Multivariable (φ, Γ)-modules and products of Galois groups

Gergely Zábrádi *

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Abstract

We show that the category of continuous representations of the dth direct power of the absolute Galois group of $\mathbb{Q}_p$ on finite dimensional $\mathbb{F}_p$-vector spaces (resp. finitely generated $\mathbb{Z}_p$-modules, resp. finite dimensional $\mathbb{Q}_p$-vector spaces) is equivalent to the category of étale (φ, Γ)-modules over a d-variable Laurent-series ring over $\mathbb{F}_p$ (resp. over $\mathbb{Z}_p$, resp. over $\mathbb{Q}_p$).

1 Introduction

This note serves as a complement to the work [11] where we relate multivariable (φ, Γ)-modules to smooth modulo p^n representations of a split reductive group G over $\mathbb{Q}_p$. The goal here is to show that the category of d-variable (φ, Γ)-modules is equivalent to the category of representations of the dth direct power of the absolute Galois group of $\mathbb{Q}_p$.

Let K be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_K$, prime element ϖ, and residue field κ. For a finite set Δ let $G_{\mathbb{Q}_p,\Delta} := \prod_{\alpha\in\Delta} \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ denote the direct power of the absolute Galois group of $\mathbb{Q}_p$ indexed by Δ. We denote by $\mathrm{Rep}_\kappa(G_{\mathbb{Q}_p,\Delta})$ (resp. by $\mathrm{Rep}_{\mathcal{O}_K}(G_{\mathbb{Q}_p,\Delta})$, resp. by $\mathrm{Rep}_K(G_{\mathbb{Q}_p,\Delta})$) the category of continuous representations of the profinite group $G_{\mathbb{Q}_p,\Delta}$ on finite dimensional κ-vector spaces (resp. finitely generated $\mathcal{O}_K$-modules, resp. finite dimensional K-vector spaces). On the other hand, for independent commuting variables X_α $(\alpha \in \Delta)$ we put

$$\begin{aligned}
E_{\Delta,\kappa} &:= \kappa[\![X_\alpha \mid \alpha\in\Delta]\!][X_\alpha^{-1} \mid \alpha\in\Delta] \ ,\\
\mathcal{O}_{\mathcal{E}_{\Delta,K}} &:= \varprojlim_h \left(\mathcal{O}_K/\varpi^h[\![X_\alpha \mid \alpha\in\Delta]\!][X_\alpha^{-1} \mid \alpha\in\Delta]\right) \ ,\\
\mathcal{E}_{\Delta,K} &:= \mathcal{O}_{\mathcal{E}_{\Delta,K}}[p^{-1}] \ .
\end{aligned}$$

Moreover, for each element $\alpha \in \Delta$ we have the partial Frobenius φ_α, and group $\Gamma_\alpha \cong \mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)$ acting on the variable X_α in the usual way and commuting with the other

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variables X_β ($\beta \in \Delta \setminus \{\alpha\}$) in the above rings. A $(\varphi_\Delta, \Gamma_\Delta)$-module over $E_{\Delta,\kappa}$ (resp. over $\mathcal{O}_{\mathcal{E}_{\Delta,K}}$, resp. over $\mathcal{E}_{\Delta,K}$) is a finitely generated $E_{\Delta,\kappa}$-module (resp. $\mathcal{O}_{\mathcal{E}_{\Delta,K}}$-module, resp. $\mathcal{E}_{\Delta,K}$-module) D together with commuting semilinear actions of the operators φ_α and groups Γ_α ($\alpha \in \Delta$). In case the coefficient ring is $E_{\Delta,\kappa}$ or $\mathcal{O}_{\mathcal{E}_{\Delta,K}}$, we say that D is étale if the map $\mathrm{id} \otimes \varphi_\alpha \colon \varphi_\alpha^* D \to D$ is an isomorphism for all $\alpha \in \Delta$. For the coefficient ring $\mathcal{E}_{\Delta,K}$ we require the stronger assumption for the étale property that D comes from an étale $(\varphi_\Delta, \Gamma_\Delta)$-module over $\mathcal{O}_{\mathcal{E}_{\Delta,K}}$ by inverting p. The main result of the paper is that $\mathrm{Rep}_\kappa(G_{\mathbb{Q}_p,\Delta})$ (resp. $\mathrm{Rep}_{\mathcal{O}_K}(G_{\mathbb{Q}_p,\Delta})$, resp. $\mathrm{Rep}_K(G_{\mathbb{Q}_p,\Delta})$) is equivalent to the category of étale $(\varphi_\Delta, \Gamma_\Delta)$-modules over $E_{\Delta,\kappa}$ (resp. over $\mathcal{O}_{\mathcal{E}_{\Delta,K}}$, resp. over $\mathcal{E}_{\Delta,K}$).

Passing from the Galois side to $(\varphi_\Delta, \Gamma_\Delta)$-modules is rather straightforward. One constructs a big ring E_Δ^{sep} as an inductive limit of completed tensor products of finite separable extensions E'_α of $E_\alpha = \mathbb{F}_p((X_\alpha))$ ($\alpha \in \Delta$) over which the action of $H_{\mathbb{Q}_p,\Delta} = \mathrm{Ker}(G_{\mathbb{Q}_p,\Delta} \twoheadrightarrow \prod_{\alpha\in\Delta} \Gamma_\Delta)$ trivializes. The other direction is more involved. In order to trivialize the action of the partial Frobenii φ_α ($\alpha \in \Delta$) using induction, the main step is to find a lattice $D_{\overline{\alpha}}^{+*}$ integral in the variable X_α for some fixed $\alpha \in \Delta$ which is an étale $(\varphi_{\Delta\setminus\{\alpha\}}, \Gamma_{\Delta\setminus\{\alpha\}})$-module over the ring $\mathbb{F}_p[\![X_\beta \mid \beta \in \Delta]\!][X_\beta^{-1} \mid \beta \in \Delta \setminus \{\alpha\}]$. This uses the ideas of Colmez [3] constructing lattices D^+ and D^{++} in usual (φ, Γ)-modules.

We remark here that Scholze [7] recently realized $G_{\mathbb{Q}_p,\Delta}$ (using Drinfeld's Lemma for diamonds) as a geometric fundamental group $\pi_1((\mathrm{Spd}\ \mathbb{Q}_p)^{|\Delta|}/\mathrm{p.Fr.})$ of the diamond $(\mathrm{Spd}\ \mathbb{Q}_p)^{|\Delta|}$ modulo the partial Frobenii φ_β ($\beta \in \Delta \setminus \{\alpha\}$) for some fixed $\alpha \in \Delta$: one can endow $E_\Delta^+ = \mathbb{F}_p[\![X_\alpha \mid \alpha \in \Delta]\!]$ with its natural compact topology, and look at the subset of its adic spectrum $\mathrm{Spa}\, E_\Delta^+$ where all X_α ($\alpha \in \Delta$) are invertible. This defines an analytic adic space over $\mathbb{F}_p$, whose perfection modulo the action of all Γ_α's is a model for $(\mathrm{Spd}\,\mathbb{Q}_p)^d$. Thus, after taking the action modulo partial Frobenii φ_β ($\beta \in \Delta \setminus \{\alpha\}$ for some fixed $\alpha \in \Delta$), the fundamental group will be $G_{\mathbb{Q}_p,\Delta}$. Now, quite generally étale local systems on diamonds are equivalent to φ-modules. This introduces the last missing Frobenius, and one ends up with an equivalence between representations of $G_{\mathbb{Q}_p,\Delta}$, and some sheaf of modules with Γ_Δ-action and commuting actions of φ_α for all $\alpha \in \Delta$. However, this will not produce an actual module over a ring, but a sheaf of modules over a sheaf of rings. One can perhaps deduce the result of this paper along these lines, but that would require some further nontrivial input (replacing the above method of finding a lattice $D_{\overline{\alpha}}^{+*}$).

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2 Algebraic properties of multivariable (φ, Γ)-modules

2.1 Definition and projectivity

For a finite set Δ (which is the set of simple roots of G in [11]) consider the Laurent series ring $E_\Delta := E_\Delta^+[X_\Delta^{-1}]$ where $E_\Delta^+ := \mathbb{F}_p[\![X_\alpha \mid \alpha \in \Delta]\!]$ and $X_\Delta := \prod_{\alpha\in\Delta} X_\alpha \in E_\Delta^+$. E_Δ^+ is a regular noehterian local ring of global dimension $|\Delta|$, therefore E_Δ is a regular noetherian ring of global dimension $|\Delta|-1$. For each index α we define the action of the partial Frobenius φ_α and of the group Γ_α with $\chi_\alpha\colon \Gamma_\alpha \xrightarrow{\sim} \mathbb{Z}_p^\times$ on E_Δ as

$$\begin{aligned} \varphi_\alpha(X_\beta) &:= \begin{cases} X_\beta & \text{if } \beta\in\Delta\setminus\{\alpha\} \\ (X_\alpha+1)^p-1 = X_\alpha^p & \text{if } \beta=\alpha \end{cases} \\ \gamma_\alpha(X_\beta) &:= \begin{cases} X_\beta & \text{if } \beta\in\Delta\setminus\{\alpha\} \\ (X_\alpha+1)^{\chi_\alpha(\gamma_\alpha)}-1 & \text{if } \beta=\alpha \end{cases} \end{aligned} \tag{1}$$

for all $\gamma_\alpha\in\Gamma_\alpha$ extending the above formulas to continuous ring endomorphisms of E_Δ in the obvious way. By an étale $(\varphi_\Delta,\Gamma_\Delta)$-module over E_Δ we mean a (unless otherwise mentioned) finitely generated module D over E_Δ together with a semilinear action of the (commutative) monoid $T_{+,\Delta} := \prod_{\alpha\in\Delta}\varphi_\alpha^{\mathbb{N}}\Gamma_\alpha$ (also denote by φ_t the action of $\varphi_t\in T_{+,\Delta}$) such that the maps

$$\mathrm{id}\otimes\varphi_t\colon \varphi_t^*D := E_\Delta\otimes_{E_\Delta,\varphi_t} D \to D$$

are isomorphisms for all elements $\varphi_t\in T_{+,\Delta}$. Here we put $\Gamma_\Delta := \prod_{\alpha\in\Delta}\Gamma_\alpha$. We denote by $\mathcal{D}^{et}(\varphi_\Delta,\Gamma_\Delta,E_\Delta)$ the category of étale $(\varphi_\Delta,\Gamma_\Delta)$-modules over E_Δ.

The category $\mathcal{D}^{et}(\varphi_\Delta,\Gamma_\Delta,E_\Delta)$ has the structure of a neutral Tannakian category: For two objects D_1 and D_2 the tensor product $D_1\otimes_{E_\Delta} D_2$ is an étale $T_{+,\Delta}$-module with the action $\varphi_t(d_1\otimes d_2) := \varphi_t(d_1)\otimes\varphi_t(d_2)$ for $\varphi_t\in T_{+,\Delta,}$, $d_i\in D_i$ $(i=1,2)$. Moreover, since E_Δ is a free module over itself via φ_t, putting $(\cdot)^* := \mathrm{Hom}_{E_\Delta}(\cdot,E_\Delta)$ we have an identification $(\varphi_t^*D)^*\cong\varphi_t^*(D^*)$. So the isomorphism $\mathrm{id}\otimes\varphi_t\colon\varphi_t^*D\to D$ dualizes to an isomorphism $D^*\to\varphi_t^*(D^*)$. The inverse of this isomorphism (for all $\varphi_t\in T_{+,\Delta}$) equips D^* with the structure of an étale $T_{+,\Delta}$-module.

Lemma 2.1. *There exists a Γ_Δ-equivariant injective resolution of E_Δ^+ as a module over itself.*

Proof. Consider the Cousin complex (see IV.2 in [6])

$$0\to E_\Delta\to E_{\Delta,(0)}\to\cdots\to\bigoplus_{\mathfrak{p}\in\mathrm{Spec}(E_\Delta),\mathrm{codim}\,\mathfrak{p}=r} J(\mathfrak{p})\to\dots$$

where $J(\mathfrak{p})$ is the injective envelope of the residue field $\kappa(\mathfrak{p})$ as a module over the local ring $E_{\Delta,\mathfrak{p}}$. This is a Γ_Δ-equivariant injective resolution since the action of Γ_Δ on $\mathrm{Spec}(E_\Delta)$ respects the codimension. □

Proposition 2.2. *Any object D in $\mathcal{D}^{et}(\varphi_\Delta,\Gamma_\Delta,E_\Delta)$ is a projective module over E_Δ.*

Proof. Since E_Δ has finite global dimension, let n be the projective dimension of D. Then by Lemma 4.1.6 in [9] we have $\mathrm{Ext}^i(D,M)=0$ for all $i>n$ and E_Δ-module M and there exists an R-module M_0 with $\mathrm{Ext}^n(D,M_0)\neq 0$. By the long exact sequence of Ext and choosing an onto module homomorphism $F\twoheadrightarrow M_0$ from a free module F we find that $\mathrm{Ext}^n(D,F)\neq 0$ whence $\mathrm{Ext}^n(D,E_\Delta)\neq 0$. However, $\mathrm{Ext}^n(D,E_\Delta)$ is a finitely generated torsion E_Δ-module for $n>0$ admitting a semilinear action of Γ_Δ. Therefore the global annihilator of $\mathrm{Ext}^n(D,E_\Delta)$ in E_Δ is a nonzero Γ_Δ-invariant ideal in E_Δ hence equals E_Δ by Lemma 2.1 in [11]. So $n=0$ and D is projective. □

Lemma 2.3. *We have $K_0(E_\Delta) \cong \mathbb{Z}$, ie. any finitely generated projective module over E_Δ is stably free.*

Proof. $E_\Delta^+ \cong \mathbb{F}_p[[X_\alpha \mid \alpha \in \Delta]]$ is a regular local ring, so it has finite global dimension and $K_0(E_\Delta^+) \cong G_0(E_\Delta^+) \cong \mathbb{Z}$ (Thm. II.7.8 in [10]). Therefore the localization $E_\Delta = E_\Delta^+[X_\Delta^{-1}]$ also has finite global dimension whence we have $K_0(E_\Delta) \cong G_0(E_\Delta)$. The statement follows noting that the map $G_0(E_\Delta^+) \to G_0(E_\Delta)$ is onto by the localization exact sequence of algebraic K-theory (Thm. II.6.4 in [10]). $\square$

Remark. I am not aware of the analogue of the Theorem of Quillen and Suslin on the freeness of projective modules over E_Δ. However, using the equivalence of categories of $\mathcal{D}^{et}(\varphi_\Delta, \Gamma_\Delta, E_\Delta)$ with $\mathrm{Rep}_{\mathbb{F}_p}(G_{\mathbb{Q}_p,\Delta})$ we shall see later on (Cor. 3.16) that any object D in $\mathcal{D}^{et}(\varphi_\Delta, \Gamma_\Delta, E_\Delta)$ is in fact free over E_Δ.

We equip E_Δ^+ with the X_Δ-adic topology. Then (E_Δ, E_Δ^+) is a Huber pair (in the sense of [7]) if we equip E_Δ with the inductive limit topology $E_\Delta = \bigcup_n X_\Delta^{-n} E_\Delta^+$. In fact, E_Δ is a complete noetherian Tate ring (op. cit.). Note that this is *not* the natural compact topology on E_Δ^+ as in the compact topology E_Δ^+ would not be open in E_Δ since the index of E_Δ^+ in $X_\Delta^{-n} E_\Delta^+$ is not finite. On the other hand, the inclusion $\mathbb{F}_p((X_\alpha)) \hookrightarrow E_\Delta$ is *not* continuous in the X_Δ-adic topology therefore we cannot apply Drinfeld's Lemma (Thm. 17.2.4 in [7]) directly in this situation.

Let D be an object in $\mathcal{D}^{et}(\varphi_\Delta, \Gamma_\Delta, E_\Delta)$. By Banach's Theorem for Tate rings (Prop. 6.18 in [8]), there is a unique E_Δ-module topology on D that we call the X_Δ-adic topology. Moreover, any E_Δ-module homomorphism is continuous in the X_Δ-adic topology.

2.2 Integrality properties

Put $\varphi_s := \prod_{\alpha \in \Delta} \varphi_\alpha \in T_{+,\Delta}$ and define $D^{++} := \{x \in D \mid \lim_{k \to \infty} \varphi_s^k(x) = 0\}$ where the limit is considered in the X_Δ-adic topology (cf. II.2.1 in [3] in case $|\Delta| = 1$). Note that φ_s is the absolute Frobenius on E_Δ, it takes any element to its pth power.

Lemma 2.4. *Let M be a finitely generated E_Δ^+-submodule in D. Then $E_\Delta^+ \varphi_s(M)$ is also finitely generated.*

Proof. If M is generated by $m_1, \dots, m_n$ then $\varphi_s(m_1), \dots, \varphi_s(m_n)$ generate $E_\Delta^+ \varphi_s(M)$. $\square$

Proposition 2.5. *D^{++} is a finitely generated E_Δ^+-submodule in D that is stable under the action of $T_{+,\Delta}$ and we have $D = D^{++}[X_\Delta^{-1}]$.*

Proof. Choose an arbitrary finitely generated E_Δ^+-submodule M of D with $M[X_\Delta^{-1}] = D$ (e.g. take $M = E_\Delta^+ e_1 + \dots + E_\Delta^+ e_n$ for some E_Δ-generating system $e_1, \dots, e_n$ of D). By Lemma 2.4 we have an integer $r \geq 0$ such that $\varphi(M) \subseteq X_\Delta^{-r} M$, since E_Δ^+ is noetherian and we have $D = \bigcup_r X_\Delta^{-r} M$. Then we have

$$\varphi_s(X_\Delta^k M) = X_\Delta^{pk} \varphi_s(M) \subseteq X_\Delta^{pk-r} M \subseteq X_\Delta^{k+1} M$$

for any integer $k \geq \frac{r+1}{p-1}$. Therefore we have $X_\Delta^{\left[\frac{r+1}{p-1}\right]+1} M \subseteq D^{++}$ whence $D^{++}[X_\Delta^{-1}] = M[X_\Delta^{-1}] = D$.

