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On a reaction–diffusion–advection system: fixed boundary or free boundary

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Abstract. This paper is devoted to the asymptotic behaviors of the solution to a reaction–diffusion–advection system in a homogeneous environment with fixed boundary or free boundary. For the fixed boundary problem, the global asymptotic stability of nonconstant semi-trivial states is obtained. It is also shown that there exists a stable nonconstant co-existence state under some appropriate conditions. Numerical simulations are given not only to illustrate the theoretical results, but also to exhibit the advection-induced difference between the left and right boundaries as time proceeds. For the free boundary problem, the spreading–vanishing dichotomy is proved, i.e., the solution either spreads or vanishes finally. Besides, the criteria for spreading and vanishing are further established.

Keywords: reaction–diffusion–advection, fixed boundary, free boundary, nonconstant steady states, spreading–vanishing.

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1 Introduction

Consider

$$\begin{cases} u_t = d_1 u_{xx} - \beta_1 u_x + u(r_1 - a_1 u - b_1 v), & (x,t) \in (0,L) \times (0,\infty), \\ v_t = d_2 v_{xx} - \beta_2 v_x + v(r_2 - a_2 v - b_2 u), & (x,t) \in (0,L) \times (0,\infty), \\ u(x,0) = u_0(x) > 0, & x \in (0,L), \\ v(x,0) = v_0(x) > 0, & x \in (0,L). \end{cases} \tag{1.1}$$

Here, $d_i, \beta_i, r_i, a_i, b_i$ are given constants, which implies in a homogeneous environment. For convenience, $i = 1,2$ in the whole text whenever it is mentioned.

Particularly, for $d_i = 0$ and $\beta_i = 0$, (1.1) is a classical ordinary differential system, and massive outstanding researches have been proposed, see [1–4, 14, 20–22] and references therein; for $d_i \in \mathbb{R}^+$ and $\beta_i \in \mathbb{R}$, (1.1) is a reaction–diffusion–advection (RDA) system. Such RDA problems were extensively used to understand the spatial behavior of populations, the dynamics of information diffusion, and so on. Up to now, many remarkable results have been achieved, see [5–10, 12, 13, 15–17, 24–27, 29], etc.

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For a population growth model, $u(x,t)$, $v(x,t)$ in (1.1) respectively represent the densities of two species at location x and time t. $d_i > 0$ denotes the random dispersal rate of the species; advection rate $\beta_i \in \mathbb{R}$ is the moving speed of individuals towards their more favorable habitats. As noted in [12], $\beta_i > 0$ means advection points towards larger x, while $\beta_i < 0$ implies advection points towards smaller x. $r_i > 0$ accounts for intrinsic growth rate; $a_i > 0$ is intra-specific interaction rate; $b_i \in \mathbb{R}$ is interspecific interaction rate. What described above means that system (1.1) is a competition model with $b_i > 0$, see [16,17,29], and a predator-prey problem for $b_1 > 0, b_2 < 0$, see [24,26,27,30]. For a information diffusion model, the meaning of $u(x,t)$, $v(x,t), d_i, \beta_i, r_i, a_i, b_i$ can refer to [22,25].

The way to formulate boundary conditions for reaction-diffusion models is based on how the flux of individuals crosses a boundary. As mentioned in [5], the flux $\overrightarrow{J} = -d\nabla u + \overrightarrow{\beta} u$ across the boundary at any given point is proportional to the density with constant of proportionality, that is

$$\left(-d\nabla u + \overrightarrow{\beta} u\right) \cdot \overrightarrow{n} = \alpha u, \tag{1.2}$$

where $d > 0$ is the diffusion rate, $\overrightarrow{\beta}$ is the advection velocity, $\overrightarrow{n}$ is the outward pointing normal vector, α is the proportionality coefficient. If $\alpha \to \infty$, then the boundary condition (1.2) becomes $u = 0$, which is a Dirichlet condition; if $\overrightarrow{\beta} = 0$ and $\alpha = 0$, we have $(-d\nabla u) \cdot \overrightarrow{n} = 0$, i.e., $\frac{\partial u}{\partial \overrightarrow{n}} = 0$, which is known as a Neumann condition; if $\overrightarrow{\beta} = 0$ and $\alpha < 0$, then (1.2) is in the form of $d\frac{\partial u}{\partial \overrightarrow{n}} - \alpha u = 0$, which is referred to a Robin condition; if $\alpha = 0$, then (1.2) becomes $d\frac{\partial u}{\partial \overrightarrow{n}} - \overrightarrow{\beta} \cdot \overrightarrow{n} u = 0$, which is called a no-flux or reflecting boundary condition, since it means that individuals encountering the boundary are always reflected back so they do not leave the domain, that is, no individual crosses the boundary.

Here we focus on the no-flux boundary conditions

$$\begin{cases} d_1 u_x(0,t) - \beta_1 u(0,t) = d_1 u_x(L,t) - \beta_1 u(L,t) = 0, & t \in (0,\infty), \\ d_2 v_x(0,t) - \beta_2 v(0,t) = d_2 v_x(L,t) - \beta_2 v(L,t) = 0, & t \in (0,\infty). \end{cases} \tag{1.3}$$

In fact, Lou et al. in [16] first qualitatively analyzed (1.1) with no-flux boundary (1.3) under $d_1 = d_2$ and

$$r_1 = r_2, \qquad a_1 = a_2 = b_1 = b_2 = 1. \tag{1.4}$$

It is shown that the movement with either smaller advection or no advection is eventually stable. Afterwards, based on the assumptions of (1.4), Zhou in [29] further investigated (1.1) with the addition of (1.3) to understand the joint effects of diffusion and advection on the outcome of competition. Overcoming the mathematical difficulties arising out of $d_1 \neq d_2$, Zhou in [29] obtained much richer observations: the movement with smaller diffusion, smaller advection and smaller ratio of advection to diffusion, or with larger diffusion and smaller advection, wins the competition.

However, for a more general model with $d_i, r_i, a_i \in \mathbb{R}^+$, $\beta_i, b_i \in \mathbb{R}$ and without assumption (1.4), there have been no results so far. Due to this reason, in this paper, we study (1.1) with fixed boundary (1.3) and present a thorough understanding: for $b_i > 0$, problem (1.1) and (1.3) may finally stabilize to a nonconstant semi-trivial steady state if $\beta_i > 0$, but admit a stable co-existence state if $\beta_1 \cdot \beta_2 < 0$; for $b_1 > 0, b_2 < 0$, the two semi-trivial steady semi-trivial steady states of (1.1) with the addition of (1.3) are both unstable. Furthermore, we give numerical simulations to find out that advection can induce great difference between the left and right boundaries as time goes on; and the problem with $b_1 > 0, b_2 < 0$ may have a co-existence state.

On the other hand, influenced by human activity, the habitat of some species often changes with time, which can be described by a free boundary. In this case, we go one step further and discuss the corresponding free boundary problem

$$\begin{cases} u_t = d_1 u_{xx} - \beta_1 u_x + u(r_1 - a_1 u - b_1 v), & 0 < x < h(t), t > 0, \\ v_t = d_2 v_{xx} - \beta_2 v_x + v(r_2 - a_2 v - b_2 u), & 0 < x < h(t), t > 0, \\ u(0,t) = v(0,t) = u(h(t),t) = v(h(t),t) = 0, & t > 0, \\ h'(t) = -\mu[u_x(h(t),t) + \rho v_x(h(t),t)], & t > 0, \\ u(x,0) = u_0(x), v(x,0) = v_0(x), h(0) = h_0, & 0 < x < h_0. \end{cases} \tag{1.5}$$

Here μ, ρ and h_0 are given positive constants. $x = h(t)$ is the free boundary, and the initial function $u_0(x), v_0(x) \in \Sigma(h_0)$ for some $h_0 > 0$, where

$$\Sigma(h_0) = \{\phi \in C^2([0,h_0]) : \phi(0) = \phi(h_0) = 0,\ \phi(x) > 0 \ \text{ in } (0,h_0)\}. \tag{1.6}$$

Wang et al. in [26] studied system (1.5) with $\beta_i = 0$ and $b_i > 0$ and obtained the long time behavior of two competing species spreading via a free boundary. Based on the assumptions of $\beta_i = 0$, $b_1 > 0$, and $b_2 < 0$, Wang in [24] investigated system (1.5) to get a spreading-vanishing dichotomy and set the criteria for spreading and vanishing, moreover, Wang in [24] gave the estimation of asymptotic spreading speed when spreading successfully.

Motivated by the works in [24,26], we study system (1.5) with $d_i > 0, \beta_i \geq 0$, $a_i > 0$, $b_1 > 0$ and $b_2 \in \mathbb{R}$. We prove that the spreading-vanishing dichotomy still holds, i.e. the solution to problem (1.5) is vanishing if $h_\infty < +\infty$, on the other hand, it is spreading if $h_\infty = +\infty$ under some proper conditions. Furthermore, we determine the criteria for spreading and vanishing.

The rest of this paper is organized as follows. In Section 2, the fixed boundary problem is analyzed, including the global asymptotic stability of nonconstant semi-trivial steady states, and the existence of a stable nonconstant co-existence state. Numerical simulations are presented to illustrate the results. Section 3 is devoted to the free boundary problem. The spreading-vanishing dichotomy is obtained and the criteria for spreading and vanishing are determined.

2 The fixed boundary problem

2.1 Existence of semi-trivial steady states

First, we consider the problem

$$\begin{cases} d_1 u_{xx} - \beta_1 u_x + u(r_1 - a_1 u) = 0, & x \in (0,L), \\ d_1 u_x(0) - \beta_1 u(0) = 0, \\ d_1 u_x(L) - \beta_1 u(L) = 0, \end{cases} \tag{2.1}$$

and the following statement is valid.

Lemma 2.1. *For any $\beta_1 \in \mathbb{R}$, and d_1, r_1, $a_1 \in \mathbb{R}^+$, problem (2.1) admits a unique positive solution $\tilde{u}$.*

Proof. For $\beta_1 \geq 0$, we rewrite problem (2.1) as

$$\begin{cases} d_1 u_{xx} - \beta_1 u_x + u(r_1 - a_1 u) = 0, & x \in (0,L), \\ d_1 u_x(0) - \beta_1 u(0) = 0, \\ d_1 u_x(L) + \beta_1 u(L) = 2\beta_1 u(L). \end{cases} \tag{2.2}$$

Let
$$\bar{u} = \frac{M_1}{a_1} e^{\frac{\beta_1}{d_1}x}, \qquad \underline{u} = \frac{M_2}{a_1} e^{\frac{\beta_1}{d_1}x},$$
where $M_1 > r_1 e^{\frac{|\beta_1|}{d_1}L}$, $0 < M_2 < r_1 e^{\frac{-|\beta_1|}{d_1}L}$. After some simple computations, we have
$$\begin{cases} d_1\bar{u}_{xx} - \beta_1\bar{u}_x + \bar{u}(r_1 - a_1\bar{u}) < 0, & x \in (0,L), \\ d_1\bar{u}_x(0) - \beta_1\bar{u}(0) = 0, \\ d_1\bar{u}_x(L) + \beta_1\bar{u}(L) = 2\beta_1\bar{u}(L), \end{cases} \tag{2.3}$$
and
$$\begin{cases} d_1\underline{u}_{xx} - \beta_1\underline{u}_x + \underline{u}(r_1 - a_1\underline{u}) > 0, & x \in (0,L), \\ d_1\underline{u}_x(0) - \beta_1\underline{u}(0) = 0, \\ d_1\underline{u}_x(L) + \beta_1\underline{u}(L) = 2\beta_1\underline{u}(L). \end{cases} \tag{2.4}$$
According to the definition of upper and lower solutions in [19], one can see that $\bar{u}$ and $\underline{u}$ are upper and lower solutions to problem (2.2).

Set
$$F(u) := u(r_1 - a_1 u), \qquad x \in (0,L),$$
and
$$G(u) := \begin{cases} 0, & x = 0, \\ 2\beta_1 u, & x = L. \end{cases}$$

For $\underline{u} \le u_2 < u_1 \le \bar{u}$, one can see that there are constants $K_1 > 0, K_2 > 0$ such that
$$F(u_1) - F(u_2) \ge -K_1(u_1 - u_2), \qquad x \in (0,L),$$
and
$$G(u_1) - G(u_2) \ge -K_2(u_1 - u_2), \qquad x = 0, L.$$
Thanks to Theorem 4.4.1 and Corollary 4.4.1 in [19], problem (2.2) has a positive solution $\tilde{u} \in C^2([0,L])$, satisfying
$$\underline{u} \le \tilde{u} \le \bar{u}.$$

The proof of uniqueness is similar to [16, Lemma 2.1]. For the readers' convenience, we outline the main ideas. Suppose $\tilde{u}_1$ and $\tilde{u}_2$ are two different positive solutions to (2.2). We may assume $\tilde{u}_1 > \tilde{u}_2 > 0$. $\tilde{u}_1$ and $\tilde{u}_2$ satisfy
$$\begin{cases} d_1\tilde{u}_{1xx} - \beta_1\tilde{u}_{1x} + \tilde{u}_1(r_1 - a_1\tilde{u}_1) = 0, & x \in (0,L), \\ d_1\tilde{u}_{1x}(0) - \beta_1\tilde{u}_1(0) = 0, \\ d_1\tilde{u}_{1x}(L) - \beta_1\tilde{u}_1(L) = 0, \end{cases} \tag{2.5}$$
and
$$\begin{cases} d_1\tilde{u}_{2xx} - \beta_1\tilde{u}_{2x} + \tilde{u}_2(r_1 - a_1\tilde{u}_2) = 0, & x \in (0,L), \\ d_1\tilde{u}_{2x}(0) - \beta_1\tilde{u}_2(0) = 0, \\ d_1\tilde{u}_{2x}(L) - \beta_1\tilde{u}_2(L) = 0, \end{cases} \tag{2.6}$$
respectively. Multiplying the first equation of (2.5) by $e^{-\frac{\beta_1}{d_1}x}\tilde{u}_2$ and the first equation of (2.6) by $e^{-\frac{\beta_1}{d_1}x}\tilde{u}_1$, subtract the resulting equations and integrate over $[0,L]$, and then we get
$$\int_0^L a_1\tilde{u}_1\tilde{u}_2 e^{-\frac{\beta_1}{d_1}x}(\tilde{u}_2 - \tilde{u}_1) = 0,$$

which contradicts $\tilde{u}_1 > \tilde{u}_2 > 0$. Therefore, we obtain the positive solution to (2.1) is unique.

