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Three solutions for second-order boundary-value problems with variable exponents

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Abstract. This paper presents several sufficient conditions for the existence of at least three weak solutions of a nonhomogeneous Neumann problem for an ordinary differential equation with $p(x)$-Laplacian operator. The technical approach is variational, based on a theorem of Bonanno and Candito. An example is also given.

Keywords: variable exponent Sobolev spaces, $p(x)$-Laplacian, three weak solutions, variational methods.

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1 Introduction

The aim of this paper is to consider the following boundary value problem involving an ordinary differential equation with $p(x)$-Laplacian operator and nonhomogeneous Neumann conditions

$$\begin{cases} -\left(|u'(x)|^{p(x)-2}u'(x)\right)' + \alpha(x)|u(x)|^{p(x)-2}u(x) = \lambda f(x,u(x)) & \text{in } (0,1), \\ |u'(0)|^{p(0)-2}u'(0) = -\mu g(u(0)), \\ |u'(1)|^{p(1)-2}u'(1) = \mu h(u(1)) \end{cases} \tag{$P^f_{\lambda,\mu}$}$$

where $p \in C([0,1],\mathbb{R})$, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, see [9, page 5] (that is $x \to f(x,t)$ is measurable for all $t \in \mathbb{R}$, $t \to f(x,t)$ is continuous for almost every $x \in [0,1]$), $g, h : \mathbb{R} \to \mathbb{R}$ are nonnegative continuous functions, λ and μ are real parameters with $\lambda > 0$ and $\mu \geq 0$, $\alpha \in L^\infty([0,1])$, with $\operatorname{ess\,inf}_{[0,1]} \alpha > 0$.

The study of various problems with the variable exponent has received considerable attention in recent years both for their interesting in applications and for the many mathematical questions arising from such problems. They can model various phenomena dealing with the

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study of nonlinear elasticity theory, electro-rheological fluids and so on (see [27,32]). The necessary framework for the study of these problems is represented by the function spaces with variable exponent $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$. For background and recent results, we refer the reader to [1,3,4,6,8–10,18,19,22–26,31] and the references therein. For example, Zhang in [31] via Leray–Schauder degree, obtained sufficient conditions for the existence of one solution for a weighted $p(x)$-Laplacian system. Bonanno and Chinnì in [3] by using a multiple critical points theorem for non-differentiable functionals, investigated the existence and multiplicity of solutions for the following problem

$$\begin{cases} -\Delta_{p(x)} u(x) = \lambda(f(x,u) + \mu g(x,u)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded domain with smooth boundary, $p \in C(\bar{\Omega})$, f and g are functions possibly discontinuous with respect to u. Cammaroto et al. in [8] by using a three critical points theorem due to Ricceri, obtained the existence of three weak solutions for the following problem

$$\begin{cases} -\Delta_{p(x)} u(x) + a(x)|u|^{p(x)-2}u = \lambda f(x,u) + \mu g(x,u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \end{cases}$$

where $a \in L^\infty(\Omega)$, $a^- = \operatorname{ess\,inf}_\Omega a(x) > 0$, n is the outward unit normal to $\partial\Omega$, $\lambda, \mu \in (0, +\infty)$ and $p \in L^\infty(\Omega)$ is such that $2 \le N < p^- = \operatorname{ess\,inf}_\Omega p(x) \le p^+ = \operatorname{ess\,sup}_\Omega p(x) < +\infty$. By using variational methods, D'Aguì in [9] established the existence of an unbounded sequence of weak solutions for the problem $(P^f_{\lambda,\mu})$ and Moschetto in [22] under suitable assumptions on the functions α, f, p and g investigated the existence of at least three solutions for the following Neumann problem

$$\begin{cases} -\Delta_{p(x)} u + \alpha(x)|u|^{p(x)-2}u = \alpha(x)f(u) + \lambda g(x,u), & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega. \end{cases}$$

We refer to [7, 14] in which the existence of infinitely many solutions for variational-hemivariational inequalities and variational-hemivariational inequalities of Kirchhoff-type, both small perturbations of nonhomogeneous Neumann boundary conditions by using the nonsmooth analysis, was discussed, respectively.

Motivated by the above facts, in the present paper, by using a three critical point theorem which is a smooth version of [2, Theorem 3.3] (see also [2, Remarks 3.9 and 3.10]) due Bonanno and Candito we study the existence of at least three non-trivial weak solution for the problem $(P^f_{\lambda,\mu})$. Our main result is Theorem 3.1. Example 3.4 illustrates Theorem 3.1. In Theorem 3.3 we present an application of Theorem 3.1. Finally, as a special case of Theorem 3.1, we obtain Theorem 3.5 considering the case $p(x) = p$ for every $x \in [0,1]$.

A special case of our main result, Theorem 3.1, is the following theorem.

Theorem 1.1. *Let f be a non-negative Carathéodory function in $[0,1] \times [0,+\infty[$. Assume that there exist positive constants $\theta_1 \ge k$, θ_2, θ_3 and $\eta \ge 1$ with $\theta_1 < \sqrt[p]{\|\alpha\|_1 k\eta}$, $\max\{\eta, \sqrt[p]{\|\alpha\|_1 k\eta}\} < \theta_2$ and $\theta_2 < \theta_3$ such that*

$$\max\left\{ \frac{\int_0^1 F(x,\theta_1)dx}{\theta_1^2}, \frac{\int_0^1 F(x,\theta_2)dx}{\theta_2^2}, \frac{\int_0^1 F(x,\theta_3)dx}{\theta_3^2 - \theta_2^2} \right\} < \frac{1}{k^2\|\alpha\|_1} \frac{\int_0^1 F(x,\eta)dx - \int_0^1 F(x,\theta_1)dx}{\eta^2}$$

where

$$k = 2\left[\frac{1}{\alpha_-^{-1}+1}\right]^{\frac{1}{2}} + \left[1-\frac{1}{\alpha_-^{-1}+1}\right]^{\frac{1}{2}}\alpha_-^{-2}.$$

Then, for every

$$\lambda \in \left(\frac{\frac{\eta^2}{2}\|\alpha\|_1}{\int_0^1 F(x,\eta)dx - \int_0^1 F(x,\theta_1)dx}, \frac{1}{2k^2}\min\left\{\frac{\theta_1^2}{\int_0^1 F(x,\theta_1)dx}, \frac{\theta_2^2}{\int_0^1 F(x,\theta_2)dx}, \frac{\theta_3^2-\theta_2^2}{\int_0^1 F(x,\theta_3)dx}\right\}\right)$$

and for every non-negative continuous functions $g, h : \mathbb{R} \to \mathbb{R}$ there exists $\delta(\lambda) > 0$ given by

$$\delta(\lambda) = \min\left\{\frac{1}{2k^2}\min\left\{\frac{\theta_1^2 - 2\lambda k^2\int_0^1 F(x,\theta_1)dx}{G(\theta_1)+H(\theta_1)}, \frac{\theta_2^2 - 2\lambda k^2\int_0^1 F(x,\theta_2)dx}{G(\theta_2)+H(\theta_2)},\right.\right.$$
$$\left.\left.\frac{(\theta_3^2-\theta_2^2) - 2\lambda k^2\int_0^1 F(x,\theta_3)dx}{G(\theta_3)+H(\theta_3)}\right\}, \frac{\frac{\eta^2}{2}\|\alpha\|_1 - \lambda\left(\int_0^1 F(x,\eta)dx - \int_0^1 F(x,\theta_1)dx\right)}{G(\eta)+H(\eta)-G(\theta_1)-H(\theta_1)}\right\}$$

such that for each $\mu \in [0, \delta(\lambda))$, the problem

$$\begin{cases} u''(x) + \alpha(x)u(x) = \lambda f(x, u(x)) & \text{in } (0,1), \\ u'(0) = -\mu g(u(0)), \\ u'(1) = \mu h(u(1)) \end{cases}$$

possesses at least three non-negative weak solutions u_1, u_2, and u_3 such that

$$\max_{x\in[0,1]} u_1(x) < \theta_1, \qquad \max_{x\in[0,1]} u_2(x) < \theta_2 \quad \text{and} \quad \max_{x\in[0,1]} u_3(x) < \theta_3.$$

The paper consists of three sections. Section 2 contains some background facts concerning the generalized Lebesgue–Sobolev spaces. The main results and their proofs are given in Section 3.