Since $T_{+\Delta}$ is commutative and the action of each φ_t ($t \in T_{+,\Delta}$) is continuous, D^{++} is stable under the action of $T_{+,\Delta}$. There is a system of neighbourhoods of 0 in D consisting of E_Δ^+-submodules therefore D^{++} is an E_Δ^+-submodule.

To prove that D^{++} is finitely generated over E_Δ^+ suppose first that D is a free module over E_Δ generated by $e_1, \dots, e_n$ and put $M := E_\Delta^+ e_1 + \dots + E_\Delta^+ e_n$. We may assume $M \subseteq D^{++}$ by replacing M with $X_\Delta^{[\frac{r+1}{p-1}]+1} M$. Moreover, further multiplying $M = E_\Delta^+ e_1 + \dots + E_\Delta^+ e_n$ by a power of X_Δ, we may assume that the matrix $A := [\varphi_s]_{e_1,\dots,e_n}$ of φ_s in the basis $e_1, \dots, e_n$ lies in $E_\Delta^{+n\times n}$ as we have $[\varphi_s]_{X_\Delta^r e_1,\dots,X_\Delta^r e_n} = X_\Delta^{(p-1)r}[\varphi_s]_{e_1,\dots,e_n}$. Now we choose the integer $r > 0$ so that it is bigger than $\mathrm{val}_{X_\alpha}(\det A)$ for all $\alpha \in \Delta$ and claim that $D^{++} \subseteq X_\Delta^{-r} M$ whence D^{++} is finitely generated over E_Δ^+ as E_Δ^+ is noetherian. Assume for contradiction that $d = \sum_{i=1}^n d_i e_i$ lies in D^{++} for some $d_i \in E_\Delta$ ($i = 1, \dots, n$) such that at least one d_i, say d_1, does not lie in $X_\Delta^{-r} E_\Delta^+$. In particular, there exists an α in Δ such that $\mathrm{val}_{X_\alpha}(d_1) < -r$. Since M is open in D and $d \in D^{++}$, there exists an integer $k > 0$ such that $\varphi_s^k(d)$ is in M which is equivalent to saying that the column vector

$$A\varphi_s(A)\dots\varphi_s^{k-1}(A)\begin{pmatrix}\varphi_s^k(d_1)\\ \vdots \\ \varphi_s^k(d_n)\end{pmatrix}$$

lies in E_Δ^{+n}. Multiplying this by the matrix built from the $(n-1)\times(n-1)$ minors of $A\varphi_s(A)\dots\varphi_s^{k-1}(A)$ we deduce that $\det(A\varphi_s(A)\dots\varphi_s^{k-1}(A))\varphi_s^k(d_1) = \det(A)^{\frac{p^k-1}{p-1}} d_1^{p^k}$ lies in E_Δ^+. We compute

$$\begin{aligned}0 \le \mathrm{val}_{X_\alpha}(\det(A)^{\frac{p^k-1}{p-1}} d_1^{p^k}) = \frac{p^k-1}{p-1}\mathrm{val}_{X_\alpha}(\det(A)) + p^k \mathrm{val}_{X_\alpha}(d_1) < \\ < \frac{p^k-1}{p-1}\mathrm{val}_{X_\alpha}(\det(A)) - p^k r < 0\end{aligned}$$

by our assumption that $r > \mathrm{val}_{X_\alpha}(\det(A))$, yielding a contradiction.

In the general case note that D is always stably free by Prop. 2.2 and Lemma 2.3. So $D_1 := D \oplus E_\Delta^k$ is a free module over E_Δ for k large enough. We make D_1 into an étale $T_{+,\Delta}$-module by the trivial action of $T_{+,\Delta}$ on E_Δ^k to deduce that D_1^{++} is finitely generated over E_Δ^+. The result follows noting that $D^{++} \subseteq D_1^{++}$ and E_Δ^+ is noetherian. $\square$

For an object D in $\mathcal{D}^{et}(\varphi_\Delta, \Gamma_\Delta, E_\Delta)$ we define

$$D^+ := \{x \in D \mid \{\varphi_s^k(x) \colon k \ge 0\} \subset D \text{ is bounded}\} .$$

Since $\varphi_s^k(X_\Delta)$ tends to 0 in the X_Δ-adic topology, we have $X_\Delta D^+ \subseteq D^{++}$, ie. $D^+ \subseteq X_\Delta^{-1} D^{++}$. In particular, D^+ is finitely generated over E_Δ^+. On the other hand, we also have $D^{++} \subseteq D^+$ by construction whence we deduce $D = D^+[X_\Delta^{-1}]$.

Lemma 2.6. *We have $\varphi_t(D^+) \subset D^+$ (resp. $\varphi_t(D^{++}) \subset D^{++}$) for all $\varphi_t \in T_{+,\Delta}$.*

Proof. For any generating system $e_1, \dots, e_n$ of D and any $\varphi_t \in T_{+,\Delta}$ there exists an integer $k = k(\varphi_t, M) > 0$ such that we have $\varphi_t(X_\Delta^k M) \subseteq X_\Delta^k E_\Delta^+ \varphi_t(M) \subseteq M$ where we put $M :=$

$E_\Delta^+ e_1 + \cdots + E_\Delta^+ e_n$ by Lemma 2.4. Indeed, X_Δ divides $\varphi_t(X_\Delta)$ in E_Δ^+, and we have $D = M[1/X_\Delta]$ by construction. The statement on D^{++} follows from the commutativity of the monoid $T_{+,\Delta}$ noting that there exists a basis of neighbouhoods of 0 in D consisting of E_Δ^+-submodules of the form M. To see that $\varphi_t(D^+) \subseteq D^+$ note that $\varphi_t(D^+)$ is bounded and we have $\varphi_s^k(\varphi_t(D^+)) = \varphi_t(\varphi_s^k(D^+)) \subset \varphi_t(D^+)$. □

Now fix an $\alpha \in \Delta$ and define $D_{\overline{\alpha}}^+ := D^+[X_{\Delta\setminus\{\alpha\}}^{-1}]$ where for any subset $S \subseteq \Delta$ we put $X_S := \prod_{\beta\in S} X_\beta$. Then $D_{\overline{\alpha}}^+$ is a finitely generated module over $E_{\overline{\alpha}}^+ := E_\Delta^+[X_{\Delta\setminus\{\alpha\}}^{-1}]$. We denote by $T_{+,\overline{\alpha}} \subset T_{+,\Delta}$ the monoid generated by φ_β ($\beta \in \Delta \setminus \{\alpha\}$) and Γ_Δ.

Lemma 2.7. *$D_{\overline{\alpha}}^+/D^+$ is X_α-torsion free: If both $X_\alpha^{n_1} d$ and $X_{\Delta\setminus\{\alpha\}}^{n_2} d$ lie in D^+ for some element $d \in D^+$, $\alpha \in \Delta$, and integers $n_1, n_2 \geq 0$ then we have $d \in D^+$. The same statement holds if we replace D^+ by D^{++}.*

Proof. At first assume that D is free as a module over E_Δ with basis $e_1, \dots, e_n$. Then the denominators of $\varphi_s^k(X_\alpha^{n_1} d) = X_\alpha^{n_1 p^k} \varphi_s^k(d)$ in the basis $e_1, \dots, e_n$ are bounded for $k \geq 0$ by assumption. Therefore the X_β-valuations of the denominators of $\varphi_s^k(d)$ are bounded for all $\beta \in \Delta\setminus\{\alpha\}$ since E_Δ^+ is a unique factorization domain. On the other hand, the X_α-valuations of these denominators are also bounded since the denominators of $\varphi_s^k(X_{\Delta\setminus\{\alpha\}}^{n_2} d) = X_{\Delta\setminus\{\alpha\}}^{n_2 p^k} \varphi_s^k(d)$ are bounded. To prove the statement we have the same argument but 'being bounded' replaced by 'tends to 0'.

Finally, by Prop. 2.2 and Lemma 2.3 $D \oplus E_\Delta^k$ is free over E_Δ and we equip it with the structure of an étale (φ, Γ)-module (trivially on E_Δ^k). The statement follows from the additivity of the constructions $D \mapsto D^+$ and $D \mapsto D_{\overline{\alpha}}^+$ in direct sums. □

Lemma 2.8. *Assume that D is generated by a single element $e_1 \in D$ over E_Δ. Then for any φ_t in $T_{+,\overline{\alpha}}$ we have $\varphi_t(e_1) = a_t e_1$ for some unit a_t in $(E_{\overline{\alpha}}^+)^\times$.*

Proof. Define $a_t \in E_\Delta$ and $a_\alpha \in E_\Delta$ so that $\varphi_t(e_1) = a_t e_1$ and $\varphi_\alpha(e_1) = a_\alpha e_1$. By the étale property both a_t and a_α are units in E_Δ, so it remains to show that $\mathrm{val}_{X_\alpha}(a_t) = 0$. We compute

$$\begin{aligned} \varphi_\alpha(a_t) a_\alpha e_1 &= \varphi_\alpha(a_t)\varphi_\alpha(e_1) = \varphi_\alpha(a_t e_1) = \varphi_\alpha(\varphi_t(e_1)) = \\ &= \varphi_t(\varphi_\alpha(e_1)) = \varphi_t(a_\alpha e_1) = \varphi_t(a_\alpha)\varphi_t(e_1) = \varphi_t(a_\alpha) a_t e_1 \end{aligned}$$

whence we deduce

$$p\,\mathrm{val}_{X_\alpha}(a_t) + \mathrm{val}_{X_\alpha}(a_\alpha) = \mathrm{val}_{X_\alpha}(\varphi_\alpha(a_t) a_\alpha) = \mathrm{val}_{X_\alpha}(\varphi_t(a_\alpha) a_t) = \mathrm{val}_{X_\alpha}(a_\alpha) + \mathrm{val}_{X_\alpha}(a_t)\ .$$

This yields $\mathrm{val}_{X_\alpha}(a_t) = 0$ as required. □

Lemma 2.9. *There exists an integer $k = k(D) > 0$ such that for any $\varphi_t \in T_{+,\overline{\alpha}}$ we have $X_\alpha^k D_{\overline{\alpha}}^+ \subseteq E_\Delta^+ \varphi_t(D_{\overline{\alpha}}^+)$.*

Proof. At first assume that D is free, choose a basis $e_1, \dots, e_n$ contained in D^+, and put $M := E_\Delta^+ e_1 + \dots E_\Delta^+ e_n$, $M_\alpha := E_{\overline{\alpha}}^+ e_1 + \cdots + E_{\overline{\alpha}}^+ e_n$. There exists an integer $k_0 > 0$ such that $D^+ \subseteq X_\Delta^{-k_0} M$. In particular, we have $D_{\overline{\alpha}}^+ \subseteq X_\alpha^{-k_0} M_{\overline{\alpha}}$. Now for a fixed $\varphi_t \in T_{+,\overline{\alpha}}$ let $A_t \in E_\Delta^{n\times n}$ be the matrix of φ_t in the basis $e_1, \dots, e_n$. Since $\varphi_t(e_i)$ lies in $D^+ \subseteq X_\alpha^{-k_0} M_{\overline{\alpha}}$, all the entries of the matrix A_t are in $X_\alpha^{-k_0}E_{\overline{\alpha}}^+$. Applying Lemma 2.8 to the single generator $e_1\wedge\dots\wedge e_n$ of $\bigwedge^n D$ we obtain $\mathrm{val}_{X_\alpha}(\det A_t)=0$. In particular, all the entries of A_t^{-1} lie in $X_\alpha^{-(n-1)k_0}E_{\overline{\alpha}}^+$ by the formula for the inverse matrix using the $(n-1)\times(n-1)$ minors in A_t. Now note that the elements $e_1,\dots,e_n$ can be written as a linear combination of $\varphi_t(e_1),\dots,\varphi_t(e_n)$ with coefficients from A_t^{-1}. Using Lemma 2.6 this shows

$$X_\alpha^{k_0}D_{\overline{\alpha}}^+\subseteq M_{\overline{\alpha}}\subseteq X_\alpha^{-(n-1)k_0}\varphi_t(M_{\overline{\alpha}})\subseteq X_\alpha^{-(n-1)k_0}D_{\overline{\alpha}}^+ \ .$$

So we may choose $k:=nk_0$ independent of φ_t.

The general case follows from Prop. 2.2 and Lemma 2.3 noting that the functor $D\mapsto D_{\overline{\alpha}}^+$ commutes with direct sums. □

In view of the above Lemma we define

$$D_{\overline{\alpha}}^{+*}:=\bigcap_{\varphi_t\in T_{+,\overline{\alpha}}}E_{\overline{\alpha}}^+\varphi_t(D_{\overline{\alpha}}^+) \ .$$

$D_{\overline{\alpha}}^{+*}$ is finitely generated over $E_{\overline{\alpha}}^+$ as it is contained in $D_{\overline{\alpha}}^+$ and $E_{\overline{\alpha}}^+$ is noetherian. On the other hand, by Lemma 2.9 we have $X_\alpha^kD_{\overline{\alpha}}^+\subseteq D_{\overline{\alpha}}^{+*}$ for some integer $k=k(D)>0$ whence, in particular, $D=D_{\overline{\alpha}}^{+*}[X_\alpha^{-1}]$.

Proposition 2.10. *$D_{\overline{\alpha}}^{+*}$ is an étale $T_{+,\overline{\alpha}}$-module over $E_{\overline{\alpha}}^+$, ie. the maps*

$$\mathrm{id}\otimes\varphi_t\colon \varphi_t^*D_{\overline{\alpha}}^{+*}=E_{\overline{\alpha}}^+\otimes_{E_{\overline{\alpha}}^+,\varphi_t}D_{\overline{\alpha}}^{+*}\to D_{\overline{\alpha}}^{+*} \tag{2}$$

are bijective for all $\varphi_t\in T_{+,\alpha}$.

Proof. At first note that we have $\varphi_t(D_{\overline{\alpha}}^{+*})\subseteq D_{\overline{\alpha}}^{+*}$ for all $\varphi_t\in T_{+,\overline{\alpha}}$ by Lemma 2.6 and the commutativity of $T_{+,\overline{\alpha}}$, so the map (2) exists. Now let $\varphi_{t_0}\in T_{+,\overline{\alpha}}$ be arbitrary. Since $E_{\overline{\alpha}}^+$ (resp. E_Δ) is a finite free module over $\varphi_{t_0}(E_{\overline{\alpha}}^+)$ (resp. over $\varphi_{t_0}(E_\Delta)$) with generators contained in E_Δ^+, we have a natural identification $\varphi_{t_0}^*D_{\overline{\alpha}}^{+*}\cong E_\Delta^+\otimes_{E_\Delta^+,\varphi_{t_0}}D_\Delta^{+*}$ (resp. $\varphi_{t_0}^*D\cong E_\Delta^+\otimes_{E_\Delta^+,\varphi_{t_0}}D$). Since E_Δ^+ is finite free (hence flat) over $\varphi_{t_0}(E_\Delta^+)$, the inclusion $D_{\overline{\alpha}}^+\subset D$ induces an inclusion $\varphi_{t_0}^*D_{\overline{\alpha}}^+\subset\varphi_{t_0}^*D$. It follows that (2) is injective since D is étale. Similarly, for each $\varphi_t\in T_{+,\overline{\alpha}}$, the map

$$\mathrm{id}\otimes\varphi_{t_0}\colon \varphi_{t_0}^*(E_{\overline{\alpha}}^+\varphi_t(D_{\overline{\alpha}}^+))\to E_{\overline{\alpha}}^+\varphi_t(D_{\overline{\alpha}}^+)$$

is injective with image $E_{\overline{\alpha}}^+\varphi_{t_0}\varphi_t(D_{\overline{\alpha}}^+)$. On the other hand, since E_Δ^+ is finite free over $\varphi_{t_0}(E_\Delta^+)$, we have $\varphi_{t_0}^*D_{\overline{\alpha}}^{+*}=\bigcap_{t\in T_{+,\overline{\alpha}}}\varphi_{t_0}^*(E_{\overline{\alpha}}^+\varphi_t(D_{\overline{\alpha}}^+))$ where the intersection is taken inside $\varphi_{t_0}^*D$. Therefore (2) is bijective as we have $D_{\overline{\alpha}}^{+*}=\bigcap_{\varphi_t\in T_{+,\overline{\alpha}}}E_{\overline{\alpha}}^+\varphi_{t_0}\varphi_t(D_{\overline{\alpha}}^+)$. □

Lemma 2.11. *There exists a finitely generated E_Δ^+-submodule $D_0\subset D_{\overline{\alpha}}^{+*}$ such that $D_0\subseteq E_\Delta^+\varphi_{\overline{\alpha}}(D_0)$ and $D_{\overline{\alpha}}^{+*}=D_0[X_{\Delta\setminus\{\alpha\}}^{-1}]$ where $\varphi_{\overline{\alpha}}:=\prod_{\beta\in\Delta\setminus\{\alpha\}}\varphi_\beta$. Moreover, we have $D_{\overline{\alpha}}^{+*}=\bigcup_{r\geq0}E_\Delta^+\varphi_{\overline{\alpha}}^r(X_{\Delta\setminus\{\alpha\}}^{-1}D_0)$.*

Proof. Put $D_1:=D^+\cap D_{\overline{\alpha}}^{+*}$. By Prop. 2.10 and the fact that $D_{\overline{\alpha}}^{+*}=D_1[X_{\Delta\setminus\{\alpha\}}^{-1}]$ we find an integer $k_0>0$ such that $X_{\Delta\setminus\{\alpha\}}^{k_0}D_1\subseteq E_\Delta^+\varphi_{\overline{\alpha}}(D_1)$. So for $k>\frac{k_0}{p-1}$ we have

$$X_{\Delta\setminus\{\alpha\}}^{-k}D_1\subseteq X_{\Delta\setminus\{\alpha\}}^{-k-k_0}E_\Delta^+\varphi_{\overline{\alpha}}(D_1)\subseteq X_{\Delta\setminus\{\alpha\}}^{-pk}E_\Delta^+\varphi_{\overline{\alpha}}(D_1)=E_\Delta^+\varphi_{\overline{\alpha}}(X_{\Delta\setminus\{\alpha\}}^{-k}D_1) \ .$$