For $\beta_1 < 0$, we can make some minor modifications to get the existence and uniqueness of the positive solution to problem (2.1), so we omit the details. $\square$

Similarly, the problem

$$\begin{cases} d_2 v_{xx} - \beta_2 v_x + v(r_2 - a_2 v) = 0, \quad x \in (0, L), \\ d_2 v_x(0) - \beta_2 v(0) = 0, \\ d_2 v_x(L) - \beta_2 v(L) = 0, \end{cases} \tag{2.7}$$

also has a unique positive solution $\tilde{v}$.

Thus, the following result follows from Lemma 2.1 directly.

Lemma 2.2. *For any β_i, $b_i \in \mathbb{R}$ and d_i, r_i, $a_i \in \mathbb{R}^+$, $i = 1, 2$, system (1.1) with the addition of (1.3) has two semi-trivial steady states, denoted by $(\tilde{u}, 0)$ and $(0, \tilde{v})$ respectively.*

Furthermore, as for $\tilde{u}$ and $\tilde{v}$, we have the following result, which is vital to our later analysis.

Lemma 2.3. *Suppose $0 < d_1 < d_2$, $a_i \in \mathbb{R}^+$, then*

(i) $0 < \frac{\tilde{u}_x}{\tilde{u}} < \frac{\beta_1}{d_1} \le \frac{\beta_2 - \beta_1}{d_2 - d_1}$, *if* $0 < \beta_1 < \beta_2$ *and* $\frac{\beta_1}{d_1} \le \frac{\beta_2}{d_2}$, $x \in (0, L)$;
$\frac{\beta_1}{d_1} < \frac{\tilde{u}_x}{\tilde{u}} < 0$, *if* $\beta_1 < 0, x \in (0, L)$;

(ii) $0 < \frac{\tilde{v}_x}{\tilde{v}} < \frac{\beta_2}{d_2} \le \frac{\beta_2 - \beta_1}{d_2 - d_1}$, *if* $0 < \beta_1 < \beta_2$ *and* $\frac{\beta_1}{d_1} \le \frac{\beta_2}{d_2}$, $x \in (0, L)$;
$\frac{\beta_2}{d_2} < \frac{\tilde{v}_x}{\tilde{v}} < 0$, *if* $\beta_2 < 0, x \in (0, L)$.

Proof. For part (i), note that $(\tilde{u}, 0)$ satisfies

$$\begin{cases} d_1 \tilde{u}_{xx} - \beta_1 \tilde{u}_x + \tilde{u}(r_1 - a_1 \tilde{u}) = 0, \quad x \in (0, L), \\ d_1 \tilde{u}_x(0) - \beta_1 \tilde{u}(0) = 0, \\ d_1 \tilde{u}_x(L) - \beta_1 \tilde{u}(L) = 0. \end{cases} \tag{2.8}$$

Set $p := \frac{\tilde{u}_x}{\tilde{u}}$. Then some straightforward computations yield

$$\begin{cases} -d_1 p_{xx} + (\beta_1 - 2d_1 p) p_x + a_1 \tilde{u} p = 0, \quad x \in (0, L), \\ p(0) = p(L) = \frac{\beta_1}{d_1}. \end{cases} \tag{2.9}$$

By using maximum principle [18, Theorem 3.6], it is clear that

$$\begin{cases} 0 < p < \frac{\beta_1}{d_1}, \quad \text{if } \beta_1 > 0, \; x \in (0, L); \\ \frac{\beta_1}{d_1} < p < 0, \quad \text{if } \beta_1 < 0, \; x \in (0, L). \end{cases} \tag{2.10}$$

According to [29, Lemma 2.4], the conditions of $0 < d_1 < d_2$, $0 < \beta_1 < \beta_2$ and $\frac{\beta_1}{d_1} \le \frac{\beta_2}{d_2}$ imply that

$$\frac{\beta_1}{d_1} \le \frac{\beta_2 - \beta_1}{d_2 - d_1}. \tag{2.11}$$

Then part (i) of Lemma 2.3 follows from (2.10) and (2.11).

Similarly, we can prove part (ii). $\square$

2.2 Local stability of semi-trivial steady states

In this subsection, assume d_i, r_i, a_i, $\in \mathbb{R}^+$ and β_i, $b_i \in \mathbb{R}$, we focus on the local stability of the semi-trivial steady states of problem (1.1) with the addition of (1.3). Beginning with $(\tilde{u}, 0)$, and its stability is governed by the equations

$$\begin{cases} u_t = d_1 u_{xx} - \beta_1 u_x + u(r_1 - 2a_1\tilde{u}) - vb_1\tilde{u}, & (x,t) \in (0,L) \times (0,\infty), \\ v_t = d_2 v_{xx} - \beta_2 v_x + v(r_2 - b_2\tilde{u}), & (x,t) \in (0,L) \times (0,\infty), \\ d_1 u_x(0,t) - \beta_1 u(0,t) = d_1 u_x(L,t) - \beta_1 u(L,t) = 0, & t > 0, \\ d_2 v_x(0,t) - \beta_2 v(0,t) = d_2 v_x(L,t) - \beta_2 v(L,t) = 0, & t > 0. \end{cases} \tag{2.12}$$

The corresponding eigenvalue problem is

$$\begin{cases} d_1 \Phi_{xx} - \beta_1 \Phi_x + \Phi(r_1 - 2a_1\tilde{u}) - \omega b_2 \tilde{u} + \lambda\Phi = 0, & x \in (0,L), \\ d_2 \omega_{xx} - \beta_2 \omega_x + \omega(r_2 - b_2\tilde{u}) + \lambda\omega = 0, & x \in (0,L), \\ d_1 \Phi_x(0) - \beta_1 \Phi(0) = d_1 \Phi_x(L) - \beta_1 \Phi(L) = 0, \\ d_2 \omega_x(0) - \beta_2 \omega(0) = d_2 \omega_x(L) - \beta_2 \omega(L) = 0. \end{cases} \tag{2.13}$$

One can find that the second equation in (2.13) is decoupled from the first. As a result, we only consider the eigenvalue problem

$$\begin{cases} d_2 \omega_{xx} - \beta_2 \omega_x + \omega(r_2 - b_2\tilde{u}) + \lambda\omega = 0, & x \in (0,L), \\ d_2 \omega_x(0) - \beta_2 \omega(0) = 0, \\ d_2 \omega_x(L) - \beta_2 \omega(L) = 0. \end{cases} \tag{2.14}$$

Similarly, in order to investigate the stability of $(0, \tilde{v})$, we consider the eigenvalue problem

$$\begin{cases} d_1 \varphi_{xx} - \beta_1 \varphi_x + \varphi(r_1 - b_1\tilde{v}) + \mu\varphi = 0, & x \in (0,L), \\ d_1 \varphi_x(0) - \beta_1 \varphi(0) = 0, \\ d_1 \varphi_x(L) - \beta_1 \varphi(L) = 0. \end{cases} \tag{2.15}$$

For convenience, the general formula of eigenvalue problems (2.14) and (2.15) is given as follows

$$\begin{cases} d\delta_{xx} - \gamma\delta_x + \delta m(x) + \sigma\delta = 0, & x \in (0,L), \\ d\delta_x(0) - \gamma\delta(0) = 0, \\ d\delta_x(L) - \gamma\delta(L) = 0. \end{cases} \tag{2.16}$$

Then we have the following statements.

Lemma 2.4. *Suppose $d \in \mathbb{R}^+$, $\gamma \in \mathbb{R}$, $m(x) \in C^1([0,L])$, then the eigenvalue problem (2.16) has a simple principle eigenvalue σ_0, and the corresponding eigenfunction $\delta_0(x)$ can be chosen as $\delta_0(x) \gg 0$.*

Proof. Consider

$$\begin{cases} Lu = du_{xx} - \gamma u_x + um(x), & x \in (0,L), \\ du_x(0) - \gamma u(0) = 0, \\ du_x(L) - \gamma u(L) = 0, \end{cases} \tag{2.17}$$

where $d \in \mathbb{R}^+$, $\gamma \in \mathbb{R}$, $m(x) \in C^1([0,L])$. Let $w = e^{-\frac{\gamma}{d}x}u$, then (2.17) becomes

$$\begin{cases} Lu = e^{\frac{\gamma}{d}x}[dw_{xx} + \gamma w_x + wm(x)], & x \in (0,L), \\ w_x(0) = w_x(L) = 0. \end{cases} \tag{2.18}$$

Denote

$$L^*w = dw_{xx} + \gamma w_x + wm(x), \quad x \in (0,L), \tag{2.19}$$

and then

$$\int_0^L wLudx = \int_0^L uL^*wdx. \tag{2.20}$$

So L^* can be seen the adjoint operator of L.

By [23, Theorem 7.6.1], problem

$$\begin{cases} L^*\delta + \sigma\delta = 0, & x \in (0,L), \\ \delta_x(0) = \delta_x(L) = 0, \end{cases} \tag{2.21}$$

has a simple principle eigenvalue σ_0, and the corresponding eigenfunction $\delta_0(x)$ can be chosen as $\delta_0(x) \gg 0$.

Thanks to [5, Corollary 2.13], we get that σ_0 is also the principle eigenvalue of

$$\begin{cases} L\delta + \sigma\delta = 0, & x \in (0,L), \\ d\delta_x(0) - \gamma\delta(0) = 0, \\ d\delta_x(L) - \gamma\delta(L) = 0. \end{cases} \tag{2.22}$$

Accordingly, Lemma 2.4 is established. □

Let λ_0 (resp. μ_0) be the principal eigenvalues of (2.14) (resp. (2.15)), and $\omega_0(x)$ (resp. $\varphi_0(x)$) be the corresponding eigenfunction satisfying $\omega_0(x)$(resp. $\varphi_0(x)$) $\gg 0$. By Lemma 2.4, $(\lambda_0, \omega_0(x))$ and $(\mu_0, \varphi_0(x))$ must exist, moreover, λ_0 and μ_0 are simple.

For simplicity, in the following, we denote $\omega_0(x)$, $\varphi_0(x)$, $\delta_0(x)$, $\frac{\partial\omega_0(x)}{\partial x}$, $\frac{\partial\varphi_0(x)}{\partial x}$ and $\frac{\partial\delta_0(x)}{\partial x}$ by ω_0, φ_0, δ_0, ω_{0x}, φ_{0x} and δ_{0x} respectively.

Lemma 2.5. *Suppose $d \in (R^+)$, $\gamma \in \mathbb{R}$, $m(x) \in C^1([0,L])$, then we have*

(i) $\frac{\delta_{0x}}{\delta_0} < \frac{\gamma}{d}$ *in* $(0,L)$, *if* $m_x \le, \not\equiv 0$ *in* $[0,L]$;

(ii) $\frac{\delta_{0x}}{\delta_0} > \frac{\gamma}{d}$ *in* $(0,L)$, *if* $m_x \ge, \not\equiv 0$ *in* $[0,L]$.

Proof. Let $h := \frac{\delta_{0x}}{\delta_0}$, then $h(0) = h(L) = \frac{\gamma}{d}$. Taking derivative of h, we derive

$$h_x = \frac{\delta_{0xx}}{\delta_0} - h^2. \tag{2.23}$$

Note that (σ_0, δ_0) satisfies (2.16), thanks to (2.23), then some direct computations yield

$$-dh_x - dh^2 + \gamma h - m(x) = \sigma_0. \tag{2.24}$$

Taking derivative of (2.24) in view of x, we obtain

$$\begin{cases} -dh_{xx} + (\gamma - 2dh)h_x = m_x, & x \in (0,L), \\ h(0) = h(L) = \frac{\gamma}{d}. \end{cases} \tag{2.25}$$

It follows from the maximum principle that $h < \gamma/d$ if $m_x \le, \not\equiv 0$; $h > \gamma/d$ if $m_x \ge, \not\equiv 0$ in $[0,L]$. □

As for the stability of semi-trivial steady states of system (1.1) with the addition of (1.3), the following result in [23] is of great concern in our subsequent analysis.

Lemma 2.6. *Suppose d_i, r_i, $a_i \in \mathbb{R}^+$, and $\beta_i, b_i \in \mathbb{R}$, then the semi-trivial steady state $(\tilde{u}, 0)$ is linearly stable (resp. unstable) if λ_0 is positive (resp. negative); the semi-trivial steady state $(0, \tilde{v})$ is linearly stable (resp. unstable) if μ_0 is positive (resp. negative).*

For clarity, we first state two propositions, which are vital to judge the stability of $(\tilde{u}, 0)$ and $(0, \tilde{v})$.

