Infinitesimal multi-mode Bargmann-state representation[a]

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We construct a representation of the Hilbert-space of a light mode (harmonic oscillator) in which an arbitrary state vector is expanded using Bargmann states $\|\alpha\rangle$ with real parameters α which are in an infinitesimal vicinity of zero. The complete Hilbert-space structure is represented in the one- and multimode case as well, making the representation able to deal with problems of continuous-variable quantum information processing.

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[a] Dedicated to the memory of our former supervisor József Janszky, who was the initiator and master of coherent-state representations in reduced dimensions.

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I. INTRODUCTION

Coherent states of the light field are of central importance in quantum optics. One significant application is to expand an arbitrary state vector of the Hilbert space in terms of coherent states. The coherent states forming an overcomplete set in the Hilbert space, such an expansion is by no means unique. A very convenient representation was given by Glauber [1] in which the expansion of a general state includes all the coherent states of the two-dimensional complex plane. This so-called *analytic* representation was used to investigate various aspects of (nonideal) continuous-variable quantum teleportation [2, 3]. A reduced set of coherent states in the complex plane, such as a straight line [4, 5] or a circle around the origin [6] is still an overcomplete set suitable to build a representation.

Our present work is intimately connected in intention to both Glauber's work and the one presented in [7] where the authors construct a coherent-state representation using only coherent states $|\alpha\rangle$ with real parameter α which are in an infinitesimal vicinity of zero. Here, to obtain a representation which is more convenient for calculations, we use Bargmann states as basis set in the stead of coherent states. This results in a suprisingly compact and elegant form for the expansion of an arbitrary state.

The Glauber function is shown to play the chief role in the representation. We give the formula for scalar product and projection, transforming the complete Hilbert-space algebra to the level of Glauber functions. As an important example we give the representation of the photon-number entangled state. An example for application is presented: using the infinitesimal representation we calculate the output of continuous-variable quantum teleportation in the general case of arbitrary input state.

II. OVERVIEW

Recall that Bargmann states are defined as

$$\|\alpha\rangle := e^{\frac{|\alpha|^2}{2}}\,|\alpha\rangle = \sum_n \frac{\alpha^n}{\sqrt{n!}}\,|n\rangle\,, \tag{1}$$

and are analytic functions of α.

As it is well known, for the states of a harmonic oscillator (mode) one can construct Glauber's analytic representation. Let us first consider the oscillator Fock state $|n\rangle$. It is easy to verify by a straightforward integration that

$$|n\rangle = \frac{1}{\pi}\int d^2\alpha\, e^{-|\alpha|^2}\,\frac{(\alpha^*)^n}{\sqrt{n!}}\,\|\alpha\rangle\,, \tag{2}$$

and therefore a general state

$$|\Phi\rangle =: \sum_n c_n\,|n\rangle \tag{3}$$

can be written as

$$|\Phi\rangle = \frac{1}{\pi}\int d^2\alpha\, e^{-|\alpha|^2} f(\alpha^*)\,\|\alpha\rangle\,, \tag{4}$$

where $f(\alpha^*)$ is an analytic function uniquely defined as

$$f(\alpha^*) := \sum_n c_n \frac{(\alpha^*)^n}{\sqrt{n!}} = \langle\alpha\|\Phi\rangle. \tag{5}$$

This analytic representation can easily be generalized for two modes. A general two mode state

$$|\Psi\rangle =: \sum_{m,n} C_{m,n}\,|m\rangle\,|n\rangle \tag{6}$$

can be written as

$$|\Psi\rangle = \frac{1}{\pi^2}\int d^2\alpha \int d^2\beta\, e^{-|\alpha|^2-|\beta|^2} F(\alpha^*,\beta^*)\,\|\alpha\rangle\,\|\beta\rangle\,, \tag{7}$$

with $F(\alpha^*,\beta^*)$ an analytic function uniquely defined as

$$F(\alpha^*,\beta^*) = \sum_{m,n} C_{m,n}\frac{(\alpha^*)^m(\beta^*)^n}{\sqrt{m!n!}} = \langle\alpha\|\langle\beta\|\Psi\rangle. \tag{8}$$