So we put $D_0 := X_{\Delta\setminus\{\alpha\}}^{-k}D_1$ so that the first part of the statement is satisfied. Iterating the inclusion $D_0 \subseteq E_\Delta^+\varphi_{\overline{\alpha}}(D_0)$ we obtain $D_0 \subseteq E_\Delta^+\varphi_{\overline{\alpha}}^r(D_0)$ for all $r \geq 1$. Finally, we compute

$$X_{\Delta\setminus\{\alpha\}}^{-p^r}D_0 \subseteq X_{\Delta\setminus\{\alpha\}}^{-p^r}E_\Delta^+\varphi_{\overline{\alpha}}^r(D_0) = E_\Delta^+\varphi_{\overline{\alpha}}^r(X_{\Delta\setminus\{\alpha\}}^{-1}D_0) \ .$$

The statement follows noting that we have $D_{\overline{\alpha}}^{+*} = D_0[X_{\Delta\setminus\{\alpha\}}^{-1}] = \bigcup_r X_{\Delta\setminus\{\alpha\}}^{-p^r}D_0$. □

3 The equivalence of categories for $\mathbb{F}_p$-representations

3.1 The functor $\mathbb{D}$

Take a copy $G_{\mathbb{Q}_p,\alpha} \cong \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ of the absolute Galois group of $\mathbb{Q}_p$ for each element $\alpha \in \Delta$ and let $G_{\mathbb{Q}_p,\Delta} := \prod_{\alpha\in\Delta} G_{\mathbb{Q}_p,\alpha}$. Let $\mathrm{Rep}_{\mathbb{F}_p}(G_{\mathbb{Q}_p,\Delta})$ be the category of continuous representations of the group $G_{\mathbb{Q}_p,\Delta}$ on finite dimensional $\mathbb{F}_p$ vectorspaces. We identify Γ_α with the Galois group $\mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)$ as a quotient of $G_{\mathbb{Q}_p,\alpha}$ via the cyclotomic character $\chi_\alpha\colon \mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p) \to \mathbb{Z}_p^\times$. Further, we denote by $H_{\mathbb{Q}_p,\alpha}$ the kernel of the natural quotient map $G_{\mathbb{Q}_p,\alpha} \to \Gamma_\alpha$ and put $H_{\mathbb{Q}_p,\Delta} := \prod_{\alpha\in\Delta} H_{\mathbb{Q}_p,\alpha} \lhd G_{\mathbb{Q}_p,\Delta}$. Putting $E_\alpha := \mathbb{F}_p((X_\alpha))$ we have the following fundamental result of Fontaine and Wintenberger (Thm. 4.16 [5]).

Theorem 3.1. *The absolute Galois group* $\mathrm{Gal}(E_\alpha^{sep}/E_\alpha)$ *is isomorphic to* $H_{\mathbb{Q}_p,\alpha}$. *Moreover,* $G_{\mathbb{Q}_p,\alpha}$ *acts on the separable closure* E_α^{sep} *via automorphisms such that the action of* $\Gamma_\alpha \cong G_{\mathbb{Q}_p,\alpha}/H_{\mathbb{Q}_p,\alpha}$ *on* $E_\alpha = (E_\alpha^{sep})^{H_{\mathbb{Q}_p,\alpha}}$ *coincides with the one given in* (1).

For each $\alpha \in \Delta$ consider a finite separable extension E'_α of E_α together with the Frobenius $\varphi_\alpha\colon E'_\alpha \to E'_\alpha$ acting by raising to the power p. We denote by $E_\alpha'^+$ the integral closure of $E_\alpha^+ = \mathbb{F}_p[[X_\alpha]]$ in E'_α. Note that E'_α is isomorphic to $\mathbb{F}_{q_\alpha}((X'_\alpha))$ for some power q_α of p and uniformizer X'_α such that we have $E_\alpha'^+ \cong \mathbb{F}_{q_\alpha}[[X'_\alpha]]$. We normalize the X_α-adic (multiplicative) valuation on E_α so that we have $|X_\alpha|_{X_\alpha} = p^{-1}$. This extends uniquely to the finite extension E'_α. Moreover, we equip the tensor product $E'_{\Delta,\circ} := \bigotimes_{\alpha\in\Delta,\mathbb{F}_p} E'_\alpha$ with a norm $|\cdot|_{prod}$ by the formula

$$|c|_{prod} := \inf\left(\max_i(\prod_{\alpha\in\Delta}|c_{\alpha,i}|_\alpha) \ \mid \ c = \sum_{i=1}^n \bigotimes_{\alpha\in\Delta} c_{\alpha,i}\right) \ . \tag{3}$$

Note that the restriction of $|\cdot|_{prod}$ to the subring $E_{\Delta,\circ}'^+ := \bigotimes_{\alpha\in\Delta,\mathbb{F}_p} E_\alpha'^+$ induces the valuation with respect to the augmentation ideal $\mathrm{Ker}(E_{\Delta,\circ}'^+ \twoheadrightarrow \bigotimes_{\alpha\in\Delta,\mathbb{F}_p}\mathbb{F}_{q_\alpha})$. The norm $|\cdot|_{prod}$ is not multiplicative in general, as the ring $\bigotimes_{\alpha\in\Delta,\mathbb{F}_p}\mathbb{F}_{q_\alpha})$ is not a domain. However, it is submultiplicative. We define $E_\Delta'^+$ as the completion of $E_{\Delta,\circ}'^+$ with respect to $|\cdot|_{prod}$ and put $E'_\Delta := E_\Delta'^+[1/X_\Delta]$. Note that E'_Δ is *not* complete with respect to $|\cdot|_{prod}$ (unless $|\Delta| = 1$) even though $E'_{\Delta,\circ} = E_{\Delta,\circ}'^+[1/X_\Delta]$ is a dense subring in E'_Δ. Since we have a containment

$$(\bigotimes_{\alpha\in\Delta,\mathbb{F}_p}\mathbb{F}_{q_\alpha})[X'_\alpha, \alpha\in\Delta] = \bigotimes_{\alpha\in\Delta,\mathbb{F}_p}\mathbb{F}_{q_\alpha}[X_\alpha] \leq_{dense} E_{\Delta,\circ}'^+$$

we may identify $E_\Delta'^+$ with the power series ring $(\bigotimes_{\alpha\in\Delta,\mathbb{F}_p}\mathbb{F}_{q_\alpha})[[X'_\alpha, \alpha \in \Delta]]$ which is the completion of the polynomial ring above. In particular, the special case $E'_\alpha = E_\alpha$ for all $\alpha \in \Delta$ yields a ring E'_Δ isomorphic to E_Δ. Therefore E_Δ is a subring of E'_Δ for all collection of finite separable extensions E'_α of E_α ($\alpha \in \Delta$). Further, φ_α acts on $E'^{+}_{\Delta,\circ}$ (and on $E'_{\Delta,\circ}$) by the Frobenius on the component in E'_α and by the identity on all the other components in E'_β, $\beta \in \Delta \setminus \{\alpha\}$. This action is continuous in the norm $|\cdot|_{prod}$ therefore extends to the completion E'^{+}_Δ and the localization E'_Δ. We have the following alternative characterization of the ring E'_Δ.

Lemma 3.2. *Put $\Delta = \{\alpha_1, \dots, \alpha_n\}$. We have*

$$E'_\Delta \cong E'_{\alpha_1} \otimes_{E_{\alpha_1}} \left(E'_{\alpha_2} \otimes_{E_{\alpha_2}} \left(\cdots (E'_{\alpha_n} \otimes_{E_{\alpha_n}} E_\Delta)\right)\right) \ .$$

Proof. By rearranging the order of tensor products we have an identification

$$E'^{+}_{\Delta,\circ} = \bigotimes_{\alpha \in \Delta, \mathbb{F}_p} (E'^{+}_\alpha \otimes_{E^+_\alpha} E^+_\alpha) \cong E'^{+}_{\alpha_1} \otimes_{E^+_{\alpha_1}} \left(E'^{+}_{\alpha_2} \otimes_{E^+_{\alpha_2}} \left(\dots (E'^{+}_{\alpha_n} \otimes_{E^+_{\alpha_n}} E^+_{\Delta,\circ})\right)\right) \ .$$

The statement follows by completing this with respect to the maximal ideal of E^+_Δ and inverting X_Δ. $\square$

We define the multivariable analogue of E^{sep} as

$$E^{sep}_\Delta := \varinjlim_{E_\alpha \leq E'_\alpha \leq E^{sep}_\alpha, \forall \alpha \in \Delta} E'_\Delta \ .$$

For any subset $S \subseteq \Delta$ we define the similar notions E'^{+}_S, E'_S, and E^{sep}_S with Δ replaced by S. We equip E^{sep}_Δ with the relative Frobenii φ_α for each $\alpha \in \Delta$ defined above on each E'_Δ. Further, E^{sep}_Δ admits an action of $G_{\mathbb{Q}_p,\Delta}$ satisfying

Proposition 3.3. *Assume that the extensions E'_α/E_α are Galois for all $\alpha \in \Delta$ and let $H' := \prod_{\alpha \in \Delta} H'_\alpha$ where $H'_\alpha := \mathrm{Gal}(E^{sep}_\alpha/E'_\alpha)$. Then we have $(E^{sep}_\Delta)^{H'_\Delta} = E'_\Delta$. In particular, the subring $(E^{sep}_\Delta)^{H_{\mathbb{Q}_p,\Delta}}$ of $H_{\mathbb{Q}_p,\Delta}$-invariants in E^{sep}_Δ equals E_Δ with the previously defined action of $\Gamma_\Delta \cong G_{\mathbb{Q}_p,\Delta}/H_{\mathbb{Q}_p,\Delta}$.*

Proof. Since X_Δ is H'_Δ-invariant and $\varinjlim$ can be interchanged with taking H'_Δ-invariants, it suffices to show that whenever

$$E_\alpha = \mathbb{F}_p((X_\alpha)) \leq E'_\alpha = \mathbb{F}_{q'_\alpha}((X'_\alpha)) \leq E''_\alpha = \mathbb{F}_{q''_\alpha}((X''_\alpha))$$

is a a sequence of finite Galois extensions for each $\alpha \in \Delta$ then we have $(E''^{+}_\Delta)^{H'_\Delta} = E'^{+}_\Delta$. The containment $(E''^{+}_\Delta)^{H'_\Delta} \supseteq E'^{+}_\Delta$ is clear. We prove the converse by induction on $|\Delta|$. Note that the ideal $\mathcal{M}_\alpha \lhd E''^{+}_\Delta$ generated by X''_α is invariant under the action of H'_Δ for any fixed α in Δ. Moreover, for any integer $k \geq 1$ the ring $E''^{+}_\alpha/\mathcal{M}^k_\alpha$ is finite dimensional over $\mathbb{F}_p$. Therefore the image of $(E''^{+}_\Delta)^{H'_\Delta}$ under the quotient map $E''^{+}_\Delta \twoheadrightarrow E''^{+}_\Delta/\mathcal{M}^k_\alpha$ is contained in

$$\begin{aligned}\left(E''^{+}_\Delta/\mathcal{M}^k_\alpha\right)^{H'_\Delta} \subseteq \left(E''^{+}_\Delta/\mathcal{M}^k_\alpha\right)^{H'_{\Delta\setminus\{\alpha\}}} = \left(E''^{+}_{\Delta\setminus\{\alpha\}} \otimes_{\mathbb{F}_p} \left(E''^{+}_\alpha/\mathcal{M}^k_\alpha\right)\right)^{H'_{\Delta\setminus\{\alpha\}}} = \\ = \left(E''^{+}_{\Delta\setminus\{\alpha\}}\right)^{H'_{\Delta\setminus\{\alpha\}}} \otimes_{\mathbb{F}_p} \left(E''^{+}_\alpha/\mathcal{M}^k_\alpha\right) = E'^{+}_{\Delta\setminus\{\alpha\}} \otimes_{\mathbb{F}_p} \left(E''^{+}_\alpha/\mathcal{M}^k_\alpha\right)\end{aligned}$$

by induction. Taking the projective limit with respect to $k \geq 1$ we deduce that $(E''^{+}_{\Delta})^{H'_\Delta}$ is contained in the power series ring

$$\left(\mathbb{F}_{q''_\alpha} \otimes_{\mathbb{F}_p} \bigotimes_{\beta\in\Delta\setminus\{\alpha\},\mathbb{F}_p} \mathbb{F}_{q'_\beta}\right)[[X''_\alpha, X'_\beta \mid \beta \in \Delta \setminus \{\alpha\}]] \subseteq E''^{+}_{\Delta} \ .$$

Now using the action of H'_α in a similar argument as above (reducing modulo the kth power of the ideal generated by all the X'_β, $\beta \in \Delta \setminus \{\alpha\}$ for all $k \geq 1$) we deduce the statement. $\square$

The subring $E^{sep}_{\Delta,\circ} \cong \bigotimes_{\alpha\in\Delta,\mathbb{F}_p} E^{sep}_\alpha$ in E^{sep}_Δ is the inductive limit of $E'_{\Delta,\circ} \subseteq E'_\Delta$ where E'_α runs through the finite separable extensions of E_α for each $\alpha \in \Delta$.

Let V be a finite dimensional representation of the group $G_{\mathbb{Q}_p,\Delta}$ over $\mathbb{F}_p$. The basechange $E^{sep}_\Delta \otimes_{\mathbb{F}_p} V$ is equipped with the diagonal semilinear action of $G_{\mathbb{Q}_p,\Delta}$ and with the Frobenii φ_α for $\alpha \in \Delta$. These all commute with each other. We define the value of the functor $\mathbb{D}$ at V by putting

$$\mathbb{D}(V) := (E^{sep}_\Delta \otimes_{\mathbb{F}_p} V)^{H_{\mathbb{Q}_p,\Delta}} \ .$$

By Lemma 3.3 $\mathbb{D}(V)$ is a module over E_Δ inheriting the action of the monoid $T_{+,\Delta}$ from the action of φ_α ($\alpha \in \Delta$) and the Galois group $G_{\mathbb{Q}_p,\Delta}$ on $E^{sep}_\Delta \otimes_{\mathbb{F}_p} V$. Our key Lemma is the following.

Lemma 3.4. *The E^{sep}_Δ-module $E^{sep}_\Delta \otimes_{\mathbb{F}_p} V$ admits a basis consisting of elements fixed by $H_{\mathbb{Q}_p,\Delta}$.*

Proof. At first consider the $E^{sep}_{\Delta,\circ}$-module $E^{sep}_{\Delta,\circ} \otimes_{\mathbb{F}_p} V$. We show by induction on $|\Delta|$ that $E^{sep}_{\Delta,\circ} \otimes_{\mathbb{F}_p} V$ admits a basis consisting of $H_{\mathbb{Q}_p,\Delta}$-invariant vectors. The statement follows from this noting that $E^{sep}_{\Delta,\circ}$ is a subring in E^{sep}_Δ therefore the required basis exists also in $E^{sep}_\Delta \otimes_{\mathbb{F}_p} V \cong E^{sep}_\Delta \otimes_{E^{sep}_{\Delta,\circ}} (E^{sep}_{\Delta,\circ} \otimes_{\mathbb{F}_p} V)$.