Proposition 2.7. $\lambda_0 > 0$ *(resp.* $\lambda_0 < 0$*) if and only if*

$$\lambda^*(d_1, d_2, \beta_1, \beta_2, r_1, r_2, a_1, b_2) > 0 \qquad (\text{resp. } \lambda^*(d_1, d_2, \beta_1, \beta_2, r_1, r_2, a_1, b_2) < 0),$$

where

$$\begin{aligned} \lambda^*(d_1, d_2, \beta_1, \beta_2, r_1, r_2, a_1, b_2) = & \int_0^L e^{-\frac{\beta_2}{d_2}x} \tilde{u}\omega_0[(r_1 - r_2) + (b_2 - a_1)\tilde{u}]dx \\ & + \int_0^L e^{-\frac{\beta_2}{d_2}x} \left(\omega_{0x} - \frac{\beta_2}{d_2}\omega_0\right)[(d_2 - d_1)\tilde{u}_x + (\beta_1 - \beta_2)\tilde{u}]dx. \end{aligned} \tag{2.26}$$

Proof. Rewrite (2.8) as

$$\begin{cases} d_2\tilde{u}_{xx} - \beta_2\tilde{u}_x + \tilde{u}(r_1 - a_1\tilde{u}) = (d_2 - d_1)\tilde{u}_{xx} + (\beta_1 - \beta_2)\tilde{u}_x, & x \in (0, L), \\ d_2\tilde{u}_x(0) - \beta_2\tilde{u}(0) = (d_2 - d_1)\tilde{u}_x(0) + (\beta_1 - \beta_2)\tilde{u}(0), \\ d_2\tilde{u}_x(L) - \beta_2\tilde{u}(L) = (d_2 - d_1)\tilde{u}_x(L) + (\beta_1 - \beta_2)\tilde{u}(L). \end{cases} \tag{2.27}$$

Note that (λ_0, w_0) satisfies

$$\begin{cases} d_2\omega_{0xx} - \beta_2\omega_{0x} + \omega_0(r_2 - b_2\tilde{u}) = -\lambda_0\omega_0, & x \in (0, L), \\ d_2\omega_{0x}(0) - \beta_2\omega_0(0) = 0, \\ d_2\omega_{0x}(L) - \beta_2\omega_0(L) = 0. \end{cases} \tag{2.28}$$

Multiplying the first equation of (2.27) by $e^{-\frac{\beta_2}{d_2}x}\omega_0$, and integrating the resulting equation over $[0, L]$, we get

$$\begin{aligned} & (d_2\tilde{u}_x - \beta_2\tilde{u})e^{-\frac{\beta_2}{d_2}x}\omega_0\Big|_0^L - \int_0^L (d_2\tilde{u}_x - \beta_2\tilde{u})e^{-\frac{\beta_2}{d_2}x}\left(\omega_{0x} - \frac{\beta_2}{d_2}\omega_0\right)dx \\ & \quad + \int_0^L e^{-\frac{\beta_2}{d_2}x}\omega_0\tilde{u}(r_1 - a_1\tilde{u})dx \\ & = [(d_2 - d_1)\tilde{u}_x + (\beta_1 - \beta_2)\tilde{u}]e^{-\frac{\beta_2}{d_2}x}\omega_0\Big|_0^L \\ & \quad - \int_0^L [(d_2 - d_1)\tilde{u}_x + (\beta_1 - \beta_2)\tilde{u}]e^{-\frac{\beta_2}{d_2}x}\left(\omega_{0x} - \frac{\beta_2}{d_2}\omega_0\right)dx. \end{aligned} \tag{2.29}$$

By the boundary conditions of (2.27), (2.29) can be simplified to

$$\begin{aligned} & \int_0^L e^{-\frac{\beta_2}{d_2}x}\omega_0\tilde{u}(r_1 - a_1\tilde{u})dx + \int_0^L [(d_2 - d_1)\tilde{u}_x + (\beta_1 - \beta_2)\tilde{u}]e^{-\frac{\beta_2}{d_2}x}\left(\omega_{0x} - \frac{\beta_2}{d_2}\omega_0\right)dx \\ & = \int_0^L (d_2\tilde{u}_x - \beta_2\tilde{u})e^{-\frac{\beta_2}{d_2}x}\left(\omega_{0x} - \frac{\beta_2}{d_2}\omega_0\right)dx. \end{aligned} \tag{2.30}$$

Multiplying the first equation of (2.28) by $e^{-\frac{\beta_2}{d_2}x}\tilde{u}$, and integrating the resulting equation over $[0,L]$, according to boundary conditions, we have

$$\begin{aligned}\lambda_0\int_0^L \tilde{u}\omega_0 e^{-\frac{\beta_2}{d_2}x}dx\\ &= \int_0^L (d_2\omega_{0x}-\beta_2\omega_0)e^{-\frac{\beta_2}{d_2}x}\left(\tilde{u}_x-\frac{\beta_2}{d_2}\tilde{u}\right)dx-\int_0^L (r_2-b_2\tilde{u})\omega_0 e^{-\frac{\beta_2}{d_2}x}\tilde{u}dx.\end{aligned} \tag{2.31}$$

Combining (2.30) and (2.31), one can find

$$\begin{aligned}\lambda_0\int_0^L \tilde{u}\omega_0 e^{-\frac{\beta_2}{d_2}x}dx &= \int_0^L e^{-\frac{\beta_2}{d_2}x}\tilde{u}\omega_0[(r_1-r_2)+(b_2-a_1)\tilde{u}]dx\\ &\quad+\int_0^L e^{-\frac{\beta_2}{d_2}x}\left(\omega_{0x}-\frac{\beta_2}{d_2}\omega_0\right)[(d_2-d_1)\tilde{u}_x+(\beta_1-\beta_2)\tilde{u}]dx.\end{aligned} \tag{2.32}$$

By the definition of $\lambda^*(d_1,d_2,\beta_1,\beta_2,r_1,r_2,a_1,b_2)$, (2.32) can be rewritten as

$$\lambda_0\int_0^L \tilde{u}\omega_0 e^{-\frac{\beta_2}{d_2}x}dx=\lambda^*(d_1,d_2,\beta_1,\beta_2,r_1,r_2,a_1,b_2). \tag{2.33}$$

Clearly, the sign of λ_0 is the same as that of λ^*, which completes the proof of Proposition 2.7. □

Similarly, we have the following proposition concerning μ_0.

Proposition 2.8. *$\mu_0>0$ (resp. $\mu_0<0$) if and only if*

$$\mu^*(d_1,d_2,\beta_1,\beta_2,r_1,r_2,b_1,a_2)>0 \qquad (\textit{resp. } \mu^*(d_1,d_2,\beta_1,\beta_2,r_1,r_2,b_1,a_2)<0),$$

where

$$\begin{aligned}&\mu^*(d_1,d_2,\beta_1,\beta_2,r_1,r_2,b_1,a_2)\\ &\quad=\int_0^L e^{-\frac{\beta_1}{d_1}x}\tilde{v}\varphi_0[(r_2-r_1)+(b_1-a_2)\tilde{v}]dx\\ &\qquad+\int_0^L e^{-\frac{\beta_1}{d_1}x}\left(\varphi_{0x}-\frac{\beta_1}{d_1}\varphi_0\right)[(d_1-d_2)\tilde{v}_x+(\beta_2-\beta_1)\tilde{v}]dx.\end{aligned} \tag{2.34}$$

Proof. By a similar method noted in the proof of Proposition 2.7, we can find

$$\begin{aligned}\mu_0\int_0^L \tilde{v}\varphi_0 e^{-\frac{\beta_1}{d_1}x}dx &= \int_0^L e^{-\frac{\beta_1}{d_1}x}\tilde{v}\varphi_0[(r_2-r_1)+(b_1-a_2)\tilde{v}]dx\\ &\quad+\int_0^L e^{-\frac{\beta_1}{d_1}x}\left(\varphi_{0x}-\frac{\beta_1}{d_1}\varphi_0\right)[(d_1-d_2)\tilde{v}_x+(\beta_2-\beta_1)\tilde{v}]dx.\end{aligned} \tag{2.35}$$

According to the definition of $\mu^*(d_1,d_2,\beta_1,\beta_2,r_1,r_2,b_1,a_2)$, (2.35) can be simplified as

$$\mu_0\int_0^L \tilde{v}\varphi_0 e^{-\frac{\beta_1}{d_1}x}dx=\mu^*(d_1,d_2,\beta_1,\beta_2,r_1,r_2,b_1,a_2), \tag{2.36}$$

which indicates that μ_0 and μ^* have the same sign. □

Now, we can establish the local stability of $(\tilde{u},0)$ and $(0,\tilde{v})$ respectively.

Lemma 2.9. *Assume $d_i, r_i, a_1 \in \mathbb{R}^+$, $\beta_i, b_2 \in \mathbb{R}$, then*

(i) if $d_1 < d_2$, $0 < \beta_1 < \beta_2$, $\frac{\beta_1}{d_1} \leq \frac{\beta_2}{d_2}$, $r_2 \leq r_1$ and $0 < a_1 \leq b_2$, $(\tilde{u}, 0)$ is stable locally;

(ii) if $d_1 < d_2$, $0 < \beta_2 < \beta_1$, $r_1 \leq r_2$ and $0 \leq b_2 \leq a_1$, $(\tilde{u}, 0)$ is unstable;

(iii) if $d_1 \leq d_2$, $\beta_1 \cdot \beta_2 < 0$, $r_1 \leq r_2$ and $0 \leq b_2 \leq a_1$, $(\tilde{u}, 0)$ is unstable;

(iv) if $d_1 < d_2$, $0 < \beta_1 < \beta_2$, $\frac{\beta_1}{d_1} \leq \frac{\beta_2}{d_2}$, $r_1 \leq r_2$ and $b_2 < 0$, $(\tilde{u}, 0)$ is unstable.

Proof. For part (i), according to part (i) of Lemma 2.3,

$$0 < \frac{\tilde{u}_x}{\tilde{u}} < \frac{\beta_1}{d_1} \leq \frac{\beta_2 - \beta_1}{d_2 - d_1}. \tag{2.37}$$

Here, $0 < \frac{\tilde{u}_x}{\tilde{u}} < \frac{\beta_1}{d_1}$ implies $\tilde{u}_x > 0$, which indicates that

$$m_x = [r_2 - b_2\tilde{u}]_x = -b_2\tilde{u}_x < 0,$$

and then

$$\frac{\omega_{0x}}{\omega_0} < \frac{\beta_2}{d_2}, \tag{2.38}$$

from Lemma 2.5.

Moreover, it directly follows that from $0 < r_2 \leq r_1$ and $0 < a_1 \leq b_2$

$$(r_1 - r_2) + (b_2 - a_1)\tilde{u} \geq 0. \tag{2.39}$$

Therefore, thanks to (2.37), (2.38) and (2.39), we get $\lambda^*(d_1, d_2, \beta_1, \beta_2, r_1, r_2, a_1, b_2) > 0$ if $0 < d_1 < d_2$, $0 < \beta_1 < \beta_2$, $\frac{\beta_1}{d_1} \leq \frac{\beta_2}{d_2}$, $0 < r_2 \leq r_1$, $0 < a_1 \leq b_2$. By Lemma 2.6 and Proposition 2.7, the proof of part (i) is finished.

Actually, $\lambda^*(d_1, d_2, \beta_1, \beta_2, r_1, r_2, a_1, b_2) < 0$ directly follows from the conditions of $0 < d_1 < d_2$, $0 < \beta_2 < \beta_1$, $0 < r_1 \leq r_2$ and $0 < b_2 \leq a_1$. Thus, we can get $(\tilde{u}, 0)$ is unstable.

For part (iii), the case of $d_1 = d_2 > 0$ is easy to check, so we only verify the case of $0 < d_1 < d_2$.

$\beta_1 \cdot \beta_2 < 0$ implies $\beta_1 > 0 > \beta_2$ or $\beta_2 > 0 > \beta_1$. We first consider $\beta_1 > 0 > \beta_2$. By the similar argument to part (i), it is easy to find

$$\begin{cases} (d_2 - d_1)\tilde{u}_x + (\beta_1 - \beta_2)\tilde{u} > 0, \\ \omega_{0x} - \frac{\beta_2}{d_2}\omega_0 < 0. \end{cases} \tag{2.40}$$

Combining (2.40) with the conditions of $0 < r_1 < r_2$ and $0 < b_2 < a_1$, we deduce

$$\lambda^*(d_1, d_2, \beta_1, \beta_2, r_1, r_2, a_1, b_2) < 0,$$

which shows that $(\tilde{u}, 0)$ is unstable by Proposition 2.7 and Lemma 2.6.

For $\beta_2 > 0 > \beta_1$, combining this condition with part (i) of Lemma 2.3 and part (ii) of Lemma 2.5, we have

$$\begin{cases} (d_2 - d_1)\tilde{u}_x + (\beta_1 - \beta_2)\tilde{u} < 0, \\ \omega_{0x} - \frac{\beta_2}{d_2}\omega_0 > 0, \end{cases} \tag{2.41}$$

which yields

$$\lambda^*(d_1, d_2, \beta_1, \beta_2, r_1, r_2, a_1, b_2) < 0,$$

if $0 < r_1 \le r_2$ and $0 < b_2 \le a_1$. Therefore, $(\tilde{u}, 0)$ is unstable.

For part (iv), in the case of $b_2 < 0$, again by part (i) of Lemma 2.3 and part (ii) of Lemma 2.5, we get

$$\begin{cases} 0 < \frac{\tilde{u}_x}{\tilde{u}} < \frac{\beta_2-\beta_1}{d_2-d_1}, & x \in (0, L), \\ 0 < \frac{\beta_2}{d_2} < \frac{w_{0x}}{w_0}, & x \in (0, L). \end{cases} \tag{2.42}$$

Since $0 < r_1 \le r_2$ and $b_2 < 0 < a_1$, we derive $\lambda^*(d_1, d_2, \beta_1, \beta_2, r_1, r_2, a_1, b_2) < 0$. Then the proof of part (iii) is completed by Proposition 2.7 and Lemma 2.6. □

Lemma 2.10. *Suppose that $d_i, r_i, a_2, b_1 \in \mathbb{R}^+$, $\beta_i \in \mathbb{R}$, then*

(i) if $d_1 < d_2$, $0 < \beta_1 < \beta_2$, $\frac{\beta_1}{d_1} \le \frac{\beta_2}{d_2}$, $r_2 \le r_1$ and $b_1 \le a_2$, $(0, \tilde{v})$ is unstable;

(ii) if $d_1 < d_2$, $0 < \beta_2 < \beta_1$, $r_1 \le r_2$ and $a_2 \le b_1$, $(0, \tilde{v})$ is stable locally;

(iii) if $d_1 \le d_2$, $\beta_1 \cdot \beta_2 < 0$, $r_2 \le r_1$ and $b_1 \le a_2$, $(0, \tilde{v})$ is unstable.

Proof. By using the similar argument to the one applied in the proof of Lemma 2.9, one can prove that $\mu^*(d_1, d_2, \beta_1, \beta_2, r_1, r_2, b_1, a_2) < 0$ for $d_1 < d_2$, $0 < \beta_1 < \beta_2$, $\frac{\beta_1}{d_1} \le \frac{\beta_2}{d_2}$, $r_2 \le r_1$, $b_1 \le a_2$; $\mu^*(d_1, d_2, \beta_1, \beta_2, r_1, r_2, b_1, a_2) > 0$ for $d_1 < d_2$, $0 < \beta_2 < \beta_1$, $r_1 \le r_2$ and $a_2 \le b_1$. Thus, part (i) and part (ii) directly follows from Proposition 2.7 and Lemma 2.6.

Part (iii), we only consider $d_1 < d_2$, because the case of $d_1 = d_2 > 0$ can be verified in a similar argument.

