Electronic Journal of Qualitative Theory of Differential Equations
2018, No. **53**, 1–21; https://doi.org/10.14232/ejqtde.2018.1.53 www.math.u-szeged.hu/ejqtde/

Permanence in a class of delay differential equations with mixed monotonicity

Dedicated to Professor László Hatvani on the occasion of his 75th birthday

István Győri[1], **Ferenc Hartung**✉[1] and **Nahed A. Mohamady**[2]

[1]Department of Mathematics, University of Pannonia, Veszprém, Hungary
[2]Department of Mathematics, Faculty of Science, Benha University, Egypt

Received 5 December 2017, appeared 26 June 2018
Communicated by Tibor Krisztin

Abstract. In this paper we consider a class of delay differential equations of the form $\dot{x}(t) = \alpha(t)h(x(t-\tau), x(t-\sigma)) - \beta(t)f(x(t))$, where h is a mixed monotone function. Sufficient conditions are presented for the permanence of the positive solutions. Our results give also lower and upper estimates of the limit inferior and the limit superior of the solutions via a special solution of an associated nonlinear system of algebraic equations. The results are generated to a more general class of delay differential equations with mixed monotonicity.

Keywords: delay differential equations, mixed monotonicity, persistence, permanence.

2010 Mathematics Subject Classification: 34K05.

1 Introduction

In this manuscript we study persistence and permanence (see definitions in Section 2) of the scalar delay equation

$$\dot{x}(t) = \alpha(t)h(x(t-\tau), x(t-\sigma)) - \beta(t)f(x(t)), \qquad t \geq 0, \tag{1.1}$$

where we assume that $h : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a *mixed monotone function*, i.e., h is monotone increasing in its first argument, and it is monotone decreasing in its second argument. Here and later $\mathbb{R}_+$ denotes the set of nonnegative reals.

Equation (1.1) is a special case of a more general scalar equation

$$\dot{x}(t) = r(t)\big(g(t, x_t) - h(x(t))\big), \qquad t \geq 0, \tag{1.2}$$

where $x_t(\theta) = x(t+\theta)$, $-\delta \leq \theta \leq 0$ is the segment function. Equation (1.2) can be considered as a population model equation with delay in the birth term $r(t)g(t, x_t)$, and no delay in the

✉Corresponding author. Email: hartung.ferenc@uni-pannon.hu

self-inhibition term $r(t)h(x(t))$. The presence of time delays in the birth term of a population model was explained, e.g., in [5, 7, 11, 19–23]. The persistence and permanence of (1.2) was studied in [13] under conditions which do not include mixed monotone functions in the birth term. Recently, persistence of the equation

$$\dot{x}(t) = \sum_{k=1}^{n} \alpha_k(t)x(t-\tau_k(t)) - \beta(t)x^2(t), \qquad t \geq 0, \tag{1.3}$$

was proved in [2] and [8] under the assumption that the coefficients α_k and β are bounded below and above by positive constants. Note that this assumption was not needed in [13], and nor we will use it in this manuscript. Persistence and permanence of classes of nonlinear delay systems was investigated, e.g., in [9, 10, 15].

Delay differential equations with mixed monotone functions in the delay term were studied, e.g., in [3, 4, 12, 16, 17]. For such equations, persistence and permanence of solutions of a class of nonlinear differential equations with multiple delays were first studied in [3]. Our manuscript extends the results of [13] for the case when the birth term in the population model contains a mixed monotone function. We note that the class of equations determined by the conditions used in [3] and this paper are different, since, e.g., one of the conditions of [3] assumes that the term f in (1.1) should be estimated below and after by linear functions, and in our paper we study cases when the nonlinearity of f can be of higher order.

The structure of the paper is the following. In Section 2 we formulate our main results related to equation (1.1). Theorem 2.4 below presents sufficient conditions implying the permanence of equation (1.1), and also, we give a method how to find the lower and upper estimate of the limit inferior and limit superior of the solutions. We extend this result to a more general scalar equation with multiple delays in Theorem 2.6. Section 3 contains applications of the main results. The proofs of the main results are presented in Section 4.