By Hilbert's Thm. 90 the $H_{\mathbb{Q}_p,\alpha}$-module $E^{sep}_\alpha \otimes_{\mathbb{F}_p} V$ is trivial for each $\alpha \in \Delta$. So we have an E^{sep}_α-basis $e^{(\alpha)}_1, \dots, e^{(\alpha)}_d$ of $E^{sep}_\alpha \otimes_{\mathbb{F}_p} V$ consisting of $H_{\mathbb{Q}_p,\alpha}$-invariant elements. Since we have an action of the direct product $H_{\mathbb{Q}_p,\Delta}$ on V, the E_α-vector space

$$V_\alpha := E_\alpha e^{(\alpha)}_1 + \cdots + E_\alpha e^{(\alpha)}_d = (E^{sep}_\alpha \otimes_{\mathbb{F}_p} V)^{H_{\mathbb{Q}_p,\alpha}}$$

admits a linear action of the group $H_{\mathbb{Q}_p,\Delta\setminus\{\alpha\}}$. Now note that the representations V and V_α of the group $H_{\mathbb{Q}_p,\Delta\setminus\{\alpha\}}$ become isomorphic over the field E^{sep}_α by construction. Since $H_{\mathbb{Q}_p,\Delta\setminus\{\alpha\}}$ acts through a finite quotient on V, there is a finite extension E'_α of E_α contained in E^{sep}_α such that we have an isomorphism $E'_\alpha \otimes_{\mathbb{F}_p} V \cong E'_\alpha \otimes_{E_\alpha} V_\alpha$ of $H_{\mathbb{Q}_p,\Delta\setminus\{\alpha\}}$-representations. Making this identification and writing $e_i := 1 \otimes e_i \in E'_\alpha \otimes_{\mathbb{F}_p} V$ (resp. $e^{(\alpha)}_i := 1 \otimes e^{(\alpha)}_i$), $i = 1, \dots, d$, for a basis $e_1, \dots, e_d$ in V (resp. for the basis $e^{(\alpha)}_1, \dots e^{(\alpha)}_d$ in V_α) by an abuse of notation, we find a matrix $B \in \mathrm{GL}_d(E'_\alpha)$ with $B\rho(h) = \rho_\alpha(h)B$ for all $h \in H_{\mathbb{Q}_p,\Delta\setminus\{\alpha\}}$ where $\rho(h) \in \mathrm{GL}_d(\mathbb{F}_p)$ (resp. $\rho_\alpha(h) \in \mathrm{GL}_d(E_\alpha)$) is the matrix of the action of h on V (resp. on V_α) in the basis $e_1, \dots, e_d$ (resp. $e^{(\alpha)}_1, \dots e^{(\alpha)}_d$). Now E'_α/E_α is a finite separable extension, so there exists a primitive element $u \in E'_\alpha$ with $E'_\alpha = E_\alpha(u)$. Hence we may write B is a sum $B = B(u) = B_0 + B_1 u + \cdots + B_{n-1}u^{n-1}$ for some matrices $B_0, B_1, \dots, B_{n-1} \in E^{d\times d}_\alpha$ with $n := |E'_\alpha : E_\alpha|$. Since $\det B \neq 0$, the polynomial $\det(B(x)) := \det(B_0 + B_1 x + \cdots + B_{n-1}x^{n-1}) \in E_\alpha[x]$ is not identically 0. As E_α is an infinite field, there exists a $u_0 \in E_\alpha$ with $\det B(u_0) \neq 0$. Now we have $\rho(h) = B(u_0)^{-1}\rho_\alpha(h)B(u_0)$ for all $h \in H_{\mathbb{Q}_p,\Delta\setminus\{\alpha\}}$, ie. the representations V and V_α of $H_{\mathbb{Q}_p,\Delta\setminus\{\alpha\}}$ are isomorphic already over E_α. This shows that there exists a basis $v_1^{(\alpha)},\dots v_d^{(\alpha)}$ in V_α such that the action of each h in $H_{\mathbb{Q}_p,\Delta\setminus\{\alpha\}}$ is given by a matrix in $\mathrm{GL}_d(\mathbb{F}_p)$ in this basis. We put

$$V_{\Delta\setminus\{\alpha\}} := \mathbb{F}_p v_1^{(\alpha)} + \dots \mathbb{F}_p v_d^{(\alpha)} \subset V_\alpha = \left(E_\alpha^{sep}\otimes_{\mathbb{F}_p} V\right)^{H_{\mathbb{Q}_p,\alpha}} =$$
$$= \left(\bigotimes_{\beta\in\Delta\setminus\{\alpha\}} 1\otimes (E_\alpha^{sep}\otimes_{\mathbb{F}_p} V)\right)^{H_{\mathbb{Q}_p,\alpha}} \subseteq \left(E_{\Delta,\circ}^{sep}\otimes_{\mathbb{F}_p} V\right)^{H_{\mathbb{Q}_p,\alpha}} .$$

By induction we find a basis $v_1,\dots,v_n$ of $E_{\Delta\setminus\{\alpha\}}^{sep}\otimes_{\mathbb{F}_p} V_{\Delta\setminus\{\alpha\}} \subseteq \left(E_{\Delta,\circ}^{sep}\otimes_{\mathbb{F}_p} V\right)^{H_{\mathbb{Q}_p,\alpha}}$ consisting of $H_{\mathbb{Q}_p,\Delta\setminus\{\alpha\}}$-invariant elements which are $H_{\mathbb{Q}_p,\alpha}$-invariant, as well, by construction. Therefore $v_1,\dots,v_n$ is an $H_{\mathbb{Q}_p,\Delta}$-invariant basis of $E_{\Delta,\circ}^{sep}\otimes_{\mathbb{F}_p} V$ as required. □

Lemma 3.5. *We have $(E_\Delta^{sep})^\times \cap E_\Delta = E_\Delta^\times$.*

Proof. Let u be arbitrary in $(E_\Delta^{sep})^\times \cap E_\Delta$. Since u is invariant under the action of $H_{\mathbb{Q}_p,\Delta}$, so is its inverse u^{-1} whence it also lies in E_Δ by Lemma 3.3. □

Lemma 3.6. *We have $\bigcap_{\alpha\in\Delta}(E_\Delta^{sep})^{\varphi_\alpha=\mathrm{id}} = \mathbb{F}_p$.*

Proof. The containment $\mathbb{F}_p \subseteq \bigcap_{\alpha\in\Delta}(E_\Delta^{sep})^{\varphi_\alpha=\mathrm{id}} \subseteq (E_\Delta^{sep})^{\varphi_s=\mathrm{id}}$ is obvious. On the other hand, let $u\in E_\Delta^{sep}$ be arbitrary such that $\varphi_\alpha(u)=u$ for all $\alpha\in\Delta$. Then we also have $u^p=\varphi_s(u)=u$ as φ_s is the absolute Frobenius on E_Δ^{sep}. Since E_Δ^{sep} is defined as an inductive limit, u lies in $E'_\Delta \cong (\bigotimes_{\alpha\in\Delta,\mathbb{F}_p}\mathbb{F}_{q_\alpha})[[X'_\alpha\mid\alpha\in\Delta]][X_\Delta]$ for some collection $E'_\alpha = \mathbb{F}_{q_\alpha}((X'_\alpha))$ $(\alpha\in\Delta)$ of finite separable extensions of E_α. Note that $\bigotimes_{\alpha\in\Delta,\mathbb{F}_p}\mathbb{F}_{q_\alpha}$ is a finite étale algebra over $\mathbb{F}_p$, in particular, it is reduced. Therefore we have $|u^p|_{prod} = |u|_{prod}^p$. We deduce $|u|_{prod}=1$ unless $u=0$. In particular, u lies in $E'^+_\Delta = (\bigotimes_{\alpha\in\Delta,\mathbb{F}_p}\mathbb{F}_{q_\alpha})[[X'_\alpha\mid\alpha\in\Delta]]$. The constant term $u_0\in\bigotimes_{\alpha\in\Delta,\mathbb{F}_p}\mathbb{F}_{q_\alpha}$ also satisfies $\varphi_\alpha(u_0)=u_0$ for all $\alpha\in\Delta$. For a fixed $\alpha\in\Delta$ we choose an $\mathbb{F}_p$-basis $d_1,\dots,d_n$ of $\bigotimes_{\beta\in\Delta\setminus\{\alpha\},\mathbb{F}_p}\mathbb{F}_{q_\beta}$ and write $u_0=\sum_{i=1}^n c_i\otimes d_i$ with $c_i\in\mathbb{F}_{q_\alpha}$. This decomposition is unique and we compute

$$\sum_{i=1}^n c_i\otimes d_i = u_0 = \varphi_\alpha(u_0) = \sum_{i=1}^n c_i^p\otimes d_i .$$

We deduce $c_i=c_i^p$, ie. $c_i\in\mathbb{F}_p$ for all $1\le i\le n$. It follows by induction on $|\Delta|$ that u_0 lies in $\mathbb{F}_p$. Now $u-u_0$ is also fixed by each φ_α $(\alpha\in\Delta)$, but we have $|u-u_0|_{prod}<1$. This implies by the discussion above that $u=u_0$ is in $\mathbb{F}_p$ as desired. □

Proposition 3.7. *$\mathbb{D}(V)$ is an étale $T_{+,\Delta}$-module over E_Δ of rank $d := \dim_{\mathbb{F}_p} V$. Moreover, we have $E_\Delta^{sep}\otimes_{E_\Delta}\mathbb{D}(V)\cong E_\Delta^{sep}\otimes_{\mathbb{F}_p} V$ and*

$$V = \bigcap_{\alpha\in\Delta}(E_\Delta^{sep}\otimes_{E_\Delta}\mathbb{D}(V))^{\varphi_\alpha=\mathrm{id}} .$$

Proof. By Lemmata 3.3 and 3.4 $\mathbb{D}(V)$ is a free module of rank d over E_Δ. Moreover, the matrix of φ_α in any basis of $\mathbb{D}(V)$ is invertible in E_Δ^{sep}, therefore also in E_Δ by Lemma 3.5. So the action of $T_{+,\Delta}$ on $\mathbb{D}(V)$ is étale. The last statement is a direct consequence of Lemmata 3.4 and 3.6. □

Lemma 3.8. *For objects V, V_1, V_2 in $\mathrm{Rep}_{\mathbb{F}_p}(G_{\mathbb{Q}_p,\Delta})$ we have $\mathbb{D}(V_1 \otimes_{\mathbb{F}_p} V_2) \cong \mathbb{D}(V_1) \otimes_{E_\Delta} \mathbb{D}(V_2)$ and $\mathbb{D}(V^*) \cong \mathbb{D}(V)^*$.*

Proof. We compute

$$\mathbb{D}(V_1 \otimes_{\mathbb{F}_p} V_2) = \left(E_\Delta^{sep} \otimes_{\mathbb{F}_p} V_1 \otimes_{\mathbb{F}_p} V_2\right)^{H_{\mathbb{Q}_p,\Delta}} \cong \Big((E_\Delta^{sep} \otimes_{\mathbb{F}_p} V_1) \otimes_{E_\Delta^{sep}} (E_\Delta^{sep} \otimes_{\mathbb{F}_p} V_2)\Big)^{H_{\mathbb{Q}_p,\Delta}} \cong$$
$$\Big((E_\Delta^{sep} \otimes_{E_\Delta} \mathbb{D}(V_1)) \otimes_{E_\Delta^{sep}} (E_\Delta^{sep} \otimes_{E_\Delta} \mathbb{D}(V_2))\Big)^{H_{\mathbb{Q}_p,\Delta}} \cong$$
$$\cong (E_\Delta^{sep} \otimes_{E_\Delta} (\mathbb{D}(V_1) \otimes_{E_\Delta} \mathbb{D}(V_2)))^{H_{\mathbb{Q}_p,\Delta}} \cong \mathbb{D}(V_1) \otimes_{E_\Delta} \mathbb{D}(V_2) \ .$$

For the second statement we have

$$\mathbb{D}(V^*) = \left(E_\Delta^{sep} \otimes_{\mathbb{F}_p} \mathrm{Hom}_{\mathbb{F}_p}(V, \mathbb{F}_p)\right)^{H_{\mathbb{Q}_p,\Delta}} \cong \mathrm{Hom}_{E_\Delta^{sep}}(E_\Delta^{sep} \otimes_{\mathbb{F}_p} V, E_\Delta^{sep})^{H_{\mathbb{Q}_p,\Delta}} \cong$$
$$\cong \mathrm{Hom}_{E_\Delta^{sep}}(E_\Delta^{sep} \otimes_{E_\Delta} \mathbb{D}(V), E_\Delta^{sep})^{H_{\mathbb{Q}_p,\Delta}} \cong (E_\Delta^{sep} \otimes_{E_\Delta} \mathrm{Hom}_{E_\Delta}(\mathbb{D}(V), E_\Delta))^{H_{\mathbb{Q}_p,\Delta}} \cong \mathbb{D}(V)^* \ .$$

□

Theorem 3.9. *$\mathbb{D}$ is a fully faithful tensor functor from the category $\mathrm{Rep}_{\mathbb{F}_p}(G_{\mathbb{Q}_p,\Delta})$ to the category $\mathcal{D}^{et}(\varphi_\Delta, \Gamma_\Delta, E_\Delta)$.*

Proof. Let $f\colon V_1 \to V_2$ be a nonzero morphism in $\mathrm{Rep}_{\mathbb{F}_p}(G_{\mathbb{Q}_p,\Delta})$. Then the E_Δ^{sep}-linear map $\mathrm{id} \otimes f\colon E_\Delta^{sep} \otimes_{\mathbb{F}_p} V_1 \to E_\Delta^{sep} \otimes_{\mathbb{F}_p} V_2$ is also nonzero. By the last statement in Prop. 3.7 it follows that $\mathbb{D}(f) \neq 0$ therefore the faithfulness.

Now let V_1 and V_2 be arbitrary objects in $\mathrm{Rep}_{\mathbb{F}_p}(G_{\mathbb{Q}_p,\Delta})$ and $\theta\colon \mathbb{D}(V_1) \to \mathbb{D}(V_2)$ be a morphism in $\mathcal{D}^{et}(\varphi_\Delta, \Gamma_\Delta, E_\Delta)$. Then by Prop. 3.7 we obtain a $G_{\mathbb{Q}_p,\Delta}$-equivariant $\mathbb{F}_p$-linear map

$$f\colon V_1 = \bigcap_{\alpha\in\Delta} (E_\Delta^{sep} \otimes_{E_\Delta} \mathbb{D}(V_1))^{\varphi_\alpha=\mathrm{id}} \to \bigcap_{\alpha\in\Delta} (E_\Delta^{sep} \otimes_{E_\Delta} \mathbb{D}(V_2))^{\varphi_\alpha=\mathrm{id}} = V_2$$

induced by θ for which we have $\theta = \mathbb{D}(f)$. Therefore $\mathbb{D}$ is full. The compatibility with tensor products is proven in Lemma 3.8. □

Remark. Note that any étale $T_{+,\Delta}$-module D in the image of the functor $\mathbb{D}$ is free as a module over E_Δ by construction.

Consider the diagonal embedding $\mathrm{diag}\colon G_{\mathbb{Q}_p} \hookrightarrow G_{\mathbb{Q}_p,\Delta}$ sending $g \in G_{\mathbb{Q}_p}$ to $(g, \ldots, g)$. This defines a functor $\widehat{\mathrm{diag}}\colon \mathrm{Rep}_{\mathbb{F}_p}(G_{\mathbb{Q}_p,\Delta}) \to \mathrm{Rep}_{\mathbb{F}_p}(G_{\mathbb{Q}_p})$ via restriction. On the other hand, we have the reduction map $\ell\colon \mathcal{D}^{et}(\varphi_\Delta, \Gamma_\Delta, E_\Delta) \to \mathcal{D}^{et}(\varphi, \Gamma, E)$ to usual (φ, Γ)-modules defined in section 2.4 of [11]. Recall that this is given by taking the quotient by the ideal generated by $(X_\alpha - X_\beta \mid \alpha, \beta \in \Delta)$ and restricting to the diagonal $\varphi = \varphi_s = \prod_{\alpha\in\Delta} \varphi_\alpha$ and $\Gamma := \{(\gamma, \ldots, \gamma)\} \leq \Gamma_\Delta$.

Corollary 3.10. *There is a natural isomorphism $\widehat{\mathrm{diag}} \cong \mathbb{V}_F \circ \ell \circ \mathbb{D}$ of functors $\mathrm{Rep}_{\mathbb{F}_p}(G_{\mathbb{Q}_p,\Delta}) \to \mathrm{Rep}_{\mathbb{F}_p}(G_{\mathbb{Q}_p})$ where $\mathbb{V}_F\colon \mathcal{D}^{et}(\varphi, \Gamma, E) \to \mathrm{Rep}_{\mathbb{F}_p}(G_{\mathbb{Q}_p})$ is Fontaine's functor from classical étale (φ, Γ)-modules to Galois representations.*

Proof. We may identify $E_\alpha \xrightarrow{\sim} E = \mathbb{F}_p((X))$ by sending $X_\alpha \to X$ for all $\alpha \in \Delta$. We extend this identification to $E_\alpha^{sep} \to E^{sep}$. So we obtain a map $\ell^{sep}\colon E_\Delta^{sep} \to E^{sep}$ sending each subring E_α^{sep} to E^{sep} via these identifications and completing on the level of each finite extension E'_Δ. We do this in a way so that the diagonal embedding of $G_{\mathbb{Q}_p} \hookrightarrow G_{\mathbb{Q}_p,\Delta}$ acts on the quotient E^{sep} in the usual way. The restriction of ℓ^{sep} to E_Δ is the map $\ell\colon E_\Delta \to E$ defined above, so the diagram

$$\begin{array}{ccc} E_\Delta & \hookrightarrow & E_\Delta^{sep} \\ \downarrow{\scriptstyle \ell} & & \downarrow{\scriptstyle \ell^{sep}} \\ E & \hookrightarrow & E^{sep} \end{array}$$

commutes. Thus for an object V in $\mathrm{Rep}_{\mathbb{F}_p}(G_{\mathbb{Q}_p,\Delta})$ we compute

$$\begin{aligned} \mathbb{V}_F \circ \ell \circ \mathbb{D}(V) = \mathbb{V}_F(E \otimes_{E_\Delta,\ell} \mathbb{D}(V)) = \mathbb{V}_F((E^{sep})^{H_{\mathbb{Q}_p}} \otimes_{E_\Delta,\ell} \mathbb{D}(V)) = \\ = \mathbb{V}_F((E^{sep} \otimes_{E_\Delta^{sep},\ell^{sep}} E_\Delta^{sep} \otimes_{E_\Delta} \mathbb{D}(V))^{H_{\mathbb{Q}_p}}) = \mathbb{V}_F((E^{sep} \otimes_{E_\Delta^{sep},\ell^{sep}} E_\Delta^{sep} \otimes_{\mathbb{F}_p} V)^{H_{\mathbb{Q}_p}}) = \\ = \mathbb{V}_F((E^{sep} \otimes_{\mathbb{F}_p} V)^{H_{\mathbb{Q}_p}}) = \mathbb{V}_F \circ \mathbb{D}_F(V) = V \mid_{\mathrm{diag}(G_{\mathbb{Q}_p})} = \widehat{\mathrm{diag}}(V) \ . \end{aligned}$$

□

3.2 The functor $\mathbb{V}$

In order to show that the functor $\mathbb{D}$ is essentially surjective, we construct its quasi-inverse $\mathbb{V}$. Let D be an object in $\mathcal{D}^{et}(\varphi_\Delta, \Gamma_\Delta, E_\Delta)$. The group $G_{\mathbb{Q}_p,\Delta}$ acts on $E_\Delta^{sep} \otimes_{E_\Delta} D$ via the formula $g(\lambda \otimes x) := g(\lambda) \otimes \chi_{cyc}(g)(x)$ ($g \in G_{\mathbb{Q}_p,\Delta}$, $\lambda \in E_\Delta^{sep}$, $x \in D$) where $\chi_{cyc}\colon G_{\mathbb{Q}_p,\Delta} \to \Gamma_\Delta$ is the quotient map. Moreover, each partial Frobenius φ_α ($\alpha \in \Delta$) acts semilinearly on $E_\Delta^{sep} \otimes_{E_\Delta} D$ via the formula $\varphi_\alpha(\lambda \otimes x) := \varphi_\alpha(\lambda) \otimes \varphi_\alpha(x)$. All these actions commute with each other by construction. We define

$$\mathbb{V}(D) := \bigcap_{\alpha \in \Delta} (E_\Delta^{sep} \otimes_{E_\Delta} D)^{\varphi_\alpha = \mathrm{id}} \ .$$

$\mathbb{V}(D)$ is a—a priori not necessarily finite dimensional—representation of $G_{\mathbb{Q}_p,\Delta}$ over $\mathbb{F}_p$.