For $\beta_1 > 0 > \beta_2$, it follows from part (ii) of Lemma 2.3 that

$$0 < -\frac{\tilde{v}_x}{\tilde{v}} < -\frac{\beta_2}{d_2} < \frac{\beta_1 - \beta_2}{d_2 - d_1},$$

that is,

$$(d_1 - d_2)\tilde{v}_x + (\beta_2 - \beta_1)\tilde{v} < 0. \tag{2.43}$$

Moreover, due to part (i) of Lemma 2.5, we deduce

$$\frac{\varphi_{0x}}{\varphi_0} > \frac{\beta_1}{d_1}, \tag{2.44}$$

since $m_x = (r_1 - b_1\tilde{v})_x = -b_1\tilde{v}_x > 0$ if $\beta_2 < 0$. So $(0, \tilde{v})$ is unstable under the conditions of $r_2 \le r_1$ and $b_1 \le a_2$.

For $\beta_2 > 0 > \beta_1$, we can use a similar argument to show $(0, \tilde{v})$ is unstable. □

2.3 The non-existence of coexistence steady state

In this subsection, we show the nonexistence of coexistence steady states under some proper conditions.

Lemma 2.11. *Suppose that $0 < d_1 < d_2$, then*

(i) if $0 < \beta_1 < \beta_2$, $\frac{\beta_1}{d_1} \le \frac{\beta_2}{d_2}$, $0 < r_2 \le r_1$, $0 < a_1 \le b_2$ and $0 < b_1 \le a_2$, system (1.1) with the addition of (1.3) has no coexistence steady state;

(ii) if $0 < \beta_2 < \beta_1$, $0 < r_1 \le r_2$, $0 < b_2 \le a_1$ and $0 < a_2 \le b_1$, system (1.1) with the addition of (1.3) has no coexistence steady state.

Proof. For part (i), arguing indirectly, we assume that system (1.1) with the addition of (1.3) has a coexistence steady state (U, V), then

$$\begin{cases} d_1 U_{xx} - \beta_1 U_x + U(r_1 - a_1 U - b_1 V) = 0, & x \in (0, L), \\ d_2 V_{xx} - \beta_2 V_x + V(r_2 - a_2 V - b_2 U) = 0, & x \in (0, L), \\ d_1 U_x(0) - \beta_1 U(0) = d_1 U_x(L) - \beta_1 U(L) = 0, \\ d_2 V_x(0) - \beta_2 V(0) = d_2 V_x(L) - \beta_2 V(L) = 0. \end{cases} \tag{2.45}$$

Define

$$f(x) := d_1 U_x - \beta_1 U, \qquad x \in [0, L]; \tag{2.46}$$

and

$$g(x) := d_2 V_x - \beta_2 V, \qquad x \in [0, L]. \tag{2.47}$$

By the boundary conditions of (2.45), we have

$$f(0) = f(L) = g(0) = g(L) = 0. \tag{2.48}$$

In the following, we show 8 claims to finish the proof.

Claim 1.

(i) If $f'(x) \geq 0$ (resp. > 0), $g'(x) \geq 0$ (resp. > 0);

(ii) If $g'(x) \leq 0$ (resp. < 0), $f'(x) \leq 0$ (resp. < 0).

Combining (2.45) with the definition of $f(x)$ and $g(x)$, we find

$$f'(x) = d_1 U_{xx} - \beta_1 U_x = U(a_1 U + b_1 V - r_1),$$

and

$$g'(x) = d_2 V_{xx} - \beta_2 V_x = V(a_2 V + b_2 U - r_2).$$

It follows that from $f'(x) \geq 0$ (resp. > 0)

$$a_1 U + b_1 V - r_1 \geq 0 \quad (\text{resp. } > 0).$$

Therefore, $0 < r_2 \leq r_1$, $0 < a_1 \leq b_2$ and $0 < b_1 \leq a_2$ directly yield

$$a_2 V + b_2 U - r_2 \geq a_1 U + b_1 V - r_1 \geq 0 \quad (\text{resp. } > 0).$$

Hence, $g'(x) \geq 0$ (resp. > 0).

Part (ii) can be proved similarly.

Claim 2.

(i) There is small $\varepsilon > 0$ such that $f(x) < 0$ in $(0, \varepsilon]$;

(ii) There is small $\delta > 0$ such that $g(x) < 0$ in $[L - \delta, L)$.

Part (i), if not, there exists some small ε_0 such that $f(x) > 0$ or $f(x) \equiv 0$ in $(0, \varepsilon_0]$. Next, we show contradictions respectively. Define

$$T := \frac{U_x}{U}, \qquad x \in [0, L]; \tag{2.49}$$

and

$$S := \frac{V_x}{V}, \qquad x \in [0, L]. \tag{2.50}$$

Due to (2.45), some straightforward calculations yield

$$\begin{cases} -d_1 T_{xx} + (\beta_1 - 2d_1 T)T_x + a_1 TU + b_1 SV = 0, & x \in (0, L), \\ -d_2 S_{xx} + (\beta_2 - 2d_2 S)S_x + a_2 SV + b_2 TU = 0, & x \in (0, L), \\ T(0) = T(L) = \frac{\beta_1}{d_1} > 0, \\ S(0) = S(L) = \frac{\beta_2}{d_2} > 0. \end{cases} \tag{2.51}$$

When $f(x) > 0$ in $(0, \varepsilon_0]$, there must be $\varepsilon_1 \in (0, \varepsilon_0]$ such that

$$f'(x) > 0, \qquad x \in (0, \varepsilon_1].$$

According to Claim 1, we have

$$g'(x) > 0, \qquad x \in (0, \varepsilon_1].$$

Due to $g(0) = 0$, we get

$$g(x) > 0, \qquad x \in (0, \varepsilon_1].$$

Denote the first zero point of $f(x)$ in $(0, L]$ by y_1, and the first zero of $g(x)$ by x_1. For brevity, we let $y_1 \le x_1$. Actually, the following expression is similar when $y_1 \ge x_1$.

Thus, we have

$$f(0) = f(y_1) = 0, \quad f(x) > 0, \quad x \in (0, y_1); \tag{2.52}$$

and

$$g(0) = 0, \quad g(y_1) \ge 0, \quad g(x) > 0, \quad x \in (0, y_1). \tag{2.53}$$

Thanks to (2.46), (2.47), (2.49), (2.50), (2.52) and (2.53), we find

$$T(0) = T(y_1) = \frac{\beta_1}{d_1}, \quad T(x) > \frac{\beta_1}{d_1} > 0, \quad x \in (0, y_1); \tag{2.54}$$

and

$$S(0) = \frac{\beta_1}{d_2}, \quad S(y_1) \ge \frac{\beta_2}{d_2} > 0, \quad S(x) > \frac{\beta_2}{d_2} > 0, \quad x \in (0, y_1). \tag{2.55}$$

From (2.54), T must attain a positive local maximum at some point denoted by z_1, $z_1 \in (0, y_1)$. By the first equation of (2.51), we get $S(z_1) < 0$, which contradicts (2.55). Consequently, the statement $f(x) > 0$ in $(0, \varepsilon_0]$ does not hold.

Actually, if $f(x) \equiv 0$ in $(0, \varepsilon_0]$, $f'(x) \equiv 0$. By Claim 1, $g'(x) \ge 0$, together with $g(0) = 0$, there must exist $\varepsilon_2 \in (0, \varepsilon_0]$ such that $g(x) \ge 0, x \in (0, \varepsilon_2)]$, deducing that

$$S \ge \frac{\beta_2}{d_2}, \qquad x \in (0, \varepsilon_2].$$

On the other hand, by (2.46) and (2.49), $f(x) \equiv 0$ in $(0, \varepsilon_0]$ guarantees $T \equiv \frac{\beta_1}{d_1}$. According to the first equation of (2.51), we derive $S < 0$ in $(0, \varepsilon_0)$, a contradiction to $S \ge \frac{\beta_2}{d_2}$ in $(0, \varepsilon_2)$.

Consequently, part (i) of Claim 2 is set up. Part (ii) can be verified by the similar method, so we omit the details.

Claim 2 implies that f and g can not be identically zero in $[0, L]$. Besides, f and g are real analytic. Therefore, all zero points of f and g are isolated.

Claim 3. f and g must change sign in $[0,L]$.

If g cannot change sign in $[0,L]$, then $g \leq, \not\equiv 0$ in $[0,L]$ by using Claim 2. Thus

$$\begin{cases} f(x) = d_1U_x - \beta_1 U < 0, & x \in (0,y_1), \\ g(x) = d_2V_x - \beta_2 V \leq, \not\equiv 0, & x \in (0,y_1), \\ f(0) = f(y_1) = 0, \\ g(0) = 0, g(y_1) \geq 0. \end{cases} \tag{2.56}$$

Combining (2.45) with (2.56), we have

$$\begin{cases} d_1U_{xx} - \beta_1 U_x + U(r_1 - a_1U - b_1V) = 0, & x \in (0,y_1), \\ d_1V_{xx} - \beta_1 V_x + V(r_2 - a_2V - b_2U) = (d_1 - d_2)V_{xx} + (\beta_2 - \beta_1)V_x, & x \in (0,y_1), \\ d_1U_x(0) - \beta_1 U(0) = d_1U_x(y_1) - \beta_1 U(y_1) = 0, \\ d_2V_x(0) - \beta_2 V(0) = 0, \\ d_2V_x(y_1) - \beta_2 V(y_1) \leq 0. \end{cases} \tag{2.57}$$

Multiplying the first equation of (2.57) by $e^{-\frac{\beta_1}{d_1}x}V$ and the second one by $e^{-\frac{\beta_1}{d_1}x}U$, subtracting the resulting equations and then integrating over $[0,y_1]$, we get

$$\begin{aligned} (d_2V_x - \beta_2 V)e^{-\frac{\beta_1}{d_1}x}|_{x=y_1} = \int_0^{y_1} & e^{-\frac{\beta_1}{d_1}x} UV[(r_1 - r_2) + (b_2 - a_1)U + (a_2 - b_1)V] \\ & + [(d_2 - d_1)V_x - (\beta_2 - \beta_1)V]e^{-\frac{\beta_1}{d_1}x}\left(U_x - \frac{\beta_1}{d_1}U\right)dx. \end{aligned} \tag{2.58}$$

Thanks to our assumptions, we have

$$e^{-\frac{\beta_1}{d_1}x} UV[(r_1 - r_2) + (b_2 - a_1)U + (a_2 - b_1)V] \geq 0. \tag{2.59}$$

From (2.56), it follows that

$$\left(U_x - \frac{\beta_1}{d_1}U\right) < 0, \qquad x \in (0,y_1), \tag{2.60}$$

and

$$\frac{V_x}{V} \leq \frac{\beta_2}{d_2}, \qquad x \in (0,y_1). \tag{2.61}$$

By part (ii) of Lemma 2.3, we get

$$\frac{V_x}{V} \leq \frac{\beta_2 - \beta_1}{d_2 - d_1}, \qquad x \in (0,y_1). \tag{2.62}$$

Accordingly, it follows from (2.59), (2.60) and (2.62) that

$$\begin{aligned} \int_0^{y_1} & e^{-\frac{\beta_1}{d_1}x} UV[(r_1 - r_2) + (b_2 - a_1)U + (a_2 - b_1)V] \\ & + [(d_2 - d_1)V_x - (\beta_2 - \beta_1)V]e^{-\frac{\beta_1}{d_1}x}\left(U_x - \frac{\beta_1}{d_1}U\right)dx \\ \geq 0. \end{aligned} \tag{2.63}$$

On the other hand, due to the last equation of (2.57), one can find

$$(d_2V_x - \beta_2V)e^{-\frac{\beta_1}{d_1}x}\Big|_{x=y_1} \leq 0. \tag{2.64}$$

(2.63) and (2.64) cause a contradiction to (2.58), which indicates that g must change sign in $[0, y_1]$. In addition, g changes sign before f.

Now, it's turn to show f also changes sign in $[0, L]$. If not, we have $f \leq, \not\equiv 0$ in $[0, L]$ by Claim 2. Denote the last zero point of g in $(0, L)$ by x_{m-1}. Here x_{m-1} must exist because g changes sign in $(0, L)$. Then

$$\begin{cases} f(x) = d_1U_x - \beta_1U \leq 0, & x \in (x_{m-1}, L), \\ g(x) = d_2V_x - \beta_2V < 0, & x \in (x_{m-1}, L), \\ f(x_{m-1}) \leq 0, f(L) = 0, \\ g(x_{m-1}) = g(L) = 0. \end{cases} \tag{2.65}$$

By some direct calculation, it follows from (2.45) and (2.65) that

$$\begin{aligned} (d_1U_x - \beta_1U)e^{-\frac{\beta_2}{d_2}x}\Big|_{x=x_{m-1}} = & \int_{x_{m-1}}^{L} e^{-\frac{\beta_2}{d_2}x}UV[(r_1 - r_2) + (b_2 - a_1)U + (a_2 - b_1)V] \\ & + [(d_2 - d_1)U_x - (\beta_2 - \beta_1)U]e^{-\frac{\beta_2}{d_2}x}\left(V_x - \frac{\beta_2}{d_2}V\right)dx. \end{aligned} \tag{2.66}$$

According to (2.65) and our assumptions, one can see that the left side of (2.66) is nonpositive but the right is positive, which gives rise to a contradiction. Therefore, f also changes sign in $[0, L]$.

Based on Claim 2 and Claim 3, g has at least one zero point in $(0, L)$ such that the sign of g must change at each side of the point. Let x_2 be the first one. Obviously, either $g \leq 0$ or $g \geq 0$ in $(0, x_2)$, so we consider these two cases:

Case i: $g \leq, \not\equiv 0$ in $(0, x_2)$; Case ii: $g \geq, \not\equiv 0$ in $(0, x_2)$.

In the following analysis, we show that $f \leq 0$ in $[0, L]$ in both cases. First, we consider Case i: $g \leq, \not\equiv 0$ in $(0, x_2)$. By Claim 2 and Claim 3, there exists x_3 $(x_2 < x_3 < L)$ such that

$$\begin{cases} g(x) \leq 0, & x \in (0, x_2), \\ g(x) \geq 0, & x \in (x_2, x_3), \\ g(0) = g(x_2) = g(x_3) = 0. \end{cases} \tag{2.67}$$

Claim 4. $f \leq 0$ in $(0, x_2]$ for Case i: $g \leq, \not\equiv 0$ in $(0, x_2)$.