III. INFINITESIMAL REPRESENTATION

A. The one-mode case

At first the infinitesimal representation will be introduced for one mode. The Hilbert space of the mode will be denoted by $\mathcal{H}$. Let us consider the following function:

$$\gamma : \mathbb{R} \to \mathcal{H};\ x \mapsto \|x\rangle\,, \tag{9}$$

which is an analytic function on the whole $\mathbb{R}$, because

$$\gamma = \sum_{n=0}^{\infty}\frac{(\mathrm{id}_{\mathbb{R}})^n}{\sqrt{n!}}\,|n\rangle\,, \tag{10}$$

where $\mathrm{id}_{\mathbb{R}}$ is the $\mathbb{R} \to \mathbb{R}$ identity function. Note that $\mathrm{id}_{\mathbb{R}}$ stands for a *function*, while $\mathrm{id}_{\mathbb{R}}(x) = x$ is the value taken up by the function at the point $x \in \mathbb{R}$.

Our infinitesimal representation is based on the derivatives of γ at zero. It is a straightforward matter to see that the first derivative

$$(D\gamma)(0) = \left(\sum_{n=0}^{\infty} \frac{n(\mathrm{id}_{\mathbb{R}})^{n-1}}{\sqrt{n!}} \left| n \right\rangle \right)(0) = \left| 1 \right\rangle . \tag{11}$$

(Here again, $D\gamma$ is the $\mathbb{R} \to \mathcal{H}$ derivative *function* of γ, while $(D\gamma)(0)$ is the derivative of γ at the point $0 \in \mathbb{R}$.) A similar calculation yields that

$$\left| n \right\rangle = \frac{(D^n\gamma)(0)}{\sqrt{n!}}, \tag{12}$$

and therefore an arbitrary one-mode state can be written in an extremely compact form as

$$\left| \Phi \right\rangle = \left(f\left(D\right) \gamma \right)(0) \tag{13}$$

with the same unique analytic function as the one appearing in Glauber's analytic representation.

To see why we call this representation infinitesimal, let us take the Δ distribution, which is a linear operator acting on functions $\varphi : \mathbb{R} \to \mathcal{H}$ as

$$\Delta\varphi := \int dx\, \delta(x)\varphi(x) = \varphi(0), \tag{14}$$

hence

$$\Delta\gamma = \int dx\, \delta(x) \left\| x \right\rangle = \left| 0 \right\rangle . \tag{15}$$

Mind that the support of the function γ is not bounded so it is *not* a test function for general distributions, still, Eq. (15) makes sense. We recall that the derivatives of distributions are *defined* by the law of partial integration:

$$(D\Delta)\varphi = \int dx \left(D\delta \right)(x)\varphi(x) :=$$

$$:= -\int dx\delta(x)(D\varphi)(x) = -\Delta(D\varphi). \tag{16}$$

Using this definition it is easy to see that Eq. (13) can also be written in the form

$$\left| \Phi \right\rangle = \left(f\left(-D\right) \Delta \right) \gamma = \int dx \left(f\left(-D\right) \delta \right)(x) \left\| x \right\rangle . \tag{17}$$

Hence it becomes clear that $\left| \Phi \right\rangle$ is expanded using only the Bargmann states in an infinitesimal interval around the vacuum, and the energy is stored in the superposition itself, since all the amplitudes which we superpose are infinitesimal.

Using the above integral expression it is straightforward to give the scalar product of two states in quite an elegant form:

$$\begin{aligned}\langle \Psi \mid \Phi \rangle &= \\ &= \int dx \int dy \, (f^*_{|\Psi\rangle}(-D)\delta)(y) \, (f_{|\Phi\rangle}(-D)\delta)(x) \, \langle y \| x \rangle = \\ &= \int dx \int dy \, (f^*_{|\Psi\rangle}(-D)\delta)(y) \, (f_{|\Phi\rangle}(-D)\delta)(x) \, e^{xy}. \end{aligned} \tag{18}$$

Integrating partially in x and y we obtain:

$$\begin{aligned}\langle \Psi \mid \Phi \rangle &= \\ &= \int dx \int dy \, \delta(y)\delta(x) \, f^*_{|\Psi\rangle}(\partial_y) f_{|\Phi\rangle}(\partial_x) \, e^{xy} = \\ &= \int dx \int dy \, \delta(y)\delta(x) \, f^*_{|\Psi\rangle}(\partial_y) f_{|\Phi\rangle}(y) \, e^{xy} = \\ &= (f^*_{|\Psi\rangle}(D) f_{|\Phi\rangle})(0). \end{aligned} \tag{19}$$

Note that the formula for the nth Fock state and hence the formula for a general state would not be so simple had we tried to construct an infinitesimal *coherent-state* representation [7], because in this case we should mind the derivatives of the normalization factor $e^{-\frac{\mathrm{id}^2_{\mathbb{R}}}{2}}$.