Lemma 3.11. *For any integer $r > 0$ we have $\bigcap_{\beta \in \Delta} (E_{\Delta\setminus\{\alpha\}}^{sep}[X_\alpha]/(X_\alpha^r))^{\varphi_\beta = \mathrm{id}} = \mathbb{F}_p[X_\alpha]/(X_\alpha^r)$.*

Proof. This follows from Lemma 3.6 noting that $\mathbb{F}_p[X_\alpha]/(X_\alpha^r)$ is a finite dimensional $\mathbb{F}_p$-vector space on which φ_β acts identically for all $\beta \in \Delta \setminus \{\alpha\}$ and we have $E_{\Delta\setminus\{\alpha\}}^{sep}[X_\alpha]/(X_\alpha^r) \cong E_{\Delta\setminus\{\alpha\}}^{sep} \otimes_{\mathbb{F}_p} \mathbb{F}_p[X_\alpha]/(X_\alpha^r)$. □

Lemma 3.12. *For any integer $r > 0$ and finitely generated $E_\alpha^+/(X_\alpha^r)$-module M we have an identification $E_{\Delta\setminus\{\alpha\}}^{sep}[X_\alpha]/(X_\alpha^r) \otimes_{E_\alpha^+/(X_\alpha^r)} M \cong E_{\Delta\setminus\{\alpha\}}^{sep} \otimes_{E_{\Delta\setminus\{\alpha\}}} M$.*

Proof. This follows from the isomorphism $E_\alpha^+/(X_\alpha^r) \cong E_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r)$. □

For a subset $S \subseteq \Delta$ we put $E_S^{sep+} := \varinjlim E_S'^{+}$ so we have $E_S^{sep} = E_S^{sep+}[X_S^{-1}]$.

Lemma 3.13. *E_S^{sep} (resp. E_S^{sep+}) is flat as a module over E_S (resp. over E_S^+) for all $S \subseteq \Delta$.*

Proof. By construction, E'_S (resp. E'^{+}_S) is finite free over E_S (resp. over E_S^+), so E_S^{sep} (resp. E_S^{sep+}) is the direct limit of flat modules hence flat. $\square$

Lemma 3.14. *We have* $(E^{sep+}_{\Delta\setminus\{\alpha\}}[[X_\alpha]][X_\Delta^{-1}])^{H_{\mathbb{Q}_p,\Delta\setminus\{\alpha\}}} = E_\Delta$.

Proof. We have $E_\Delta = E^+_{\Delta\setminus\{\alpha\}}[[X_\alpha]][X_\Delta^{-1}]$ where $E^+_{\Delta\setminus\{\alpha\}} = (E^{sep+}_{\Delta\setminus\{\alpha\}})^{H_{\mathbb{Q}_p,\Delta\setminus\{\alpha\}}}$ by Lemma 3.3 and $H_{\mathbb{Q}_p,\Delta\setminus\{\alpha\}}$ acts trivially on both X_α and X_Δ, so acts on the power series ring $E^+_{\Delta\setminus\{\alpha\}}[[X_\alpha]]$ coefficientwise. $\square$

Our main result in this section is the following

Theorem 3.15. *The functors* $\mathbb{D}$ *and* $\mathbb{V}$ *are quasi-inverse equivalences of categories between the Tannakian categories* $\mathrm{Rep}_{\mathbb{F}_p}(G_{\mathbb{Q}_p,\Delta})$ *and* $\mathcal{D}^{et}(\varphi_\Delta, \Gamma_\Delta, E_\Delta)$.

Corollary 3.16. *Any object* D *in* $\mathcal{D}^{et}(\varphi_\Delta, \Gamma_\Delta, E_\Delta)$ *is a free module over* E_Δ.

Proof. This follows from the essential surjectivity of $\mathbb{D}$ using the remark after Thm. 3.9. $\square$

Proof of Thm. 3.15. This is a long proof that we divide into 5 steps.

Step 1. Reducing the statement to the essential surjectivity of $\mathbb{D}$. By Thm. 3.9 the functor $\mathbb{D}$ is fully faithful and we have $\mathbb{V}\circ\mathbb{D}(V)\cong V$ naturally in V for any object V in $\mathrm{Rep}_{\mathbb{F}_p}(G_{\mathbb{Q}_p,\Delta})$ by Prop. 3.7. Moreover, by Lemma 3.8 $\mathbb{D}$ is compatible with tensor products and duals. So it remains to show that $\mathbb{D}$ is essentially surjective. We proceed by induction on $|\Delta|$. For $|\Delta| = 1$ this is a classical result of Fontaine (see e.g. Thm. 2.21 in [5]). Suppose that $|\Delta| > 1$, fix $\alpha\in\Delta$, and pick an object D in $\mathcal{D}^{et}(\varphi_\Delta, \Gamma_\Delta, E_\Delta)$.

Step 2. The goal here is to trivialize the φ_β*-action (*$\beta\in\Delta\setminus\{\alpha\}$*) on* $D^{+*}_{\overline{\alpha}}/X_\alpha^r$ *uniformly in* r *by tensoring up with* $E^{sep}_{\Delta\setminus\{\alpha\}}$. By Prop. 2.10 $D^{+*}_{\overline{\alpha}}$ is an étale $T_{+,\overline{\alpha}}$-module over $E^+_{\overline{\alpha}}$. Reducing mod X_α^r for an integer $r>0$ we deduce that $D^{+*}_{\overline{\alpha},r} := D^{+*}_{\overline{\alpha}}/X_\alpha^r D^{+*}_{\overline{\alpha}}$ is an étale $T_{+,\overline{\alpha}}$-module over $E^+_{\overline{\alpha}}/(X_\alpha^r)\cong E_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r)$. Since each φ_β ($\beta\in\Delta\setminus\{\alpha\}$) acts trivially on the variable X_α, we have a natural isomorphism of functors

$$E_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r)\otimes_{E_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r),\varphi_t}\cdot\cong E_{\Delta\setminus\{\alpha\}}\otimes_{E_{\Delta\setminus\{\alpha\}},\varphi_t}\cdot$$

for all $t\in T_{+,\overline{\alpha}}$. Hence $D^{+*}_{\overline{\alpha},r}$ is an object in $\mathcal{D}^{et}(\varphi_{\Delta\setminus\{\alpha\}}, \Gamma_{\Delta\setminus\{\alpha\}}, E_{\Delta\setminus\{\alpha\}})$ since $E_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r)$ is finitely generated as a module over $E_{\Delta\setminus\{\alpha\}}$. By induction, we can trivialize $D^{+*}_{\overline{\alpha},r}$ over $E^{sep}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r)$: the natural map

$$E^{sep}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r)\otimes_{\mathbb{F}_p[X_\alpha]/(X_\alpha^r)}\bigcap_{\beta\in\Delta\setminus\{\alpha\}}\left(E^{sep}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r)\otimes_{E^+_{\overline{\alpha}}/(X_\alpha^r)} D^{+*}_{\overline{\alpha},r}\right)^{\varphi_\beta=\mathrm{id}}\overset{\sim}{\to}$$

$$\overset{\sim}{\to} E^{sep}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r)\otimes_{E^+_{\overline{\alpha}}/(X_\alpha^r)} D^{+*}_{\overline{\alpha},r}\cong E^{sep}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r)\otimes_{E^+_{\overline{\alpha}}} D^{+*}_{\overline{\alpha}} \qquad (4)$$

is an isomorphism for all $r>0$ using Lemmata 3.11 and 3.12. Our key Lemma is the following consequence of Prop. 2.10.

Lemma 3.17. *There exists a finitely generated* E^+_Δ*-submodule* $M\leq D^{+*}_{\overline{\alpha}}$ *such that*

$$\bigcap_{\beta\in\Delta\setminus\{\alpha\}}\left(E^{sep}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r)\otimes_{E^+_{\overline{\alpha}}} D^{+*}_{\overline{\alpha},}\right)^{\varphi_\beta=\mathrm{id}} \qquad (5)$$

is contained in the image of the map

$$E^{sep+}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r)\otimes_{E^+_\Delta} M\to E^{sep+}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r)\otimes_{E^+_\Delta} D^{+*}_{\overline{\alpha}}\cong E^{sep}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r)\otimes_{E^+_{\overline{\alpha}}} D^{+*}_{\overline{\alpha}} \quad (6)$$

induced by the inclusion $M\leq D^{+*}_{\overline{\alpha}}$ for all $r>0$. Moreover, M can be chosen in such a way that (6) *is injective.*

Proof. We show that $M:=X^{-k}_{\Delta\setminus\{\alpha\}}(D^+\cap D^{+*}_{\overline{\alpha}})$ will do for k large enough. Since D^+ is finitely generated over E^+_Δ, so is M by noetherianity. Using Lemma 2.11 we choose $k>0$ so that we have $D^{+*}_{\overline{\alpha}}=\bigcup_{l\geq 0}E^+_\Delta\varphi^l_{\overline{\alpha}}(M)$, ie. we put $M:=X^{-1}_{\Delta\setminus\{\alpha\}}D_0$. For any fixed $r>0$ there exists an integer $l_r\geq 0$ such that (5) is contained in

$$E^{sep+}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r)\otimes_{E^+_\Delta} X^{-p^{l_r}+1}_{\Delta\setminus\{\alpha\}}M\subseteq E^{sep+}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r)\otimes_{E^+_\Delta} E^+_\Delta\varphi^{l_r}_{\overline{\alpha}}(M)= \\ =E^{sep+}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r)\varphi^{l_r}_{\overline{\alpha}}(E^{sep+}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r)\otimes_{E^+_\Delta} M)\ .$$

Now if x lies in (5), then we have $\varphi^{l_r}_{\overline{\alpha}}(x)=x$. On the other hand, x lies in

$$E'_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r)\varphi^{l_r}_{\overline{\alpha}}(E'_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r)\otimes_{E^+_\Delta} M)$$

for some finite separable extensions E'_β/E_β for $\beta\in\Delta\setminus\{\alpha\}$ and $E'_{\Delta\setminus\{\alpha\}}:=\widehat{\bigotimes}_{\beta\in\Delta\setminus\{\alpha\},\mathbb{F}_p}E'_\beta$. Therefore x lies in fact in $E'_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r)\otimes_{E^+_\Delta} M$ by the injectivity of the map

$$\mathrm{id}\otimes\varphi^{l_r}_{\overline{\alpha}}\colon E'_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r)\otimes_{E'_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r),\varphi^{l_r}_{\overline{\alpha}}}(E'_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r)\otimes_{E^+_{\overline{\alpha}}} D^{+*}_{\overline{\alpha}})\to \\ \to E'_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r)\otimes_{E^+_{\overline{\alpha}}} D^{+*}_{\overline{\alpha}}$$

($D^{+*}_{\overline{\alpha}}$ is étale) noting that the absolute Frobenius $\varphi_{\overline{\alpha}}\colon E'_{\Delta\setminus\{\alpha\}}\to E'_{\Delta\setminus\{\alpha\}}$ is injective since the ring $E'_{\Delta\setminus\{\alpha\}}$ is the localization of a power series ring over a finite étale algebra over $\mathbb{F}_p$, in particular, it is reduced.

Finally, by Lemma 2.7 $D^{+*}_{\overline{\alpha}}/M$ has no X_α-torsion as $D^{+*}_{\overline{\alpha}}/M\cong D^{+*}_{\overline{\alpha}}+X^{-k}_{\Delta\setminus\{\alpha\}}D^+/(X^{-k}_{\Delta\setminus\{\alpha\}}D^+)$ is contained in $D^+_{\overline{\alpha}}/(X^{-k}_{\Delta\setminus\{\alpha\}}D^+)\cong D^+_{\overline{\alpha}}/D^+$. Therefore the map (6) is injective. □

Step 3. The goal here is to show the following compatibility of our construction with projective limits with respect to r.

Lemma 3.18. *We have*

$$\varprojlim_r\left(E^{sep+}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r)\otimes_{E^+_\Delta} M\right)\cong E^{sep+}_{\Delta\setminus\{\alpha\}}[\![X_\alpha]\!]\otimes_{E^+_\Delta} M\ ,$$

$$\varprojlim_r\left(E^{sep}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r)\otimes_{E^+_{\overline{\alpha}}} D^{+*}_{\overline{\alpha}}\right)\cong E^{sep}_{\Delta\setminus\{\alpha\}}[\![X_\alpha]\!]\otimes_{E^+_{\overline{\alpha}}} D^{+*}_{\overline{\alpha}}\quad \text{, and}$$

$$\varprojlim_r\left(E^{sep}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r)\otimes_{\mathbb{F}_p[X_\alpha]/(X_\alpha^r)}\bigcap_{\beta\in\Delta\setminus\{\alpha\}}\left(E^{sep}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r)\otimes_{E^+_{\overline{\alpha}}/(X_\alpha^r)} D^{+*}_{\overline{\alpha},r}\right)^{\varphi_\beta=\mathrm{id}}\right)\cong \\ \cong E^{sep}_{\Delta\setminus\{\alpha\}}[\![X_\alpha]\!]\otimes_{\mathbb{F}_p[\![X_\alpha]\!]}\bigcap_{\beta\in\Delta\setminus\{\alpha\}}\left(E^{sep}_{\Delta\setminus\{\alpha\}}(\!(X_\alpha)\!)\otimes_{E_\Delta} D\right)^{\varphi_\beta=\mathrm{id}}\ .$$

Proof. Since M is contained in D, M has no X_α-torsion. In particular, M is flat as a module over the local ring $\mathbb{F}_p[\![X_\alpha]\!]$. Now we deduce that M and $E^{sep+}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r)$ are Tor-independent over E_Δ^+ by Lemma 3.13 since we have the identification

$$E^{sep+}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r)\otimes_{E_\Delta^+} M \cong E^{sep+}_{\Delta\setminus\{\alpha\}}\otimes_{E^+_{\Delta\setminus\{\alpha\}}}\left(\mathbb{F}_p[X_\alpha]/(X_\alpha^r)\otimes_{\mathbb{F}_p[\![X_\alpha]\!]} M\right) .$$

On the other hand, M is finitely generated over E_Δ^+, so we short exact sequences

$$0\to M_1\to (E_\Delta^+)^{k_0}\overset{f_0}{\to} M\to 0 \qquad \text{and} \qquad 0\to M_2\to (E_\Delta^+)^{k_1}\to M_1\to 0$$

by noetherianity. In order to simplify notation write $(\cdot)_r$ for $E^{sep+}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r)\otimes_{E_\Delta^+}\cdot$ to obtain an exact sequence

$$(M_2)_r\to (E_\Delta^+)_r^{k_1}\overset{f_{1,r}}{\to}(E_\Delta^+)_r^{k_0}\overset{f_{0,r}}{\to}(M)_r\to 0$$

for all $r>0$ using the Tor-indepence above. Now since the natural map $(N)_{r_1}\to(N)_{r_2}$ is surjective for any E_Δ^+-module N and $r_1\geq r_2>0$ by the right exactness of $\cdot\otimes_{E_\Delta^+}N$, the natural map $\mathrm{Ker}(f_{0,r_1})\to\mathrm{Ker}(f_{0,r_2})$ is also surjective (applying this in case $N=M_1$ and a diagram chasing). So the Mittag-Leffler property is satisfied for these projective systems showing that the map $\varprojlim_r f_{0,r}$ is surjective with kernel $\varprojlim_r \mathrm{Ker}(f_{0,r})=\varprojlim_r \mathrm{Im}(f_{1,r})$. Applying the same trick as above with $N=M_2$ we deduce that the projective system $\mathrm{Ker}(f_{1,r})$ also satisfies the Mittag-Leffler property showing that $\varprojlim_r f_{1,r}$ has image $\varprojlim_r \mathrm{Im}(f_{1,r})$. In particular, $\varprojlim_r (M)_r$ is the cokernel of the map $\varprojlim_r f_{1,r}\colon (E^{sep+}_{\Delta\setminus\{\alpha\}}[\![X_\alpha]\!])^{k_1}\to(E^{sep+}_{\Delta\setminus\{\alpha\}}[\![X_\alpha]\!])^{k_0}$ and so is $E^{sep+}_{\Delta\setminus\{\alpha\}}[\![X_\alpha]\!]\otimes_{E_\Delta^+}M$ as claimed. The second statement follows in the exactly same way.

