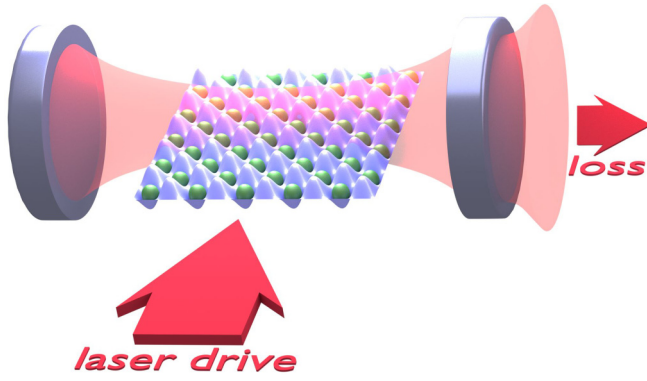
FIG. 1. Illustration of the coupled cavity Bose-Hubbard model setup. An atomic cloud is loaded into a square optical lattice, which is inside a single-mode high-Q Fabry-Pérot resonator. The period of the cavity mode is approximately equal to that of the optical lattice. The cavity is pumped by the side through the light scattered by the atoms from the laser drive. The system is open, and photons leak out from the cavity, resulting in heating and decoherence.

and Nunnenkamp studied the time evolution of the density matrix by integrating the master equation [32]. They found that in the bad-cavity limit, which is close to the experimental situation in Ref. [11], the steady state is an infinite-temperature state. In the experiments, however, they found a more or less coherent evolution during the time of the measurements.

In this paper we investigate the time limitation on the coherence in cavity Bose-Hubbard models. We consider the cavity photon loss to be the dominant dissipative process. Since the outcoupled photons can be directed to a photodetector, the fluctuations associated with the loss process can be equally well conceived as the measurement-induced backaction on the quantum system. We derive simple analytical formulas in terms of experimentally measurable quantities to quantify the time limitation of the Hamiltonian description. In Sec. II, we introduce one example of a cavity Bose-Hubbard model and derive the Heisenberg-Langevin equations that take into account the fluctuations of the cavity mode. In Sec. III, we derive an effective model for the lattice bosons by adiabatically eliminating the cavity field. In Sec. IV, we calculate and compare the effects of dissipation noise in the (i) superfluid-type and (ii) Mott-type phases. Finally, we summarize the results in Sec. V.

II. CAVITY BOSE-HUBBARD MODEL

An atomic degenerate Bose gas trapped in an optical lattice inside a high-finesse optical cavity gives rise to a system with competing short- and long-range interactions. The lattice dynamics of the ultracold atoms including on-site collisional interactions and tunneling between adjacent sites corresponds to the usual Bose-Hubbard model [33,34]. The optical lattice depth set by the power of its driving laser controls the strength of these short-range effects. On top of the bosonic lattice model, there is a long-range atom-atom interaction mediated by the cavity field in photon-scattering processes. In order to be specific, we consider the geometry of the experimental setup in Ref. [11], illustrated in Fig 1. A sample of bosonic atoms is placed inside a two-dimensional optical lattice, which overlaps

with a mode of an optical cavity. A single mode is selected in which the wavelength is very close to that of the optical lattice. The atoms are illuminated from the side by means of a far-detuned laser source which is close to resonance with the cavity mode. The atoms scatter photons coherently between the laser and the commonly coupled cavity mode [6]. The collective coupling of the atoms to the cavity mode leads to an indirect and infinite-range interaction between the atoms, which can be controlled independently from other parameters of the lattice dynamics by means of a small detuning of the drive frequency from the cavity resonance. The effective Hamiltonian of the system reads (see Ref. [11])

$$H = H_{at} + H_{ph} + H_I, \tag{1a}$$

where

$$H_{at} = -J \sum_{\langle i,j \rangle} (b_i^\dagger b_j + b_j^\dagger b_i) + \frac{U_s}{2} \sum_j n_j (n_j - 1), \tag{1b}$$

$$H_{ph} = -\hbar \Delta_C a^\dagger a, \tag{1c}$$

$$H_I = \hbar \eta (a^\dagger + a) \sum_j (-1)^j n_j. \tag{1d}$$

The first term is the standard Bose-Hubbard model describing the atomic dynamics inside the optical lattice. The atomic annihilation and creation operators at site i are b_i and $b_i^\dagger$, respectively. The tunneling is characterized by J, and the strength of the on-site interaction is denoted by U_s. The second term represents the oscillator of the cavity field, with a being the photon annihilation operator and $\Delta_C = \omega_L - \omega_C$ being the cavity detuning, i.e., the frequency difference of the pumping laser and the empty cavity. The interaction term H_I is a driving of the cavity mode which depends on the atom density $n_j = b_j^\dagger b_j$ and, as a key element, on the phase factor $(-1)^j$. The phase comes from the fact that the atoms scatter the laser light into the cavity with an amplitude depending on the sign of the cavity-mode function at the given site. This sign alternates between adjacent sites of the optical lattice, discriminating "even" and "odd" sites in a checkerboard pattern. The strength of the interaction is modeled by the parameter η, which is proportional to the pumping amplitude of the laser and inversely proportional to cavity detuning Δ_C.

