On the size of k-cross-free families

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April 10, 2017

Abstract

Two subsets A, B of an n-element ground set X are said to be *crossing*, if none of the four sets $A \cap B$, $A \setminus B$, $B \setminus A$ and $X \setminus (A \cup B)$ are empty. It was conjectured by Karzanov and Lomonosov forty years ago that if a family $\mathcal{F}$ of subsets of X does not contain k pairwise crossing elements, then $|\mathcal{F}| = O_k(n)$. For $k = 2$ and 3, the conjecture is true, but for larger values of k the best known upper bound, due to Lomonosov, is $|\mathcal{F}| = O_k(n \log n)$. In this paper, we improve this bound by showing that $|\mathcal{F}| = O_k(n \log^* n)$ holds, where $\log^*$ denotes the iterated logarithm function.

1 Introduction

As usual, denote $[n] := \{1, \ldots, n\}$ and let $2^{[n]}$ be the family of all subsets of $[n]$. Two sets $A, B \in 2^{[n]}$ are said to be *crossing*, if $A \setminus B$, $B \setminus A$, $A \cap B$ and $[n] \setminus (A \cup B)$ are all non-empty.

We say that a family $\mathcal{F} \subset 2^{[n]}$ is *k-cross-free* if it does not contain k pairwise crossing sets. The following conjecture was made by Karzanov and Lomonosov [12], [11] and later by Pevzner [14]; see also Conjecture 3 in [4], Section 9.

Conjecture 1. *Let $k \geq 2$ and n be positive integers, and let $\mathcal{F} \subset 2^{[n]}$ be a k-cross-free family. Then $|\mathcal{F}| = O_k(n)$.*

Here and in the rest of this paper, $f(n) = O_k(n)$ means that $f(n) \leq c_k n$ for a suitable constant $c_k > 0$, which may depend on the parameter k.

It was shown by Edmonds and Giles [8] that every 2-cross-free family $\mathcal{F} \subset 2^{[n]}$ has at most $4n - 2$ members. Pevzner [14] proved that every 3-cross-free family on an n-element underlying set has at most $6n$ elements, and Fleiner [9] established the weaker bound $10n$, using a simpler argument. For $k > 3$, Conjecture 1 remains open. The best known general upper bound for the size of a k-cross-free family is $O_k(n \log n)$, which can be obtained by the following elegant argument, due to Lomonosov.

Let $\mathcal{F} \subset 2^{[n]}$ be a maximal k-cross-free family. Notice that for any set $A \in \mathcal{F}$, the complement of A also belongs to $\mathcal{F}$. Thus, the subfamily

$$\mathcal{F}' = \{A \in \mathcal{F} : |A| < n/2\} \cup \{A \in \mathcal{F} : |A| = n/2 \text{ and } 1 \in A\}$$

contains precisely half of the members of $\mathcal{F}$. For every s, $1 \leq s \leq n/2$, any two s-element members of $\mathcal{F}'$ that have a point in common, are crossing. Since $\mathcal{F}'$ has no k pairwise crossing members, every element of $[n]$ is contained in at most $k-1$ members of $\mathcal{F}'$ of size s. Thus, the number of s-element members is at most $(k-1)n/s$, and

$$|\mathcal{F}'| = |\mathcal{F}|/2 \leq 1 + \sum_{s=1}^{n/2} (k-1)n/s = O_k(n \log n).$$

The main result of the present note represents the first improvement on this 40 years old bound. Let $\log_{(i)} n$ denote the function $\log \ldots \log n$, where the log is iterated i times, and let $\log^* n$ denote the *iterated logarithm of n*, that is, the largest positive integer i such that $\log_{(i)} n > 1$.

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†Moscow Institute of Physics and Technology, Research partially supported by the grant N 15-01-03530 of the Russian Foundation for Basic Research.

Theorem 2. *Let $k \geq 2$ and n be positive integers, and let $\mathcal{F} \subset 2^{[n]}$ be a k-cross-free family. Then $|\mathcal{F}| = O_k(n \log^* n)$.*

Conjecture 1 has been proved in the following special case. Let $\mathcal{F}$ be a k-cross-free family consisting of contiguous subintervals of the cyclic sequence $1, 2, \ldots, n$. It was shown by Capoyleas and Pach [5] that in this case

$$|\mathcal{F}| \leq 4(k-1)n - 2\binom{2k-1}{2},$$

provided that $n \geq 2k - 1$. This bound cannot be improved.