For the third statement note that the isomorphism (4) and the surjectivity of the map $E^{sep}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^{r_1})\otimes_{E_\alpha^+}D_\alpha^{+*}\to E^{sep}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^{r_2})\otimes_{E_\alpha^+}D_\alpha^{+*}$ implies that the map

$$\bigcap_{\beta\in\Delta\setminus\{\alpha\}}\left(E^{sep}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^{r_1})\otimes_{E_\alpha^+/(X_\alpha^{r_1})}D^{+*}_{\alpha,r}\right)^{\varphi_\beta=\mathrm{id}}\to$$
$$\to\bigcap_{\beta\in\Delta\setminus\{\alpha\}}\left(E^{sep}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^{r_2})\otimes_{E_\alpha^+/(X_\alpha^{r_2})}D^{+*}_{\alpha,r}\right)^{\varphi_\beta=\mathrm{id}}$$

is also onto for all $r_1\geq r_2$. Therefore the natural map

$$\bigcap_{\beta\in\Delta\setminus\{\alpha\}}\left(E^{sep}_{\Delta\setminus\{\alpha\}}[\![X_\alpha]\!]\otimes_{E_\alpha^+}D_\alpha^{+*}\right)^{\varphi_\beta=\mathrm{id}}=$$
$$=\varprojlim_r\bigcap_{\beta\in\Delta\setminus\{\alpha\}}\left(E^{sep}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^{r})\otimes_{E_\alpha^+/(X_\alpha^{r})}D^{+*}_{\alpha,r}\right)^{\varphi_\beta=\mathrm{id}}\to$$
$$\to\bigcap_{\beta\in\Delta\setminus\{\alpha\}}\left(E^{sep}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha)\otimes_{E_\alpha^+/(X_\alpha)}D^{+*}_{\alpha,r}\right)^{\varphi_\beta=\mathrm{id}}$$

is also onto using the second statement of the Lemma. On the other hand, the kernel of this map equals

$$\bigcap_{\beta\in\Delta\setminus\{\alpha\}}\left(E^{sep}_{\Delta\setminus\{\alpha\}}[\![X_\alpha]\!]\otimes_{E^+_{\alpha}} D^{+*}_{\alpha}\right)^{\varphi_\beta=\mathrm{id}}\cap X_\alpha E^{sep}_{\Delta\setminus\{\alpha\}}[\![X_\alpha]\!]\otimes_{E^+_{\alpha}} D^{+*}_{\alpha}=$$

$$=X_\alpha\bigcap_{\beta\in\Delta\setminus\{\alpha\}}\left(E^{sep}_{\Delta\setminus\{\alpha\}}[\![X_\alpha]\!]\otimes_{E^+_{\alpha}} D^{+*}_{\alpha}\right)^{\varphi_\beta=\mathrm{id}}$$

since X_α is fixed by each φ_β and $E^{sep}_{\Delta\setminus\{\alpha\}}[\![X_\alpha]\!]\otimes_{E^+_{\alpha}} D^{+*}_{\alpha}$ has no X_α-torsion. This shows, in particular, that $\bigcap_{\beta\in\Delta\setminus\{\alpha\}}\left(E^{sep}_{\Delta\setminus\{\alpha\}}[\![X_\alpha]\!]\otimes_{E^+_{\alpha}} D^{+*}_{\alpha}\right)^{\varphi_\beta=\mathrm{id}}$ is finitely generated over $\mathbb{F}_p[\![X_\alpha]\!]$ by the topological Nakayama Lemma (see [1]). Moreover, it is torsion-free hence free as $E^{sep}_{\Delta\setminus\{\alpha\}}[\![X_\alpha]\!]\otimes_{E^+_{\alpha}} D^{+*}_{\alpha}$ has no X_α-torsion either. In particular,

$$E^{sep}_{\Delta\setminus\{\alpha\}}[\![X_\alpha]\!]\otimes_{\mathbb{F}_p[\![X_\alpha]\!]}\bigcap_{\beta\in\Delta\setminus\{\alpha\}}\left(E^{sep}_{\Delta\setminus\{\alpha\}}(\!(X_\alpha)\!)\otimes_{E_\Delta} D\right)^{\varphi_\beta=\mathrm{id}}$$

is X_α-adically complete and the result follows. □

Step 4. The goal here is to obtain a $(\varphi_\alpha,\Gamma_\alpha)$-module D_α over E_α (by trivializing the action of each φ_β, $\beta\in\Delta\setminus\{\alpha\}$) which is at the same time a linear representation of the group $G_{\mathbb{Q}_p,\Delta\setminus\{\alpha\}}$. We take projective limits of the inclusions in Lemma 3.17 with respect to r to conclude (using Lemma 3.18) that

$$\bigcap_{\beta\in\Delta\setminus\{\alpha\}}\left(E^{sep}_{\Delta\setminus\{\alpha\}}[\![X_\alpha]\!]\otimes_{E^+_{\alpha}} D^{+*}_{\alpha}\right)^{\varphi_\beta=\mathrm{id}}$$

is contained in the image of the map

$$E^{sep+}_{\Delta\setminus\{\alpha\}}[\![X_\alpha]\!]\otimes_{E^+_{\Delta}} M\to E^{sep}_{\Delta\setminus\{\alpha\}}[\![X_\alpha]\!]\otimes_{E^+_{\alpha}} D^{+*}_{\alpha}\ .$$

Note that $M[X_\Delta^{-1}]=D^{+*}_{\alpha}[X_\Delta^{-1}]=D^{+*}_{\alpha}[X_\alpha^{-1}]=D$ and φ_β acts trivially on X_α. So inverting X_Δ above we deduce that

$$D_\alpha:=\bigcap_{\beta\in\Delta\setminus\{\alpha\}}\left(E^{sep}_{\Delta\setminus\{\alpha\}}(\!(X_\alpha)\!)\otimes_{E_\Delta} D\right)^{\varphi_\beta=\mathrm{id}}$$

is contained in the image of the map

$$E^{sep+}_{\Delta\setminus\{\alpha\}}[\![X_\alpha]\!][X_\Delta^{-1}]\otimes_{E_\Delta} D\hookrightarrow E^{sep}_{\Delta\setminus\{\alpha\}}(\!(X_\alpha)\!)\otimes_{E_\Delta} D\ .$$

On the other hand, by (4) and the third statement of Lemma 3.18 we have an isomorphism

$$E^{sep}_{\Delta\setminus\{\alpha\}}(\!(X_\alpha)\!)\otimes_{\mathbb{F}_p(\!(X_\alpha)\!)} D_\alpha\xrightarrow{\sim}E^{sep}_{\Delta\setminus\{\alpha\}}(\!(X_\alpha)\!)\otimes_{E_\Delta} D\ . \tag{7}$$

Lemma 3.19. *The finite dimensional $\mathbb{F}_p(\!(X_\alpha)\!)$-vector space D_α has the structure of an étale $(\varphi_\alpha,\Gamma_\alpha)$-module. At the same time it is a (linear) representation of the group $G_{\mathbb{Q}_p,\Delta\setminus\{\alpha\}}$. These two actions commute with each other.*

Proof. The operator φ_α and the groups Γ_α and $G_{\mathbb{Q}_p,\Delta\setminus\{\alpha\}}$ act naturally on D_α. For the étaleness of the action of φ_α on D_α note that we have $\mathbb{F}_p(\!(X_\alpha)\!)\otimes_{\mathbb{F}_p(\!(X_\alpha)\!),\varphi_\alpha} D\cong D$ by the étale property of φ_α on D and that φ_β acts trivially on $\mathbb{F}_p(\!(X_\alpha)\!)$. So we compute

$$\begin{aligned}\mathbb{F}_p(\!(X_\alpha)\!)\otimes_{\mathbb{F}_p(\!(X_\alpha)\!),\varphi_\alpha} D_\alpha &= \mathbb{F}_p(\!(X_\alpha)\!)\otimes_{\mathbb{F}_p(\!(X_\alpha)\!),\varphi_\alpha}\bigcap_{\beta\in\Delta\setminus\{\alpha\}}\left(E^{sep}_{\Delta\setminus\{\alpha\}}(\!(X_\alpha)\!)\otimes_{E_\Delta} D\right)^{\varphi_\beta=\mathrm{id}} =\\ &= \bigcap_{\beta\in\Delta\setminus\{\alpha\}}\left(\mathbb{F}_p(\!(X_\alpha)\!)\otimes_{\mathbb{F}_p(\!(X_\alpha)\!),\varphi_\alpha} E^{sep}_{\Delta\setminus\{\alpha\}}(\!(X_\alpha)\!)\otimes_{E_\Delta} D\right)^{\varphi_\beta=\mathrm{id}} =\\ &= \bigcap_{\beta\in\Delta\setminus\{\alpha\}}\left(E^{sep}_{\Delta\setminus\{\alpha\}}(\!(X_\alpha)\!)\otimes_{E_\Delta}\mathbb{F}_p(\!(X_\alpha)\!)\otimes_{\mathbb{F}_p(\!(X_\alpha)\!),\varphi_\alpha} D\right)^{\varphi_\beta=\mathrm{id}} \cong\\ &\cong \bigcap_{\beta\in\Delta\setminus\{\alpha\}}\left(E^{sep}_{\Delta\setminus\{\alpha\}}(\!(X_\alpha)\!)\otimes_{E_\Delta} D\right)^{\varphi_\beta=\mathrm{id}} = D_\alpha\ .\end{aligned}$$

∎

Step 5. We show the essential surjectivity of $\mathbb{D}$ *here.* Now we apply $\mathbb{V}_{F,\alpha}=(E^{sep}_\alpha\otimes_{\mathbb{F}_p(\!(X_\alpha)\!)}\cdot)^{\varphi_\alpha=\mathrm{id}}$ on D_α to obtain a finite dimensional $\mathbb{F}_p$-representation V of $G_{\mathbb{Q}_p,\Delta}$. Moreover, we have $\dim_{\mathbb{F}_p} V=\dim_{\mathbb{F}_p(\!(X_\alpha)\!)} D_\alpha=\mathrm{rk}_{E_\Delta} D$ by the isomorphism (7) since $\mathbb{V}_{F,\alpha}$ is rank-preserving by Fontaine's classical result. Using again the isomorphism (7) and the containment $D_\alpha\subset E^{sep+}_{\Delta\setminus\{\alpha\}}[\![X_\alpha]\!][X_\Delta^{-1}]\otimes_{E_\Delta} D$ we conclude an injective map

$$E^{sep+}_{\Delta\setminus\{\alpha\}}[\![X_\alpha]\!][X_\Delta^{-1}]\otimes_{\mathbb{F}_p(\!(X_\alpha)\!)} D_\alpha\hookrightarrow E^{sep+}_{\Delta\setminus\{\alpha\}}[\![X_\alpha]\!][X_\Delta^{-1}]\otimes_{E_\Delta} D$$

and applying $E^{sep}_\alpha\otimes_{\mathbb{F}_p(\!(X_\alpha)\!)}\cdot$ another injective composite map

$$\begin{aligned}&E^{sep}_\Delta\otimes_{\mathbb{F}_p} V\hookrightarrow\\ \hookrightarrow&\left(E^{sep+}_{\Delta\setminus\{\alpha\}}[\![X_\alpha]\!][X_\Delta^{-1}]\otimes_{\mathbb{F}_p(\!(X_\alpha)\!)} E^{sep}_\alpha\right)\otimes_{\mathbb{F}_p} V\cong\\ \cong&\, E^{sep+}_{\Delta\setminus\{\alpha\}}[\![X_\alpha]\!][X_\Delta^{-1}]\otimes_{\mathbb{F}_p(\!(X_\alpha)\!)} E^{sep}_\alpha\otimes_{\mathbb{F}_p(\!(X_\alpha)\!)} D_\alpha=\\ =&\, E^{sep}_\alpha\otimes_{\mathbb{F}_p(\!(X_\alpha)\!)} E^{sep+}_{\Delta\setminus\{\alpha\}}[\![X_\alpha]\!][X_\Delta^{-1}]\otimes_{\mathbb{F}_p(\!(X_\alpha)\!)} D_\alpha\hookrightarrow\\ \hookrightarrow&\left(E^{sep}_\alpha\otimes_{\mathbb{F}_p(\!(X_\alpha)\!)} E^{sep+}_{\Delta\setminus\{\alpha\}}[\![X_\alpha]\!][X_\Delta^{-1}]\right)\otimes_{E_\Delta} D\ .\end{aligned}$$

Taking $G_{\mathbb{Q}_p,\Delta}$-invariants of this inclusion we deduce an inclusion $\mathbb{D}(V)\hookrightarrow D$ using Lemma 3.14. However, this is an isomorphism by Prop. 2.1 in [11] as $\mathbb{D}(V)$ and D have the same rank. ∎

Remarks. 1. Even though we have constructed V in the proof of the above theorem by a different procedure from just putting $V:=\mathbb{V}(D)$, we still have an isomorphism $V\cong\mathbb{V}(\mathbb{D}(V))\cong\mathbb{V}(D)$ by Prop. 3.7.

2. If κ is a finite extension of $\mathbb{F}_p$, then we have an equivalence of categories between $\mathrm{Rep}_\kappa(G_{\mathbb{Q}_p,\Delta})$ and $\mathcal{D}^{et}(\varphi_\Delta,\Gamma_\Delta,\kappa\otimes_{\mathbb{F}_p}E_\Delta)$. Indeed, we have a natural isomorphism $(\kappa\otimes_{\mathbb{F}_p}E^{sep}_\Delta)\otimes_\kappa\cdot\cong E^{sep}_\Delta\otimes_{\mathbb{F}_p}\cdot$ as functors on $\mathrm{Rep}_\kappa(G_{\mathbb{Q}_p,\Delta})$.

4 The case of p-adic representations

4.1 Cohomological preliminaries

We will need the following multivariable analogue of Hilbert's Theorem 90 (additive form).

Proposition 4.1. *The continuous group cohomology $H^1_{cont}(H_{\mathbb{Q}_p,\Delta}, E_\Delta^{sep})$ vanishes.*

Proof. By Prop. 3.3 it suffices to show that for finite Galois extensions E'_α/E_α (for all $\alpha \in \Delta$) with Galois group $H'_\alpha := \mathrm{Gal}(E'_\alpha/E_\alpha)$ we have $H^1(H', E'_\Delta) = \{1\}$ where we put $H' := \prod_{\alpha\in\Delta} H'_\alpha$. Choose a normal basis $e_1, \dots, e_{n_\alpha} \in E'_\alpha$ over E_α for each $\alpha \in \Delta$. By Lemma 3.2 the set $\{\prod_{\alpha\in\Delta} e_{i_\alpha} \mid 1 \leq i_\alpha \leq n_\alpha,\ \alpha \in \Delta\}$ is a basis of the free E_Δ-module E'_Δ. In particular, $E'_\Delta \cong E_\Delta[H']$ is induced as an H'-module whence the cohomology group $H^1(H', E'_\Delta)$ is trivial. $\square$

Let D be an abelian group admitting an action of the commutative monoid $\prod_{\alpha\in\Delta} \varphi_\alpha^{\mathbb{N}}$. Fix a total ordering $<$ on Δ and consider the complex

$$\Phi^\bullet(D)\colon 0 \to D \to \bigoplus_{\alpha\in\Delta} D \to \cdots \to \bigoplus_{\{\alpha_1,\dots,\alpha_r\}\in\binom{\Delta}{r}} D \to \cdots \to D \to 0$$

where for all $0 \leq r \leq |\Delta| - 1$ the map $d^{\beta_1,\dots,\beta_{r+1}}_{\alpha_1,\dots,\alpha_r}\colon D \to D$ from the component in the rth term corresponding to $\{\alpha_1, \dots, \alpha_r\} \subseteq \Delta$ to the component corresponding to the $(r+1)$-tuple $\{\beta_1, \dots, \beta_{r+1}\} \subseteq \Delta$ is given by

$$d^{\beta_1,\dots,\beta_{r+1}}_{\alpha_1,\dots,\alpha_r} = \begin{cases} 0 & \text{if } \{\alpha_1,\dots,\alpha_r\} \not\subseteq \{\beta_1,\dots,\beta_{r+1}\} \\ (-1)^\varepsilon(\mathrm{id} - \varphi_\beta) & \text{if } \{\beta_1,\dots,\beta_{r+1}\} = \{\alpha_1,\dots,\alpha_r\} \cup \{\beta\}\ , \end{cases}$$

where $\varepsilon = \varepsilon(\alpha_1, \dots, \alpha_r, \beta)$ is the number of elements in the set $\{\alpha_1, \dots, \alpha_r\}$ smaller than β. Since the operators $(\mathrm{id} - \varphi_\beta)$ commute with each other, $\Phi^\bullet(D)$ is a chain complex of abelian groups. Note that for each $\alpha \in \Delta$ we have a complex

$$\Phi^\bullet_\alpha(D)\colon 0 \to D \overset{\mathrm{id} - \varphi_\alpha}{\to} D \to 0$$

such that $\Phi^\bullet(E_\Delta^{sep})$ is a kind of completed tensor product of the complexes $\Phi^\bullet_\alpha(E_\alpha^{sep})$. More precisely, the tensor product over $\mathbb{F}_p$ of the complexes $\Phi^\bullet(E_\alpha^{sep})$ is the complex $\Phi^\bullet(E_{\Delta,\circ}^{sep})$ which is therefore acyclic in nonzero degrees with 0th cohomology equal to $\mathbb{F}_p$ by the Künneth formula. Note that there are no higher Tor's as the tensor product is taken over the field $\mathbb{F}_p$. We need the following completed version of this observation.

Proposition 4.2. *The complex $\Phi^\bullet(E_\Delta^{sep})$ is acyclic in nonzero degrees with 0th cohomology equal to $\mathbb{F}_p$.*

The following Lemma is well-known.

Lemma 4.3. *For any finite separable extension E'_α/E_α the map $\mathrm{id} - \varphi_\alpha\colon X'_\alpha E'^+_\alpha \to X'_\alpha E'^+_\alpha$ is bijective.*

Proof. The kernel of $\mathrm{id} - \varphi_\alpha$ is $\mathbb{F}_p$ which is not contained in $X'_\alpha E'^+_\alpha$. On the other hand, $\sum_{n=0}^\infty \varphi_\alpha^n$ converges on this set and is therefore an inverse to $\mathrm{id} - \varphi_\alpha$ by formal reasons. $\square$

Our key is the following

Lemma 4.4. *For all $\alpha \in S \subseteq \Delta$ the map* $\mathrm{id} - \varphi_\alpha \colon E_S^{sep} \to E_S^{sep}$ *is surjective with kernel* $E_{S\setminus\{\alpha\}}^{sep}$.