B. The multi-mode case

The representation is readily generalized to the case when we have more than one modes, by taking the direct product of one-dimensional representations.

As an example let us consider the two-mode case. The representation is now based on the function

$$\gamma^{(2)} : \mathbb{R} \times \mathbb{R} \to \mathcal{H} \times \mathcal{H}; \; (x_1, x_2) \mapsto \|x_1\rangle \, \|x_2\rangle \,, \tag{20}$$

and a state $|\Psi\rangle$ is represented as

$$|\Psi\rangle = \left(f\left(\partial_1, \partial_2\right) \gamma^{(2)} \right)(0). \tag{21}$$

In a very similar way as in Eqs. (18) and (19) it is easy to see that the scalar product of two states

$|\Psi\rangle$ and $|\Phi\rangle$ can now be expressed in four different ways as

$$\langle\Psi \mid \Phi\rangle = \Big(f^*_{|\Psi\rangle}(\partial_1,\partial_2) f_{|\Phi\rangle}(\,.\,,.)\Big)(0) = \tag{22a}$$

$$= \Big(f_{|\Phi\rangle}(\partial_1,\partial_2) f^*_{|\Psi\rangle}(\,.\,,.)\Big)(0) = \tag{22b}$$

$$= \Big(f^*_{|\Psi\rangle}(\partial_1,.) f_{|\Phi\rangle}(\,.\,,\partial_2)\Big)(0) = \tag{22c}$$

$$= \Big(f^*_{|\Psi\rangle}(\,.\,,\partial_2) f_{|\Phi\rangle}(\partial_1,\,.)\Big)(0). \tag{22d}$$

This feature may be exploited for calculational convenience in some situations.

As an example for this representation let us consider a two-mode state of great importance in quantum information science, the photon-number entangled state:

$$|\Psi_{\mathrm{EPR}}\rangle = \sqrt{1-s^2}\sum_n s^n\,|n\rangle\,|n\rangle\,. \tag{23}$$

For this state it is easy to find that the analytic representation is given by the function $f_{\mathrm{EPR}}(\alpha^*,\beta^*) = \sqrt{1-s^2}\exp\left(s\,\alpha^*\beta^*\right)$, so that in infinitesimal representation the state reads

$$|\Psi_{\mathrm{EPR}}\rangle = \Big(\sqrt{1-s^2}\exp\left(s\,\partial_1\partial_2\right)\gamma^{(2)}\Big)(0). \tag{24}$$

IV. CONTINUOUS TELEPORTATION

As an important application of the infinitesimal representation let us consider the continuous teleportation in the setup first presented independently in [8] and [9]. Three modes are involved in the scheme the representation of which is therefore based on the function

$$\gamma^{(3)}: \mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathcal{H}\times\mathcal{H}\times\mathcal{H};$$

$$(x_1,x_2,x_3) \mapsto \|x_1\rangle\,\|x_2\rangle\,\|x_3\rangle\,. \tag{25}$$

Mode 1 contains an arbitrary state, with Glauber function f_{in}, to be teleported, while mode 2 and 3 contains the entangled resource usually considered to be the state (23).

So the initial state of the three modes up to a normalization factor reads

$$|\Psi_{\mathrm{i}}\rangle \propto \Big(f_{\mathrm{in}}(\partial_1)\exp\left(s\,\partial_2\partial_3\right)\gamma^{(3)}\Big)(0). \tag{26}$$

The Bell measurement projects the state of modes 1 and 2 on the state [2]

$$|\Psi_{\mathrm{B}}(A)\rangle_{12} = \frac{1}{\pi}\int_{\mathbb{C}} \mathrm{d}^2\lambda\; e^{\lambda^*A-\lambda A^*}\,|\lambda+A\rangle_1\,|\lambda^*-A^*\rangle_2 \tag{27}$$

where A is the result of the measurement. Using Eq. (8), the Glauber function of this state is found to be

$$f_{\mathrm{B}}(\alpha^*,\beta^*) = \langle\alpha\|\langle\beta\|\Psi_{\mathrm{B}}(A)\rangle = \\ = \exp(2(A\alpha^* - A^*\beta^*) + \alpha^*\beta^*). \quad (28)$$

To obtain this result, one has to apply Glauber's useful identity:

$$f(\beta^*) = \frac{1}{\pi}\int \mathrm{d}^2\lambda\, f(\lambda^*)\, e^{-|\lambda|^2+\lambda\beta^*}. \quad (29)$$

This identity holds for analytic $f:\mathbb{C}\to\mathbb{C}$ functions, and is easy to prove by writing the integrational variable in polar coordinates as $\lambda = r\,e^{i\phi}$ and integrating over the real variables r and ϕ.