2 Preliminaries

Our main tool to discuss the existence of three solutions for the problem ($P^f_{\lambda,\mu}$) is the following three critical point theorem due Bonanno and Candito, see [2, Theorem 3.3 and Remarks 3.9 and 3.10].

Let X be a nonempty set and $\Phi, \Psi : X \to \mathbb{R}$ be two functions. For all $r,\ r_1,\ r_2 > \inf_X \Phi,\ r_2 > r_1,\ r_3 > 0$, we define

$$\begin{aligned} \varphi(r) &:= \inf_{u\in\Phi^{-1}(-\infty,r)} \frac{(\sup_{u\in\Phi^{-1}(-\infty,r)}\Psi(u)) - \Psi(u)}{r - \Phi(u)}, \\ \beta(r_1,r_2) &:= \inf_{u\in\Phi^{-1}(-\infty,r_1)} \sup_{v\in\Phi^{-1}[r_1,r_2)} \frac{\Psi(v)-\Psi(u)}{\Phi(v)-\Phi(u)}, \\ \gamma(r_2,r_3) &:= \frac{\sup_{u\in\Phi^{-1}(-\infty,r_2+r_3)}\Psi(u)}{r_3}, \\ \alpha(r_1,r_2,r_3) &:= \max\{\varphi(r_1), \varphi(r_2), \gamma(r_2,r_3)\}. \end{aligned}$$

Theorem 2.1 ([2, Theorem 3.3]). *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a convex, coercive and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact, such that*

(a_1) $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$;

(a_2) *for every λ as in the conclusion and for every u_1 and u_2 which are local minima for the functional $\Phi - \lambda\Psi$ such that $\Psi(u_1) \geq 0$ and $\Psi(u_2) \geq 0$, one has*

$$\inf_{s\in[0,1]} \Psi(su_1 + (1-s)u_2) \geq 0.$$

Assume that there are three positive constants r_1, r_2, r_3 with $r_1 < r_2$, such that

(a_3) $\varphi(r_1) < \beta(r_1, r_2)$;

(a_4) $\varphi(r_2) < \beta(r_1, r_2)$;

(a_5) $\gamma(r_2, r_3) < \beta(r_1, r_2)$.

Then, for each $\lambda \in \left(\frac{1}{\beta(r_1,r_2)}, \frac{1}{\alpha(r_1,r_2,r_3)}\right)$ the functional $\Phi - \lambda\Psi$ admits three distinct critical points u_1, u_2, u_3 such that $u_1 \in \Phi^{-1}(-\infty, r_1)$, $u_2 \in \Phi^{-1}[r_1, r_2)$ and $u_3 \in \Phi^{-1}(-\infty, r_2 + r_3)$.

We refer the interested reader to the papers [5, 15–17, 20] in which Theorem 2.1 has been successfully employed to obtain the existence of at least three solutions for boundary value problems.

For the reader's convenience, we state some basic properties of variable exponent Sobolev spaces and introduce some notations. For more details, we refer the reader to [11–13, 21, 27, 29].

We assume that the function $p \in C([0,1], \mathbb{R})$ satisfies the condition

$$1 < p^- := \min_{x\in[0,1]} p(x) \leq p^+ := \max_{x\in[0,1]} p(x). \tag{2.1}$$

The variable exponent Lebesgue spaces are defined as follows

$$L^{p(x)}([0,1]) := \left\{ u : [0,1] \to \mathbb{R} \text{ measurable and } \int_0^1 |u(x)|^{p(x)} dx < +\infty \right\}.$$

equipped with the norm

$$\|u\|_{L^{p(x)}([0,1])} = \inf\left\{ \beta > 0 : \int_0^1 \left| \frac{u(x)}{\beta} \right|^{p(x)} dx \leq 1 \right\}.$$

The space $\left(L^{p(x)}([0,1]), \|u\|_{L^{p(x)}([0,1])}\right)$ is a Banach space called a variable exponent Lebesgue space. Define the Sobolev space with variable exponent

$$W^{1,p(x)}([0,1]) = \left\{ u \in L^{p(x)}([0,1]) : u' \in L^{p(x)}([0,1]) \right\}$$

equipped with the norm

$$\|u\|_{W^{1,p(x)}([0,1])} := \|u\|_{L^{p(x)}([0,1])} + \|u'\|_{L^{p(x)}([0,1])}. \tag{2.2}$$

It is well known (see [13]) that, in view of (2.1), the spaces $L^{p(x)}([0,1])$ and $W^{1,p(x)}([0,1])$, with corresponding norms, are separable, reflexive and uniformly convex Banach spaces. Moreover, since $\alpha \in L^{\infty}([0,1])$ and $\alpha_- := \operatorname{ess\,inf}_{\in[0,1]} \alpha(x) > 0$ the norm

$$\|u\|_\alpha := \inf\left\{\beta > 0 : \int_0^1 \left(\left|\frac{u'(x)}{\beta}\right|^{p(x)} + \alpha(x)\left|\frac{u(x)}{\beta}\right|^{p(x)}\right) dx \leq 1\right\},$$

on $W^{1,p(x)}([0,1])$ is equivalent to that introduced in (2.2).

Next, we refer to the following embedding result of G. D'Agui [9]:

Proposition 2.2 ([9, Proposition 2.1]). *For all $u \in W^{1,p(x)}([0,1])$, one has*

$$\|u\|_{C^0[0,1]} \leq k\|u\|_\alpha \tag{2.3}$$

where

$$k = \begin{cases} 2\left[\dfrac{1}{\alpha_-^{\frac{p^+}{p^-(1-p^+)}}+1}\right]^{\frac{1}{p^+}} + \left[1 - \dfrac{1}{\alpha_-^{\frac{1}{1-p^+}}+1}\right]^{\frac{1}{p^+}} \alpha_-^{\frac{2}{1-p^+}} & \text{if } \alpha_- < 1, \\ 2\left[\dfrac{1}{\alpha_-^{\frac{1}{1-p^+}}+1}\right]^{\frac{1}{p^+}} + \left[1 - \dfrac{1}{\alpha_-^{\frac{1}{1-p^+}}+1}\right]^{\frac{1}{p^+}} \alpha_-^{\frac{2}{1-p^+}} & \text{if } \alpha_- \geq 1. \end{cases}$$

Now, we present the following propositions which will be used later.

Proposition 2.3 ([13, 19]). *Set $\rho(u) = \int_0^1 (|u'(x)|^{p(x)} + \alpha(x)|u(x)|^{p(x)})dx$. For $u \in X$ we have*

(i) $\|u\|_\alpha < (=;>)1 \Leftrightarrow \rho(u) < (=;>)1$,

(ii) $\|u\|_\alpha < 1 \Rightarrow \|u\|_\alpha^{p^+} \leq \rho(u) \leq \|u\|_\alpha^{p^-}$,

(iii) $\|u\|_\alpha > 1 \Rightarrow \|u\|_\alpha^{p^-} \leq \rho(u) \leq \|u\|_\alpha^{p^+}$.