2 Main results

Consider the scalar nonlinear delay equation

$$\dot{x}(t) = \alpha(t)h(x(t-\tau), x(t-\sigma)) - \beta(t)f(x(t)), \qquad t \geq 0, \tag{2.1}$$

with the associated initial condition

$$x(t) = \varphi(t), \qquad -\delta \leq t \leq 0. \tag{2.2}$$

Let $\tau, \sigma \geq 0$ and $\delta := \max\{\tau, \sigma\} > 0$ be fixed, and $C := C([-\delta, 0], \mathbb{R})$, $C_+ := \{\psi \in C([-\delta, 0], \mathbb{R}_+) : \psi(0) > 0\}$.

In the following, we state our conditions which will be needed in later results:

(H1) $\alpha, \beta \in C(\mathbb{R}_+, \mathbb{R}_+)$ with $\beta(t) > 0$ for $t > 0$, $\int_0^\infty \beta(s)ds = \infty$ and

$$0 < \inf_{t>0} \frac{\alpha(t)}{\beta(t)} \leq \sup_{t>0} \frac{\alpha(t)}{\beta(t)} < \infty; \tag{2.3}$$

(H2) $h \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$ with $h(x, y) > 0$ for $x > 0, y \geq 0$, and $h(x_1, y_1) \leq h(x_2, y_2)$ for $x_1, x_2, y_1, y_2 \in \mathbb{R}_+$ with $x_1 \leq x_2, y_1 \geq y_2$, i.e., h is a mixed monotone function;

(H3) $f \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfies $0 = f(0) < f(x_1) < f(x_2)$ for $0 < x_1 < x_2$,

$$\lim_{x \to 0^+} \frac{f(x)}{h(x,y)} = 0, \quad \text{for all fixed } y \geq 0, \tag{2.4}$$

and

$$\lim_{x \to \infty} \frac{f(x)}{h(x,y)} = \infty, \quad \text{for all fixed } y \geq 0; \tag{2.5}$$

(H4) $\frac{f(x)}{h(x,y)}$ is strictly monotone increasing in x for a fixed $y \geq 0$, and it is strictly monotone increasing in y for a fixed $x > 0$, and

$$\lim_{y \to \infty} \frac{f(x)}{h(x,y)} = \infty \quad \text{for all fixed } x > 0. \tag{2.6}$$

Note that our assumptions imply the existence of a solution of the initial value problem (IVP) (2.1)–(2.2) for any $\varphi \in C_+$, but the solution may not be unique. Any fixed solution corresponding to the initial function will be denoted by $x(\varphi)(t)$. Lemma 2.3 below will imply that any solution $x(\varphi)$ is defined on $\mathbb{R}_+$ under the conditions (H1)–(H3).

We say that equation (2.1) is *persistent* if for $\varphi \in C_+$ any solution $x(t) = x(\varphi)(t)$ of the IVP (2.1)–(2.2) satisfies

$$\liminf_{t \to \infty} x(t) > 0.$$

Equation (2.1) is *permanent* if there exist constants $0 < m \leq M$ such that

$$m \leq \liminf_{t \to \infty} x(t) \leq \limsup_{t \to \infty} x(t) \leq M$$

for any $\varphi \in C_+$ and for any corresponding solution $x(t) = x(\varphi)(t)$ of the IVP (2.1)–(2.2). Note that sometimes this property is called in the literature as uniform permanence.

Our first result shows that assumptions (H1) and (H2) imply that all solutions corresponding to an initial function $\varphi \in C_+$ remain positive for $t > 0$.