A *geometric graph* G is a graph drawn in the plane so that its vertices are represented by points in general position in the plane and its edges are represented by (possibly crossing) straight-line segments between these points. Two edges of G are said to be *crossing* if the segments representing them have a point in common.

Conjecture 3. *Let $k \geq 2$ and n be positive integers, and let G be a geometric graph with n vertices, containing no k pairwise crossing edges. Then the number of edges of G is $O_k(n)$.*

The result of Capoyleas and Pach mentioned above implies that Conjecture 3 holds for geometric graphs G, where the points representing the vertices of G form the vertex set of a convex n-gon. It is also known to be true for $k \leq 4$; see [3], [1], [2]. For $k > 4$, it was proved by Valtr [16] that if a geometric graph on n vertices contains no k pairwise crossing edges then it its number of edges is $O_k(n \log n)$ edges.

A bipartite variant of Conjecture 1 was proved by Suk [15]. He showed that if $\mathcal{F} \subset 2^{[n]}$ does not contain $2k$ sets $A_1, \ldots, A_k$ and $B_1, \ldots, B_k$ such that A_i and B_j are crossing for all $i, j \in [k]$, then $|\mathcal{F}| \leq (2k-1)^2 n$.

The notion of k-cross-free families was first introduced by Karzanov [11] in the context of multicommodity flow problems. Let $G = (V, E)$ be a graph, $X \subset V$. A multiflow f is a fractional packing of paths in G. We say that f locks a subset $A \subset X$ in G if the total value of all paths between A and $X \setminus A$ is equal to the minimum number of edges separating A from $X \setminus A$ in G. A family $\mathcal{F}$ of subsets of X is called *lockable* if for every graph G with the above property there exists a multiflow f that locks every member $A \in \mathcal{F}$. The celebrated locking theorem of Karzanov and Lomonosov [12] states that a set family is lockable if and only if it is 3-*cross-free*. This is a useful extension of the Ford-Fulkerson theorem for network flows, and it generalizes some previous results of Cherkasky [6] and Lovász [13]; see also [10].

2 The proof of Theorem 2

In this section, we prove our main theorem. Throughout the proof, floors and ceilings are omitted whenever they are not crucial, and log stands for the base 2 logarithm. Also, for convenience, we shall use the following extended definition of binomial coefficients: if x is a real number and k is a positive integer,

$$\binom{x}{k} = \begin{cases} \frac{x(x-1)\ldots(x-k+1)}{k!} & \text{if } x \geq k-1 \\ 0 & \text{if } x < k-1. \end{cases}$$

Let us remark that the function $f(x) = \binom{x}{k}$ is monotone increasing and convex.

A pair of sets, $A, B \in 2^{[n]}$, are said to be *weakly crossing*, if $A \setminus B$, $B \setminus A$ and $A \cap B$ are all non-empty. Clearly, if A and B are crossing, then A and B are weakly crossing as well. We call a set family $\mathcal{F} \subset 2^{[n]}$ *weakly k-cross-free* if it does not contain k pairwise weakly crossing sets.

As our first step of the proof, we show that if $\mathcal{F} \subset 2^{[n]}$ is a k-cross-free family, then we can pass to a weakly k-cross-free family $\mathcal{F}' \subset 2^{[n]}$ by losing a factor of at most 2 in the cardinality.

Lemma 4. *Let $\mathcal{F} \subset 2^{[n]}$ be a k-cross-free family. Then there exists a weakly k-cross-free family $\mathcal{F}' \subset 2^{[n]}$ such that $|\mathcal{F}'| \geq |\mathcal{F}|/2$.*

Proof. Let

$$\mathcal{F}' = \{A \in \mathcal{F}' : 1 \notin A\} \cup \{[n] \setminus A : A \in \mathcal{F}', 1 \in A\}.$$

Clearly, we have $|\mathcal{F}'| \geq |\mathcal{F}|/2$.