Remark 2.4 ([9, Remark 2.2]). It is worth mentioning that if $\alpha_- \geq 1$, the constant k does not exceed 2. Instead, when $\alpha_- < 1$, k depends on α_- and in particular is less than $2\left(1 + \frac{1}{\alpha_-}\right)$.

We introduce the functions $F : [0,1] \times \mathbb{R} \to \mathbb{R}$, $G : \mathbb{R} \to \mathbb{R}$ and $H : \mathbb{R} \to \mathbb{R}$, corresponding to the functions f, g and h as follows

$$F(x,t) = \int_0^t f(x,\xi)d\xi \quad \text{for all } (x,t) \in [0,1] \times \mathbb{R},$$
$$G(t) = \int_0^t g(\xi)d\xi \quad \text{for all } t \in \mathbb{R}$$

and

$$H(t) = \int_0^t h(\xi)d\xi \quad \text{for all } t \in \mathbb{R}.$$

We say that a function $u \in W^{1,p(x)}([0,1])$ is a *weak solution* of problem $(P^f_{\lambda,\mu})$ if

$$\int_0^1 |u'(x)|^{p(x)-2}u'(x)v'(x)dx + \int_0^1 \alpha(x)|u(x)|^{p(x)-2}u(x)\, v(x)\, dx \\ - \lambda \int_0^1 f(x,u(x))v(x)dx - \mu(g(u(0))v(0) + h(u(1))v(1)) = 0$$

holds for all $v \in W^{1,p(x)}([0,1])$.

3 Main results

We fix four positive constants $\theta_1 \geq k$, θ_2, θ_3 and $\eta \geq 1$, put

$$\begin{aligned}\delta_{\lambda,g,h} := \min\Bigg\{ & \frac{1}{k^{p^-}p^+}\min\Bigg\{\frac{\theta_1^{p^-} - \lambda k^{p^-}p^+\int_0^1 F(x,\theta_1)dx}{G(\theta_1)+H(\theta_1)}, \\ & \frac{\theta_2^{p^-} - \lambda k^{p^-}p^+\int_0^1 F(x,\theta_2)dx}{G(\theta_2)+H(\theta_2)}, \frac{(\theta_3^{p^-}-\theta_2^{p^-}) - \lambda k^{p^-}p^+\int_0^1 F(x,\theta_3)dx}{G(\theta_3)+H(\theta_3)}\Bigg\}, \\ & \frac{\frac{\eta^{p^+}}{p^-}\|\alpha\|_1 - \lambda\left(\int_0^1 F(x,\eta)dx - \int_0^1 F(x,\theta_1)dx\right)}{G(\eta)+H(\eta)-G(\theta_1)-H(\theta_1)}\Bigg\}.\end{aligned} \tag{3.1}$$

We present our main result as follows.

Theorem 3.1. *Let f be a non-negative Carathéodory function in $[0,1]\times[0,+\infty[$. Assume that there exist positive constants $\theta_1 \geq k$, θ_2, θ_3 and $\eta \geq 1$ with $\theta_1 < \sqrt[p^-]{\|\alpha\|_1}k\eta$ and*

$$\max\left\{\eta,\ \sqrt[p^-]{\frac{p^+\|\alpha\|_1}{p^-}}k\eta^{\frac{p^+}{p^-}}\right\} < \theta_2 < \theta_3$$

such that

(A$_1$) $$\max\left\{\frac{\int_0^1 F(x,\theta_1)dx}{\theta_1^{p^-}}, \frac{\int_0^1 F(x,\theta_2)dx}{\theta_2^{p^-}}, \frac{\int_0^1 F(x,\theta_3)dx}{\theta_3^{p^-}-\theta_2^{p^-}}\right\} < \frac{p^-}{k^{p^-}p^+\|\alpha\|_1}\frac{\int_0^1 F(x,\eta)dx - \int_0^1 F(x,\theta_1)dx}{\eta^{p^+}}.$$

Then, for every

$$\lambda \in \left(\frac{\frac{\eta^{p^+}}{p^-}\|\alpha\|_1}{\int_0^1 F(x,\eta)dx - \int_0^1 F(x,\theta_1)dx}, \frac{1}{p^+k^{p^-}}\min\left\{\frac{\theta_1^{p^-}}{\int_0^1 F(x,\theta_1)dx}, \frac{\theta_2^{p^-}}{\int_0^1 F(x,\theta_2)dx}, \frac{\theta_3^{p^-}-\theta_2^{p^-}}{\int_0^1 F(x,\theta_3)dx}\right\}\right)$$

and for every non-negative continuous functions $g, h : \mathbb{R} \to \mathbb{R}$ there exists $\delta_{\lambda,g,h} > 0$ given by (3.1) such that for each $\mu \in [0, \delta_{\lambda,g,h})$, the problem $(P^f_{\lambda,\mu})$ possesses at least three non-negative weak solutions u_1, u_2, and u_3 such that

$$\max_{x\in[0,1]} u_1(x) < \theta_1, \qquad \max_{x\in[0,1]} u_2(x) < \theta_2 \quad \textit{and} \quad \max_{x\in[0,1]} u_3(x) < \theta_3.$$

Proof. Without loss of generality, we can assume $f(x,t) = f(x,0)$ for all $(x,t) \in [0,1]\times]-\infty,0[$. We apply Theorem 2.1 to our problem. Let X be the Sobolev space $W^{1,p(x)}([0,1])$. Fix λ, g and μ as in the conclusion. In order to apply Theorem 2.1 to our problem, we define Φ, Ψ for every $u \in X$ by

$$\Phi(u) := \int_0^1 \frac{1}{p(x)}\left(|u'(x)|^{p(x)} + \alpha(x)|u(x)|^{p(x)}\right)dx \tag{3.2}$$

and

$$\Psi(u) := \int_0^1 F(x, u(x))dx + G(u(0)) + H(u(1)), \tag{3.3}$$

and put $I_\lambda(u) = \Phi(u) - \lambda\Psi(u)$ for every $u \in X$. Note that the weak solutions of $(P^f_{\lambda,\mu})$ are exactly the critical points of I_λ. The functionals Φ and Ψ satisfy the regularity assumptions of Theorem 2.1. Indeed, Φ is Gâteaux differentiable and sequentially weakly lower semicontinuous and its Gâteaux derivative is the functional $\Phi'(u) \in X^*$, given by

$$\Phi'(u)(v) = \int_0^1 |u(x)'|^{p(x)-2}u'(x)v'(x)dx + \int_0^1 \alpha(x)|u(x)|^{p(x)-2}u(x)\, v(x)\, dx$$

for every $v \in X$. We prove that Φ' admits a continuous inverse on X^*. Assuming $\|u\|_\alpha > 1$, by Proposition 2.3 we have

$$\Phi'(u)(u) = \int_0^1 |u(x)'|^{p(x)} + \alpha(x)|u(x)|^{p(x)}\, dx \geq \|u\|_\alpha^{p^-},$$

and since $p^- > 1$, it follows that Φ' is coercive. Since Φ' is the Fréchet derivative of Φ, it follows that Φ' is continuous and bounded. Using the elementary inequality [28]

$$|x-y|^\gamma \leq 2^\gamma(|x|^{\gamma-2}x - |y|^{\gamma-2}y)(x-y) \quad \text{if } \gamma \geq 2,$$