Lemma 2.1. *Assume that α, β satisfy (H1), h satisfies (H2) and $f \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfies $0 = f(0) < f(x_1) < f(x_2)$ for $0 < x_1 < x_2$. Then, for any $\varphi \in C_+$, we have that $x(\varphi)(t) > 0$ for $t \in \mathbb{R}_+$.*

In the proof of the next lemma and in our main theorem we need to solve a system of two nonlinear algebraic equations of the form

$$g(x,y) = \underline{m} \tag{2.7}$$
$$g(y,x) = \overline{m}, \tag{2.8}$$

where $\underline{m} > 0$, $\overline{m} > 0$, $g \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$. We say that (x,y) is a *positive solution* of the system (2.7)–(2.8) if $x > 0$ and $y > 0$. A pair of positive numbers (x^*, y^*) is called a *dominant positive solution* of the system (2.7)–(2.8) if (x^*, y^*) is a positive solution, and it satisfies $x^* \leq x$ and $y^* \geq y$ for any positive solutions (x,y) of the system (2.7)–(2.8).

The next lemma gives conditions under which the algebraic system (2.7)–(2.8) has a positive solution, and also shows the existence of a unique dominant positive solution. Also, we investigate solvability of a related system of inequalities.

Lemma 2.2. *Suppose* $\underline{m} > 0$, $\overline{m} > 0$, $g \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$ *is strictly monotone increasing in* x *and* y, *i.e., for any fixed* $x \geq 0$ *and* $y \geq 0$ *the functions* $g(x, \cdot)$ *and* $g(\cdot, y)$ *are strictly monotone increasing. Moreover, assume that* g *satisfies*

$$g(0, y) = 0 \quad \textit{for all fixed } y \geq 0, \tag{2.9}$$

$$\lim_{x\to\infty} g(x, y) = \infty \quad \textit{for all fixed } y \geq 0, \tag{2.10}$$

and

$$\lim_{y\to\infty} g(x, y) = \infty \quad \textit{for all fixed } x > 0. \tag{2.11}$$

Then

(i) the system (2.7)–(2.8) has at least one positive solution, and it has a dominant positive solution.

(ii) The system of inequalities

$$g(x, y) \geq \underline{m} \tag{2.12}$$

$$g(y, x) \leq \overline{m} \tag{2.13}$$

has infinitely many positive solutions (x, y), *i.e., solutions with* $x > 0$ *and* $y > 0$. *In addition, for any* $M > 0$ *and* $\varepsilon > 0$ *the system (2.12)–(2.13) has a positive solution* (x, y) *satisfying* $x \geq M$ *and* $y \leq \varepsilon$.

Moreover, for any positive solution (x, y) *of the system of inequalities (2.12)–(2.13) it follows*

$$x \geq x^* \quad \textit{and} \quad y \leq y^*,$$

where (x^*, y^*) *is the dominant positive solution of the system (2.7)–(2.8).*

Our next results shows that equation (2.1) is persistent under our conditions.

Lemma 2.3. *Assume that (H1)–(H3) are satisfied. Then, for any* $\varphi \in C_+$, *any solution* $x(\varphi)(t)$ *of the IVP (2.1)–(2.2) satisfies*

$$\inf_{t\geq 0} x(\varphi)(t) > 0 \tag{2.14}$$

and

$$\sup_{t\geq 0} x(\varphi)(t) < \infty. \tag{2.15}$$

We need the following notations in the next theorem:

$$\underline{m} := \liminf_{t\to\infty} \frac{\alpha(t)}{\beta(t)} > 0, \qquad \overline{m} := \limsup_{t\to\infty} \frac{\alpha(t)}{\beta(t)} < \infty. \tag{2.16}$$

Now we are ready to formulate our main result, which claims that equation (2.1) is persistent under our conditions. We refer to [16] and [17] for related results for delay equations with a single delay in the mixed monotone term and a linear function f.