Note that two sets $A, B \in [n]$ are crossing if and only if A and $[n] \setminus B$ are crossing. Hence, $\mathcal{F}'$ does not contain k pairwise crossing sets. But no set in $\mathcal{F}'$ contains 1, so we cannot have $A \cup B = [n]$ for any $A, B \in \mathcal{F}'$. Thus, $A, B \in \mathcal{F}'$ are crossing if and only if A and B are weakly crossing. Hence, $\mathcal{F}'$ satisfies the conditions of the lemma. $\square$

Now Theorem 2 follows trivially from the combination of Lemma 4 and the following theorem.

Theorem 5. *Let $k \geq 2$ and n be positive integers and let $\mathcal{F} \subset 2^{[n]}$ be a weakly k-cross-free family. Then $|\mathcal{F}| = O_k(n \log^* n)$.*

The rest of this section is devoted to the proof of this theorem. Let us briefly sketch the idea of the proof while introducing some of the main notation.

Let $\mathcal{F}$ be a weakly k-cross-free family. First, we shall divide the elements of $\mathcal{F}$ into $\log n$ parts according to their sizes: for $i = 0, \dots, \log n$, let $\mathcal{F}_i := \{X \in \mathcal{F} : 2^i < |X| \leq 2^{i+1}\}$. We might refer to the families $\mathcal{F}_i$ as *blocks*. Next, we show that, as the block $\mathcal{F}_i$ is weakly k-cross-free, it must have the following property: a positive proportion of $\mathcal{F}_i$ can be covered by a collection of chains Γ_i with the maximal elements of these chains forming an antichain. These chains are going to be the objects of main interest in our proof.

We show that if $\mathcal{F}$ has too many elements, then we can find k chains $\mathcal{C}_1 \subset \Gamma_{i_1}, \dots, \mathcal{C}_k \subset \Gamma_{i_k}$ for some $i_1 < \dots < i_k$, and k elements $C_{j,1} \subset \dots \subset C_{j,k}$ in each chain $\mathcal{C}_j$ such that $C_{j,l} \subset C_{j',l'}$ if $j \leq j'$ and $l \leq l'$, and $C_{j,l}$ and $C_{j',l'}$ are weakly crossing otherwise. But then we arrive to a contradiction since the k sets $C_{1,k}, C_{2,k-1}, \dots, C_{k,1}$ are pairwise weakly crossing.

Now let us show how to execute this argument precisely.

Proof of Theorem 5. Without loss of generality, we can assume that $\mathcal{F}$ does not contain the empty set and 1-element sets, since by deleting them we decrease the size of $\mathcal{F}$ by at most $n+1$.

Let us remind the reader of the definition of blocks: for $i = 0, 1, \dots, \log n$, we have

$$\mathcal{F}_i := \{X \in \mathcal{F} : 2^i < |X| \leq 2^{i+1}\}.$$

The next claim gives an upper bound on the size of an antichain in $\mathcal{F}_i$.

Claim 6. *If $\mathcal{A} \subset \mathcal{F}_i$ is an antichain, then*

$$|\mathcal{A}| \leq \frac{(k-1)n}{2^i}.$$

Proof. Suppose that there exists $x \in [n]$ such that x is contained in k sets from $\mathcal{A}$. Then these k sets are pairwise weakly crossing. Hence, every element of $[n]$ is contained in at most $k-1$ of the sets in $\mathcal{A}$, which implies that

$$(k-1)n \geq \sum_{A \in \mathcal{A}} |A| \geq |\mathcal{A}| 2^i.$$

$\square$

In the next claim, we show that a positive proportion of $\mathcal{F}_i$ can be covered by chains whose maximal elements form an antichain. We shall use the following notation concerning chains. If $\mathcal{C}$ is a chain of size l, denote its elements by $\mathcal{C}(1) \subset \dots \subset \mathcal{C}(l)$. Accordingly, let $\min \mathcal{C} = \mathcal{C}(1)$ and $\max \mathcal{C} = \mathcal{C}(l)$.