for all $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$, $N \geq 1$, we obtain for all $u, v \in X$ such that $u \neq v$,

$$\langle \Phi'(u) - \Phi'(v), u - v\rangle > 0,$$

which means that Φ' is strictly monotone. Thus Φ' is injective. Consequently, thanks to the Minty–Browder theorem [30], the operator Φ' is an surjection and has an inverse $\Phi'^{-1} : X^* \to X$, and one has Φ'^{-1} is continuous. On the other hand, it is well known that Ψ is a differentiable functional whose differential at the point $u \in X$ is

$$\Psi'(u)(v) = \int_0^1 f(x, u(x))v(x)dx + g(u(0))v(0) + h(u(1))v(1)$$

for any $v \in X$ as well as it is sequentially weakly upper semicontinuous. Furthermore $\Psi' : X \to X^*$ is a compact operator. Put

$$r_1 := \frac{1}{p^+}\left(\frac{\theta_1}{k}\right)^{p^-}, \qquad r_2 := \frac{1}{p^+}\left(\frac{\theta_2}{k}\right)^{p^-}, \qquad r_3 := \frac{1}{p^+}\left(\frac{\theta_3^{p^-} - \theta_2^{p^-}}{k^{p^-}}\right)$$

and $w(x) = \eta$ for all $x \in [0, 1]$. We clearly observe that $w \in X$. Hence, we have definitively,

$$\begin{aligned}\Phi(w) &= \int_0^1 \frac{1}{p(x)}\left(|w'(x)|^{p(x)} + \alpha(x)|w(x)|^{p(x)}\right) dx\\ &= \int_0^1 \frac{1}{p(x)}\alpha(x)|w(x)|^{p(x)}dx \leq \frac{\eta^{p^+}}{p^-}\|\alpha\|_1\end{aligned}$$

and

$$\begin{aligned}\Phi(w) &= \int_0^1 \frac{1}{p(x)}\left(|w'(x)|^{p(x)} + \alpha(x)|w(x)|^{p(x)}\right) dx\\ &= \int_0^1 \frac{1}{p(x)}\alpha(x)|w(x)|^{p(x)}dx \geq \frac{\eta^{p^-}}{p^+}\|\alpha\|_1.\end{aligned}$$

From the conditions $\theta_3 > \theta_2$, $\theta_1 < \sqrt[p^-]{\|\alpha\|_1}k\eta$ and

$$\sqrt[p^-]{\frac{p^+\|\alpha\|_1}{p^-}}k\eta^{\frac{p^+}{p^-}} < \theta_2,$$

we get $r_3 > 0$ and $r_1 < \Phi(w) < r_2$. By Proposition 2.3 and the fact that $\max\left\{r_1^{1/p^-}, r_1^{1/p^+}\right\} = r_1^{1/p^-}$, we deduce

$$\{u \in X : \Phi(u) < r_1\} \subseteq \left\{u \in X : \|u\|_\alpha < r_1^{1/p^-}\right\} = \left\{u \in X : \|u\|_\alpha < \frac{\theta_1}{k}\right\}.$$

Moreover, due to (2.3), we have

$$|u(x)| \le \|u\|_\infty \le k\|u\|_\alpha \le \theta_1, \quad \forall x \in [0,1].$$

Hence,

$$\left\{u \in X : \|u\|_\alpha < \frac{\theta_1}{k}\right\} \subseteq \{u \in X : \|u\|_\infty \le \theta_1\}$$

and this ensures

$$\begin{aligned}\Psi(u) &\le \sup_{u\in\Phi^{-1}(-\infty,r_1)} \left[\int_0^1 F(x,u(x))dx + G(u(0)) + H(u(1))\right] \\ &\le \int_0^1 \max_{|t|\le\theta_1} F(x,t)dx + \max_{|t|\le\theta_1}[G(t)+H(t)] \\ &= \int_0^1 \max_{|t|\le\theta_1} F(x,t)dx + G(\theta_1) + H(\theta_1)\end{aligned}$$

for every $u \in X$ such that $\Phi(u) < r_1$. Since we assumed that f is non-negative, one has

$$\sup_{\Phi(u)<r_1} \Psi(u) \le \int_0^1 F(x,\theta_1)dx + G(\theta_1) + H(\theta_1).$$

In a similar way, we have

$$\sup_{\Phi(u)<r_2} \Psi(u) \le \int_0^1 F(x,\theta_2)dx + G(\theta_2) + H(\theta_2)$$

and

$$\sup_{\Phi(u)<r_2+r_3} \Psi(u) \le \int_0^1 F(x,\theta_3)dx + G(\theta_3) + H(\theta_3).$$

Therefore, since $0 \in \Phi^{-1}(-\infty, r_1)$ and $\Phi(0) = \Psi(0) = 0$, one has

$$\begin{aligned}\varphi(r_1) &= \inf_{u\in\Phi^{-1}(-\infty,r_1)} \frac{(\sup_{u\in\Phi^{-1}(-\infty,r_1)}\Psi(u)) - \Psi(u)}{r_1 - \Phi(u)} \\ &\le \frac{\sup_{u\in\Phi^{-1}(-\infty,r_1)}\Psi(u)}{r_1} \\ &= \frac{\sup_{u\in\Phi^{-1}(-\infty,r_1)}\left[\int_0^1 F(x,u(x))dx + \frac{\mu}{\lambda}(G(u(0)) + H(u(1)))\right]}{r_1} \\ &\le \frac{\int_0^1 F(x,\theta_1)dx + \frac{\mu}{\lambda}(G(\theta_1)+H(\theta_1))}{\frac{1}{p^+}\left(\frac{\theta_1}{k}\right)^{p^-}},\end{aligned}$$

$$\begin{aligned}
\varphi(r_2) &\le \frac{\sup_{u\in\Phi^{-1}(-\infty,r_2)}\Psi(u)}{r_2}\\
&= \frac{\sup_{u\in\Phi^{-1}(-\infty,r_2)}\left[\int_0^1 F(x,u(x))dx + \frac{\mu}{\lambda}(G(u(0))+H(u(1)))\right]}{r_2}\\
&\le \frac{\int_0^1 F(x,\theta_2)dx + \frac{\mu}{\lambda}(G(\theta_2)+H(\theta_2))}{\frac{1}{p^+}\left(\frac{\theta_2}{k}\right)^{p^-}}
\end{aligned}$$

and

$$\begin{aligned}
\gamma(r_2,r_3) &\le \frac{\sup_{u\in\Phi^{-1}(-\infty,r_2+r_3)}\Psi(u)}{r_3}\\
&= \frac{\sup_{u\in\Phi^{-1}(-\infty,r_2+r_3)}\left[\int_0^1 F(x,u(x))dx + \frac{\mu}{\lambda}(G(u(0))+H(u(1)))\right]}{r_3}\\
&\le \frac{\int_0^1 F(x,\theta_3)dx + \frac{\mu}{\lambda}(G(\theta_3)+H(\theta_3))}{\frac{1}{p^+}\left(\frac{\theta_3^{p^-}-\theta_2^{p^-}}{k^{p^-}}\right)}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\Psi(w) &= \int_0^1 F(x,w(x))dx + \frac{\mu}{\lambda}(G(w)+H(w))\\
&= \int_0^1 F(x,\eta)dx + \frac{\mu}{\lambda}(G(\eta)+H(\eta)).
\end{aligned}$$