Proof. We may assume $S = \Delta$. The inclusion $E_{\Delta\setminus\{\alpha\}}^{sep} \subseteq \mathrm{Ker}(\mathrm{id} - \varphi_\alpha)$ is clear. For a collection $E_\beta \leq E'_\beta = \mathbb{F}_{q_\beta}((X'_\beta))$ $(\beta \in \Delta)$ of finite separable extensions the ring E'_Δ is embedded into $(E'_{\Delta\setminus\{\alpha\}} \otimes_{\mathbb{F}_p} \mathbb{F}_{q_\alpha})((X'_\alpha))$. By comparing the coefficients we find that $(E'_{\Delta\setminus\{\alpha\}} \otimes_{\mathbb{F}_p} \mathbb{F}_{q_\alpha})((X'_\alpha))^{\varphi_\alpha = \mathrm{id}} = E'_{\Delta\setminus\{\alpha\}}$.

For the surjectivity pick an element c in $E'_\Delta \subset E_\Delta^{sep}$ for some collection of finite separable extensions $E_\beta \leq E'_\beta = \mathbb{F}_{q_\beta}((X'_\beta))$ $(\beta \in \Delta)$. There exists an integer $k \geq 0$ such that c lies in $X_\Delta^{-k} E_\Delta'^{+} = \widehat{\bigotimes}_{\beta\in\Delta,\mathbb{F}_p} X_\beta^{-k} E_\beta'^{+}$. So we may write c as a convergent sum $c = \sum_{n=1}^\infty c_{\overline{\alpha},n} \otimes c_{\alpha,n}$ such that $c_{\overline{\alpha},n} \in X_{\Delta\setminus\{\alpha\}}^{-k} E_{\Delta\setminus\{\alpha\}}'^{+}$ with $c_{\overline{\alpha},n} \to 0$ and $c_{\alpha,n} \in X_\alpha^{-k} E_\alpha'^{+}$. Now the images of the elements $c_{\alpha,n}$ $(n \geq 1)$ under the map $E'_\alpha / X'_\alpha E_\alpha'^{+}$ are contained in the finite set $X_\alpha^{-k} E_\alpha'^{+} / X'_\alpha E_\alpha'^{+}$, so by Lemma 4.3 there exists an finite separable extension $E'_\alpha \leq E''_\alpha$ such that $c_{\alpha,n} = d_{\alpha,n} - \varphi_\alpha(d_{\alpha,n})$ for some $d_{\alpha,n} \in E''_\alpha$ for all $n \geq 1$. Moreover, the X_α-adic valuation of $d_{\alpha,n}$ is bounded by that of the X_α-adic valuation of $c_{\alpha,n}$ showing that the sum $d := \sum_{n=1}^\infty c_{\overline{\alpha},n} \otimes d_{\alpha,n}$ defines an element in E_Δ^{sep} with $c = d - \varphi_\alpha(d)$. $\square$

Proof of Prop. 4.2. We proceed by induction on $|\Delta|$. The case $|\Delta| = 1$ is clear, so suppose $n := |\Delta| > 1$ and we have proven the statement for any proper subset $S \subsetneq \Delta = \{\alpha_1, \dots, \alpha_n\}$. Let $c = (c_S)_{S \in \binom{\Delta}{r}} \in \bigoplus_{S\in\binom{\Delta}{r}} E_\Delta^{sep}$ be a cocycle in degree r. By Lemma 4.4 we find an element $x = (x_U)_{U\in\binom{\Delta}{r-1}}$ with $d_U = 0$ for all U with $\alpha_n \nmid U$ such that $(c - d^{r-1}(x))_S = 0$ for all $S \in \binom{\Delta}{r}$ with $\alpha_n \in S$. Indeed, the map $\cdot \cup \{\alpha_n\} \colon \binom{\Delta\setminus\{\alpha_n\}}{r-1} \to \{S \in \binom{\Delta}{r} \mid \alpha_n \in S\}$ is a bijection and by our assumption that x is concentrated into $\binom{\Delta\setminus\{\alpha_n\}}{r-1} \subset \binom{\Delta}{r-1}$ only the $S \setminus \{\alpha\}$-component of x contributes to the S component of $d^{r-1}(x)$ for $\alpha_n \in S$. So by replacing c with $c - d^{r-1}(x)$ we may assume without loss of generality that $c_S = 0$ for all S containing α_n. In particular, for $S' \in \binom{\Delta\setminus\{\alpha_n\}}{r}$ we compute

$$0 = (d^r(c))_{S'\cup\{\alpha_n\}} = (-1)^r(\mathrm{id} - \varphi_{\alpha_n})(c_{S'}) + \sum_{\beta\in S'} (-1)^{\varepsilon(\beta,S)}(\mathrm{id} - \varphi_\beta)(c_{S'\cup\{\alpha_n\}\setminus\{\beta\}}) =$$

$$= (-1)^r(\mathrm{id} - \varphi_{\alpha_n})(c_{S'}) \ .$$

Using Lemma 4.4 again this yields $c_{S'} \in E_{\Delta\setminus\{\alpha_n\}}^{sep}$ for all $S' \in \binom{\Delta}{r}$. Now the statement follows by induction. $\square$

The association $D \mapsto \Phi^\bullet(D)$ is an exact functor from the category of abelian groups with an action of $\prod_{\alpha\in\Delta} \varphi_\alpha^{\mathbb{N}}$ to the category of chain complexes of abelian groups. In particular, for any short exact sequence $0 \to D_1 \to D_2 \to D_3 \to 0$, we have a short exact sequence $0 \to \Phi^\bullet(D_1) \to \Phi^\bullet(D_2) \to \Phi^\bullet(D_3) \to 0$ of chain complexes. This yields a long exact sequence

$$0 \to h^0\Phi^\bullet(D_1) \to h^0\Phi^\bullet(D_2) \to h^0\Phi^\bullet(D_3) \to h^1\Phi^\bullet(D_1) \to h^1\Phi^\bullet(D_2) \to h^1\Phi^\bullet(D_3) \to \cdots$$

of abelian groups.

4.2 The multivariable p-adic coefficient ring

Our goal in this section is to lift E_Δ and E_Δ^{sep} to characteristic 0 so we can classify p-adic representations of $G_{\mathbb{Q}_p,\Delta}$. Recall [5] that $\mathcal{O}_\mathcal{E} \cong \varprojlim_h \mathbb{Z}/(p^h)((X))$ is constructed as a Cohen ring of $E \cong \mathbb{F}_p((X))$. Via the embedding $X \mapsto [\varepsilon]-1$ these are subrings of $\tilde{B}$ which is defined as $\tilde{B} := W(\widehat{E^{sep}})[p^{-1}]$ where $W(\widehat{E^{sep}})$ is the ring of p-typical Witt vectors of the completion $\widehat{E^{sep}}$ (with respect to the X-adic topology) of the separable closure E^{sep}. Here $[\varepsilon]$ denotes the Teichmüller representative of the sequence $\varepsilon = (\varepsilon_n)_n \in \varprojlim_{x\mapsto x^p} \mathcal{O}_{\mathbb{C}_p} \cong \widehat{E^{sep}}^+$ of p-power roots of unity with $\varepsilon_1 \neq 1$. Note that $\widehat{E^{sep}}$ is an algebraically closed field of characteristic p which is, in fact, isomorphic to the tilt $\mathbb{C}_p^\flat = \mathrm{Frac}(\varprojlim_{x\mapsto x^p} \mathcal{O}_{\mathbb{C}_p}/(p))$ of $\mathbb{C}_p$ in the modern terminology. Further, for any finite extension E'/E contained in E^{sep} there exists a unique finite unramified extension $\mathcal{E}'$ of $\mathcal{E} = \mathcal{O}_\mathcal{E}[p^{-1}]$ contained in $\tilde{B}$ with residue field E' (Prop. 4.20 in [5]).

We define the ring $\mathcal{O}_{\mathcal{E}_\Delta}$ as the projective limit $\varprojlim_h \left(\mathbb{Z}/(p^h)[[X_\alpha \mid \alpha\in\Delta]][X_\Delta^{-1}]\right)$ and put $\mathcal{E}_\Delta := \mathcal{O}_{\mathcal{E}_\Delta}[p^{-1}]$ so we have $\mathcal{O}_{\mathcal{E}_\Delta}/(p) \cong E_\Delta$. The Iwasawa algebra $\mathcal{O}^+_{\mathcal{E}_\Delta} = \mathbb{Z}_p[[X_\alpha \mid \alpha\in\Delta]] \leq \mathcal{O}_{\mathcal{E}_\Delta}$ is isomorphic to the completed tensor product of the one-variable Iwasawa algebras $\mathcal{O}^+_{\mathcal{E}_\alpha} := \mathbb{Z}_p[[X_\alpha]]$ ($\alpha\in\Delta$) over $\mathbb{Z}_p$. This motivates the way we can lift E'_Δ to characteristic 0 for a collection E'_α/E_α ($\alpha\in\Delta$) of finite separable extensions. We define

$$\mathcal{O}^+_{\mathcal{E}'_\Delta} := \widehat{\bigotimes_{\alpha\in\Delta,\mathbb{Z}_p}} \mathcal{O}_{\mathcal{E}'_\alpha}$$

as a completed tensor product. If we write $E'_\alpha = \mathbb{F}_{q_\alpha}((X'_\alpha))$ ($\alpha\in\Delta$) then we may identify $\mathcal{O}^+_{\mathcal{E}'_\Delta}$ with the power series ring $\left(\bigotimes_{\alpha\in\Delta,\mathbb{Z}_p} W(\mathbb{F}_{q_\alpha})\right)[[X'_\alpha \mid \alpha\in\Delta]]$ over the finite étale $\mathbb{Z}_p$-algebra $\bigotimes_{\alpha\in\Delta,\mathbb{Z}_p} W(\mathbb{F}_{q_\alpha})$. We define $\mathcal{O}_{\mathcal{E}'_\Delta}$ as the p-adic completion $\widehat{\mathcal{O}^+_{\mathcal{E}'_\Delta}[X_\Delta^{-1}]} = \varprojlim_h \mathcal{O}^+_{\mathcal{E}'_\Delta}[X_\Delta^{-1}]/(p^h)$ and put $\mathcal{E}'_\Delta := \mathcal{O}_{\mathcal{E}'_\Delta}[p^{-1}]$. We have the following alternative characterization of $\mathcal{O}_{\mathcal{E}'_\Delta}$.

Lemma 4.5. *Writing $\Delta = \{\alpha_1,\dots,\alpha_n\}$ we have*

$$\mathcal{O}_{\mathcal{E}'_\Delta} \cong \mathcal{O}_{\mathcal{E}'_{\alpha_1}} \otimes_{\mathcal{O}_{\mathcal{E}_{\alpha_1}}} (\cdots(\mathcal{O}_{\mathcal{E}'_{\alpha_n}} \otimes_{\mathcal{O}_{\mathcal{E}_{\alpha_n}}} \mathcal{O}_{\mathcal{E}_\Delta}))\ .$$

In particular, $\mathcal{O}_{\mathcal{E}'_\Delta}$ is a free module of rank $\prod_{i=1}^n |E'_{\alpha_i} : E_{\alpha_i}|$ over $\mathcal{O}_{\mathcal{E}_\Delta}$.

Proof. Each $\mathcal{O}_{\mathcal{E}'_{\alpha_i}}$ is naturally a subring in $\mathcal{O}_{\mathcal{E}'_\Delta}$ and so is $\mathcal{O}_{\mathcal{E}_\Delta}$. Therefore there is a ring homomorphism from the right hand side to the left hand side which is an isomorphism modulo p by Lemma 3.2. The first statement follows from the p-adic completeness of both sides.

Since $\mathcal{O}_{\mathcal{E}_{\alpha_i}}$ is a complete discrete valuation ring, $\mathcal{O}_{\mathcal{E}'_{\alpha_i}}$ is finite free over $\mathcal{O}_{\mathcal{E}_{\alpha_i}}$ of rank $|E'_{\alpha_i} : E_{\alpha_i}|$ ($i=1,\dots,n$). Therefore the second statement. ☐

Now we define $\mathcal{E}_\Delta^{ur} := \varinjlim \mathcal{E}'_\Delta$ and $\mathcal{O}_{\mathcal{E}_\Delta^{ur}} := \varinjlim \mathcal{O}_{\mathcal{E}'_\Delta}$ where E'_α runs over the finite subextensions of E_α in E_α^{sep} for all $\alpha\in\Delta$. Further, we denote by $\widehat{\mathcal{E}_\Delta^{ur}}$ (resp. by $\mathcal{O}_{\widehat{\mathcal{E}_\Delta^{ur}}}$) the p-adic completion of $\mathcal{E}_\Delta^{ur}$ (resp. of $\mathcal{O}_{\mathcal{E}_\Delta^{ur}}$). We have $\mathcal{O}_{\widehat{\mathcal{E}_\Delta^{ur}}}/(p) \cong E_\Delta^{sep}$ by construction. The group $G_{\mathbb{Q}_p,\Delta}$ acts naturally on $\widehat{\mathcal{E}_\Delta^{ur}}$ (resp. on $\mathcal{O}_{\widehat{\mathcal{E}_\Delta^{ur}}}$). Moreover, for each $\alpha\in\Delta$ we have the Frobenius lift φ_α on $\tilde{B}_\alpha$ (the copy of $\tilde{B}$ indexed by α) which acts on $[\varepsilon]$ by raising to the pth power (as it is a Teichmüller representative). So we have $\varphi_\alpha(X_\alpha) = (X_\alpha+1)^p - 1$. For each finite extension E'_α/E_α we have $\varphi(\mathcal{E}'_\alpha) \subset \mathcal{E}'_\alpha$, so this defines an action of φ_α on the rings $\mathcal{E}_\Delta^{ur}$, $\mathcal{O}_{\mathcal{E}_\Delta^{ur}}$, $\widehat{\mathcal{E}_\Delta^{ur}}$, and $\mathcal{O}_{\widehat{\mathcal{E}_\Delta^{ur}}}$ for all $\alpha \in \Delta$. These operators commute with each other and with the action of the group $G_{\mathbb{Q}_p,\Delta}$.

Proposition 4.6. *We have*

$$\widehat{\mathcal{E}_\Delta^{ur}}^{H_{\mathbb{Q}_p,\Delta}} = \mathcal{E}_\Delta \ , \qquad \bigcap_{\alpha\in\Delta} \widehat{\mathcal{E}_\Delta^{ur}}^{\varphi_\alpha=\mathrm{id}} = \mathbb{Q}_p \ , \ \textit{and}$$
$$\mathcal{O}_{\widehat{\mathcal{E}_\Delta^{ur}}}^{H_{\mathbb{Q}_p,\Delta}} = \mathcal{O}_{\mathcal{E}_\Delta} \ , \qquad \bigcap_{\alpha\in\Delta} \mathcal{O}_{\widehat{\mathcal{E}_\Delta^{ur}}}^{\varphi_\alpha=\mathrm{id}} = \mathbb{Z}_p \ .$$

Proof. The statements on $\widehat{\mathcal{E}_\Delta^{ur}}$ follow from those on $\mathcal{O}_{\widehat{\mathcal{E}_\Delta^{ur}}}$ as p is φ_α- and $H_{\mathbb{Q}_p,\Delta}$-invariant for all $\alpha \in \Delta$. Moreover, the latter statements are consequences of Prop. 3.3, resp. Lemma 3.6 using devissage. □

4.3 The equivalence of categories

We denote by $\mathrm{Rep}_{\mathbb{Z}_p}(G_{\mathbb{Q}_p,\Delta})$ (resp. by $\mathrm{Rep}_{\mathbb{Q}_p}(G_{\mathbb{Q}_p,\Delta})$) the category of continuous representations of $G_{\mathbb{Q}_p,\Delta}$ on finitely generated $\mathbb{Z}_p$-modules (resp. on finite dimensional $\mathbb{Q}_p$-vector spaces). Let T (resp. V) be an object in $\mathrm{Rep}_{\mathbb{Z}_p}(G_{\mathbb{Q}_p,\Delta})$ (resp. in $\mathrm{Rep}_{\mathbb{Q}_p}(G_{\mathbb{Q}_p,\Delta})$). We define

$$\mathbb{D}(T) := \left(\mathcal{O}_{\widehat{\mathcal{E}_\Delta^{ur}}} \otimes_{\mathbb{Z}_p} T\right)^{H_{\mathbb{Q}_p,\Delta}} \qquad \left(\text{resp. } \mathbb{D}(V) := \left(\widehat{\mathcal{E}_\Delta^{ur}} \otimes_{\mathbb{Q}_p} V\right)^{H_{\mathbb{Q}_p,\Delta}}\right) \ .$$

By Prop. 4.6 $\mathbb{D}(T)$ (resp. $\mathbb{D}(V)$) is a module over $\mathcal{O}_{\mathcal{E}_\Delta}$ (resp. over $\mathcal{E}_\Delta$). Moreover, it admits an action of the monoid $T_{+,\Delta}$: the action of φ_α ($\alpha \in \Delta$) is trivial on T (resp. on V) and therefore comes from the action on $\mathcal{O}_{\widehat{\mathcal{E}_\Delta^{ur}}}$ (resp. on $\widehat{\mathcal{E}_\Delta^{ur}}$) defined above. The action of $\Gamma_\Delta = G_{\mathbb{Q}_p,\Delta}/H_{\mathbb{Q}_p,\Delta}$ comes from the diagonal action of $G_{\mathbb{Q}_p,\Delta}$ on $\mathcal{O}_{\widehat{\mathcal{E}_\Delta^{ur}}} \otimes_{\mathbb{Z}_p} T$ (resp. on $\widehat{\mathcal{E}_\Delta^{ur}} \otimes_{\mathbb{Q}_p} V$).

Proposition 4.7. *Let T be an object in* $\mathrm{Rep}_{\mathbb{Z}_p}(G_{\mathbb{Q}_p,\Delta})$. *The natural map*

$$\mathcal{O}_{\widehat{\mathcal{E}_\Delta^{ur}}} \otimes_{\mathcal{O}_{\mathcal{E}_\Delta}} \mathbb{D}(T) \to \mathcal{O}_{\widehat{\mathcal{E}_\Delta^{ur}}} \otimes_{\mathbb{Z}_p} T$$

is an isomorphism.