The output of the teleportation is given as the projection:

$$|\Psi_{\mathrm{out}}\rangle \propto \langle\Psi_{\mathrm{B}}\mid\Psi_{\mathrm{i}}\rangle\,, \quad (30)$$

which is readily calculated using Eq. (22b):

$$|\Psi_{\mathrm{out}}\rangle \propto f_{\mathrm{in}}(\partial_1)\,\exp\left(s\,\partial_2\partial_3\right)\times \\ \times \exp(2(A^*x_1 - Ax_2) + x_1x_2)\,\gamma(x_3)\Big|_{x_1,x_2,x_3=0}. \quad (31)$$

This expression can be simplified using the following straightforward identity

$$f(\partial_1)e^{x_1x_2}\Big|_{x_1=0} = f(x_2) \quad (32)$$

for any $f:\mathbb{R}\to\mathbb{R}$ analytic function. We obtain

$$|\Psi_{\mathrm{out}}\rangle \propto \\ \propto f_{\mathrm{in}}(\partial_1)\,\exp\left((x_1-2A)s\partial_3 + 2A^*x_1\right)\,\gamma(x_3)\Big|_{x_1,x_3=0} = \\ = f_{\mathrm{in}}(s\partial_3 + 2A^*)\,\exp\left(-2As\partial_3\right)\,\gamma(x_3)\Big|_{x_3=0}, \quad (33)$$

whence the Glauber function of the output state can be read as

$$f_{\mathrm{out}}(\alpha^*) = f_{\mathrm{in}}(s\alpha^* + 2A^*)\,\exp\left(-2As\alpha^*\right). \quad (34)$$

To verify this result let us take Eq. (9) of [3]:

$$|\Psi_{\mathrm{out}}\rangle \propto \\ \propto \frac{1}{\pi}\int \mathrm{d}^2\lambda\; f(\lambda^* + 2A^*)\; e^{-\left(\frac{1}{g^2}+\frac{1}{2}\right)|\lambda|^2 - 2\lambda^* A}\,|\lambda\rangle_3\,. \quad (35)$$

which is the output state of the same teleportation process with

$$g = \sqrt{\frac{s}{1-s}}. \tag{36}$$

Now we calculate the Glauber function of this state, which, of course, can not be the kernel of the integral (35), because that is not an analytic $\mathbb{C} \to \mathbb{C}$ function:

$$\begin{aligned} f_{\text{out}}(\alpha^*) = \langle \alpha \| \Psi_{\text{out}} \rangle \propto \\ \propto \frac{1}{\pi} \int \mathrm{d}^2\lambda\, f_{\text{in}}(\lambda^* + 2A^*)\, e^{-\left(\frac{1}{g^2}+1\right)|\lambda|^2 - 2\lambda^* A + \lambda\alpha^*} = \\ = \frac{1}{\pi} \int \mathrm{d}^2\lambda\, f_{\text{in}}(\lambda^* + 2A^*)\, e^{-\frac{1}{s}|\lambda|^2 - 2\lambda^* A + \lambda\alpha^*}. \end{aligned} \tag{37}$$

This is simplified using a generalization of Eq. (29):

$$\frac{1}{\pi} \int \mathrm{d}^2\lambda\, f(\lambda^*)\, e^{-c|\lambda|^2 + \lambda\beta^*} = \frac{1}{\sqrt{c}} f\left(\frac{\beta^*}{c}\right). \tag{38}$$

We obtain

$$f_{\text{out}}(\alpha^*) = f_{\text{in}}(s\alpha^* + 2A^*)\, \exp\left(-2As\alpha^*\right), \tag{39}$$

which reproduces the result (34).

As a conclusion we observe that the above calculation has heavily relied on the fact that the photon-number entangled state has an exponential Glauber function (it is a Gaussian state). The infinitesimal representation is particularly useful in problems invoking such kind of states.

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