Because of the very large detuning of the driving laser with respect to all atomic resonances, the atoms scatter photons only in a coherent manner, and the spontaneous emission can be neglected. However, one must take into account that the photons leak out from the cavity through the mirrors and couple to the free-space modes. The loss process can be treated within the Markov approximation with an intensity loss rate of 2κ. The Heisenberg equations of motion of the operators are complemented by a damping term and a corresponding Langevin-type quantum noise term. The Heisenberg-Langevin equations are

$$\partial_t a = i\hbar^{-1}[H,a] - \kappa a + \xi = (i\Delta_C - \kappa)a - i\eta\Delta n + \xi, \tag{2a}$$

$$\partial_t b_j = i\hbar^{-1}[H,b_j] = i\hbar^{-1} J \sum_\varepsilon b_{j+\varepsilon} - i\hbar^{-1} U_s b_j^\dagger b_j b_j$$

$$- i\eta(a^\dagger + a)(-1)^j b_j, \tag{2b}$$

where ∂_t denotes the time derivative and $\Delta n = \sum_j (-1)^j n_j$ is the imbalance between the particle numbers on the even and odd sites. It can be seen from the form of this term that the atoms act collectively on the cavity-mode amplitude. In Eq. (2b), the sum for ε is over the four neighboring sites of site j. The noise term in Eq. (2a) is a δ-correlated white noise with zero mean value: $\langle \xi(t) \rangle = 0$. Furthermore, let us assume that the temperature is very low compared to optical frequencies; that is, normal-ordered correlations are zero. In particular, $\langle \xi^\dagger(t') \xi(t) \rangle = 0$. All other correlations are determined by the bosonic commutation relations,

$$[\xi(t), \xi^\dagger(t')] = 2\kappa \delta(t - t'), \tag{3}$$

in second order. The goal of this paper is to reveal the dynamical consequences of this noise term during the initial short evolution time.

III. ADIABATIC DYNAMICS OF THE ATOMS

Since the timescale of the cavity-field relaxation is the shortest, i.e., $\kappa, \Delta_C \gg J, U_s$, the cavity mode can be slaved to the slow atomic lattice dynamics. Performing such an adiabatic elimination of the cavity dynamics results in a renormalization of the parameters describing the atomic motion in the dynamical lattice. Furthermore, damping and decoherence are introduced into the atomic time evolution. These latter effects are modeled also by a Langevin-like noise, now as part of the atomic motion, which, in general, is not a white noise.

Let us start by integrating out the fast cavity-field equations of motion (2a), while keeping the slow atomic operators constant,

$$a(t) = \frac{\eta \, \Delta n}{\Delta_c + i\kappa} + \Sigma(t), \tag{4a}$$

where the first term is the adiabatic steady state of the photon field, and the noise is

$$\Sigma(t) = i \int \frac{d\omega}{2\pi} \frac{\xi(\omega) e^{-i\omega t}}{\omega + \Delta_C + i\kappa}, \tag{4b}$$

which is a white noise filtered through the cavity mode. The commutation relation can be derived from that of the original equation (3),

$$[\Sigma(t), \Sigma^\dagger(t')] = e^{i\Delta_c(t-t')} e^{-\kappa|t-t'|}. \tag{5}$$

Combining Eqs. (4) with Eq. (2b), we arrive at the adiabatic dynamics

$$\partial_t b_j = i\hbar^{-1} J \sum_\varepsilon b_{j+\varepsilon} - i\hbar^{-1} U_s b_j^\dagger b_j b_j$$
$$- i\eta \frac{2\Delta_C \, \eta \, \Delta n}{\Delta_C^2 + \kappa^2} (-1)^j b_j - i\eta \, R(t) (-1)^j b_j(t), \tag{6}$$

where the last noise term includes the product of atomic operators and the noise $R(t) = \Sigma(t) + \Sigma^\dagger(t)$, which is a self-adjoint operator. The mean value of $R(t)$ is zero, and the second-order correlations are evaluated similarly to Eq. (5) with the following result:

$$\langle R(t) R(t') \rangle = \langle \Sigma(t) \Sigma^\dagger(t') \rangle = e^{i\Delta_c(t-t')} e^{-\kappa|t-t'|}. \tag{7}$$

When the photon decay κ^{-1} is much shorter than the other timescales of the problem, we can approximate Eqs. (5) and (7) by a δ-correlated noise,

$$\langle R(t) R(t') \rangle \approx D_R \, \delta(t - t'), \tag{8}$$

with

$$D_R = \frac{2\kappa}{\Delta_C^2 + \kappa^2}. \tag{9}$$

This approximation corresponds to the broad bandwidth reservoir assumption used for Markovian decay.

We note that the appearance of fluctuations due to the lossy cavity mode is accompanied by nonadiabatic drift terms, the so-called cavity cooling or heating, depending on the sign of the detuning Δ_C [35]. However, this is beyond the adiabatic approximation and can be safely neglected in the limit $\kappa \gg \omega_R$.

IV. MANIFESTATION OF THE MULTIPLICATIVE NOISE IN THE DIFFERENT PHASES

In this section we analyze the effects of the noise created by the leakage of cavity photons on the atomic dynamics. We assume that this effect is small and that the noise can be considered an additional effect on top of the quantum fluctuations arising from the interactions within the many-body system.

The study has to be done separately for the distinct cases corresponding to the different possible thermodynamic phases [18–24]. In fact, the model allows for four different ground-state phases depending on the preservation or breaking of two different symmetries. The first is the global U(1) symmetry associated with particle number conservation, the breaking of which corresponds to the presence of off-diagonal long-range order. The second is the $\mathbb{Z}_2$ symmetry of even and odd lattice sites, which the atom-cavity interaction (1d) can break spontaneously. When the $\mathbb{Z}_2$ symmetry is not broken, all averages on the even and odd lattice sites are equal. When the $\mathbb{Z}_2$ symmetry is broken, the expectation values of operators on the even and odd sublattices can be different. The four phases are the *superfluid phase* with broken U(1) and unbroken $\mathbb{Z}_2$ symmetry, the *supersolid phase* with both symmetries broken, the *Mott phase* with both phases unbroken, and finally, the *charge-density wave (CDW) phase* with broken $\mathbb{Z}_2$ and unbroken U(1) symmetries.