Theorem 2.4. *Assume that (H1)–(H4) are satisfied. Let* $\underline{m}$ *and* $\overline{m}$ *be defined by (2.16). Then for any* $\varphi \in C_+$, *any solution* $x(t) = x(\varphi)(t)$ *of the IVP (2.1)–(2.2) satisfies*

$$\underline{x}^* \leq \liminf_{t\to\infty} x(t) \leq \limsup_{t\to\infty} x(t) \leq \overline{x}^*, \tag{2.17}$$

where $(\underline{x}^*, \overline{x}^*)$ *is the dominant positive solution of the algebraic system*

$$f(x) = \underline{m} h(x, y) \tag{2.18}$$

$$f(y) = \overline{m} h(y, x). \tag{2.19}$$

The next result claims that if, in addition to the conditions of Theorem 2.4, we assume that $\lim_{t\to\infty}\frac{\alpha(t)}{\beta(t)}$ exists, then all positive solutions converge to the same limit.

Corollary 2.5. *Assume that (H1)–(H4) are satisfied. Moreover, suppose that the limit*

$$m := \lim_{t\to\infty}\frac{\alpha(t)}{\beta(t)} > 0 \tag{2.20}$$

exists and the algebraic system

$$f(x) = mh(x,y) \tag{2.21}$$
$$f(y) = mh(y,x) \tag{2.22}$$

has a unique positive solution. Then for any $\varphi \in C_+$, any solution $x(t) = x(\varphi)(t)$ of the IVP (2.1)–(2.2) *satisfies*

$$\lim_{t\to\infty} x(t) = x^*, \tag{2.23}$$

where x^* is the unique positive solution of the algebraic equation

$$f(x) = mh(x,x). \tag{2.24}$$

Note that it is an interesting open question whether the statement remains true if the algebraic system (2.21)–(2.22) has several positive solutions.

Our results can be extended to equations of the form

$$\dot{x}(t) = H(t, x(t-\tau_1), \dots, x(t-\tau_k), x(t-\sigma_1), \dots, x(t-\sigma_\ell)) - F(t, x(t)), \qquad t \geq 0. \tag{2.25}$$

Here $\tau_1, \dots, \tau_k, \sigma_1, \dots, \sigma_\ell \geq 0$, and $\delta := \max\{\tau_1, \dots, \tau_k, \sigma_1, \dots, \sigma_\ell\} > 0$. We assume that

(H0) $H \in C(\mathbb{R}_+^{k+\ell+1}, \mathbb{R}_+)$, $F \in C(\mathbb{R}_+^2, \mathbb{R}_+)$, and there exist functions $H_0 \in C(\mathbb{R}_+^{k+\ell}, \mathbb{R}_+)$, $f \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\alpha_1, \alpha_2, \beta_1, \beta_2 \in C(\mathbb{R}_+, \mathbb{R}_+)$ such that

$$\begin{aligned} \alpha_1(t)H_0(u_1, \dots, u_k, v_1, \dots, v_\ell) &\leq H(t, u_1, \dots, u_k, v_1, \dots, v_\ell) \\ &\leq \alpha_2(t)H_0(u_1, \dots, u_k, v_1, \dots, v_\ell) \end{aligned} \tag{2.26}$$

for all $t, u_1, \dots, u_k, v_1, \dots, v_\ell \in \mathbb{R}_+$, and

$$\beta_1(t)f(u) \leq F(t,u) \leq \beta_2(t)f(u), \qquad t, u \in \mathbb{R}_+. \tag{2.27}$$

(H2*) $H_0(u_1, \dots, u_k, v_1, \dots, v_\ell) > 0$ for $u_1, \dots, u_k > 0$ and $v_1, \dots, v_\ell \geq 0$, and the function $H_0(u_1, \dots, u_k, v_1, \dots, v_\ell)$ is monotone increasing in the variables $u_1, \dots, u_k$, and it is monotone decreasing in the variables $v_1, \dots, v_\ell$.

We define the function

$$h(u,v) := H_0(u, \dots, u, v, \dots, v). \tag{2.28}$$

Then the following result is an easy consequence of the proof of the main Theorem 2.4.