If not, f has at least one zero point in $(0, L)$ such that the sign of f must change at each side of the point. Let y_2 be the first one. Thus,

$$\begin{cases} f(x) = d_1U_x - \beta_1U \leq 0, & x \in (0, y_2), \\ g(x) = d_2V_x - \beta_2V \leq 0, & x \in (0, y_2), \\ f(0) = f(y_2) = 0, \\ g(0) = 0, \ g(y_2) \leq 0. \end{cases} \tag{2.68}$$

Combining (2.45) with (2.68), some direct calculations yield

$$\begin{aligned} (d_2V_x - \beta_2V)e^{-\frac{\beta_1}{d_1}x}\Big|_{x=y_2} = & \int_{0}^{y_2} e^{-\frac{\beta_1}{d_1}x}UV[(r_1 - r_2) + (b_2 - a_1)U + (a_2 - b_1)V] \\ & + [(d_2 - d_1)V_x - (\beta_2 - \beta_1)V]e^{-\frac{\beta_1}{d_1}x}\left(U_x - \frac{\beta_1}{d_1}U\right)dx. \end{aligned} \tag{2.69}$$

By our assumptions and the last equation of (2.68), we have

$$(d_2V_x - \beta_2 V)e^{-\frac{\beta_1}{d_1}x}\Big|_{x=y_2} \le 0, \tag{2.70}$$

and

$$\begin{aligned}\int_0^{y_2} & e^{-\frac{\beta_1}{d_1}x} UV[(r_1-r_2)+(b_2-a_1)U+(a_2-b_1)V] \\ & + [(d_2-d_1)V_x - (\beta_2-\beta_1)V]e^{-\frac{\beta_1}{d_1}x}\left(U_x - \frac{\beta_1}{d_1}U\right)dx \\ > 0. &\end{aligned} \tag{2.71}$$

The contradiction completes the proof of Claim 4.

Claim 5. $f \le 0$ in $(x_2, x_3]$ with $f(x_3) < 0$ for Case i: $g \le, \not\equiv 0$ in $(0, x_2)$.

First, we show $f \le 0$ in $[x_2, x_3]$. Otherwise, there is y_3 and z_2 with $x_2 \le y_3 < z_2 \le x_3$ such that

$$\begin{cases} f(x) \le 0, & x \in (x_2, y_3), \\ f(y_3) = 0, & \\ f(x) > 0, & x \in (y_3, z_2). \end{cases} \tag{2.72}$$

(2.72) implies that there exists small $\epsilon_1 > 0$ such that f is increasing in $(y_3, y_3+\epsilon_1)$. On the other hand, since $g(x_3) = 0$ and $g(x) \ge 0$ in (x_2, x_3), there exists small $\epsilon_2 > 0$ such that g is diminishing in $(x_3 - \epsilon_2, x_3)$. By Claim 1, f is also diminishing in $(x_3 - \epsilon_2, x_3)$.

Consequently, f must has at least one positive local maximum value point in (y_3, x_3). Let z_3 be the closest to y_3. Then we have

$$\begin{cases} f(x) = d_1U_x - \beta_1 U > 0, & x \in (y_3, z_3], \\ f'(z_3) = 0, & \end{cases} \tag{2.73}$$

Note that

$$f'(x) = d_1 U_{xx} - \beta_1 U_x, \tag{2.74}$$

and

$$T'(x) = \left[\frac{U_x}{U}\right]_x = \frac{U_{xx}}{U} - \left(\frac{U_x}{U}\right)^2. \tag{2.75}$$

By (2.73) and (2.74), we find

$$\begin{cases} \frac{U_x}{U} > \frac{\beta_1}{d_1}, & x \in (y_3, z_3], \\ f'(z_3) = d_1U_{xx}(z_3) - \beta_1 U_x(z_3) = 0. & \end{cases} \tag{2.76}$$

Due to (2.75) and (2.76), we deduce

$$T'(z_3) = \frac{U_{xx}}{U}\Big|_{x=z_3} - \left(\frac{U_x}{U}\right)^2\Big|_{x=z_3} = \frac{\beta_1 U_x}{d_1 U}\Big|_{x=z_3} - \left(\frac{U_x}{U}\right)^2\Big|_{x=z_3} < 0. \tag{2.77}$$

Then, (2.49), (2.72), (2.73) and (2.77) directly yield

$$\begin{cases} T(x) > \frac{\beta_1}{d_1}, \; x \in (y_3, z_3), \\ T(y_3) = \frac{\beta_1}{d_1}, \\ T'(z_3) < 0. \end{cases} \tag{2.78}$$

From (2.78), we can obtain that T has a positive local maximum value point saying z_4 in (y_3, z_3). By the first equation of (2.51), we get

$$S(z_4) < 0. \tag{2.79}$$

On the other hand, since $g(x) > 0$ in (x_2, x_3),

$$S(z_4) > \frac{\beta_2}{d_2}. \tag{2.80}$$

The contradiction caused by (2.79) and (2.80) shows that $f(x) \leq 0$ in $[x_2, x_3]$. Considering f is diminishing in a small neighborhood of x_3, we derive $f(x_3) < 0$.

After x_3, we can find the next zero point of g. Denote it by $x_4 \in (x_3, L]$. It is obvious either $g \leq 0$ or $g \geq 0$ in $(x_3, x_4]$.

Claim 6. $f \leq 0$ in $(x_3, x_4]$ with $f(x_4) < 0$ for $g \leq, \not\equiv 0$ in $(0, x_2)$.

Actually, when $g \geq 0$ in $(x_3, x_4]$ happens, we can deduce $f \leq 0$ in $(x_3, x_4]$ with $f(x_4) < 0$ in the similar way to Claim 5. Next, we show $f \leq 0$ in $(x_3, x_4]$ if $g \leq 0$ in $(x_3, x_4]$ happens. If not, there exists $y_4 \in (x_3, y_4]$ such that

$$\begin{cases} f(x) = d_1 U_x - \beta_1 U < 0, & x \in (x_3, y_4), \\ g(x) = d_2 V_x - \beta_2 V \leq 0, & x \in (x_3, y_4), \\ g(x_3) = 0, g(y_4) \leq 0, \\ f(x_3) < 0, f(y_4) = 0. \end{cases} \tag{2.81}$$

Some direct calculations yield

$$\begin{aligned} &(d_1 U_x - \beta_1 U) e^{-\frac{\beta_2}{d_2}x} V\Big|_{x=x_3} + (d_2 V_x - \beta_2 V) e^{-\frac{\beta_2}{d_2}x} U\Big|_{x=y_4} \\ &= \int_{x_3}^{y_4} e^{-\frac{\beta_2}{d_2}x} UV[(r_1 - r_2) + (b_2 - a_1)U + (a_2 - b_1)V] \\ &\quad + [(d_2 - d_1)U_x - (\beta_2 - \beta_1)U] e^{-\frac{\beta_2}{d_2}x} \left(V_x - \frac{\beta_2}{d_2} V \right) dx, \end{aligned} \tag{2.82}$$

By the similar method to Claim 4, we arrive at

$$(d_1 U_x - \beta_1 U) e^{-\frac{\beta_2}{d_2}x} V\Big|_{x=x_3} + (d_2 V_x - \beta_2 V) e^{-\frac{\beta_2}{d_2}x} U\Big|_{x=y_4} < 0, \tag{2.83}$$

and

$$\begin{aligned} &\int_{x_3}^{y_4} e^{-\frac{\beta_2}{d_2}x} UV[(r_1 - r_2) + (b_2 - a_1)U + (a_2 - b_1)V] \\ &\quad + [(d_2 - d_1)U_x - (\beta_2 - \beta_1)U] e^{-\frac{\beta_2}{d_2}x} \left(V_x - \frac{\beta_2}{d_2} V \right) dx \\ &\geq 0. \end{aligned} \tag{2.84}$$

The contradiction ends the proof of Claim 6.

Claim 7. $f \leq 0$ in $[0, L]$ for $g \leq, \not\equiv 0$ in $(0, x_2)$.

Due to Claim 2, Claim 3 and the isolated properties of the zero points for g, g has finitely many zero points. Consequently, by repeating the above analysis, we can obtain that $f \leq 0$ in $[0, L]$ for $g \leq, \not\equiv 0$ in $(0, x_2)$.

Claim 8. $f \le 0$ in $[0, L]$

Actually, for $g \ge, \not\equiv 0$ in $(0, x_2)$, we can show that $f \le 0$ in $[0, L]$ by the similar method to the case of $g \le, \not\equiv 0$ in $(0, x_2)$, so Claim 8 can be verified directly.

$f \le 0$ in $[0, L]$ contradicts Claim 3, which shows that the coexistence steady state (U, V) of system (1.1) with the addition of (1.3) does not exist if $0 < r_2 \le r_1$, $0 < a_1 \le b_2$, $0 < b_1 \le a_2$, $0 < d_1 < d_2$, $0 < \beta_1 < \beta_2$ and $\frac{\beta_1}{d_1} < \frac{\beta_2}{d_2}$.

For part (ii), we can apply the similar arguments to part (i) to show the non-existence of co-existence steady states if $0 < r_1 \le r_2$, $0 < b_2 \le a_1$, $0 < a_2 \le b_1$, $0 < d_1 < d_2$ and $0 < \beta_2 < \beta_1$, so we omit the details. □

2.4 Global dynamics for problem (1.1) with the addition of (1.3)

In this subsection, we show the asymptotic behaviors of solution to problem (1.1) with the addition of (1.3). For convenience, we list the following conditions:

C1. $0 < \beta_1 < \beta_2, 0 < r_2 \le r_1, 0 < a_1 \le b_2, 0 < b_1 \le a_2$;

C2. $0 < \beta_2 < \beta_1, 0 < r_1 \le r_2, 0 < a_2 \le b_1, 0 < b_2 \le a_1$;

C3. $\beta_1 \cdot \beta_2 < 0, r_1 = r_2, 0 < b_2 \le a_1, 0 < b_1 \le a_2$.

Theorem 2.12. *Suppose that* $0 < d_1 < d_2$,

(i) if $\frac{\beta_1}{d_1} \le \frac{\beta_2}{d_2}$ *and C1 holds, the semi-trivial steady state* $(\tilde{u}, 0)$ *is globally asymptotically stable;*

(ii) if C2 holds, the semi-trivial steady state $(0, \tilde{v})$ *is globally asymptotically stable.*

Proof. By using the theory of monotone dynamical system [23], part (i) of Theorem 2.12 directly follows from part (i) of lemma 2.9, part (i) of Lemma 2.10 and part (i) of Lemma 2.11; part (ii) of Theorem 2.12 directly follows from part (ii) of lemma 2.9, part (ii) of Lemma 2.10 and part (ii) of Lemma 2.11. □

Actually, interchanging the labels of d_1 and d_2, β_1 and β_2, r_1 and r_2, a_1 and a_2, b_1 and b_2, we can get the parallel results.

Theorem 2.13. *Suppose that* $0 < d_2 < d_1$,

(i) if $\frac{\beta_2}{d_2} \le \frac{\beta_1}{d_1}$ *and C2 holds, the semi-trivial steady state* $(0, \tilde{v})$ *is globally asymptotically stable;*

(ii) if C1 holds, the semi-trivial steady state $(\tilde{u}, 0)$ *is globally asymptotically stable.*

Remark 2.14. Actually, if $r_1 = r_2 > 0, a_1 = b_2 > 0, a_2 = b_1 > 0$, part (i) of Theorem 2.13 supports statement (1) of Theorem 1.1 in [29]; part (ii) of Theorem 2.13 is consistent with [29, Theorem 1.2].

Actually, let

$$r_1 = r_2 = r > 0, \qquad a_1 = b_2 = a > 0, \qquad a_2 = b_1 = b > 0,$$

and

$$au = \widehat{u}, \qquad bv = \widehat{v}.$$

According to system (1.1) with the addition of (1.3), $(\widehat{u}, \widehat{v})$ satisfies

$$\begin{cases} \widehat{u}_t = d_1 \widehat{u}_{xx} - \beta_1 \widehat{u}_x + \widehat{u}(r - \widehat{u} - \widehat{v}), & (x,t) \in (0,L) \times (0,\infty), \\ \widehat{v}_t = d_2 \widehat{v}_{xx} - \beta_2 \widehat{v}_x + \widehat{v}(r - \widehat{u} - \widehat{v}), & (x,t) \in (0,L) \times (0,\infty), \\ d_1 \widehat{u}_x(0,t) - \beta_1 \widehat{u}(0,t) = d_1 \widehat{u}_x(L,t) - \beta_1 \widehat{u}(L,t) = 0, & t \in (0,\infty), \\ d_2 \widehat{v}_x(0,t) - \beta_2 \widehat{v}(0,t) = d_2 \widehat{v}_x(L,t) - \beta_2 \widehat{v}(L,t) = 0, & t \in (0,\infty), \\ \widehat{u}(x,0) = \widehat{u}_0(x), \widehat{v}(x,0) = \widehat{v}_0(x), & x \in [0,L]. \end{cases} \tag{2.85}$$

System (2.85) is just the model in [29]. As a result, Remark 2.14 is obvious.

Theorem 2.15. *Suppose $0 < d_1 \le d_2$ and C3 holds, problem (1.1) with the addition of (1.3) has two coexistence steady states (U_1, V_1) and (U_2, V_2) with $U_1 \le U_2$ and $V_1 \ge V_2$. Furthermore, any positive solution $U(x,t) = (u(x,t), v(x,t))$ to problem (1.1) with the addition of (1.3) satisfies $\lim_{t\to+\infty} d(U(x,t), [U_1, U_2] \times [V_2, V_1]) = 0$ uniformly for $x \in [0, L]$.*

Proof. By the continuous-time version of [28, Theorem 2.4.1], Theorem 2.15 follows from part (iii) of Lemma 2.9 and part (iii) of Lemma 2.10. □

Again interchanging the labels of d_1 and d_2, β_1 and β_2, a_1 and a_2, b_1 and b_2, the parallel results for Theorem 2.15 is as follows.

Theorem 2.16. *Suppose $0 < d_2 \le d_1$ and C3 holds, then we have the same results as in Theorem 2.15.*

Combining Theorem 2.15 with Theorem 2.16, we get the following more general conclusion.