Claim 7. *For every $i \geq 0$, there exists a collection Γ_i of chains in $\mathcal{F}_i$ such that $\{\max \mathcal{C} : \mathcal{C} \in \Gamma_i\}$ is an antichain and*

$$\sum_{\mathcal{C} \in \Gamma_i} |\mathcal{C}| \geq \frac{|\mathcal{F}_i|}{k-1}.$$

Proof. Let $\mathcal{M}$ be the family of maximal elements of $\mathcal{F}_i$ with respect to containment. For each $M \in \mathcal{M}$, let $\mathcal{H}_M \subset \mathcal{F}_i$ be a family of sets contained in M such that the system $\{\mathcal{H}_M\}_{M \in \mathcal{M}}$ forms a partition of $\mathcal{F}_i$.

Note that any two sets in $\mathcal{H}_M$ have a nontrivial intersection, as every $A \in \mathcal{H}_M$ satisfies $A \subset M$ and $|A| > |M|/2$. Hence, $\mathcal{H}_M$ cannot contain an antichain of size k, otherwise, these k sets would be pairwise weakly crossing. Therefore, by Dilworth's theorem [7], $\mathcal{H}_M$ contains a chain $\mathcal{C}_M$ of size at least $|\mathcal{H}_M|/(k-1)$. The collection $\Gamma_i = \{\mathcal{C}_M : M \in \mathcal{M}\}$ meets the requirements of the Claim. $\square$

Let Γ_i be a collection of chains in $\mathcal{F}_i$ satisfying the conditions in Claim 7. As the maximal elements of the chains in Γ_i form an antichain, Claim 6 gives the following upper bound on the size of Γ_i:

$$|\Gamma_i| \leq \frac{(k-1)n}{2^i}. \tag{1}$$

From now on, fix some positive real numbers a, b with $a \leq b \leq \log n$ and consider the union of blocks $\mathcal{F}_{a,b} = \bigcup_{a<i\leq b} \mathcal{F}_i$. Analogously, let $\Gamma_{a,b} = \bigcup_{a<i\leq b} \Gamma_i$. Allowing a and b to be not necessarily integers will serve as a slight convenience. In what follows, we bound the size of $\mathcal{F}_{a,b}$.

For each chain $\mathcal{C} \in \Gamma_{a,b}$, define a set $Y(\mathcal{C})$ by picking an arbitrary element from each of the difference sets $\mathcal{C}(j+1) \setminus \mathcal{C}(j)$ for $j = 1, \ldots, |\mathcal{C}| - 1$, and from $\mathcal{C}_1$, as well. Clearly, we have $|Y(\mathcal{C})| = |\mathcal{C}|$. For every $y \in [n]$, let $d(y)$ be the number of chains $\mathcal{C}$ in $\Gamma_{a,b}$ such that $y \in Y(\mathcal{C})$. Note that

$$\sum_{y\in[n]} d(y) = \sum_{\mathcal{C}\in\Gamma_{a,b}} |Y(\mathcal{C})| = \sum_{\mathcal{C}\in\Gamma_{a,b}} |\mathcal{C}| \geq \frac{|\mathcal{F}_{a,b}|}{k-1}, \tag{2}$$

where the last inequality holds by Claim 7.

We will bound the size of $\mathcal{F}_{a,b}$ by arguing that one cannot have k different elements of $[n]$ appearing in $Y(\mathcal{C})$ for many different sets $\mathcal{C} \in \Gamma_{a,b}$ without violating the condition that $\mathcal{F}$ is weakly k-cross-free. Thus, $\sum_{y\in[n]} d(y)$ must be small. For this, we need the following definition.

Definition 8. *Let $y \in [n]$. Consider a k-tuple of chains $(\mathcal{C}_1, \ldots, \mathcal{C}_k)$ in $\Gamma_{a,b}$, where $\mathcal{C}_i \in \Gamma_{j_i}$ for a strictly increasing sequence $j_1 < \ldots < j_k$. We say that $(C_1, \ldots, C_k)$ is* good *for y if*

(i) *$y \in Y(\mathcal{C}_i)$ for $i \in [k]$,*

(ii) *if $C_i \in \mathcal{C}_i$ is the smallest set such that $y \in C_i$, then $C_1 \subset \ldots \subset C_k$.*

Next, we show that if $d(y)$ is large, then y is good for many k-tuples of chains. Let $g(y)$ denote the number of good k-tuples for y.