Theorem 2.6. *Suppose (H0) holds, the functions α_1, α_2, β_1 and β_2 defined in (H0) satisfy (H1) with $\alpha = \alpha_i$ and $\beta = \beta_j$ ($i = 1,2$, $j = 1,2$), the function H_0 defined in (H0) satisfies (H2*), the functions f defined in (H0) and h defined by* (2.28) *satisfy (H3) and (H4). Let*

$$\underline{m} := \liminf_{t\to\infty}\frac{\alpha_1(t)}{\beta_2(t)} > 0, \qquad \overline{m} := \limsup_{t\to\infty}\frac{\alpha_2(t)}{\beta_1(t)} < \infty. \tag{2.29}$$

Then for any $\varphi \in C_+$, any solution $x(t) = x(\varphi)(t)$ of the IVP (2.25) *and* (2.2) *satisfies*

$$\underline{x}^* \leq \liminf_{t\to\infty} x(t) \leq \limsup_{t\to\infty} x(t) \leq \overline{x}^*, \tag{2.30}$$

where $(\underline{x}^*, \overline{x}^*)$ is the dominant positive solution of the algebraic system (2.18)–(2.19).

3 Applications and examples

In this section, we give some applications of our main results. We present two equations with multiple delays and mixed monotonicity. In the first model the associated nonlinear system of algebraic equations has a unique positive solution. In the second model, depending on some system parameters, the associated algebraic system has one, two or three positive solutions. We also present numerical examples to test the accuracy of the obtained estimates for the limit inferior and limit superior of the solutions.

First, we consider the nonlinear delay differential equation:

$$\dot{x}(t) = \alpha(t)\frac{\gamma_1 + \gamma_2 x(t-\tau)}{\gamma_3 + \gamma_4 x(t-\sigma)} - \beta(t)x^p(t), \qquad t \geq 0, \tag{3.1}$$

with the initial condition

$$x(t) = \varphi(t), \qquad -\delta \leq t \leq 0, \tag{3.2}$$

where $\delta := \max\{\tau, \sigma\} > 0$ and $\varphi \in C_+$.

We show that, under natural conditions, Theorem 2.4 can be applied to prove the boundedness of the positive solutions. Equation (3.1) can be written in the form (2.1) with $h(x,y) := \frac{\gamma_1+\gamma_2 x}{\gamma_3+\gamma_4 y}$ and $f(x) := x^p$. We suppose the functions α and β satisfy (H1). We assume $\gamma_1 \geq 0, \gamma_2 > 0, \gamma_3 > 0$ and $\gamma_4 > 0$. Then h satisfies (H2). We check that all conditions of Theorem 2.4 are satisfied for equation (3.1). Our assumptions shows that condition (H1) is satisfied. The function $h(x,y) = \frac{\gamma_1+\gamma_2 x}{\gamma_3+\gamma_4 y}$ clearly satisfies condition (H2). Suppose $p > 1$. For condition (H3), we have $f(0) = 0$ and f is strictly monotone increasing. Since

$$\lim_{x\to 0^+} \frac{f(x)}{h(x,y)} = \lim_{x\to 0^+} x^p \frac{\gamma_3 + \gamma_4 y}{\gamma_1 + \gamma_2 x} = 0 \quad \text{for all fixed } y \geq 0,$$

$$\lim_{x\to \infty} \frac{f(x)}{h(x,y)} = \lim_{x\to \infty} x^p \frac{\gamma_3 + \gamma_4 y}{\gamma_1 + \gamma_2 x} = \infty \quad \text{for all fixed } y \geq 0,$$

and

$$\lim_{y\to \infty} \frac{f(x)}{h(x,y)} = \lim_{y\to \infty} x^p \frac{\gamma_3 + \gamma_4 y}{\gamma_1 + \gamma_2 x} = \infty \quad \text{for all fixed } x > 0.$$