Theorem 2.17. *Assume $d_1, d_2 \in \mathbb{R}^+$ and C3 holds, then we have the same results as in Theorem 2.15.*

Remark 2.18. Particularly, if $a_1 = b_2 > 0, a_2 = b_1 > 0$, Theorem 2.16 and Theorem 2.17 support Theorem 1.3 and Theorem 1.3* in [29], respectively.

2.5 Numerical simulations

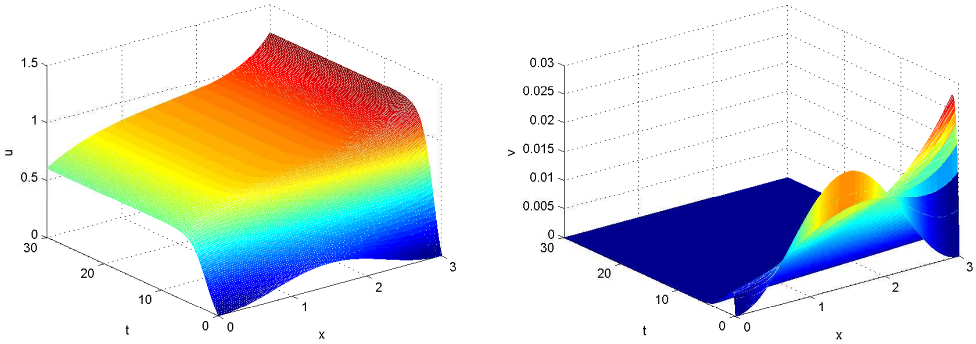
Next we present some numerical simulations to indicate our results obtained above. Furthermore, for $b_1 > 0, b_2 < 0$, we observe that problem (1.1) with the addition of (1.3) may have a co-existence state.

Throughout this subsection, we fix

$$u_0(x) = 0.1 - 0.1\cos\left(\frac{2\pi x}{L}\right),$$
$$v_0(x) = 0.01 - 0.01\cos\left(\frac{2\pi x}{L}\right),$$

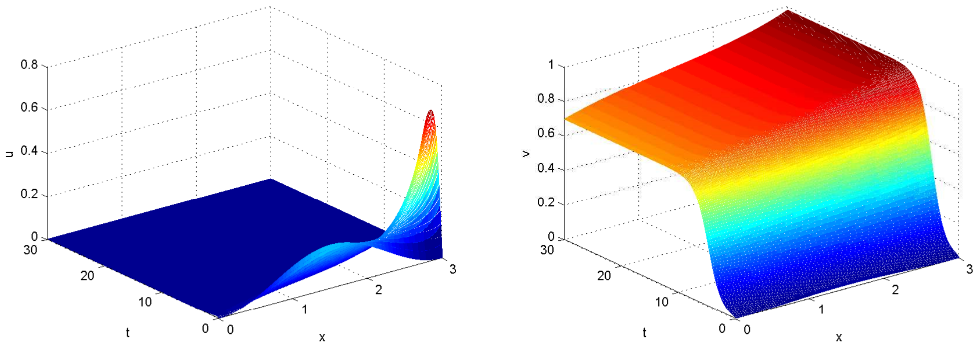
and

$$L = 3, \qquad d_1 = 0.3, \qquad d_2 = 1.1.$$



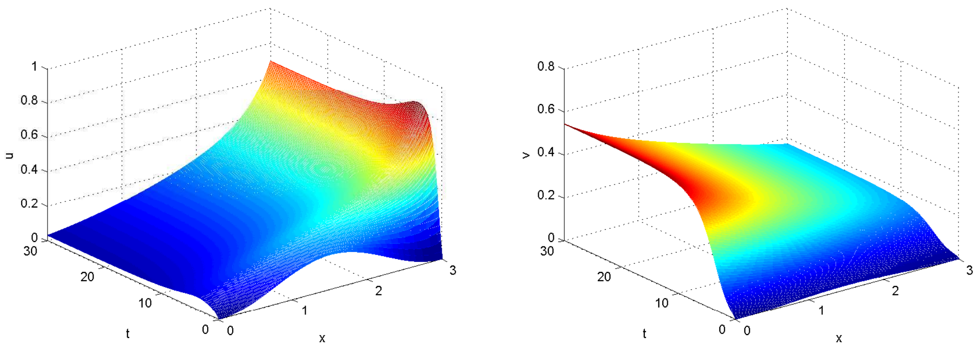
(a) (b)

Figure 2.1: The asymptotic behaviors of $u(x,t)$ in (a) and $v(x,t)$ in (b) with $\beta_1 = 0.2, \beta_2 = 1, r_1 = 1, r_2 = 0.5, a_1 = 1.1, a_2 = 2.1, b_1 = 1.2, b_2 = 2.$



(a) (b)

Figure 2.2: The asymptotic behaviors of $u(x,t)$ in (a) and $v(x,t)$ in (b) with $\beta_1 = 1, \beta_2 = 0.2, r_1 = 0.5, r_2 = 1, a_1 = 2.1, a_2 = 1.2, b_1 = 2.1, b_2 = 1.2.$



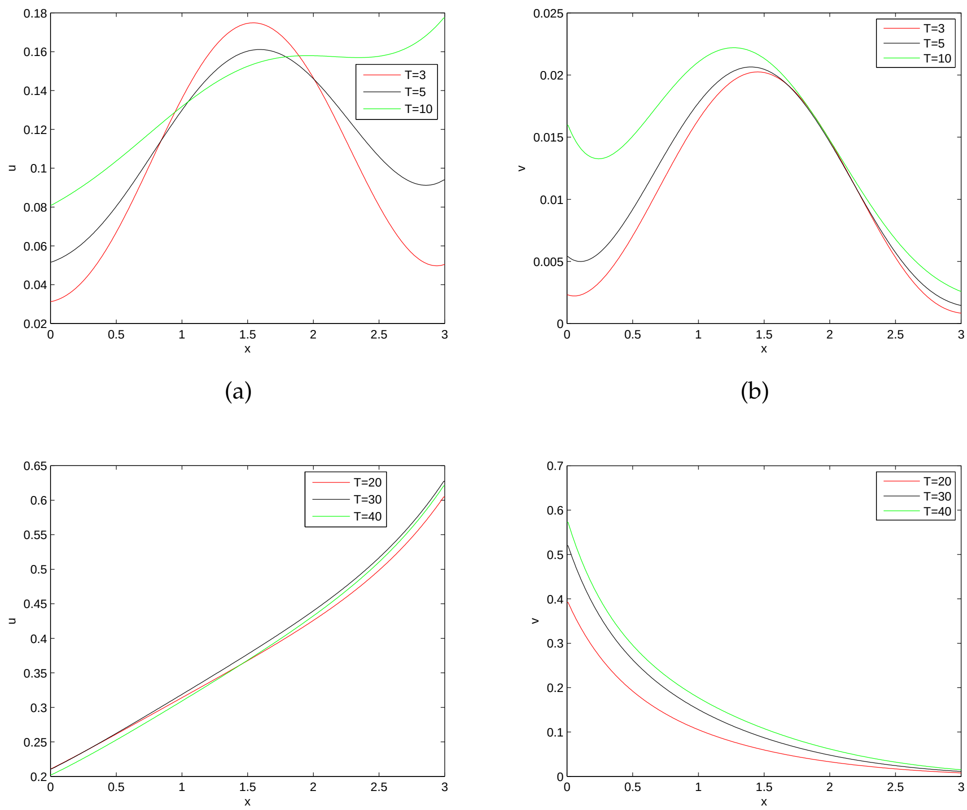
(a) (b)

Figure 2.3: The asymptotic behaviors of $u(x,t)$ in (a) and $v(x,t)$ in (b) with $\beta_1 = 0.5, \beta_2 = -0.5, r_1 = r_2 = 1, a_1 = 2.1, a_2 = 2.3, b_1 = 1.2, b_2 = 1.1.$

Example 2.19. Figure 2.1 (resp. Figure 2.2) describes part (i) (resp. (ii)) of Theorem 2.12, and Figure 2.3 displays Theorem 2.15. From Figure 2.1, one can observe that the $u(x,t)$ in (a) stabilizes to a positive nonconstant state but $v(x,t)$ in (b) decays to zero quickly; from Figure 2.2, $u(x,t)$ drops to zero fast but $v(x,t)$ stabilizes to a positive nonconstant state. As shown in Figure 2.3, $u(x,t)$ and $v(x,t)$ both stabilize to positive nonconstant states as time goes on.

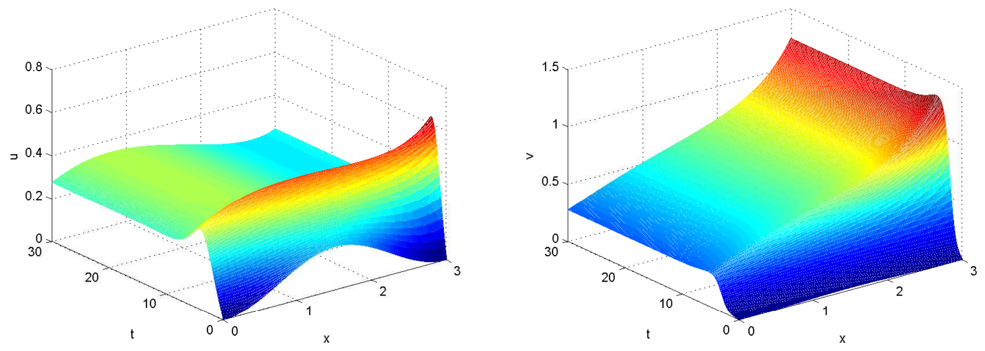
Furthermore, from Figure 2.3, one can see that advection can induce great difference between the left and right boundaries as time proceeds. To illustrate the variation on $u(x,t)$ and $v(x,t)$ with x from the left boundary to the right, the curves of $u(x)$ and $v(x)$ with $t = 3, 5, 10, 20, 30, 40$ are presented in Figure 2.4. One can observe that $u(x)$ in (a) and $v(x)$ in (b) goes up and down as x increases when t is fixed on 3, 5 or 10; whereas $u(x)$ in (c) grows and $v(x)$ in (d) decreases with the increase of x when t is fixed on 20, 30 or 40. This is because the positive advection towards the right boundary but the negative advection towards the left boundary.

Example 2.20. Based on the the conditions in part (iv) of Lemma 2.9 and part (i) of Lemma 2.10, the numerical results are illustrated in Figure 2.5. We get that $u(x,t)$ in (a) and $v(x,t)$ in (b) both stabilize to positive nonconstant states as time goes on. This implies that $(\tilde{u}, 0)$ and $(0, \tilde{v})$ are both unstable, which is consistent with part (iv) of Lemma 2.9 and part (i) of Lemma 2.10.



(a) (b) (c) (d)

Figure 2.4: The curves of $u(x)$ and $v(x)$ with respect to x when t is fixed. Here $\beta_1, \beta_2, r_1, r_2, a_1, a_2, b_1, b_2$ take the same values as them in Figure 2.3.



(a) (b)

Figure 2.5: The illustration of $u(x,t)$ in (a) and $v(x,t)$ in (b) with $\beta_1 = 0.2, \beta_2 = 1,\ r_1 = r_2 = 1,\ a_1 = 1.2,\ a_2 = 2.1,\ b_1 = 1.1,\ b_2 = -2$.

3 The free boundary problem

3.1 Existence and uniqueness

By a similar argument in [7,8], we have the following result.

Lemma 3.1. *For any given u_0, v_0 satisfying (1.6) and any $\alpha \in (0,1)$, there is a $T > 0$ such that problem (1.5) admits a unique solution*

$$(u,v,h) \in [C^{1+\alpha,(1+\alpha)/2}(D_T)]^2 \times C^{1+\alpha/2}([0,T]), \tag{3.1}$$

where $D_T := \{(x,t) : x \in [0,h(t)], t \in [0,T]\}$, T only depends on $h_0, \alpha, \|u_0\|_{C([0,h_0])}$ and $\|v_0\|_{C([0,h_0])}$. Moreover, there exists a constant C such that

$$0 < u(x,t), v(x,t), h'(t) \le C \quad \text{for } (x,t) \in (0,h(t)) \times [0,\infty).$$

Next, we discuss the steady states of the problem (1.5),

$$\begin{cases} -d_1 u_{xx} + \beta_1 u_x = u(r_1 - a_1 u - b_1 v), & 0 < x < \infty, \\ -d_2 v_{xx} + \beta_2 v_x = v(r_2 - a_2 v - b_2 u), & 0 < x < \infty, \\ u(0) = v(0) = 0. \end{cases} \tag{3.2}$$

Lemma 3.2. *Assume that $d_i > 0,\ r_i > 0,\ a_i > 0,\ b_i > 0,\ 0 \le \beta_i < 2\sqrt{d_i r_i}$ for $i = 1,2$ and $a_2 r_1 > b_1 r_2,\ a_1 r_2 > b_2 r_1$, then problem (3.2) has a positive solution. Moreover, any positive solution (u,v) of (3.2) satisfies*

$$\underline{u}(x) \le u(x) \le \bar{u}(x), \quad \underline{v}(x) \le v(x) \le \bar{v}(x), \quad \text{for } x \in [0,\infty),$$

where $\underline{u}(x), \bar{u}(x), \underline{v}(x)$ and $\bar{v}(x)$ will be given in the following proof.

Proof. Let $\bar{u}$ be the unique solution of

$$\begin{cases} -d_1 u_{xx} + \beta_1 u_x = u(r_1 - a_1 u), & 0 < x < \infty, \\ u(0) = 0, \end{cases} \tag{3.3}$$

with $0 \le \beta_1 < 2\sqrt{d_1 r_1}$. By [30, Theorem 2.4], $\lim_{x\to\infty} \bar{u}(x) = r_1/a_1$.

Similarly, the problem

$$\begin{cases} -d_2 v_{xx} + \beta_2 v_x = v(r_2 - a_2 v), & 0 < x < \infty, \\ v(0) = 0, \end{cases} \tag{3.4}$$

with $0 \le \beta_2 < 2\sqrt{d_2 r_2}$, has a unique solution $\bar{v}$, then $\lim_{x\to\infty} \bar{v}(x) = r_2/a_2$. Since $0 \le \beta_1 < 2\sqrt{d_1 r_1}$ and $a_2 r_1 > b_1 r_2$, the problem

$$\begin{cases} -d_1 u_{xx} + \beta_1 u_x = u(r_1 - a_1 u - b_1 \bar{v}), & 0 < x < \infty, \\ u(0) = 0, \end{cases} \tag{3.5}$$

has a unique solution $\underline{u}$, then $\lim_{x\to\infty} \underline{u}(x) = \frac{a_2 r_1 - b_1 r_2}{a_1 a_2}$.