For each $u \in \Phi^{-1}(-\infty, r_1)$ one has

$$\begin{aligned}
\beta(r_1,r_2) \ge& \frac{\int_0^1 F(x,\eta)dx - \int_0^1 F(x,\theta_1)dx + \frac{\mu}{\lambda}\left(G(\eta)+H(\eta)-G(\theta_1)-H(\theta_1)\right)}{\Phi(w)-\Phi(u)}\\
\ge& \frac{\int_0^1 F(x,\eta)dx - \int_0^1 F(x,\theta_1)dx + \frac{\mu}{\lambda}\left(G(\eta)+H(\eta)-G(\theta_1)-H(\theta_1)\right)}{\frac{\eta^{p^+}}{p^-}\|\alpha\|_1}.
\end{aligned}$$

Due to (A_1) we get

$$\alpha(r_1,r_2,r_3) < \beta(r_1,r_2).$$

Now, we show that the functional I_λ satisfies the assumption (a_2) of Theorem 2.1. Let u_1 and u_2 be two local minima for I_λ. Then u_1 and u_2 are critical points for I_λ, and so, they are weak solutions for the problem $(P^f_{\lambda,\mu})$. We want to prove that they are non-negative. Let u_0 be a (non-trivial) weak solution of the problem $(P^f_{\lambda,\mu})$. Arguing by a contradiction, assume that the set $\mathcal{A} = \{x \in [0,1] : u_0(x) < 0\}$ is non-empty and of positive measure. Put $\bar{v}(x) = \min\{0, u_0(x)\}$ for all $x \in [0,1]$. Clearly, $\bar{v} \in X$ and one has

$$\begin{aligned}
&\int_0^1 |u_0'(x)|^{p(x)-2}u_0'(x)\bar{v}'(x)dx + \int_0^1 \alpha(x)|u_0(x)|^{p(x)-2}u_0(x)\bar{v}(x)dx\\
&\qquad - \lambda\int_0^1 f(x,u_0(x))\bar{v}(x)dx - \mu g(u_0(0))\bar{v}(0) - \mu h(u_0(1))\bar{v}(1) = 0.
\end{aligned}$$

Since we could assume that f is non-negative, and g and h are non-negative, for fixed $\lambda > 0$ and $\mu \geq 0$ and by choosing $\bar{v}(x) = u_0(x)$ one has

$$\int_{\mathcal{A}} |u_0'(x)|^{p(x)}dx + \int_{\mathcal{A}} \alpha(x)|u_0(x)|^{p(x)}dx$$
$$= \lambda \int_{\mathcal{A}} f(x, u_0(x))u_0(x)dx + \mu g(u_0(0))(u_0(0) + \mu h(u_0(1))(u_0(1) \leq 0.$$

Hence $\|u_0\|_{w^{1,p(x)}(\mathcal{A})} = 0$ which is an absurd. Hence, our claim is proved. Then, we observe $u_1(x) \geq 0$ and $u_2(x) \geq 0$ for every $x \in [0,1]$. Thus, it follows that $(\lambda f + \mu(g+h))(x, su_1 + (1-s)u_2) \geq 0$ for all $s \in [0,1]$, and consequently, $\Psi(su_1 + (1-s)u_2) \geq 0$, for every $s \in [0,1]$. Hence, Theorem 2.1 implies that for every

$$\lambda \in \left(\frac{\frac{\eta^{p^+}}{p^-}\|\alpha\|_1}{\int_0^1 F(x,\eta)dx - \int_0^1 F(x,\theta_1)dx}, \frac{1}{p^+ k^{p^-}} \min\left\{ \frac{\theta_1^{p^-}}{\int_0^1 F(x,\theta_1)dx}, \frac{\theta_2^{p^-}}{\int_0^1 F(x,\theta_2)dx}, \frac{\theta_3^{p^-} - \theta_2^{p^-}}{\int_0^1 F(x,\theta_3)dx} \right\} \right)$$

and $\mu \in [0, \delta_{\lambda,g})$, the functional I_λ has three critical points u_i, $i = 1,2,3$, in X such that $\Phi(u_1) < r_1$, $\Phi(u_2) < r_2$ and $\Phi(u_3) < r_2 + r_3$, that is,

$$\max_{x\in[0,1]} u_1(x) < \theta_1, \qquad \max_{x\in[0,1]} u_2(x) < \theta_2 \quad \text{and} \quad \max_{x\in[0,1]} u_3(x) < \theta_3.$$

Then, taking into account the fact that the weak solutions of the problem $(P^f_{\lambda,\mu})$ are exactly critical points of the functional I_λ we have the desired conclusion. □

Remark 3.2. We observe that, in Theorem 3.1, no asymptotic conditions on f and g are needed and only algebraic conditions on f are imposed to guarantee the existence of the weak solutions.

For positive constants $\theta_1 \geq k$, θ_4 and $\eta \geq 1$, set

$$\delta'_{\lambda,g,h} := \min\left\{ \frac{1}{k^{p^-}p^+} \min\left\{ \frac{\theta_1^{p^-} - \lambda k^{p^-} p^+ \int_0^1 F(x,\theta_1)dx}{G(\theta_1) + H(\theta_1)}, \right.\right.$$
$$\left. \frac{\theta_4^{p^-} - 2\lambda k^{p^-} p^+ \int_0^1 F(x, \frac{1}{\sqrt[p^-]{2}}\theta_4)dx}{2\left(G\left(\frac{1}{\sqrt[p^-]{2}}\theta_4\right) + H\left(\frac{1}{\sqrt[p^-]{2}}\theta_4\right)\right)}, \frac{\theta_4^{p^-} - 2\lambda k^{p^-} p^+ \int_0^1 F(x,\theta_4)dx}{2(G(\theta_4) + H(\theta_4))} \right\}, \tag{3.4}$$
$$\left. \frac{\frac{\eta^{p^+}}{p^-}\|\alpha\|_1 - \lambda\left(\int_0^1 F(x,\eta)dx - \int_0^1 F(x,\theta_1)dx\right)}{G(\eta) + H(\eta) - G(\theta_1) - H(\theta_1)} \right\}.$$

Now, we deduce the following straightforward consequence of Theorem 3.1.

Theorem 3.3. *Let f be a non-negative Carathéodory function in $[0,1] \times [0,+\infty[$. Assume that there exist positive constants $\theta_1 \geq k$, θ_4 and $\eta \geq 1$ with*

$$\theta_1 < \min\left\{ \eta^{\frac{p^+}{p^-}}, \sqrt[p^-]{\|\alpha\|_1 k\eta} \right\} \quad \textit{and} \quad \max\left\{ \eta, \sqrt[p^-]{\frac{2p^+\|\alpha\|_1}{p^-}} k\eta^{\frac{p^+}{p^-}} \right\} < \theta_4$$

such that

$$(\mathrm{A}_2) \qquad \max\left\{\frac{\int_0^1 F(x,\theta_1)dx}{\theta_1^{p^-}}, \frac{2\int_0^1 F(x,\theta_4)dx}{\theta_4^{p^-}}\right\} < \frac{p^-}{p^- + k^{p^-}p^+\|\alpha\|_1}\frac{\int_0^1 F(x,\eta)dx}{\eta^{p^+}}.$$