Proof. This is very similar to the proof of Prop. 2.30 in [5]. We proceed in two steps. Assume first that T is killed by a power p^h of p. We use induction on h. The case $h=1$ is done in Prop. 3.7. Now for $h>1$ we have a short exact sequence $0 \to T_1 \to T \to T_2 \to 0$ of objects in $\mathrm{Rep}_{\mathbb{Z}_p}(G_{\mathbb{Q}_p,\Delta})$ such that $pT_1 = 0$ and $p^{h-1}T_2$. Since $\mathcal{O}_{\widehat{\mathcal{E}_\Delta^{ur}}}$ has no p-torsion, it is flat as $\mathbb{Z}_p$-module. Therefore we obtain a short exact sequence

$$0 \to \mathcal{O}_{\widehat{\mathcal{E}_\Delta^{ur}}} \otimes_{\mathbb{Z}_p} T_1 \to \mathcal{O}_{\widehat{\mathcal{E}_\Delta^{ur}}} \otimes_{\mathbb{Z}_p} T \to \mathcal{O}_{\widehat{\mathcal{E}_\Delta^{ur}}} \otimes_{\mathbb{Z}_p} T_2 \to 0 \ .$$

Now we have an identification $\mathcal{O}_{\widehat{\mathcal{E}_\Delta^{ur}}} \otimes_{\mathbb{Z}_p} T_1 \cong E_\Delta^{sep} \otimes_{\mathbb{F}_p} T_1 \cong E_\Delta^{sep} \otimes_{E_\Delta} \mathbb{D}(T_1)$. In particular, as a representation of $H_{\mathbb{Q}_p,\Delta}$ we have $\mathcal{O}_{\widehat{\mathcal{E}_\Delta^{ur}}} \otimes_{\mathbb{Z}_p} T_1 \cong (E_\Delta^{sep})^{\dim_{\mathbb{F}_p} T_1}$. In particular, Prop. 4.1 yields

$H^1_{cont}(H_{\mathbb{Q}_p,\Delta}, \mathcal{O}_{\widehat{\mathcal{E}^{ur}_\Delta}} \otimes_{\mathbb{Z}_p} T_1) = \{1\}$. By the long exact sequence of continuous $H_{\mathbb{Q}_p,\Delta}$-cohomology we deduce the exactness of the sequence

$$0 \to \mathbb{D}(T_1) \to \mathbb{D}(T) \to \mathbb{D}(T_2) \to 0 \ .$$

Now we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{\widehat{\mathcal{E}^{ur}_\Delta}} \otimes_{\mathcal{O}_{\mathcal{E}_\Delta}} \mathbb{D}(T_1) & \longrightarrow & \mathcal{O}_{\widehat{\mathcal{E}^{ur}_\Delta}} \otimes_{\mathcal{O}_{\mathcal{E}_\Delta}} \mathbb{D}(T) & \longrightarrow & \mathcal{O}_{\widehat{\mathcal{E}^{ur}_\Delta}} \otimes_{\mathcal{O}_{\mathcal{E}_\Delta}} \mathbb{D}(T_2) & \longrightarrow & 0 \\ & & \downarrow \wr & & \downarrow & & \downarrow \wr & & \\ 0 & \longrightarrow & \mathcal{O}_{\widehat{\mathcal{E}^{ur}_\Delta}} \otimes_{\mathbb{Z}_p} T_1 & \longrightarrow & \mathcal{O}_{\widehat{\mathcal{E}^{ur}_\Delta}} \otimes_{\mathbb{Z}_p} T & \longrightarrow & \mathcal{O}_{\widehat{\mathcal{E}^{ur}_\Delta}} \otimes_{\mathbb{Z}_p} T_2 & \longrightarrow & 0 \end{array}$$

with exact rows. Thus the vertical map in the middle is an isomorphism by induction using the 5-lemma.

The general case follows from this by taking the projective limit of the isomorphisms above for T/p^hT as h tends to infinity. $\square$

An étale $T_{+,\Delta}$-module over $\mathcal{O}_{\mathcal{E}_\Delta}$ is a finitely generated $\mathcal{O}_{\mathcal{E}_\Delta}$-module D together with a semilinear action of the monoid $T_{+,\Delta}$ such that for all $\varphi_t \in T_{+,\Delta}$ the map

$$\mathrm{id} \otimes \varphi_t \colon \varphi_t^* D := \mathcal{O}_{\mathcal{E}_\Delta} \otimes_{\mathcal{O}_{\mathcal{E}_\Delta},\varphi_t} D \to D$$

is an isomorphism. We denote by $\mathcal{D}^{et}(\varphi_\Delta, \Gamma_\Delta, \mathcal{O}_{\mathcal{E}_\Delta})$ the category of étale $T_{+,\Delta}$-modules over $\mathcal{O}_{\mathcal{E}_\Delta}$. As in the mod p case, $\mathcal{D}^{et}(\varphi_\Delta, \Gamma_\Delta, \mathcal{O}_{\mathcal{E}_\Delta})$ has the structure of a neutral Tannakian category. If D is finitely generated $\mathcal{O}_{\mathcal{E}_\Delta}$ module that is killed by a power p^h of p we define the generic length of D as $\mathrm{length}_{gen} D := \sum_{i=1}^h \mathrm{rk}_{E_\Delta}\, p^{i-1}D/p^iD$ where rk_{E_Δ} denotes the generic rank (ie. dimension over $\mathrm{Frac}(E_\Delta)$ of the localisation at (0)).

Corollary 4.8. *The functor $\mathbb{D}$ is exact. $\mathbb{D}(T)$ is an object in $\mathcal{D}^{et}(\varphi_\Delta, \Gamma_\Delta, \mathcal{O}_{\mathcal{E}_\Delta})$ for any T in $\mathrm{Rep}_{\mathbb{Z}_p}(G_{\mathbb{Q}_p,\Delta})$. Moreover, if T is killed by a power of p then the we have* $\mathrm{length}_{gen}\mathbb{D}(T) = \mathrm{length}_{\mathbb{Z}_p} T$.

Proof. If T is an object in $\mathrm{Rep}_{\mathbb{Z}_p}(G_{\mathbb{Q}_p,\Delta})$ such that $p^hT = 0$, then we have $H^1(H_{\mathbb{Q}_p,\Delta}, \mathcal{O}_{\widehat{\mathcal{E}^{ur}_\Delta}} \otimes_{\mathbb{Z}_p} T) = \{1\}$ by induction on h using the long exact sequence of continuous $H_{\mathbb{Q}_p,\Delta}$-cohomology. So the exactness of $\mathbb{D}$ on finite length objects in $\mathrm{Rep}_{\mathbb{Z}_p}(G_{\mathbb{Q}_p,\Delta})$ follows the same way as in the proof of Prop. 4.7 in the special case when $pT_1 = 0$. Now if $0 \to T_1 \to T_2 \to T_3 \to 0$ is an arbitrary short exact sequence in $\mathrm{Rep}_{\mathbb{Z}_p}(G_{\mathbb{Q}_p,\Delta})$ then we have an exact sequence

$$0 \to T_1[p^h] \to T_2[p^h] \to T_3[p^h] \overset{\partial_h}{\to} T_1/p^hT_1 \to T_2/p^hT_2 \to T_3/p^hT_3 \to 0$$

of finite length objects for all $h \geq 1$. Applying $\mathbb{D}$ yields an exact sequence

$$0 \to \mathbb{D}(T_1[p^h]) \to \mathbb{D}(T_2[p^h]) \to \mathbb{D}(T_3[p^h]) \to \mathbb{D}(T_1/p^hT_1) \to \mathbb{D}(T_2/p^hT_2) \to \mathbb{D}(T_3/p^hT_3) \to 0$$

for all $h \geq 1$. Since T_i is finitely generated over $\mathbb{Z}_p$, we have $T_i[p^h] = (T_i)_{tors}$ for $h \geq h_0$ large enough ($i = 1, 2, 3$). In particular, the connecting map $T_i[p^{(n+1)h}] \overset{p^h\cdot}{\to} T_i[p^{nh}]$ is the zero map for $h \geq h_0$ and $i = 1, 2, 3$. Thus the Mittag–Leffler property is satisfied for both $\mathrm{Im}(\partial_h)_h$ and

$\mathrm{Coker}(\partial_h)_h$ as the map $T_1/p^{h+1}T_1 \to T_1/p^hT_1$ is surjective for all $h \geq 1$. Hence taking the projective limit we obtain an exact sequence $0 \to \mathbb{D}(T_1) \to \mathbb{D}(T_2) \to \mathbb{D}(T_3) \to 0$ as claimed.

The statement on the generic length follows from the exactness using Prop. 3.7 and induction on h such that $p^hT = 0$. In particular, $\mathbb{D}(T)$ is finitely generated over $\mathcal{O}_{\mathcal{E}_\Delta}$ if T has finite length. Now if T is not necessarily of finite length then we apply the exactness of $\mathbb{D}$ on the exact sequence $0 \to T[p] \to T \overset{p\cdot}{\to} T \to T/pT \to 0$ we obtain that $\mathbb{D}(T/pT) = \mathbb{D}(T)/p\mathbb{D}(T)$ which is finitely generated over E_Δ. Therefore $\mathbb{D}(T)$ is finitely generated over $\mathcal{O}_{\mathcal{E}_\Delta}$ by the p-adic completeness of $\mathbb{D}(T)$ (by definition we have $\varprojlim_h \mathbb{D}(T/p^hT) = \mathbb{D}(T)$).

Finally, the étale property for finite length modules follows by induction on the length from the case $h = 1$ (Prop. 3.7) and in general by taking the projective limit. $\square$

Conversely, let D be an object in $\mathcal{D}^{et}(\varphi_\Delta, \Gamma_\Delta, \mathcal{O}_{\mathcal{E}_\Delta})$. We define

$$\mathbb{T}(D) := \bigcap_{\alpha\in\Delta} \left(\mathcal{O}_{\widehat{\mathcal{E}_\Delta^{ur}}} \otimes_{\mathcal{O}_{\mathcal{E}_\Delta}} D\right)^{\varphi_\alpha=\mathrm{id}} .$$

This is a $\mathbb{Z}_p$-module admitting a diagonal action of $G_{\mathbb{Q}_p,\Delta}$ via the formula $g(\lambda\otimes d) := g(\lambda)\otimes \chi(g)(d)$ where $\chi\colon G_{\mathbb{Q}_p,\Delta} \twoheadrightarrow \Gamma_\Delta$ is the quotient map.

Proposition 4.9. *For any object D in $\mathcal{D}^{et}(\varphi_\Delta, \Gamma_\Delta, \mathcal{O}_{\mathcal{E}_\Delta})$, the natural map*

$$\mathcal{O}_{\widehat{\mathcal{E}_\Delta^{ur}}} \otimes_{\mathbb{Z}_p} \mathbb{T}(D) \to \mathcal{O}_{\widehat{\mathcal{E}_\Delta^{ur}}} \otimes_{\mathcal{O}_{\mathcal{E}_\Delta}} D$$

is an isomorphism.

Proof. This is completely analogous to the proof of Prop. 2.31 in [5]. We proceed in two steps. At first assume that $p^hD = 0$ for some integer $h \geq 1$. Consider the exact sequence $0 \to D[p] \to D \to D/D[p] \to 0$ and apply the exact functor $\Phi^\bullet \circ (\mathcal{O}_{\widehat{\mathcal{E}_\Delta^{ur}}} \otimes_{\mathcal{O}_{\mathcal{E}_\Delta}} \cdot)$ to obtain an exact sequence

$$0 \to \Phi^\bullet(\mathcal{O}_{\widehat{\mathcal{E}_\Delta^{ur}}} \otimes_{\mathcal{O}_{\mathcal{E}_\Delta}} D[p]) \to \Phi^\bullet(\mathcal{O}_{\widehat{\mathcal{E}_\Delta^{ur}}} \otimes_{\mathcal{O}_{\mathcal{E}_\Delta}} D) \to \Phi^\bullet(\mathcal{O}_{\widehat{\mathcal{E}_\Delta^{ur}}} \otimes_{\mathcal{O}_{\mathcal{E}_\Delta}} D/D[p]) \to 0 .$$

By Thm. 3.15 $D[p]$ is in the image of the functor $\mathbb{D}$ whence $\mathcal{O}_{\widehat{\mathcal{E}_\Delta^{ur}}} \otimes_{\mathcal{O}_{\mathcal{E}_\Delta}} D[p]$ is isomorphic to $(E_\Delta^{sep})^{\mathrm{rk}_{E_\Delta} D[p]}$ as a $\prod_{\alpha\in\Delta}\varphi_\alpha^{\mathbb{N}}$-module using Prop. 3.7. In particular, $h^1\Phi^\bullet(\mathcal{O}_{\widehat{\mathcal{E}_\Delta^{ur}}} \otimes_{\mathcal{O}_{\mathcal{E}_\Delta}} D[p]) = 0$ by Prop. 4.2. This yields an exact sequence

$$0 \to \mathbb{T}(D[p]) \to \mathbb{T}(D) \to \mathbb{T}(D/D[p]) \to 0 ,$$

and the statement follows the same way as in the proof of Prop. 4.7.

The general case follows by taking the limit. $\square$

Now note that $\mathbb{T}(D)$ is finitely generated over $\mathbb{Z}_p$: this is obvious in the case when $p^hD = 0$ using induction on h and in the general case by Nakayama's lemma as we have $\mathbb{T}(D) = \varprojlim_h \mathbb{T}(D/p^hD)$ by construction. So we deduce

Theorem 4.10. *The functors $\mathbb{D}$ and $\mathbb{T}$ are quasi-inverse equivalences of categories between the Tannakian categories $\mathrm{Rep}_{\mathbb{Z}_p}(G_{\mathbb{Q}_p,\Delta})$ and $\mathcal{D}^{et}(\varphi_\Delta, \Gamma_\Delta, \mathcal{O}_{\mathcal{E}_\Delta})$.*

Finally, an étale $T_{+,\Delta}$-module over $\mathcal{E}_\Delta$ is a finitely generated $\mathcal{E}_\Delta$-module D together with a semilinear action of the monoid $T_{+,\Delta}$ such that there exists an object D_0 in $\mathcal{D}^{et}(\varphi_\Delta, \Gamma_\Delta, \mathcal{O}_{\mathcal{E}_\Delta})$ with an isomorphism $D \cong D_0[p^{-1}] = \mathcal{E}_\Delta \otimes_{\mathcal{O}_{\mathcal{E}_\Delta}} D_0$. We denote by $\mathcal{D}^{et}(\varphi_\Delta, \Gamma_\Delta, \mathcal{E}_\Delta)$ the category of étale $T_{+,\Delta}$-modules over $\mathcal{E}_\Delta$. As before, $\mathcal{D}^{et}(\varphi_\Delta, \Gamma_\Delta, \mathcal{E}_\Delta)$ has the structure of a neutral Tannakian category. We have the following characteristic 0 version of the category equivalence:

Theorem 4.11. *The functors*

$$\begin{aligned} V &\mapsto \mathbb{D}(V) := \left(\widehat{\mathcal{E}_\Delta^{ur}} \otimes_{\mathbb{Q}_p} V\right)^{H_{\mathbb{Q}_p,\Delta}} \\ D &\mapsto \mathbb{V}(D) := \bigcap_{\alpha\in\Delta} \left(\widehat{\mathcal{E}_\Delta^{ur}} \otimes_{\mathcal{E}_\Delta} D\right)^{\varphi_\alpha = \mathrm{id}} \end{aligned}$$

are quasi-inverse equivalences of categories between the Tannakian categories $\mathrm{Rep}_{\mathbb{Q}_p}(G_{\mathbb{Q}_p,\Delta})$ *and* $\mathcal{D}^{et}(\varphi_\Delta, \Gamma_\Delta, \mathcal{E}_\Delta)$.

Proof. Since $G_{\mathbb{Q}_p,\Delta}$ is compact, any finite dimensional $\mathbb{Q}_p$-representation V contains a $G_{\mathbb{Q}_p,\Delta}$-invariant lattice T. The statement follows from Thm. 4.10 by inverting p on both sides. The compatibility with tensor products and duals follows the same way as in characteristic p. □

Remarks.

1. If A is a $\mathbb{Z}_p$-algebra which is finitely generated as a module over $\mathbb{Z}_p$, then we have an equivalence of categories between $\mathrm{Rep}_A(G_{\mathbb{Q}_p,\Delta})$ and $\mathcal{D}^{et}(\varphi_\Delta, \Gamma_\Delta, A \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathcal{E}_\Delta})$. Indeed, we have a natural isomorphism $(A \otimes_{\mathbb{Z}_p} \mathcal{O}_{\widehat{\mathcal{E}_\Delta^{ur}}}) \otimes_A \cdot \cong \mathcal{O}_{\widehat{\mathcal{E}_\Delta^{ur}}} \otimes_{\mathbb{Z}_p} \cdot$ as functors on $\mathrm{Rep}_A(G_{\mathbb{Q}_p,\Delta})$. Similarly, if K is a finite extension of $\mathbb{Q}_p$, then we have an equivalence of categories between $\mathrm{Rep}_K(G_{\mathbb{Q}_p,\Delta})$ and $\mathcal{D}^{et}(\varphi_\Delta, \Gamma_\Delta, K \otimes_{\mathbb{Q}_p} \mathcal{E}_\Delta)$.

2. It is expected that there is a similar equivalence of categories for representations of the $|\Delta|$th direct power of the group $\mathrm{Gal}(\overline{\mathbb{Q}_p}/F)$ for a finite extension $F/\mathbb{Q}_p$. However, at this point it is not clear what type of (φ, Γ)-modules one should consider. The usual cyclotomic (φ, Γ)-modules do not seem to be well-suited for the purpose of the p-adic and mod p Langlands programme. On the other hand, the Lubin–Tate setting may not work properly in characteristic p due to the non-existence of the distinguished left inverse ψ of φ. To work over the character variety of the group $\mathcal{O}_F$ [2] seems, however, to be a good candidate.

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