Claim 9. *For every $y \in [n]$, we have*

$$g(y) \geq \binom{d(y)/(k-1)^2}{k}.$$

Proof. Let $d = d(y)$ and let $\mathcal{C}_1, \ldots, \mathcal{C}_d \in \Gamma_{a,b}$ be the chains such that $y \in Y(\mathcal{C}_i)$. Also, for $i = 1, \ldots, d$, let C_i be the smallest set in $\mathcal{C}_i$ containing y, and let $\mathcal{H} = \{C_1, \ldots, C_d\}$.

The family $\mathcal{H}$ is intersecting. Therefore, it cannot contain an antichain of size k, as any two elements of such an antichain are weakly crossing. Applying Dilworth's theorem [7], we obtain that $\mathcal{H}$ contains a chain of size at least $s = \lceil d/(k-1) \rceil$. Without loss of generality, let $C_1 \subset \ldots \subset C_s$ be such a chain.

For any $a \leq i \leq b$, $\mathcal{F}_i$ contains at most $k-1$ members of the sequence $C_1, \ldots, C_s$. Otherwise, if $\mathcal{C}_{j_1}, \ldots, \mathcal{C}_{j_k} \in \Gamma_i$ for some $1 \leq j_1 < \ldots < j_k \leq s$, the maximal elements $\max \mathcal{C}_{j_1}, \ldots, \max \mathcal{C}_{j_k}$ are pairwise weakly crossing, because these sets form an antichain and contain y.

This implies that the sets $C_1, \ldots, C_s$ are contained in at least $r = \lceil s/(k-1) \rceil \geq d/(k-1)^2$ different blocks. Thus, we can assume that there exist $i_1 < \ldots < i_r$ and $j_1 < \ldots < j_r$ such that $C_{i_l} \in \mathcal{F}_{j_l}$ for $l \in [r]$.

Then any k-element subset of $\{\mathcal{C}_{i_1}, \ldots, \mathcal{C}_{i_r}\}$ is a good k-tuple for y, resulting in at least

$$\binom{r}{k} \geq \binom{d(y)/(k-1)^2}{k}$$

good k-tuples for y. □

Now we give an upper bound on the total number of k-tuples that may be good for some $y \in [n]$. A k-tuple of chains in $\Gamma_{a,b}$ is called *nice* if it is good for some $y \in [n]$. Let N be the number of nice k-tuples.

Claim 10. *We have*

$$N < \frac{2(k-1)^k n}{2^a} \binom{b}{k-1}.$$

Proof. Let $\mathcal{C} \in \Gamma_{a,b}$. Let us count the number of nice k-tuples $(\mathcal{C}_1, \ldots, \mathcal{C}_k)$ for which $\mathcal{C} = \mathcal{C}_1$. Note that in a nice k-tuple $(\mathcal{C}_1, \ldots, \mathcal{C}_k)$, the set $\min \mathcal{C}_1$ is contained in $\max \mathcal{C}_1, \ldots, \max \mathcal{C}_k$.

But then, for any positive integer i satisfying $a < i \leq b$, there are at most $k-1$ chains in Γ_i that can belong to a nice k-tuple with first element $\mathcal{C}$. Indeed, suppose that there exist k chains $\mathcal{D}_1, \ldots, \mathcal{D}_k$ in Γ_i that all appear in a nice k-tuple with their first element being $\mathcal{C}$. Then $\{\max \mathcal{D}_1, \ldots, \max \mathcal{D}_k\}$ is an intersecting antichain: it is intersecting because $\max \mathcal{D}_j$ contains $\min \mathcal{C}$ for $j \in [k]$, and it is an antichain, by the definition of Γ_i. Thus, any two sets among $\max \mathcal{D}_1, \ldots, \max \mathcal{D}_k$ are weakly crossing, a contradiction.

Hence, the number of nice k-tuples $(C_1, \ldots, C_k)$ for which $\mathcal{C}_1 = \mathcal{C}$ is at most $\binom{b}{k-1}(k-1)^{k-1}$, as there are at most $\binom{b}{k-1}$ choices for $j_2 < \ldots < j_k \leq b$ such that $C_l \in \Gamma_{j_l}$ for $l = 2, \ldots, k$, and there are at most $k-1$ further choices for each chain C_l in Γ_{j_l}.