Then (2.4), (2.5) and (2.6) are satisfied. It is clear that $\frac{f(x)}{h(x,y)}$ is strictly monotone increasing in x for a fixed $y \geq 0$ and is strictly monotone increasing in y for a fixed $x > 0$. Thus all conditions of Theorem 2.4 are satisfied for equation (3.1). The associated algebraic system (2.18)–(2.19) has the form

$$x^p = \underline{m}\frac{\gamma_1 + \gamma_2 x}{\gamma_3 + \gamma_4 y} \tag{3.3}$$

$$y^p = \overline{m}\frac{\gamma_1 + \gamma_2 y}{\gamma_3 + \gamma_4 x}. \tag{3.4}$$

An application of Lemma 2.2 with $g(x,y) = x^p \frac{\gamma_3+\gamma_4 y}{\gamma_1+\gamma_2 x}$ gives immediately the existence of the dominant positive solution of the system (3.3)–(3.4) if $p > 1$, $\gamma_1 \geq 0$, $\gamma_2 > 0$, $\gamma_3 > 0$, $\gamma_4 > 0$, $\underline{m} > 0$ and $\overline{m} > 0$. In the next lemma we prove that if $p > 2$, then the positive solution of the system (3.3)–(3.4) is unique.

Lemma 3.1. *Assume that $p > 2$, $\gamma_1 \geq 0$, $\gamma_2 > 0$, $\gamma_3 > 0$, $\gamma_4 > 0$, $\underline{m} > 0$ and $\overline{m} > 0$. Then the system (3.3)–(3.4) has a unique positive solution (x^*, y^*).*

Proof. If (x, y) satisfies (3.3) and (3.4), then

$$\frac{x^p}{y^p} = \mu \frac{\gamma_1 + \gamma_2 x}{\gamma_3 + \gamma_4 y} \frac{\gamma_3 + \gamma_4 x}{\gamma_1 + \gamma_2 y},$$

where $\mu := \frac{\underline{m}}{\overline{m}}$. Thus

$$\frac{(\gamma_1 + \gamma_2 y)(\gamma_3 + \gamma_4 y)}{y^p} = \mu \frac{(\gamma_1 + \gamma_2 x)(\gamma_3 + \gamma_4 x)}{x^p}. \tag{3.5}$$

Define the function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ by $\omega(u) := \gamma_1\gamma_3 u^{-p} + (\gamma_1\gamma_4 + \gamma_2\gamma_3)u^{1-p} + \gamma_2\gamma_4 u^{2-p}$. Then ω is strictly decreasing, and we can see that $\lim_{u\to 0^+} \omega(u) = \infty$ and $\lim_{u\to\infty} \omega(u) = 0$. Thus ω is invertible and its inverse satisfies $\lim_{u\to 0^+} \omega^{-1}(u) = \infty$ and $\lim_{u\to\infty} \omega^{-1}(u) = 0$. From (3.5), we get

$$\omega(y) = \mu\omega(x)$$

or equivalently

$$y = \omega^{-1}(\mu\omega(x)). \tag{3.6}$$

Equation (3.3) is equivalent to

$$x^p(\gamma_3 + \gamma_4 y) = \underline{m}(\gamma_1 + \gamma_2 x),$$

or

$$\gamma_3 + \gamma_4\omega^{-1}(\mu\omega(x)) = \gamma_1\underline{m}x^{-p} + \gamma_2\underline{m}x^{1-p}. \tag{3.7}$$

Since $p > 2$, then the left hand side of (3.7) is strictly increasing on $(0, \infty)$ with

$$\lim_{x\to 0^+} (\gamma_3 + \gamma_4\omega^{-1}(\mu\omega(x))) = \gamma_3 > 0, \qquad \lim_{x\to\infty} (\gamma_3 + \gamma_4\omega^{-1}(\mu\omega(x))) = \infty,$$

and also the right hand side of (3.7) is strictly decreasing with

$$\lim_{x\to\infty} (\gamma_1\underline{m}x^{-p} + \gamma_2\underline{m}x^{1-p}) = 0, \qquad \lim_{x\to 0^+} (\gamma_1\underline{m}x^{-p} + \gamma_2\underline{m}x^{1-p}) = \infty.$$

We conclude that equation (3.7) has a unique positive solution x^*. Hence the system (3.3)–(3.4) has a unique positive solution (x^*, y^*), where y^* is defined by (3.6). □

The above discussion and Theorem 2.4 imply the following result for equation (3.1).