The broken U(1) symmetry phases emerge when the on-site atom-atom interaction is small compared to tunneling, $U_s \ll J$. In this weakly interacting limit the equations of motion of the atomic field operators can be truncated at second order in the usual perturbative way and can be eventually dealt by Bogoliubov transformation [36]. In the other limit, when $U_s \gg J$, one goes to the strongly interacting phases, where U(1) is unbroken. Here the Mott phase and the CDW phase are described after a canonical transformation to the low-energy degrees of freedom [37,38].

A. Superfluid and supersolid phases

In the weakly interacting limit, where $U_s \ll J$, Bose-Einstein condensation occurs for sufficiently low temperatures. The wavefunction of the condensate is proportional to the

expectation value of the atomic ladder operators, where the coefficient is the square root of the average atomic density of the condensate. Close to zero temperature almost all the atoms are condensed, and the condensate density can be approximated by the total density. It is convenient to separate the mean values from the fluctuations,

$$b_i(t) = [\sqrt{n}\beta_i + \delta b_i(t)]e^{-i\mu t/\hbar}, \qquad (10)$$

where $n = N/M$ is the total density of atoms, i.e., the total number of atoms N divided by the total number of sites M. Furthermore, μ plays the role of the chemical potential. When inserting Eq. (10) into Eq. (6) and collecting the terms proportional to $\sqrt{n}$, we arrive at the Gross-Pitaevskii equation (GPE),

$$0 = -J\sum_\varepsilon \beta_{j+\varepsilon} - \mu\beta_j - (-1)^j Y^2\Theta\beta_j + g|\beta_j|^2\beta_j. \quad (11)$$

We introduced the dimensionless parameter $Y^2 = -2\hbar\Delta_C\eta^2 N/(\Delta_C^2 + \kappa^2)$ characterizing the power of the pumping laser and the scaled on-site interaction strength $g = nU_s$. Furthermore, we introduced the supersolid order parameter Θ, which is the difference between the even- and odd-site condensate densities, which are normalized to unity in agreement with Eq. (10),

$$\Theta = \frac{1}{M}\sum_j (-1)^j|\beta_j|^2, \qquad (12)$$

$$1 = \frac{1}{M}\sum_j |\beta_j|^2. \qquad (13)$$

Equations (11), (12), and (13) form a closed set for the chemical potential μ, the supersolid order parameter Θ, and the condensate wavefunction β_j. They have a remarkably simple solution in both the superfluid and supersolid phases. In both phases,

$$\Theta = \sqrt{1 - \frac{z^2 J^2}{(g-\mu)^2}}, \qquad (14)$$

where z is the coordination number of the lattice. For the square lattice, $z = 4$. Furthermore, by introducing $\beta_e = \sqrt{1+\Theta}$ for the even sites and $\beta_o = \sqrt{1-\Theta}$ for the odd sites, the solution becomes

$$\beta_j = \begin{cases} \beta_e & \text{for even } j, \\ \beta_o & \text{for odd } j. \end{cases} \qquad (15)$$

Finally, the following equation has to be fulfilled:
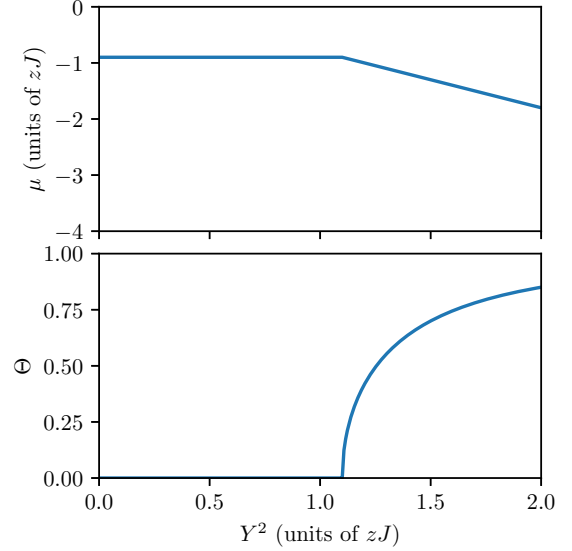
$$0 = \Theta(\mu + Y^2 - 2g). \qquad (16)$$

FIG. 2. The chemical potential and the order parameters as a function of the pump power Y^2. We measure the chemical potential and Y^2 in units of zJ and use $g = 0.1zJ$.

from which either $\Theta = 0$ or $\mu = 2g - Y^2$. The superfluid phase is characterized by $\Theta = 0$, i.e., with a homogeneous condensate $\beta_e = \beta_o = 1$. In this phase, from Eq. (14), we get $\mu = g - zJ$. In the supersolid phase $\mu = 2g - Y^2$, and from Eq. (14), we arrive at $\Theta = [1 - z^2 J^2/(Y^2 - g)^2]^{1/2}$. The critical pumping power separating the two phases is at

$$Y_c^2 = zJ + g. \qquad (17)$$

The chemical potential and the order parameters are plotted in Fig. 2.