Then, for every

$$\lambda \in \left(\frac{\frac{p^- + k^{p^-}p^+\|\alpha\|_1}{p^+p^-k^{p^-}}\eta^{p^+}}{\int_0^1 F(x,\eta)dx}, \frac{1}{p^+k^{p^-}}\min\left\{\frac{\theta_1^{p^-}}{\int_0^1 F(x,\theta_1)dx}, \frac{\theta_4^{p^-}}{2\int_0^1 F(x,\theta_4)dx}\right\}\right)$$

and for every non-negative continuous functions $g, h : \mathbb{R} \to \mathbb{R}$ there exists $\delta'_{\lambda,g,h} > 0$ given by (3.4) *such that for each $\mu \in [0, \delta'_{\lambda,g,h})$, the problem $(P^f_{\lambda,\mu})$ possesses at least three non-negative weak solutions u_1, u_2 and u_3 such that*

$$\max_{x\in[0,1]} u_1(x) < \theta_1, \qquad \max_{x\in[0,1]} u_2(x) < \frac{1}{\sqrt[p^-]{2}}\theta_4 \quad \textit{and} \quad \max_{x\in[0,1]} u_3(x) < \theta_4.$$

Proof. Choose $\theta_2 = \frac{1}{\sqrt[p^-]{2}}\theta_4$ and $\theta_3 = \theta_4$. So, from (A_2) one has

$$\begin{aligned}\frac{\int_0^1 F(x,\theta_2)dx}{\theta_2^{p^-}} &= \frac{2\int_0^1 F\left(x, \frac{1}{\sqrt[p^-]{2}}\theta_4\right)dx}{\theta_4^{p^-}} \leq \frac{2\int_0^1 F(x,\theta_4)dx}{\theta_4^{p^-}} \\ &< \frac{p^-}{p^- + k^{p^-}p^+\|\alpha\|_1}\frac{\int_0^1 F(x,\eta)dx}{\eta^{p^+}}\end{aligned} \tag{3.5}$$

and

$$\frac{\int_0^1 F(x,\theta_3)dx}{\theta_3^{p^-} - \theta_2^{p^-}} = \frac{2\int_0^1 F(x,\theta_4)dx}{\theta_4^{p^-}} < \frac{p^-}{p^- + k^{p^-}p^+\|\alpha\|_1}\frac{\int_0^1 F(x,\eta)dx}{\eta^{p^+}}. \tag{3.6}$$

Moreover, taking into account that $\theta_1 < \eta^{\frac{p^+}{p^-}}$, by using (A_2) we have

$$\begin{aligned}&\frac{p^-}{k^{p^-}p^+\|\alpha\|_1}\frac{\int_0^1 F(x,\eta)dx - \int_0^1 F(x,\theta_1)dx}{\eta^{p^+}} \\ &> \frac{p^-}{k^{p^-}p^+\|\alpha\|_1}\frac{\int_0^1 F(x,\eta)dx}{\eta^{p^+}} - \frac{p^-}{k^{p^-}p^+\|\alpha\|_1}\frac{\int_0^1 F(x,\theta_1)dx}{\theta_1^{p^-}} \\ &> \frac{p^-}{k^{p^-}p^+\|\alpha\|_1}\left(\frac{\int_0^1 F(x,\eta)dx}{\eta^{p^+}} - \frac{p^-}{p^- + k^{p^-}p^+\|\alpha\|_1}\frac{\int_0^1 F(x,\eta)dx}{\eta^{p^+}}\right) \\ &= \frac{p^-}{p^- + k^{p^-}p^+\|\alpha\|_1}\frac{\int_0^1 F(x,\eta)dx}{\eta^{p^+}}.\end{aligned}$$

Hence, from (A_2), (3.5) and (3.6), it is easy to see that the assumption (A_1) of Theorem 3.1 is satisfied, and it follows the conclusion. □

We now present the following example to illustrate Theorem 3.3.

Example 3.4. We consider the problem

$$\begin{cases} -\left(|u'(x)|^{p(x)-2}u'(x)\right)' + \alpha(x)|u(x)|^{p(x)-2}u(x) = \lambda f(u(x)) & \text{in } (0,1), \\ |u'(0)|^{p(0)-2}u'(0) = -\mu g(u(0)), \\ |u'(1)|^{p(1)-2}u'(1) = \mu h(u(1)) \end{cases} \tag{3.7}$$

where $p(x) = x^2 + 4$ for every $x \in [0,1]$, $\alpha(x) = x^2 + 1$ for every $x \in [0,1]$ and

$$f(t) = \begin{cases} 7t^6, & \text{if } t \le 1, \\ 6t + e^{1-t}, & \text{if } t > 1. \end{cases}$$

We have

$$F(t) = \begin{cases} t^7, & \text{if } t \le 1, \\ 3t^2 - e^{1-t} - 1, & \text{if } t > 1. \end{cases}$$

By simple calculations, we obtain $k = \frac{3\sqrt[5]{16}}{2}$, $\alpha^- = 1$, $\alpha^+ = 2$, $p^- = 4$ and $p^+ = 5$. Taking $\theta_1 = \frac{1}{10}$, $\theta_4 = 10^4$ and $\eta = 1$, then all conditions in Theorem 3.3 are satisfied. Therefore, it follows that for each

$$\lambda \in \left(\frac{12 + 20k^4}{60k^4}, \frac{10^3}{5k^4}\right) \approx (0.33763, 4.299)$$

and for every non-negative continuous functions $g, h : \mathbb{R} \to \mathbb{R}$ there exists $\delta > 0$ such that, for each $\mu \in [0, \delta)$, the problem (3.7) possesses at least three non-negative weak solutions u_1, u_2 and u_3 such that

$$\max_{x\in[0,1]} u_1(x) < \frac{1}{10}, \; \max_{x\in[0,1]} u_2(x) < \frac{1}{\sqrt[4]{2}} 10^4 \text{ and } \max_{x\in[0,1]} u_3(x) < 10^4.$$

We want to point out a simple consequence of Theorem 3.3, in which the function f has separated variables.

Theorem 3.5. *Let $f_1 \in L^1([0,1])$ and $f_2 \in C(\mathbb{R})$ be two functions. Put $\tilde{F}(t) = \int_0^t f_2(\xi)d\xi$ for all $t \in \mathbb{R}$ and assume that there exist positive constants $\theta_1 \ge k$, θ_4 and $\eta \ge 1$ with*

$$\theta_1 < \min\left\{\eta^{\frac{p^+}{p^-}}, \sqrt[p^-]{\|\alpha\|_1 k\eta}\right\} \quad \text{and} \quad \max\left\{\eta, \sqrt[p^-]{\frac{2p^+\|\alpha\|_1}{p^-}k\eta^{\frac{p^+}{p^-}}}\right\} < \theta_4$$

such that

(A_3) *$f_1(x) \ge 0$ for each $x \in [0,1]$ and $f_2(t) \ge 0$ for each $t \in [0, +\infty[$;*

(A_4)
$$\max\left\{\frac{\sup_{|t|\le\theta_1} \tilde{F}(t)}{\theta_1^{p^-}}, \frac{2\sup_{|t|\le\theta_4} \tilde{F}(t)}{\theta_4^{p^-}}\right\} < \frac{p^-}{p^- + k^{p^-} p^+ \|\alpha\|_1} \frac{\tilde{F}(\eta)}{\eta^{p^+}}.$$