Corollary 3.2. *Assume the functions α and β satisfy (H1), $p > 2$, $\gamma_1 \geq 0, \gamma_2 > 0, \gamma_3 > 0$ and $\gamma_4 > 0$. Then, for any initial function $\varphi \in C_+$, the solution $x(\varphi)(t)$ of the IVP (3.1)–(3.2) satisfies*

$$\underline{x}^* \leq \liminf_{t\to\infty} x(\varphi)(t) \leq \limsup_{t\to\infty} x(\varphi)(t) \leq \overline{x}^*,$$

where $(\underline{x}^*, \overline{x}^*)$ is the unique positive solution of the algebraic system (3.3)–(3.4), where $\underline{m} := \liminf_{t\to\infty} \frac{\alpha(t)}{\beta(t)}$ and $\overline{m} := \limsup_{t\to\infty} \frac{\alpha(t)}{\beta(t)}$.

Example 3.3. Consider the differential equation

$$\dot{x}(t) = \sqrt{t}(2 + \cos t)\frac{1 + x(t-2)}{1 + 2x(t-1)} - \sqrt{t}x^3(t), \qquad t \geq 0. \tag{3.8}$$

Note that the conditions of Corollary 3.2 are satisfied for (3.8). Therefore, the associated algebraic system

$$x^3 = \underline{m}\frac{1+x}{1+2y} \tag{3.9}$$

$$y^3 = \overline{m}\frac{1+y}{1+2x}, \tag{3.10}$$

where $\underline{m} := \liminf_{t\to\infty} \frac{\alpha(t)}{\beta(t)} = 1$ and $\overline{m} := \limsup_{t\to\infty} \frac{\alpha(t)}{\beta(t)} = 3$ has a unique positive solution. We solve the system (3.9)–(3.10) numerically by the fixed point iteration

$$x_{k+1} = \sqrt[3]{\frac{1+x_k}{1+2y_k}} \tag{3.11}$$

$$y_{k+1} = \sqrt[3]{\frac{3(1+y_k)}{1+2x_k}} \tag{3.12}$$

starting from the initial value $(x_0, y_0) = (0, 0)$. The first 14 terms of this sequence are displayed in Table 3.1. We can observe that the sequence is convergent, and its limit is equal to $(\underline{x}^*, \overline{x}^*) \approx (0.77329, 1.41745)$. Therefore Corollary 3.2 yields

$$\underline{x}^* \le \liminf_{t\to\infty} x(\varphi)(t) \le \limsup_{t\to\infty} x(\varphi)(t) \le \overline{x}^*. \tag{3.13}$$

We plotted the numerical solution of equation (3.8) in Figure 3.1 corresponding to the constant initial functions $\varphi(t) = 0.2, \varphi(t) = 1$ and $\varphi(t) = 1.8$, respectively. The horizontal lines in Figure 3.1 correspond to the lower and upper bounds 0.77329 and 1.41745, respectively. We also observe that the difference of any two solutions converges to zero, i.e., the solutions are asymptotically equivalent. The numerical results demonstrate the theoretical bounds (3.13).

k	x_k	y_k
0	0.00000	0.00000
1	1.00000	1.44225
2	0.80149	1.34668
3	0.78717	1.39327
4	0.77859	1.40761
5	0.77539	1.41356
6	0.77412	1.41591
7	0.77362	1.41684
8	0.77342	1.41721
9	0.77334	1.41735
10	0.77331	1.41741
11	0.77330	1.41743
12	0.77329	1.41744
13	0.77329	1.41745
14	0.77329	1.41745

Table 3.1: Fixed point iteration defined by (3.11)–(3.12)