Using again the substitution equation (10) in Eq. (6) but now keeping the terms linear in the atomic operators and the noise, we get the equation of motion for the fluctuations,

$$i\hbar\partial_t\delta b_j = -J\sum_\varepsilon \delta b_{j+\varepsilon} - \mu\delta b_j$$
$$- (-1)^j Y^2\left[\Theta\delta b_j + \frac{\beta_j}{M}\sum_k (-1)^k\beta_k(\delta b_k + \delta b_k^\dagger)\right]$$
$$+ g\beta_j^2(2\delta b_j + \delta b_j^\dagger) + (-1)^j\hbar\sqrt{n}\eta R(t)\beta_j. \quad (18)$$

We restrict ourselves to the low-energy excitations and neglect the wave-number dependence of the fluctuations. That is, we keep only the two relevant modes of the symmetry breaking: the fluctuations of the ladder operators of the even and odd sites. The two coupled equations become

$$i\hbar\partial_t\delta b_e = [(g-\mu) + \Theta(g-Y^2)]\delta b_e - Jz\delta b_o + \beta_e^2\left(g - \tfrac{1}{2}Y^2\right)(\delta b_e + \delta b_e^\dagger) + \tfrac{1}{2}Y^2\beta_e\beta_o(\delta b_o + \delta b_o^\dagger) + \sqrt{n}\hbar\eta\beta_e R(t), \quad (19a)$$

$$i\hbar\partial_t\delta b_o = -Jz\delta b_e + [(g-\mu) - \Theta(g-Y^2)]\delta b_o + \tfrac{1}{2}Y^2\beta_e\beta_o(\delta b_e + \delta b_e^\dagger) + \beta_o^2\left(g - \tfrac{1}{2}Y^2\right)(\delta b_o + \delta b_o^\dagger) - \sqrt{n}\hbar\eta\beta_o R(t). \quad (19b)$$

These Bogoliubov equations are constant-coefficient inhomogeneous linear differential equations with a Langevin-type noise added as driving. Another two equations describing the time evolution of $\delta b_e^\dagger$ and $\delta b_o^\dagger$ have to be added to get a closed

set of equations for the fluctuations. These latter two equations are obtained by taking the Hermitian conjugates of Eqs. (19). First, let us gather the fluctuation operators to the formal vector $\mathbf{w} = (\delta b_e, \delta b_o, \delta b_e^\dagger, \delta b_o^\dagger)$. Then, the four Bogoliubov equations

become

$$i\hbar\partial_t \mathbf{w} = M_{4\times 4}\,\mathbf{w} + \Xi(t), \tag{20}$$

where the 4×4 coefficient matrix $M_{4\times 4}$ and the noise vector Ξ can be read off from Eqs. (19). The solution is obtained by finding the normal modes, i.e., diagonalizing the coefficient matrix. The eigenvalues of $M_{4\times 4}$ come in $\pm$ pairs, and two of the four eigenvalues are identically zero in the entire Bose condensed phase [36]. The normal mode with zero eigenvalue, which we call the zero mode, corresponds to phase fluctuations of the condensate. The corresponding eigenvector is proportional to the condensate wavefunction, and therefore, the zero mode does not describe any of the fluctuations orthogonal to the condensate. Furthermore, its operator is anti-Hermitian and is decoupled from the dynamics of the other normal mode with eigenvalues $\pm\lambda$. This latter excitation describes density waves with a period of two lattice sites. At the transition point between

$$M = \begin{pmatrix} 2(g-\mu) + \frac{z^2 J^2}{(g-\mu)^2}(g-Y^2) & \frac{z^2 J^2}{(g-\mu)^2}(g-Y^2) \\ \frac{-z^2 J^2}{(g-\mu)^2}(g-Y^2) & -2(g-\mu) - \frac{z^2 J^2}{(g-\mu)^2}(g-Y^2) \end{pmatrix}, \tag{23}$$

while the effective noise operator is

$$\zeta = \sqrt{2n}\hbar\eta\,\frac{z\,J}{g-\mu}\,R(t). \tag{24}$$

Note that this noise operator inherits the properties of $R(t)$; hence, it is Hermitian, and its correlation is determined by Eq. (8).

The noise leads to a diffusionlike process that depletes the superfluid or supersolid ground states by heating the condensate atoms into the orthogonal fluctuation mode. In the following we calculate the rate by which the atoms leave the Bose condensed state. The matrix M is diagonalized by its real right and left eigenvectors, $M\mathbf{r}^{(k)} = \omega_k \mathbf{r}^{(k)}$ and $(\mathbf{l}^{(k)})^T M = \omega_k (\mathbf{l}^{(k)})^T$. Their scalar product is conveniently normalized, $(\mathbf{l}^{(k)}, \mathbf{r}^{(k)}) = 1$. Multiplying Eq. (22) from the left with $(\mathbf{l}^{(k)})^T$, we obtain the equation of motion of the normal modes $\rho_k = (\mathbf{l}^{(k)}, \mathbf{v})$ that reads

$$i\hbar\partial_t \rho_k = \hbar\omega_k \rho_k + Q_k, \tag{25}$$

where $Q_k(t) = (\mathbf{l}^{(k)}, \Xi'(t))$ is the projection of the noise vector to each mode.

The normal-mode frequencies (eigenvalues of the coefficient matrix M) are $\omega_{1,2} = \pm\hbar^{-1}\lambda$, with

$$\lambda = 2\sqrt{(g-\mu)^2 + z^2 J^2 \frac{g-Y^2}{g-\mu}}. \tag{26}$$

The eigenvalues are plotted in Fig. 3. The corresponding normal modes form a Hermitian adjoint pair $\rho_2 = \rho_1^\dagger$. Their correlation function can be directly calculated from the formal solution of Eq. (25),

$$\langle\rho_k(t)\rho_l(t)\rangle = [\rho_k(0)\rho_l(0)]e^{-i(\omega_k+\omega_l)t} - D_R 2n\eta^2 \frac{z^2 J^2}{(g-\mu)^2}$$

$$\times \frac{1 - e^{-i(\omega_k+\omega_l)t}}{i(\omega_k+\omega_l)}\sum_{i,j}(-1)^{i+j}l_i^{(k)}l_j^{(l)}. \tag{27}$$