Then, for every

$$\lambda \in \left(\frac{\frac{p^- + k^{p^-} p^+\|\alpha\|_1}{p^+ p^- k^{p^-}} \eta^{p^+}}{\tilde{F}(\eta)\int_0^1 f_1(x)dx}, \frac{1}{p^+ k^{p^-} \int_0^1 f_1(x)dx} \min\left\{\frac{\theta_1^{p^-}}{\sup_{|t|\le\theta_1} \tilde{F}(t)}, \frac{\theta_4^{p^-}}{2\sup_{|t|\le\theta_4} \tilde{F}(t)}\right\}\right)$$

and for every non-negative continuous functions $g, h : \mathbb{R} \to \mathbb{R}$ there exists $\delta_\lambda > 0$ given by

$$\delta_\lambda = \min \left\{ \frac{1}{k^{p^-} p^+} \min \left\{ \frac{\theta_1^{p^-} - \lambda k^{p^-} p^+ \sup_{|t| \le \theta_1} \tilde{F}(t) \int_0^1 f_1(x)dx}{G(\theta_1) + H(\theta_1)}, \frac{\theta_4^{p^-} - 2\lambda k^{p^-} p^+ \sup_{|t| \le \frac{1}{\sqrt[p^-]{2}}\theta_4} \tilde{F}(t) \int_0^1 f_1(x)dx}{2(G(\frac{1}{\sqrt[p^-]{2}}\theta_4) + H(\frac{1}{\sqrt[p^-]{2}}\theta_4))}, \frac{\theta_4^{p^-} - 2\lambda k^{p^-} p^+ \sup_{|t| \le \theta_4} \tilde{F}(t) \int_0^1 f_1(x)dx}{2(G(\theta_4) + H(\theta_4))} \right\}, \frac{\frac{\eta^{p^+}}{p^-}\|\alpha\|_1 - \lambda \int_0^1 f_1(x)dx \left(\tilde{F}(\eta) - \sup_{|t| \le \theta_1} \tilde{F}(t) \right)}{[G(\eta) + H(\eta) - G(\theta_1) - H(\theta_1)]} \right\}$$

such that for every $\mu \in [0, \delta_\lambda)$ the problem

$$\begin{cases} -\left(|u'(x)|^{p(x)-2} u'(x) \right)' + \alpha(x)|u(x)|^{p(x)-2}u(x) = \lambda f_1(x) f_2(u) & \text{in } (0,1), \\ |u'(0)|^{p(0)-2}u'(0) = -\lambda g(u(0)), \\ |u'(1)|^{p(1)-2}u'(1) = \lambda h(u(1)) \end{cases}$$

possesses at least three non-negative weak solutions u_1, u_2 and u_3 such that

$$\max_{x \in [0,1]} u_1(x) < \theta_1, \qquad \max_{x \in [0,1]} u_2(x) < \frac{1}{\sqrt[p^-]{2}}\theta_4 \quad \text{and} \quad \max_{x \in [0,1]} u_3(x) < \theta_4.$$

Proof. Set $f(x,u) = f_1(x) f_2(u)$ for each $(x,u) \in [0,1] \times \mathbb{R}$. Since

$$F(x,t) = f_1(x)\tilde{F}(t),$$

from (A_4) we obtain (A_2). □

Next, we present a simple consequence of Theorem 3.3 in the case f does not depend upon x.

Theorem 3.6. *Assume that there exist positive constants $\theta_1 \ge k$, θ_4 and $\eta \ge 1$ with*

$$\theta_1 < \min \left\{ \eta^{\frac{p^+}{p^-}}, \sqrt[p^-]{\|\alpha\|_1 k\eta} \right\} \quad \text{and} \quad \max \left\{ \eta, \sqrt[p^-]{\frac{2p^+\|\alpha\|_1}{p^-} k\eta^{\frac{p^+}{p^-}}} \right\} < \theta_4$$

such that

(A_5) *$f(t) \ge 0$ for each $t \in [0, +\infty[$;*

(A_6)
$$\max \left\{ \frac{F(\theta_1)}{\theta_1^{p^-}}, \frac{2F(\theta_4)}{\theta_4^{p^-}} \right\} < \frac{p^-}{p^- + k^{p^-} p^+ \|\alpha\|_1} \frac{F(\eta)}{\eta^{p^+}}.$$

Then, for every

$$\lambda \in \left(\frac{\frac{p^- + k^{p^-} p^+ \|\alpha\|_1}{p^+ p^- k^{p^-}} \eta^{p^+}}{F(\eta)}, \frac{1}{p^+ k^{p^-}} \min \left\{ \frac{\theta_1^{p^-}}{F(\theta_1)}, \frac{\theta_4^{p^-}}{2F(\theta_4)} \right\} \right)$$

and for every non-negative continuous functions $g, h : \mathbb{R} \to \mathbb{R}$ there exists $\delta'_\lambda > 0$ given by

$$\delta'_\lambda = \min\left\{ \frac{1}{k^{p^-} p^+} \min\left\{ \frac{\theta_1^{p^-} - \lambda k^{p^-} p^+ F(\theta_1)}{G(\theta_1) + H(\theta_1)}, \right.\right.$$
$$\left.\left. \frac{\theta_4^{p^-} - 2\lambda k^{p^-} p^+ F\left(\frac{1}{\sqrt[p^-]{2}}\theta_4\right)}{2\left(G\left(\frac{1}{\sqrt[p^-]{2}}\theta_4\right) + H\left(\frac{1}{\sqrt[p^-]{2}}\theta_4\right)\right)}, \frac{\theta_4^{p^-} - 2\lambda k^{p^-} p^+ F(\theta_4)}{2(G(\theta_4) + H(\theta_4))} \right\}, \frac{\frac{\eta^{p^+}}{p^-}\|\alpha\|_1 - \lambda\left(F(\eta) - F(\theta_1)\right)}{G(\eta) + H(\eta) - G(\theta_1) - H(\theta_1)} \right\}$$

such that for every $\mu \in [0, \delta'_\lambda)$ the problem

$$\begin{cases} -\left(|u'(x)|^{p(x)-2}u'(x)\right)' + \alpha(x)|u(x)|^{p(x)-2}u(x) = \lambda f(u(x)) & \text{in } (0,1), \\ |u'(0)|^{p(0)-2}u'(0) = -\lambda g(u(0)), \\ |u'(1)|^{p(1)-2}u'(1) = \lambda h(u(1)) \end{cases}$$

possesses at least three non-negative weak solutions u_1, u_2 and u_3 such that

$$\max_{x\in[0,1]} u_1(x) < \theta_1, \qquad \max_{x\in[0,1]} u_2(x) < \frac{1}{\sqrt[p^-]{2}}\theta_4 \quad \text{and} \quad \max_{x\in[0,1]} u_3(x) < \theta_4.$$

The following result is a consequence of Theorem 3.3 when $\mu = 0$.

Theorem 3.7. *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that $tf(x,t) > 0$ for all $(x,t) \in [0,1] \times (\mathbb{R}\setminus\{0\})$. Assume that*

(A_7) $\lim_{t\to 0} \dfrac{f(x,t)}{|t|^{p^- -1}} = \lim_{|t|\to\infty} \dfrac{f(x,t)}{|t|^{p^- -1}} = 0.$

Then, for every $\lambda > \overline{\lambda}$ where

$$\overline{\lambda} = \frac{p^- + k^{p^-} p^+ \|\alpha\|_1}{p^+ p^- k^{p^-}}$$
$$\times \max\left\{ \inf_{\eta \geq 1} \frac{\eta^{p^+}}{\int_0^1 F(x,\eta)dx}; \inf_{0<\eta<1} \frac{\eta^{p^-}}{\int_0^1 F(x,\eta)dx}; \inf_{-1<\eta<0} \frac{(-\eta)^{p^-}}{\int_0^1 F(x,\eta)dx}; \inf_{\eta\leq -1} \frac{(-\eta)^{p^+}}{\int_0^1 F(x,\eta)dx} \right\},$$

the problem ($P^f_{\lambda,\mu}$), in the case $\mu = 0$ possesses at least four distinct non-trivial solutions.