Clearly, the number of choices for $\mathcal{C} = \mathcal{C}_1$ is at most the size of $\Gamma_{a,b}$, which is

$$|\Gamma_{a,b}| = \sum_{a<i\leq b} |\Gamma_i| \leq \sum_{a<i\leq b} \frac{(k-1)n}{2^i} < \frac{2(k-1)n}{2^a};$$

see (1) for the first inequality. Hence, the total number of nice k-tuples is at most

$$\frac{2(k-1)^k n}{2^a}\binom{b}{k-1}.$$

□

The next claim is the key observation in our proof. It tells us that a k-tuple of chains cannot be good for k different elements of $[n]$.

Claim 11. *There are no k different elements $y_1, \ldots, y_k \in [n]$ and a k-tuple $(\mathcal{C}_1, \ldots, \mathcal{C}_k)$ such that $(\mathcal{C}_1, \ldots, \mathcal{C}_k)$ is good for $y_1, \ldots, y_k$.*

Proof. Suppose that there exist such a k-tuple $(\mathcal{C}_1, \ldots, \mathcal{C}_k)$ and k elements $y_1, \ldots, y_k$. For $i, j \in [k]$, let $C_{i,j}$ be the smallest set in $\mathcal{C}_i$ that contains y_j. By the definition of a good k-tuple, we have $C_{1,j} \subset \ldots \subset C_{k,j}$ for $j \in [k]$. Also, the sets $C_{1,1}, \ldots, C_{1,k}$ are distinct elements of the chain $\mathcal{C}_1$, so, without loss of generality, we can assume that $C_{1,1} \subset C_{1,2} \subset \ldots \subset C_{1,k}$.

First, we show that this assumption forces $C_{i,1} \subset \ldots \subset \mathcal{C}_{i,k}$ for all $i \in [k]$, as well. To this end, it is enough to prove that we cannot have $C_{i,j'} \subset C_{i,j}$ for some $1 \leq j < j' \leq k$. Indeed, suppose that $C_{i,j'} \subset C_{i,j}$. Then $y_j \in C_{i,j}$, but $y_j \notin C_{i,j'}$. However, $y_j \in C_{1,j}$ and $C_{1,j} \subset C_{1,j'} \subset C_{i,j'}$, contradiction.

Next, we show that any two sets in the family

$$\mathcal{H} := \left\{C_{i,k+1-i} : i \in [k]\right\}$$

are weakly crossing. Every element of $\mathcal{H}$ contains $C_{1,1}$, so $\mathcal{H}$ is an intersecting family. Our task is reduced to showing that $\mathcal{H}$ is an antichain. Suppose that $C_{i,k+1-i} \subset C_{i',k+1-i'}$ for some $i, i' \in [k]$, $i \neq i'$. Then we must have $i < i'$. Otherwise, $|C_{i,k+1-i}| > |C_{i',k+1-i'}|$, as $C_{i,k+1-i} \in \mathcal{F}_{j_i}$ and $C_{i',k+1-i'} \in \mathcal{F}_{j_{i'}}$ hold for some $j_{i'} < j_i$. But if $i < i'$, we have $y_{k+1-i} \in C_{i,k+1-i}$ and $y_{k+1-i} \notin C_{i',k+1-i'}$, so $C_{i,k+1-i} \not\subset C_{i',k+1-i'}$.

Thus, any two sets of the k-element family $\mathcal{H}$ are weakly crossing, which is a contradiction. □

Let M be the number of pairs $(y, (C_1, \ldots, C_k))$ such that $(C_1, \ldots, C_k)$ is a good k-tuple for $y \in [n]$. Let us double count M.