the superfluid and the supersolid phases, this excitation also becomes soft; therefore, we call it the soft mode. In other words, we introduce δb_z and δb_s, corresponding to the zero and soft modes, respectively, as

$$(\delta b_e, \delta b_o) = \boldsymbol{\gamma}\,\delta b_z + \boldsymbol{\varepsilon}\,\delta b_s, \tag{21}$$

where $\boldsymbol{\gamma} = (\beta_e, \beta_o)/\sqrt{2}$ and $\boldsymbol{\varepsilon} = (\beta_o, -\beta_e)/\sqrt{2}$ are orthogonal vectors. These two vectors span the two-dimensional space, from which only the direction orthogonal to the condensate, i.e., the soft mode, is relevant for us. The soft mode decouples from the purely anti-Hermitian zero mode, and its Bogoliubov equation is obtained directly from Eq. (20) simply by taking the appropriate submatrix of $M_{4\times 4}$ and subvector of Ξ,

$$i\hbar\partial_t \mathbf{v} = M\,\mathbf{v} + \Xi'(t), \tag{22}$$

with the reduced vector $\mathbf{v} = (\delta b_s, \delta b_s^\dagger)^T$ and noise $\Xi'(t) = (\zeta, -\zeta)^T$. The 2×2 coefficient matrix reads

The fluctuation number is expressed with the normal modes

$$\langle\delta b_s^\dagger \delta b_s\rangle = \sum_{k,l}\langle\rho_k\rho_l\rangle r_2^{(k)} r_1^{(l)}. \tag{28}$$

The summation over the eigenvector components gives a factor of -1. According to Eq. (27), the system obeys an exponential relaxation law towards the steady state when the eigenvalue λ has an imaginary part, such as in Ref. [39]. This finding is similar to what was found in Ref. [32] in the bad-cavity limit and for the case of vanishing on-site interactions. For short times, $t \ll \lambda^{-1}$, the incoherent population, Eq. (28), builds up linearly in time; hence, the rate characterizing the diffusion is

$$\frac{\langle\delta b_s^\dagger \delta b_s\rangle}{t} = D_R 2n\eta^2 \frac{z^2 J^2}{(g-\mu)^2} = \frac{2\kappa z J}{M\hbar|\Delta_C|}\frac{Y^2 z J}{(g-\mu)^2}. \tag{29}$$
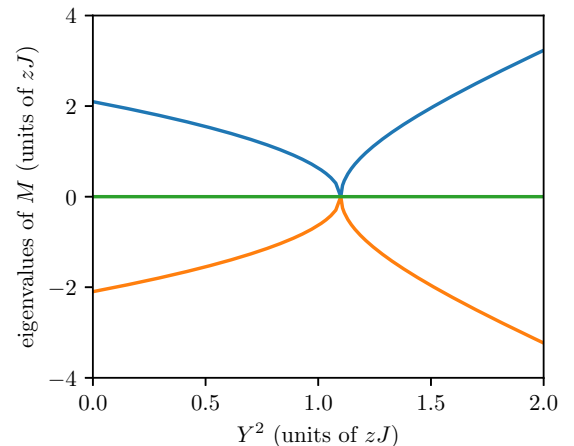
FIG. 3. The eigenvalues of the Bogoliubov problem as a function of the pumping power Y^2.
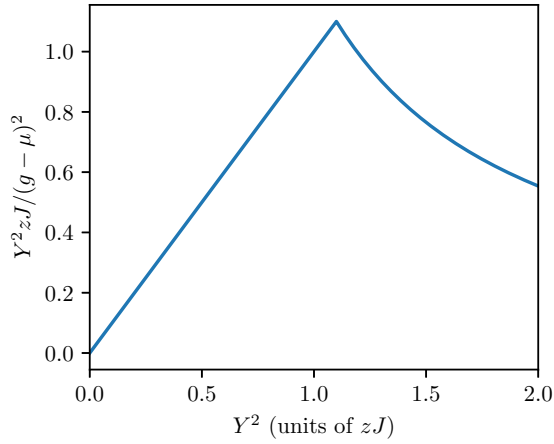
FIG. 4. The effective diffusion rate (29), by which the atoms leave the superfluid and supersolid ground states. More precisely, we plot the dimensionless quantity $Y^2 z J/(g - \mu)^2$ as a function Y^2. The diffusion rate exhibits a cusp at the transition point between the superfluid and supersolid phases.

The right-hand side of Eq. (29) is written as a product of two factors. The first one sets the dimension and the order of magnitude of the diffusion, while the second factor is dimensionless and is on the order of unity. This latter quantity is plotted in Fig. 4. In the superfluid phase it is proportional to Y^2, while in the supersolid phase it goes to zero according to $1/Y^2$. The first factor has the dimension of s^{-1} and is a combination of three different frequency scales: the photon escape rate κ, the magnitude of the cavity detuning $|\Delta_C|$, and the optical-lattice tunneling rate $z J/\hbar$. In the experiments, the first two of the three frequencies are usually chosen to be close to each other, while $z J$ is much smaller than the other two. Furthermore, the number of lattice sites illuminated by the cavity mode is on the order of a thousand ($M \sim 10^3$). Therefore, we estimate $2\kappa z J/(M\hbar|\Delta_C|) \sim 10^{-2} z J$–$10^{-3} z J$. Clearly, this magnitude of the diffusion process allows for considering the tunneling dynamics to be coherent on a long enough timescale. Notably, some of the technical noise sources can be more restrictive in this respect. However, one must be aware that with the cavity detuning Δ_C going to zero, the photon outcoupling process can generate a large number of photons within the cavity. These results are in complete agreement with the model without the additional optical lattice [40]; the noise leads to the departure of the system from its quantum ground state, and for long times it relaxes exponentially to a steady state with an excess noise depletion of the condensate [41] in the cavity-cooling regime, while it relaxes exponentially to the infinite-temperature state in the cavity-heating case.