Denote $\underline{v}$ is the unique solution of

$$\begin{cases} -d_2 v_{xx} + \beta_2 v_x = v(r_2 - a_2 v - b_2 \bar{u}), & 0 < x < \infty, \\ u(0) = 0, \end{cases} \tag{3.6}$$

with $0 \le \beta_2 < 2\sqrt{d_2 r_2}$. Since $a_1 r_2 > b_2 r_1$, still by [30, Theorem 2.4], $\lim_{x\to\infty} \underline{v}(x) = \frac{a_1 r_2 - b_2 r_1}{a_1 a_2}$.

The above proof implies that $\underline{u}(x), \underline{v}(x), \bar{u}(x)$ and $\bar{v}(x)$ are the coupled ordered lower and upper solutions of (3.2). Clearly, for any $l > 0, \underline{u}(x), \underline{v}(x), \bar{u}(x)$ and $\bar{v}(x)$ are also the coupled ordered lower and upper solutions of

$$\begin{cases} -d_1 u_{xx} + \beta_1 u_x = u(r_1 - a_1 u - b_1 v), & 0 < x < l, \\ -d_2 v_{xx} + \beta_2 v_x = v(r_2 - a_2 v - b_2 u), & 0 < x < l, \\ u(0) = \underline{u}(0), v(0) = \underline{v}(0), \\ u(l) = \bar{u}(l), v(l) = \bar{v}(l). \end{cases}$$

By the standard upper and lower solutions method, we see that the problem has at least one positive solution (u_l, v_l), satisfying

$$\underline{u}(x) \le u_l(x) \le \bar{u}(x), \quad \underline{v}(x) \le v_l(x) \le \bar{u}(x), \quad \text{for } x \in [0, l].$$

According to the local estimation and compactness argument, we can conclude that $(u_l, v_l) \to (u, v)$ in $[C^2_{loc}([0,\infty))]^2$, and (u, v) satisfies (3.2). □

Next, we can obtain the similar result in the case: $b_2 < 0$.

Lemma 3.3. *Assume that $d_i > 0, r_i > 0, a_i > 0, b_1 > 0, b_2 < 0, 0 \le \beta_i < 2\sqrt{d_i r_i}$ for $i = 1, 2$ and $a_1 a_2 r_1 > a_1 b_1 r_2 - b_1 b_2 r_1$. Then problem (3.2) has a positive solution. Besides, any positive solution (u, v) of (3.2) satisfies*

$$\underline{u}(x) \le u(x) \le \bar{u}(x), \quad \underline{v}(x) \le v(x) \le \bar{v}(x), \quad \textit{for } x \in [0, \infty),$$

where $\bar{u}(x), \bar{v}(x), \underline{u}(x)$ and $\underline{v}(x)$ are the positive solutions of the following problems, respectively.

$$-d_1 \bar{u}_{xx} + \beta_1 \bar{u}_x = \bar{u}(r_1 - a_1 \bar{u}), \quad 0 < x < \infty, \quad \bar{u}(0) = 0, \tag{3.7}$$

$$-d_2 \bar{v}_{xx} + \beta_2 \bar{v}_x = \bar{v}(r_2 - a_2 \bar{v} - b_2 \bar{u}), \quad 0 < x < \infty, \quad \bar{v}(0) = 0, \tag{3.8}$$

$$-d_1 \underline{u}_{xx} + \beta_1 \underline{u}_x = \underline{u}(r_1 - a_1 \underline{u} - b_1 \bar{v}), \quad 0 < x < \infty, \quad \underline{u}(0) = 0, \tag{3.9}$$

$$-d_2 \underline{v}_{xx} + \beta_2 \underline{v}_x = \underline{v}(r_2 - a_2 \underline{v} - b_2 \underline{u}), \quad 0 < x < \infty, \quad \underline{v}(0) = 0. \tag{3.10}$$

3.2 Conditions for spreading and vanishing

It follows from Lemma 3.1 that $x = h(t)$ is monotonically increasing. Then there exits $h_\infty \in (0, +\infty]$ such that $h_\infty = \lim_{t\to+\infty} h(t)$.

Definition 3.4. The solution is vanishing if $h_\infty < +\infty$ and

$$\lim_{t\to+\infty} \|u(\cdot,t)\|_{C([0,h(t)])} = \lim_{t\to+\infty} \|v(\cdot,t)\|_{C([0,h(t)])} = 0,$$

while the solution is spreading if $h_\infty = +\infty$ and

$$\liminf_{t\to+\infty} u(x,t) > 0, \ \liminf_{t\to+\infty} v(x,t) > 0 \text{ uniformly in any compact subset of } (0,+\infty).$$

3.2.1 Vanishing case

Lemma 3.5. *Let (u, v, h) be the solution of problem (1.5). If $h_\infty < +\infty$, then there exists a constant K such that*

$$\|u(\cdot,t), v(\cdot,t)\|_{C^1([0,h(t)])} \le K, \quad \forall t > 1,$$
$$\lim_{t\to\infty} h'(t) = 0.$$

Proof. The proof of this lemma is similar to [24, Theorem 3.1]. We give the details for the readers' convenience. Define a transformation

$$(x,t) \to (y,t), \quad y = \frac{x}{h(t)}, \quad 0 \le y < \infty.$$

Let $u(x,t) := U(y,t), v(x,t) := V(y,t)$ and set

$$F(U,V) = U(r_1 - a_1U - b_1V), \qquad G(U,V) = V(r_2 - a_2V - b_2U),$$

then the problem (1.5) becomes

$$\begin{cases} U_t - \alpha_1 U_{yy} - \gamma_1 U_y = F(U,V), & 0 < y < 1, \ t > 0, \\ V_t - \alpha_2 V_{yy} - \gamma_2 V_y = G(U,V), & 0 < y < 1, \ t > 0, \\ U(0,t) = V(0,t) = U(1,t) = V(1,t) = 0, & t > 0, \\ U(y,0) = u_0(y), V(y,0) = v_0(y), & 0 \le y \le 1, \end{cases}$$

where $\alpha_i(t) := \frac{d_i}{h^2(t)}$ and $\gamma_i(y,t) := \frac{h'(t)y - \beta_i}{h(t)}$ for $i = 1, 2$.

Denote that $h_n(t) = h(t+n), U_n(y,t) = U(y,t+n), V_n(y,t) = V(y,t+n), (\alpha_i)_n(t) = \alpha_i(t+n), (\gamma_i)_n(y,t) = \gamma_i(y,t+n)$ for $i = 1, 2$. Then U_n satisfies

$$\begin{cases} (U_n)_t - (\alpha_1)_n (U_n)_{yy} - (\gamma_1)_n (U_n)_y = F_n(y,t), & 0 < y < 1, \ t > 0, \\ (U_n)(0,t) = U_n(1,t) = 0, & t > 0, \\ U_n(y,0) = u(h(n)y, n), & 0 \le y \le 1, \end{cases}$$

where $F_n = U_n(r_1 - a_1U_n - b_1V_n)$.

Using the L^p estimate and embedding theorem, there exists a positive constant K such that

$$\|U_n\|_{C^{1+\alpha, \frac{1+\alpha}{2}}([0,1]\times[1,3])} \le K,$$

for all $n \geq 0$. This implies $\|U\|_{C^{1+\alpha,\frac{1+\alpha}{2}}(E_n)} \leq K$ for all $n \geq 0$, where $E_n = [0,1] \times [n+1, n+3]$. Similarly, we get $\|V\|_{C^{1+\alpha,\frac{1+\alpha}{2}}(E_n)} \leq K$ for all $n \geq 0$. Because these rectangles E_n overlap and K is independent of n, then $\|U, V\|_{C^{1,0}([0,1]\times[1,\infty))} \leq K$.

Since $u_x = h^{-1}(t)U_y$ and $v_x = h^{-1}(t)V_y$, then

$$\|u(\cdot,t), v(\cdot,t)\|_{C^1([0,h(t)])} \leq K, \quad \forall t > 1.$$

Due to the Stefan condition and $0 < h'(t) \leq C_2$, we derive that $\|h'\|_{C^{\frac{\alpha}{2}}([1,\infty))} \leq L$. Since $h_\infty < \infty$, then $\lim_{t\to\infty} h'(t) = 0$. □

In view of [27, Theorem 2.2] and Lemma 3.5, we get the following theorem.

Theorem 3.6. *Let (u, v, h) be the solution of problem (1.5). If $h_\infty < +\infty$, then*

$$\lim_{t\to\infty} \|u(\cdot,t), v(\cdot,t)\|_{C([0,h(t)])} = 0. \tag{3.11}$$

3.2.2 Spreading case

Theorem 3.7. *Suppose that $d_i > 0, r_i > 0, a_i > 0, b_i > 0, 0 \leq \beta_i < 2\sqrt{d_i r_i}$ for $i = 1, 2$ and $a_2 r_1 > b_1 r_2, a_1 r_2 > b_2 r_1$. If $h_\infty = \infty$, then the solution (u, v) of the problem (1.5) satisfies*

$$\liminf_{t\to\infty} u(x,t) \geq \underline{u}(x),\ \limsup_{t\to\infty} u(x,t) \leq \bar{u}(x),\ \textit{locally uniformly for } x \in [0,\infty),$$

$$\liminf_{t\to\infty} v(x,t) \geq \underline{v}(x),\ \limsup_{t\to\infty} v(x,t) \leq \bar{v}(x),\ \textit{locally uniformly for } x \in [0,\infty).$$

where $\bar{u}, \bar{v}, \underline{u}$ and $\underline{v}$ are given in the proof of Lemma 3.2.

The proof of this theorem is similar to the proof of [24, Theorem 3.5].

Theorem 3.8. *Suppose that $d_i > 0, r_i > 0, a_i > 0, b_1 > 0, b_2 < 0, 0 \leq \beta_i < 2\sqrt{d_i r_i}$ for $i = 1, 2$ and $a_1 a_2 r_1 > a_1 b_1 r_2 - b_1 b_2 r_1$. If $h_\infty = \infty$, then the solution (u, v) of the problem (1.5) satisfies*

$$\liminf_{t\to\infty} u(x,t) \geq \underline{u}(x),\ \limsup_{t\to\infty} u(x,t) \leq \bar{u}(x),\ \textit{locally uniformly for } x \in [0,\infty),$$

$$\liminf_{t\to\infty} v(x,t) \geq \underline{v}(x),\ \limsup_{t\to\infty} v(x,t) \leq \bar{v}(x),\ \textit{locally uniformly for } x \in [0,\infty),$$

where $\bar{u}, \bar{v}, \underline{u}$ and $\underline{v}$ are given in the proof of Lemma 3.3.

3.3 The criteria for spreading and vanishing

Here we first give the comparison principle. The proof is similar to the proof of [7, Lemma 3.5].

Lemma 3.9. *Let $\bar{h} \in C^1([0,\infty)), \bar{u}, \bar{v} \in C(\overline{D}) \times C^{2,1}(D)$, with $D := \{(x,t) \in R^2 : 0 < x < \bar{h}(t), t > 0\}$. Assume that $d_i > 0, r_i > 0, a_i > 0, b_i > 0$ and $(\bar{u}, \bar{v}, \bar{h})$ satisfies*

$$\begin{cases} \bar{u}_t - d_1\bar{u}_{xx} + \beta_1\bar{u}_x \geq \bar{u}(r_1 - a_1\bar{u}), & 0 < x < \bar{h}(t), \\ \bar{v}_t - d_2\bar{v}_{xx} + \beta_2\bar{v}_x \geq \bar{v}(r_2 - a_2\bar{v}), & 0 < x < \bar{h}(t), \\ \bar{u}(0,t) \geq 0, \bar{v}(0,t) \geq 0, \bar{u}(\bar{h}(t),t) = 0, \bar{v}(\bar{h}(t),t) = 0, & t > 0, \\ \bar{h}'(t) \geq -\mu[\bar{u}_x(\bar{h}(t),t) + \rho\bar{v}_x(\bar{h}(t),t)], & t > 0. \end{cases} \tag{3.12}$$

Assume that $d_i > 0, r_i > 0, a_i > 0, b_1 > 0, b_2 < 0$ and $(\bar{u}, \bar{v}, \bar{h})$ satisfies

$$\begin{cases} \bar{u}_t - d_1\bar{u}_{xx} + \beta_1\bar{u}_x \geq \bar{u}(r_1 - a_1\bar{u}), & 0 < x < \bar{h}(t), \\ \bar{v}_t - d_2\bar{v}_{xx} + \beta_2\bar{v}_x \geq \bar{v}(r_2 - a_2\bar{v} - b_2\bar{u}), & 0 < x < \bar{h}(t), \\ \bar{u}(0,t) \geq 0, \bar{v}(0,t) \geq 0, \bar{u}(\bar{h}(t),t) = 0, \bar{v}(\bar{h}(t),t) = 0, & t > 0, \\ \bar{h}'(t) \geq -\mu[\bar{u}_x(\bar{h}(t),t) + \rho\bar{v}_x(\bar{h}(t),t)], & t > 0. \end{cases} \tag{3.13}$$

If $\bar{h}(0) \geq h_0, \bar{u}(x,0) \geq u_0(x)$ and $\bar{v}(x,0) \geq v_0(x)$ on $[0,h_0]$, then the solution (u,v,h) of problem (1.5) satisfies $\bar{h}(t) \geq h(t)$ on $[0,\infty)$ and $\bar{u} \geq u, \bar{v} \geq v$ on $[0,h(t)] \times [0,\infty)$.