Proof. Set

$$f_1(x,t) = \begin{cases} f(x,t), & \text{if } (x,t) \in [0,1] \times [0,+\infty), \\ 0, & \text{otherwise}, \end{cases}$$

and

$$f_2(x,t) = \begin{cases} -f(x,-t), & \text{if } (x,t) \in [0,1] \times [0,+\infty), \\ 0, & \text{otherwise}, \end{cases}$$

and define $F_1(x,t) := \int_0^t f_1(x,\xi)d\xi$ for every $(x,t) \in [0,1] \times \mathbb{R}$. Fix $\lambda > \lambda^*$, and let $\eta \geq 1$ such that

$$\lambda > \frac{\left(\frac{p^- + k^{p^-} p^+ \|\alpha\|_1}{p^+ p^- k^{p^-}}\right) \eta^{p^+}}{\int_0^1 F_1(x,\eta)dx}.$$

From
$$\lim_{t\to 0^+}\frac{f_1(x,t)}{t^{p^--1}}=\lim_{t\to+\infty}\frac{f_1(x,t)}{t^{p^--1}}=0,$$
there is $\theta_1\geq k$ such that
$$\theta_1<\min\left\{\eta^{\frac{p^+}{p^-}},\ \sqrt[p^-]{\|\alpha\|_1 k\eta}\right\}\ \text{and}\ \frac{\int_0^1 F_1(x,\theta_1)dx}{\theta_1^{p^-}}<\frac{1}{\lambda k^{p^-}p^+}$$
and there is $\theta_4>0$ such that
$$\max\left\{\eta,\ \sqrt[p^-]{\frac{2p^+\|\alpha\|_1}{p^-}k\eta^{\frac{p^+}{p^-}}}\right\}<\theta_4\ \text{and}\ \frac{\int_0^1 F_1(x,\theta_4)dx}{\theta_4^{p^-}}<\frac{1}{2\lambda k^{p^-}p^+}.$$
Then, (A_2) in Theorem 3.3 is satisfied,
$$\lambda\in\left(\frac{\frac{p^-+k^{p^-}p^+\|\alpha\|_1}{p^+p^-k^{p^-}}\eta^{p^+}}{\int_0^1 F_1(x,\eta)dx},\ \frac{1}{p^+k^{p^-}}\min\left\{\frac{\theta_1^{p^-}}{\int_0^1 F_1(x,\theta_1)dx},\ \frac{\theta_4^{p^-}}{2\int_0^1 F_1(x,\theta_4)dx}\right\}\right).$$

Hence, the problem $(P_\lambda^{f_1})$, in the case $\mu=0$ admits two positive solutions u_1, u_2, which are positive solutions of the problem $(P^f_{\lambda,\mu})$, in the case $\mu=0$. Next, arguing in the same way, from
$$\lim_{t\to 0^+}\frac{f_2(x,t)}{t^{p^--1}}=\lim_{t\to+\infty}\frac{f_2(x,t)}{t^{p^--1}}=0,$$
we ensure the existence of two positive solutions u_3, u_4 for the problem $(P_\lambda^{f_2})$, in the case $\mu=0$. Clearly, $-u_3$, $-u_4$ are negative solutions of the problem $(P^f_{\lambda,\mu})$, in the case $\mu=0$. □

Remark 3.8. We explicitly observe that in Theorem 3.7 no symmetric condition on f is assumed. However, whenever f is an odd continuous non-zero function such that $f(x,t)\geq 0$ for all $(x,t)\in[0,1]\times[0,+\infty)$, (A_7) can be replaced by

(A_8) $\lim_{t\to 0^+}\dfrac{f(x,t)}{t^{p^--1}}=\lim_{t\to\infty}\dfrac{f(x,t)}{t^{p^--1}}=0,$

ensuring the existence of at least four distinct non-trivial solutions the problem $(P^f_{\lambda,\mu})$, in the case $\mu=0$ for every $\lambda>\lambda^*$ where
$$\lambda^*=\inf_{\eta\geq 1}\frac{\frac{p^-+k^{p^-}p^+\|\alpha\|_1}{p^+p^-k^{p^-}}\eta^{p^+}}{\int_0^1 F(x,\eta)dx}.$$

We end this paper by presenting the following version of Theorem 3.1, in the case $p(x)=p$ for every $x\in[0,1]$ and $\alpha(x)=1$ for every $x\in[0,1]$.

Theorem 3.9. *Let f be a non-negative Carathéodory function in $[0,1]\times[0,+\infty[$. Let $p(x)=p>1$ for every $x\in[0,1]$. Assume that there exist positive constants $\theta_1\geq k$, θ_2, θ_3 and $\eta\geq 1$ with $\theta_1<k\eta$, $\max\{\eta,k\eta\}<\theta_2$ and $\theta_2<\theta_3$ such that*

(A_9) $\max\left\{\dfrac{\int_0^1 F(x,\theta_1)dx}{\theta_1^p},\ \dfrac{\int_0^1 F(x,\theta_2)dx}{\theta_2^p},\ \dfrac{\int_0^1 F(x,\theta_3)dx}{\theta_3^p-\theta_2^p}\right\}<\dfrac{1}{k^p}\dfrac{\int_0^1 F(x,\eta)dx-\int_0^1 F(x,\theta_1)dx}{\eta^p}$

where $k = 3\left[\frac{1}{2}\right]^{\frac{1}{p}}$. *Then, for every*

$$\lambda \in \left(\frac{\frac{\eta^p}{p}}{\int_0^1 F(x,\eta)dx - \int_0^1 F(x,\theta_1)dx}, \frac{1}{pk^p} \min\left\{ \frac{\theta_1^p}{\int_0^1 F(x,\theta_1)dx}, \frac{\theta_2^p}{\int_0^1 F(x,\theta_2)dx}, \frac{\theta_3^p - \theta_2^p}{\int_0^1 F(x,\theta_3)dx} \right\} \right)$$

for every non-negative continuous functions $g, h : \mathbb{R} \to \mathbb{R}$ *there exists* $\delta_{\lambda,g} > 0$ *given by*

$$\delta_{\lambda,g} = \min\left\{ \frac{1}{pk^p} \min\left\{ \frac{\theta_1^p - \lambda k^p p \int_0^1 F(x,\theta_1)dx}{G(\theta_1) + H(\theta_1)}, \frac{\theta_2^p - \lambda k^p p \int_0^1 F(x,\theta_2)dx}{G(\theta_2) + H(\theta_2)}, \right.\right.$$
$$\left.\left. \frac{(\theta_3^p - \theta_2^p) - \lambda k^p p \int_0^1 F(x,\theta_3)dx}{G(\theta_3) + H(\theta_3)} \right\}, \frac{\frac{\eta^p}{p} - \lambda\left(\int_0^1 F(x,\eta)dx - \int_0^1 F(x,\theta_1)dx\right)}{G(\eta) + H(\eta) - G(\theta_1) - H(\theta_1)} \right\}$$

such that for every $\mu \in [0, \delta_{\lambda,g})$ *the problem*

$$\begin{cases} -\left(|u'(x)|^{p-2}u'(x)\right)' + |u(x)|^{p-2}u(x) = \lambda f(x,u(x)) & \text{in } (0,1), \\ |u'(0)|^{p-2}u'(0) = -\lambda g(u(0)), \\ |u'(1)|^{p-2}u'(1) = \lambda h(u(1)) \end{cases}$$

possesses at least three non-negative weak solutions u_1, u_2, *and* u_3 *such that*

$$\max_{x\in[0,1]} u_1(x) < \theta_1, \qquad \max_{x\in[0,1]} u_2(x) < \theta_2 \quad \text{and} \quad \max_{x\in[0,1]} u_3(x) < \theta_3.$$

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