B. Mott and CDW phases

When the strength of the on-site repulsion is much larger than the amplitude of the hopping ($U \gg J$), multiple occupations of a site become energetically penalized. For commensurate fillings, i.e., when the number of atoms is an integer times the number of lattice sites, each lattice site becomes populated with exactly the same number of atoms. At the same time, particle number fluctuations become suppressed at each

individual site. Depending on the strength of the transverse laser driving, the system can be in a Mott state or in a CDW state. For weak pumping, the Mott state is realized, where each site has the same number of particles. In contrast, when the pumping strength is large, the CDW state is the ground state, where all even sites have the same occupation and all odd sites have the same occupation but the two are different. In both the Mott and CDW phases, the low-energy excitations are no longer the Bogoliubov quasiparticles but, rather, particles and holes of a strongly correlated system. These quasiparticle excitations have to be introduced separately for each phase. To be specific, we are going to study the excitations over the Mott state with one particle per site and over the CDW state with one particle per two sites. The generalization of the theory to other Mott and CDW states is straightforward, and the conclusions do not change qualitatively. For a more transparent presentation, we also completely neglect the effect of the kinetic energy, which can be reintroduced with the help of perturbation theory, as was done for the Mott-superfluid transition in Refs. [37,38].

Without tunneling, the Hamiltonian (1) becomes

$$H = H_{\text{ph}} + \sum_j \left[\frac{U_s}{2} n_j(n_j - 1) - \mu n_j \right.$$
$$\left. + \hbar\eta(a^\dagger + a)(-1)^j n_j \right]. \quad (30)$$

In the one particle per site Mott lobe, we follow the route of Ref. [37] and truncate the single-site Hilbert space for the three lowest occupied states, $|0\rangle_j, |1\rangle_j, |2\rangle_j$. We introduce the three Schwinger boson operators creating these states,

$$|1\rangle_j = t_{1,j}^\dagger |\text{vac}\rangle, \quad (31a)$$

$$|0\rangle_j = t_{0,j}^\dagger |\text{vac}\rangle, \quad (31b)$$

$$|2\rangle_j = t_{2,j}^\dagger |\text{vac}\rangle. \quad (31c)$$

The state $|\text{vac}\rangle$ is a fictitious vacuum state, not present in our original Hilbert space. In order to exclude the unphysical states, these Schwinger bosons must share a single excitation at each site; therefore, we have the constraint

$$t_{0,j}^\dagger t_{0,j} + t_{1,j}^\dagger t_{1,j} + t_{2,j}^\dagger t_{2,j} = 1. \quad (32)$$

In the present case, the ground state is a tensor product state of the singly occupied sites,

$$|\text{GS}\rangle = \prod_j |1\rangle_j. \quad (33)$$

That is, in the ground state, the occupation of the t_1 boson is 1 at each site, while the other bosons have zero occupations. Low-energy excited states have these other bosonic excitations mixed in with some amplitude much smaller than unity, while the $t_{1,j}$ bosons still have an occupation very close to unity. Thus, by using Eq. (32), the Hamiltonian can be written as

$$H = H_{\text{ph}} + \sum_j [\tilde{\mu}_j t_{0,j}^\dagger t_{0,j} + (U - \tilde{\mu}_j) t_{2,j}^\dagger t_{2,j} - \tilde{\mu}_j], \quad (34)$$

with

$$\tilde{\mu}_j = \mu - \hbar\eta(-1)^j(a^\dagger + a). \quad (35)$$

Equation (34) is quadratic in the excitations, i.e., both the hole-type excitations created by $t_{0,j}^\dagger$ and the particlelike excitations of $t_{2,j}^\dagger$. In the Mott phase, the effect of the photon field is through renormalizing the (chemical) potential. Integrating out the photon field, we still have Eqs. (4), but now with

$$\Delta n = \sum_j (-1)^j (1 - t_{0,j}^\dagger t_{0,j} + t_{2,j}^\dagger t_{2,j}). \tag{36}$$

To lowest order in the single-particle-per-site Mott phase, we can approximate $\Delta n \approx \sum_j (-1)^j = 0$. That is, in the Mott phase there is no classical part of the cavity field $a_{ss} = 0$. Therefore, the chemical potential is renormalized only by a noise,

$$\tilde{\mu}_j = \mu - \hbar\eta(-1)^j R(t). \tag{37}$$

Therefore, in the Mott phase, for zero tunneling, the effect of cavity decay is to make the chemical potential noisy. As the Mott phase is gapped, there is no effect of the noise on the ground state while we stay away from the phase boundary. The fluctuation of the chemical potential affects, on the other hand, the particle- and holelike excitations of the Mott phase: the corresponding quasiparticle resonances get broadened by the amount of

$$\Delta = \frac{2\kappa\eta^2}{\kappa^2 + \Delta_C^2}. \tag{38}$$

When a small enough tunneling is introduced, the situation can be more complicated, as the noise can also dress the quasiparticles. This effect must be small, as it has to vanish for zero tunneling. In Ref. [32] the authors found anomalous and normal diffusion towards the steady state similar to the standard optical lattice Mott insulator [42].

The present analysis was performed for the first Mott sate, i.e., the one with $n = 1$ particle per site. This analysis can simply be generalized to any of the Mott states with n particles per site by keeping the most relevant three states, namely, $|n - 1\rangle_j$, $|n\rangle_j$, and $|n + 1\rangle_j$, at each site. Equations (33)–(36) have to be modified, but in a straightforward manner. Nevertheless, the final result for the noisy chemical potential, Eq. (37), holds for each Mott state.