Let $\lambda_1^{(i)}(l)$ be the principle eigenvalue of the following problem for $i = 1,2$

$$\begin{cases} -d_i\phi_{xx} + \beta_i\phi_x = \lambda_1^{(i)}(l)\phi, & 0 < x < l, \\ \phi(0) = \phi(l) = 0. \end{cases} \tag{3.14}$$

It is well known that $\lambda_1^{(i)}(l) = \left(\frac{\beta_i}{2d_i}\right)^2 + d_i\left(\frac{\pi}{l}\right)^2$ is a strictly decreasing and continuous function in l and

$$\lim_{l\to 0}\lambda_1^{(i)}(l) = \infty, \qquad \lim_{l\to\infty}\lambda_1^{(i)}(l) = \left(\frac{\beta_i}{2d_i}\right)^2.$$

Theorem 3.10. *Suppose that $d_i > 0, r_i > 0, a_i > 0, b_1 > 0, b_2 \in \mathbb{R}, 0 \leq \beta_i < 2\sqrt{d_i r_i}$ for $i = 1,2$. If $h_\infty < \infty$, then $h_\infty \leq h^* = \min\{L_{r_i}, i = 1,2\}$, where L_{r_i} satisfies $\lambda_1^{(i)}(L_{r_i}) = r_i$.*

Proof. Due to Theorem 3.6, $\lim_{t\to\infty}\|u(\cdot,t), v(\cdot,t)\|_{C^1([0,h(t)])} = 0$ if $h_\infty < \infty$. Assume $h_\infty > h^*$ to get a contradiction. If $h_\infty > L_{r_1}$, then there exists $\varepsilon > 0$ such that $h_\infty > L_{r_1 - b_1\varepsilon}$. For such ε, there exists $T_0 \gg 1$ such that $h(T_0) = l > L_{r_1 - b_1\varepsilon}$ and $v(x,t) \leq \varepsilon$ for $t \geq T_0, 0 \leq x \leq h(t)$.

Let $z = z(x,t)$ be the unique solution of

$$\begin{cases} z_t - d_1 z_{xx} + \beta_1 z_x = z(r_1 - a_1 z - b_1\varepsilon), & 0 < x < l, t \geq T_0, \\ z(0,t) = z(l,t) = 0, & t \geq T_0, \\ z(x,T_0) = u(x,T_0), & 0 \leq x \leq l. \end{cases}$$

Applying the comparison principle, $z(x,t) \leq u(x,t)$ for $t \geq T_0, 0 \leq x \leq l$. Since $l > L_{r_1 - b_1\varepsilon}$, then $\|z(\cdot,t) - Z(\cdot)\|_{C([0,l])} \to 0$ as $t \to \infty$, where $Z(x)$ is the unique positive solution of

$$\begin{cases} -d_1 Z_{xx} + \beta_1 Z_x = Z(r_1 - a_1 Z - b_1\varepsilon), & 0 < x < l, \\ Z(0) = Z(l) = 0. \end{cases}$$

$\liminf_{t\to\infty} u(x,t) \geq \lim_{t\to\infty} z(x,t) = Z(x) > 0$ in $(0,l)$. This is a contradiction. If $h_\infty > L_{r_2}$, we can get a contradiction by using the similar argument. □

Lemma 3.11. *Suppose that $r_i > 0, a_i > 0, b_i > 0, 0 \leq \beta_i < 2\sqrt{d_i r_i}, d_i = 1$ for $i = 1,2$ and $h_0 < h^*$, then there exists $\underline{\mu} > 0$ depending on u_0 and v_0 such that $h_\infty < \infty$ if $\mu \leq \underline{\mu}$.*

Proof. Since $h_0 < h^*$, then $\lambda_1^{(i)}(h_0) = \left(\frac{\beta_i}{2d_i}\right)^2 + d_i\left(\frac{2\pi}{h_0}\right)^2 > r_i$ $(i = 1,2)$. We can choose δ, γ small such that

$$\frac{1}{h_0(1+\delta)}\left(\frac{d_i\pi^2}{h_0(1+\delta)} - \delta\gamma h_0\right) + \frac{\beta_i^2}{4} - \gamma - r_i > 0.$$

Define

$$\begin{aligned} \bar{h}(t) &:= h_0\left(1+\delta-\frac{\delta}{2}e^{-\gamma t}\right), && t\geq 0,\\ U(y) &:= \sin(\pi y), && 0\leq y\leq 1,\\ \bar{u}(x,t) &:= Me^{\frac{\beta_1}{2}x-\gamma t}U\left(\frac{x}{\bar{h}(t)}\right), && 0\leq x\leq \bar{h}(t),\\ \bar{v}(x,t) &:= Me^{\frac{\beta_2}{2}x-\gamma t}U\left(\frac{x}{\bar{h}(t)}\right), && 0\leq x\leq \bar{h}(t), \end{aligned}$$

where M is a positive constant to be determined.

Direct computations yield

$$\bar{u}_t - d_1\bar{u}_{xx} + \beta_1\bar{u}_x - \bar{u}(r_1 - a_1\bar{u}) \geq \bar{u}\left[\frac{d_1\pi^2}{\bar{h}^2} + \frac{\beta_1^2}{4} - r_1 - \gamma - \frac{h_0\delta\gamma e^{-\gamma t}\pi y}{2\bar{h}\sin\pi y}\cos\pi y\right].$$

Since $\cos\pi y\leq 0$ for $1/2\leq y\leq 1$, we have for $\bar{h}(t)/2\leq x\leq \bar{h}(t)$

$$\bar{u}_t - d_1\bar{u}_{xx} + \beta_1\bar{u}_x - \bar{u}(r_1 - a_1\bar{u}) \geq \bar{u}\left[\frac{d_1\pi^2}{\bar{h}^2} + \frac{\beta_1^2}{4} - r_1 - \gamma\right] \geq 0.$$

Note that $0\leq\cos\pi y\leq 1, y\leq\frac{2}{\pi}\sin\pi y$ for $0\leq y\leq 1/2$, and $e^{-\gamma t}\leq 1$ for $t\geq 0$, then for $t>0$ and $0\leq x\leq\bar{h}(t)/2$,

$$\bar{u}_t - d_1\bar{u}_{xx} + \beta_1\bar{u}_x - \bar{u}(r_1 - a_1\bar{u}) \geq \bar{u}\left(\frac{d_1\pi^2}{\bar{h}^2} + \frac{\beta_1^2}{4} - r_1 - \gamma - \frac{h_0\delta\gamma}{\bar{h}}\right).$$

It follows that for $t>0$ and $0\leq x\leq\bar{h}(t)/2$,

$$\bar{u}_t - d_1\bar{u}_{xx} + \beta_1\bar{u}_x - \bar{u}(r_1 - a_1\bar{u}) \geq \bar{u}\left[\frac{1}{h_0(1+\delta)}\left(\frac{d_1\pi^2}{h_0(1+\delta)} - \delta\gamma h_0\right) + \frac{\beta_1^2}{4} - r_1 - \gamma\right] \geq 0.$$

In conclusion, for $t>0$ and $0\leq x\leq\bar{h}(t)$,

$$\bar{u}_t - d_1\bar{u}_{xx} + \beta_1\bar{u}_x - \bar{u}(r_1 - a_1\bar{u}) \geq 0.$$

Similarly, for $t>0$ and $0\leq x\leq\bar{h}(t)$,

$$\bar{v}_t - d_2\bar{v}_{xx} + \beta_2\bar{v}_x - \bar{v}(r_2 - a_2\bar{v}) \geq 0.$$

On the other hand, $\bar{u}(0,t)=\bar{v}(0,t)=\bar{u}(\bar{h}(t),t)=\bar{v}(\bar{h}(t),t)=0$. If we choose M sufficiently large such that

$$\bar{u}(x,0)\geq u_0(x), \qquad \bar{v}(x,0)\geq v_0(x), \quad \text{for } x\in[0,\bar{h}(0)].$$

For

$$\mu\leq\underline{\mu} := \frac{\delta\gamma(2+\delta)h_0^2}{2\pi M(1+\rho)},$$

then

$$\bar{h}'(t) + \mu[\bar{u}_x(\bar{h}(t),t) + \rho\bar{v}_x(\bar{h}(t),t)] \geq 0.$$

By Lemma 3.9, we have

$$\bar{h}(t)\geq h(t), \qquad \forall t\geq 0.$$

So we obtain $h_\infty\leq h_0(1+\delta)$. Then we complete the proof. □

Lemma 3.12. *Suppose that $d_i > 0, r_i > 0, a_i > 0, b_i \in \mathbb{R}, 0 \leq \beta_i < 2\sqrt{d_i r_i}$ for $i = 1,2$ and $h_0 < h^*$, there exists $\bar{\mu} > 0$ such that $h_\infty = \infty$ if $\mu \geq \bar{\mu}$.*

Proof. Due to the boundedness of u and v, there exists δ^* such that

$$u(r_1 - a_1 u - b_1 v) \geq -\delta^* u, \qquad v(r_2 - a_2 v - b_2 u) \geq -\delta^* v.$$

Consider the following problem

$$\begin{cases} w_t - d_1 w_{xx} + \beta_1 w_x = -\delta^* w, & 0 < x < r(t), t > 0, \\ z_t - d_2 z_{xx} + \beta_2 z_x = -\delta^* z, & 0 < x < r(t), t > 0, \\ w(0,t) = z(0,t) = 0, & t > 0, \\ w(r(t),t) = z(r(t),t) = 0, & t > 0, \\ r'(t) = -\mu[w_x(r(t),t) + \rho z_x(r(t),t)], & t > 0, \\ w(x,0) = u_0(x), z(x,0) = v_0(x), r(0) = h_0, & 0 < x < h_0. \end{cases} \tag{3.15}$$

Similar to Lemma 3.1, such problem admits a unique global solution (w,z,r). Applying the comparison principle, it follows that

$$u(x,t) \geq w(x,t), \quad v(x,t) \geq z(x,t), \quad h(t) \geq r(t), \quad \text{for } x \in [0, r(t)], \quad t > 0. \tag{3.16}$$

Next, we prove that for all large $\mu, r(1) \geq h^*$. Choose a smooth function $\underline{r}(t)$ such that

$$\underline{r}(0) = h_0/2, \quad \underline{r}(1) = h^*, \quad \underline{r}'(t) > 0, \quad \text{for } t > 0.$$

Consider the following initial-boundary value problem

$$\begin{cases} \underline{w}_t - d_1 \underline{w}_{xx} + \beta_1 \underline{w}_x = -\delta^* \underline{w}, & 0 < x < \underline{r}(t), t > 0, \\ \underline{z}_t - d_2 \underline{z}_{xx} + \beta_2 \underline{z}_x = -\delta^* \underline{z}, & 0 < x < \underline{r}(t), t > 0, \\ \underline{w}(0,t) = \underline{z}(0,t) = 0, & t > 0, \\ \underline{w}(\underline{r}(t),t) = \underline{z}(\underline{r}(t),t) = 0, & t > 0, \\ \underline{w}(x,0) = \underline{w}_0(x), \underline{z}(x,0) = \underline{z}_0(x), & 0 < x < h_0/2, \end{cases} \tag{3.17}$$

here $(\underline{w}_0(x), \underline{z}_0(x))$ satisfies

$$\begin{aligned} &0 < \underline{w}_0(x) \leq u_0(x) \quad \text{on } [0, h_0/2], \qquad \underline{w}_0(0) = \underline{w}_0(h_0/2) = 0, \qquad \underline{w}_0'(h_0/2) < 0, \\ &0 < \underline{z}_0(x) \leq v_0(x) \quad \text{on } [0, h_0/2], \qquad \underline{z}_0(0) = \underline{z}_0(h_0/2) = 0, \qquad \underline{z}_0'(h_0/2) < 0. \end{aligned}$$

The standard theory for parabolic equations ensures that (3.17) has an unique positive solution $(\underline{w}, \underline{z})$ and $\underline{w}_x(\underline{r}(t),t) < 0, \underline{z}_x(\underline{r}(t),t) < 0$ for all $t \in [0,1]$ due to the Hopf Lemma. Then there exists a constant $\bar{\mu} > 0$ such that for all $\mu \geq \bar{\mu}$,

$$\underline{r}'(t) \leq -\mu[\underline{w}_x(\underline{r}(t),t) + \rho \underline{z}_x(\underline{r}(t),t)] \quad \text{for } t \in [0,1]. \tag{3.18}$$

Since the choice of initial values and (3.15)–(3.18), we have

$$r(t) \geq \underline{r}(t), \quad w(x,t) \geq \underline{w}(x,t), z(x,t) \geq \underline{z}(x,t), \quad \text{for } x \in [0, \underline{r}(t)], \quad t \in [0,1],$$

which implies $r(1) \geq \underline{r}(1) = h^*$. In view of (3.16), $h_\infty > h(1) \geq h^*$. Together with Theorem 3.10, derives the desired result. □

Theorem 3.13. *Assume that $r_i > 0, a_i > 0, b_i > 0, 0 \le \beta_i < 2\sqrt{d_i r_i}, d_i = 1$ for $i = 1, 2$ and $h_0 < h^*$, there exists $\mu^* \ge \mu_* > 0$ such that vanishing happens ($h_\infty < \infty$) if $0 < \mu \le \mu_*$ or $\mu = \mu^*$, and spreading happens ($h_\infty = \infty$) if $\mu > \mu^*$.*

Proof. Define $\Gamma := \{\mu > 0 : h_\infty \le h^*\}$. Due to Lemma 3.11, $\Gamma \ne \emptyset$. In view of Lemma 3.12, $\mu^* := \sup \Gamma \in [\underline{\mu}, \bar{\mu}]$. By the definition of μ^* and Theorem 3.10, we get that $h_\infty = \infty$ if $\mu > \mu^*$.

Next, we prove that $h_\infty < \infty$ if $\mu = \mu^*$. If not, $h_\infty = \infty$ for $\mu = \mu^*$. So there exists T such that $h(T) > h^*$. Since the solution (u, v, h) depends on μ, we write (u_μ, v_μ, h_μ) instead of (u, v, h). By the continuous dependence of (u_μ, v_μ, h_μ) on μ, for small $\varepsilon > 0$, $h_\mu(T) > h^*$ for all $[\mu^* - \varepsilon, \mu^* + \varepsilon]$. Then $\sup \Gamma \le \mu^* - \varepsilon$, which contradicts to the definition of μ^*. Hence $\mu^* \in \Gamma$.

Denote $\Lambda := \{\nu > 0 : \nu \ge \underline{\mu}$ such that $h_\infty \le h^*$ for all $0 < \mu \le \nu\}$ and $\mu_* := \sup \Lambda \le \mu^*$. Using the similar way to the above, we obtain that $\mu_* \in \Lambda$. The proof is completed. □

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