On one hand, Claim 11 implies that $M \leq (k-1)N$. Plugging in our upper bound of Claim 10 for N, we get

$$M \leq (k-1)N < \frac{2(k-1)^{k+1}n}{2^a}\binom{b}{k-1} \leq \frac{2n(k-1)^{k+1}b^{k-1}}{2^a(k-1)!}.$$

For simplicity, write $c_1(k) = 2(k-1)^{k+1}/(k-1)!$, then our inequality becomes

$$M \leq \frac{c_1(k)nb^{k-1}}{2^a}. \tag{3}$$

On the other hand, we have

$$M = \sum_{y\in[n]} g(y),$$

where $g(y)$, as before, stands for the number of good k-tuples for y. Applying Claim 9, we can bound the right-hand side from below, as follows.

$$\sum_{y\in[n]} g(y) \geq \sum_{y\in[n]} \binom{d(y)/(k-1)^2}{k}.$$

Exploiting the convexity of the function $\binom{x}{k}$, Jensen's inequality implies that the right-hand side is at least

$$n\binom{\sum_{y\in[n]} d(y)/(k-1)^2 n}{k}.$$

Finally, using (2), we obtain

$$M \geq n\binom{|\,\mathcal{F}_{a,b}\,|/(k-1)^3 n}{k}. \tag{4}$$

Suppose that $|\,\mathcal{F}_{a,b}\,| > 2k(k-1)^3 n$. In this case, we have

$$\binom{|\,\mathcal{F}_{a,b}\,|/(k-1)^3 n}{k} > \left(\frac{|\mathcal{F}_{a,b}|}{2(k-1)^3 n}\right)^k \frac{1}{k!}.$$

Writing $c_2(k) = 1/2^k(k-1)^{3k}k!$, we can further bound the right-hand side of (4) and arrive at the inequality

$$M > \frac{c_2(k)|\,\mathcal{F}_{a,b}\,|^k}{n^{k-1}}. \tag{5}$$

Comparing (3) and (5), we obtain

$$\frac{c_1(k)nb^{k-1}}{2^a} > \frac{c_2(k)|\,\mathcal{F}_{a,b}\,|^k}{n^{k-1}},$$

which yields the following upper bound for the size of $\mathcal{F}_{a,b}$:

$$|\mathcal{F}_{a,b}| < n\left(\frac{c_1(k)}{c_2(k)}\right)^{1/k} \frac{b^{(k-1)/k}}{2^{a/k}}.$$

Recall that (5) and the last inequality hold under the assumption that $|\,\mathcal{F}_{a,b}\,| > 2k(k-1)^3 n$. Hence, writing $c_3(k) = (c_1(k)/c_2(k))^{1/k}$, we get that

$$|\mathcal{F}_{a,b}| < \max\left\{2k(k-1)^3 n, \frac{c_3(k)nb^{(k-1)/k}}{2^{a/k}}\right\} \tag{6}$$

holds without any assumption.

We finish the proof by choosing an appropriate sequence $\{a_i\}_{i=0}^s$ and applying the bound (6) for the families $\mathcal{F}_{a_i,a_{i+1}}$.

Define the sequence $\{a_i\}_{i=0,1,\ldots}$ such that $a_0 = 0$, $a_1 = k^2$ and $a_{i+1} = 2^{a_i/(k-1)}$ for $i = 1, 2, \ldots$. Let s be the smallest positive integer such that $a_s > \log n$. Clearly, we have $s = O_k(\log^*(n))$. Also,

$$|\mathcal{F}_{a_0,a_1}| = |\mathcal{F}_{0,k^2}| \leq \sum_{l=1}^{2^{k^2}} \frac{(k-1)n}{l} = O_k(n),$$

as $\mathcal{F}$ has at most $(k-1)n/l$ elements of size l for $l \in [n]$, by the weakly k-cross-free property. Finally, for $i = 1, \ldots, s-1$, (6) yields that

$$|\mathcal{F}_{a_i,a_{i+1}}| < \max\left\{2k(k-1)^3 n, \frac{c_3(k)na_{i+1}^{(k-1)/k}}{2^{a_i/k}}\right\} = \max\{2k(k-1)^3, c_3(k)\}n.$$

The proof of Theorem 5 can be completed by noting that

$$|\,\mathcal{F}\,| = \sum_{i=0}^{s-1} |\mathcal{F}_{a_i,a_{i+1}}| \leq O_k(n) + s\max\{2k(k-1)^3, c_3(k)\}n = O_k(n\log^* n).$$

□

Acknowledgement. We are grateful to Peter Frankl for his useful remarks. In particular, he pointed out that with more careful computation Claim 9 can be improved to

$$g(y) \geq (k-1)^k \binom{d(y)/(k-1)^2}{k}.$$

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