In order to generalize the previous analysis further to the CDW phase, we need to introduce two sublattices in a checkerboard setting. One sublattice has n_e particles on each site, while on the other sublattice each site has n_o atoms. We shall refer to these sublattices as the even and odd sublattices, respectively. A unit cell now contains two neighboring sites, each from a different sublattice. The Hamiltonian reads

$$H = H_{\rm ph} + \sum_m \left\{ \frac{U_s}{2}[n_{m,e}(n_{m,e} - 1) + n_{m,o}(n_{m,o} - 1)] \right.$$
$$\left. - \mu(n_{m,e} + n_{m,o}) + \hbar\eta(a + a^\dagger)(n_{m,e} - n_{m,o}) \right\}. \tag{39}$$

The sum goes over the unit cells indexed by m. The number of unit cells is half the number of sites. For simplicity we work with the $n_e = 1$ and $n_o = 0$ CDW phase. As the occupation number of each site on the even sublattice is unity, we use the same three states and the same three Schwinger bosons as in Eqs. (31). On the odd sublattice, we have no atoms at all

in the ground state, and we use only two Schwinger bosons corresponding to the empty and singly occupied sites. That is, if site j is the even site in unit cell m,

$$|1\rangle_{m,e} = t_{1,m}^\dagger|{\rm vac}\rangle, \quad |0\rangle_{m,e} = t_{0,m}^\dagger|{\rm vac}\rangle, \quad |2\rangle_{m,e} = t_{2,m}^\dagger|{\rm vac}\rangle. \tag{40a}$$

However, if j is on the odd sublattice inside unit cell m,

$$|0\rangle_m = s_{0,m}^\dagger|{\rm vac}\rangle, \quad |1\rangle_m = s_{1,m}^\dagger|{\rm vac}\rangle. \tag{40b}$$

We have two constraints for each unit cell, namely,

$$t_{0,m}^\dagger t_{0,m} + t_{1,m}^\dagger t_{1,m} + t_{2,m}^\dagger t_{2,m} = 1, \tag{41a}$$

$$s_{0,m}^\dagger s_{0,m} + s_{1,m}^\dagger s_{1,m} = 1. \tag{41b}$$

Now Eq. (39) is expressed with the new Schwinger bosons, also using the constraints (41),

$$H = H_{\rm ph} + \sum_m \left\{ -(\mu + \Delta\mu)s_{1,m}^\dagger s_{1,m} + (\mu - \Delta\mu)t_{0,m}^\dagger t_{0,m} \right.$$
$$\left. + [U_s - (\mu - \Delta\mu)]t_{2,m}^\dagger t_{2,m} - (\mu - \Delta\mu) \right\}, \tag{42}$$

with

$$\Delta\mu = \hbar\eta(a + a^\dagger) = \frac{2\hbar\eta^2\Delta_C}{\Delta_C^2 + \kappa^2}\Delta n + \hbar\eta R, \tag{43}$$

where the photon field is adiabatically eliminated in order to reach the final result. In this phase, $\Delta n = N$ since the odd sites are empty, and $\Delta\mu$ is negative (note that Δ_C is negative). In fact, this CDW phase is stable while the energy of the $s_{1,m}$ particle excitations on the odd sites is positive, i.e., while $|\Delta\mu| > \mu$. Below that, the system relaxes to the $n = 1$ Mott phase. Notice that the noise term randomly pulls $\Delta\mu$, and the boundary of the phase gets smoothed out.

V. DISCUSSION AND SUMMARY

In this paper, we studied the dynamics of a lattice Bose gas, which apart from the external optical lattice, is immersed in the optical field of a single-mode high-Q Fabry-Pérot interferometer. The wavelength of the resonator is tuned close to that of the optical lattice in order to allow the lattice gas to form a grating. This atomic grating can either scatter light into the cavity mode with constructive interference, thus allowing a classical cavity field to build up, or scatter the light destructively, thereby destroying the cavity field. There is a phase transition in between these two regimes separating two phases with different lattice periodicity.

We derived analytical expressions for the superfluid-supersolid phase transition in the weak-coupling limit and an analytical formula for the excitation energy of the critical density wave. In the experimental realization of the cavity Bose-Hubbard model, the optical resonator is an inherently open quantum system. Photons from the laser pump are scattered into the cavity and finally leak out through the cavity mirrors. The corresponding dissipation process leads to a quantum noise that can excite the system out of its ground states. A fundamental question is what time limitation this intrinsic quantum process imposes on the validity of the

ground-state description, which is substantial to all studies relying on a Hamiltonian description of the cavity Bose-Hubbard model. We calculated analytically the effects of photon-loss dissipation in the superfluid and Mott phases. We showed that the dissipative quantum noise has substantially different effects in the two phases. In the superfluid phase, the noise heats the system out from its ground state with the timescale $2\kappa z J/(M\hbar|\Delta_C|)$. This timescale is a slow one except for very small detunings.

In the Mott-type phase the gap protects the population in the ground state against the photon field fluctuation noise and allows only a much slower approach towards equilibrium.

However, the noise blurs also the phase boundaries and affects the excitation spectrum by the quasiparticle resonance broadening by an amount of $2\kappa\,\eta^2/(\kappa^2 + \Delta_C^2)$.

[1] S. Nussmann, M. Hijlkema, B. Weber, F. Rohde, G. Rempe, and A. Kuhn, Phys. Rev. Lett. **95**, 173602 (2005).

[2] M. Khudaverdyan, W. Alt, I. Dotsenko, T. Kampschulte, K. Lenhard, A. Rauschenbeutel, S. Reick, K. Schörner, A. Widera, and D. Meschede, New J. Phys. **10**, 073023 (2008).

[3] R. J. Thompson, G. Rempe, and H. J. Kimble, Phys. Rev. Lett. **68**, 1132 (1992).

[4] C. Hamsen, K. N. Tolazzi, T. Wilk, and G. Rempe, Phys. Rev. Lett. **118**, 133604 (2017).

[5] A. Kubanek, A. Ourjoumtsev, I. Schuster, M. Koch, P. W. H. Pinkse, K. Murr, and G. Rempe, Phys. Rev. Lett. **101**, 203602 (2008).

[6] H. Ritsch, P. Domokos, F. Brennecke, and T. Esslinger, Rev. Mod. Phys. **85**, 553 (2013).

[7] K. Baumann, C. Guerlin, F. Brennecke, and T. Esslinger, Nature (London) **464**, 1301 (2010).

[8] K. Baumann, R. Mottl, F. Brennecke, and T. Esslinger, Phys. Rev. Lett. **107**, 140402 (2011).

[9] J. Klinder, H. Kessler, M. Wolke, L. Mathey, and A. Hemmerich, Proc. Natl. Acad. Sci. USA **112**, 3290 (2015).

[10] F. Brennecke, S. Ritter, T. Donner, and T. Esslinger, Science **322**, 235 (2008).

[11] R. Landig, L. Hruby, N. Dogra, M. Landini, R. Mottl, T. Donner, and T. Esslinger, Nature (London) **532**, 476 (2016).

[12] J. Klinder, H. Kessler, M. R. Bakhtiari, M. Thorwart, and A. Hemmerich, Phys. Rev. Lett. **115**, 230403 (2015).

[13] I. B. Mekhov and H. Ritsch, J. Phys. B **45**, 102001 (2012).

[14] T. J. Elliott, W. Kozlowski, S. F. Caballero-Benitez, and I. B. Mekhov, Phys. Rev. Lett. **114**, 113604 (2015).

[15] W. Kozlowski, S. F. Caballero-Benitez, and I. B. Mekhov, Sci. Rep. **7**, 42597 (2017).

[16] S. Fernández-Vidal, G. De Chiara, J. Larson, and G. Morigi, Phys. Rev. A **81**, 043407 (2010).

[17] Y. Li, L. He, and W. Hofstetter, Phys. Rev. A **87**, 051604 (2013).

[18] N. Dogra, F. Brennecke, S. D. Huber, and T. Donner, Phys. Rev. A **94**, 023632 (2016).

[19] Y. Chen, Z. Yu, and H. Zhai, Phys. Rev. A **93**, 041601 (2016).

[20] A. E. Niederle, G. Morigi, and H. Rieger, Phys. Rev. A **94**, 033607 (2016).

[21] B. Sundar and E. J. Mueller, Phys. Rev. A **94**, 033631 (2016).

[22] J. Panas, A. Kauch, and K. Byczuk, Phys. Rev. B **95**, 115105 (2017).

[23] T. Flottat, L. de Forges de Parny, F. Hébert, V. G. Rousseau, and G. G. Batrouni, Phys. Rev. B **95**, 144501 (2017).

[24] R. Liao, H.-J. Chen, D.-C. Zheng, and Z.-G. Huang, Phys. Rev. A **97**, 013624 (2018).

[25] B. Blass, H. Rieger, G. Rósz, and F. Iglói, arXiv:1711.10961.

[26] S. F. Caballero-Benitez, G. Mazzucchi, and I. B. Mekhov, Phys. Rev. A **93**, 063632 (2016).

[27] E. G. Dalla Torre, Y. Shchadilova, E. Y. Wilner, M. D. Lukin, and E. Demler, Phys. Rev. A **94**, 061802 (2016).

[28] W. Zheng and N. R. Cooper, Phys. Rev. A **97**, 021601 (2018).

[29] C. Kollath, A. Sheikhan, S. Wolff, and F. Brennecke, Phys. Rev. Lett. **116**, 060401 (2016).

[30] A. Sheikhan, F. Brennecke, and C. Kollath, Phys. Rev. A **93**, 043609 (2016).

[31] F. Mivehvar, H. Ritsch, and F. Piazza, Phys. Rev. Lett. **118**, 073602 (2017).

[32] E. I. Rodríguez Chiacchio and A. Nunnenkamp, Phys. Rev. A **97**, 033618 (2018).

[33] M. P. A. Fisher, P. B. Weichman, G. Grinstein, and D. S. Fisher, Phys. Rev. B **40**, 546 (1989).

[34] D. Jaksch, C. Bruder, J. I. Cirac, C. W. Gardiner, and P. Zoller, Phys. Rev. Lett. **81**, 3108 (1998).

[35] D. Nagy, P. Domokos, A. Vukics, and H. Ritsch, Eur. Phys. J. D **55**, 659 (2009).

[36] Y. Castin, in *Coherent Atomic Matter Waves*, edited by R. Kaiser, C. Westbrook, and F. David (Springer, Berlin, 2001), pp. 1–136.

[37] E. Altman and A. Auerbach, Phys. Rev. Lett. **89**, 250404 (2002).

[38] S. D. Huber, E. Altman, H. P. Büchler, and G. Blatter, Phys. Rev. B **75**, 085106 (2007).

[39] G. Szirmai, D. Nagy, and P. Domokos, Phys. Rev. A **81**, 043639 (2010).

[40] D. Nagy, G. Kónya, G. Szirmai, and P. Domokos, Phys. Rev. Lett. **104**, 130401 (2010).

[41] G. Szirmai, D. Nagy, and P. Domokos, Phys. Rev. Lett. **102**, 080401 (2009).

[42] D. Poletti, P. Barmettler, A. Georges, and C. Kollath, Phys. Rev. Lett. **111**, 195301 (2013).