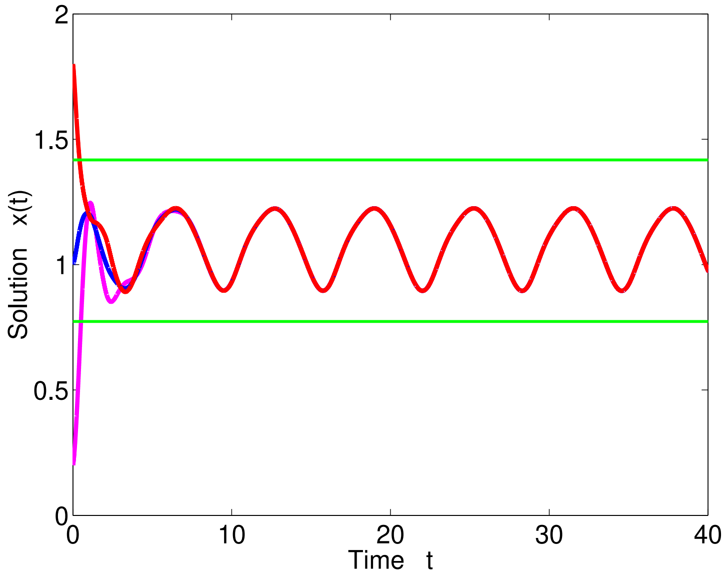

Figure 3.1: Numerical solution of equation (3.8).

Next, we consider the nonlinear delay differential equation:

$$\dot{x}(t) = \alpha(t)x(t-\tau)e^{-\gamma x(t-\sigma)} - \beta(t)x^p(t), \qquad t \geq 0, \tag{3.14}$$

with the initial condition (3.2).

Equation (3.14) can be written in the form (2.1) with $h(x,y) := xe^{-\gamma y}$ and $f(x) := x^p$. We assume that α and β satisfy (H1), $\gamma > 0$ and $p > 1$. The function h clearly satisfies condition (H2). For condition (H3), we have $f(0) = 0$ and f is strictly monotone increasing. We have

$$\lim_{x\to 0^+} \frac{f(x)}{h(x,y)} = \lim_{x\to 0^+} \frac{x^p}{xe^{-\gamma y}} = \lim_{x\to 0^+} x^{p-1}e^{\gamma y} = 0 \quad \text{for all fixed } y \geq 0,$$

$$\lim_{x\to\infty} \frac{f(x)}{h(x,y)} = \lim_{x\to\infty} x^{p-1}e^{\gamma y} = \infty \quad \text{for all fixed } y \geq 0,$$

and

$$\lim_{y\to\infty} \frac{f(x)}{h(x,y)} = \lim_{y\to\infty} x^{p-1}e^{\gamma y} = \infty \qquad \text{for all fixed } x > 0.$$

Then (2.4), (2.5) and (2.6) are satisfied. Also it is clear that $\frac{f(x)}{h(x,y)} = x^{p-1}e^{\gamma y}$ is strictly monotone increasing in x for a fixed $y \geq 0$, and it is strictly monotone increasing in y for a fixed $x > 0$. Thus all conditions of Theorem 2.4 are satisfied for equation (3.14).

The system of algebraic equations (2.18)–(2.19) related to (3.14) equals to

$$x^p = \underline{m}xe^{-\gamma y} \tag{3.15}$$
$$y^p = \overline{m}ye^{-\gamma x}. \tag{3.16}$$

Lemma 2.2 with $g(x,y) = x^{p-1}e^{\gamma y}$ implies that (3.15)–(3.16) has a dominant positive solution under the above conditions. The following lemma gives necessary and sufficient conditions under which the system has either one, two or three positive solutions.

Lemma 3.4. *Assume $p > 1$, $\gamma > 0$, $\underline{m} > 0$ and $\overline{m} > 0$. For $\overline{m} > \left(\frac{p-1}{\gamma}e\right)^{p-1}$ let $0 < x_1 < x_2$ be the roots of the equation*

$$xe^{-\frac{\gamma}{p-1}x} = \left(\frac{p-1}{\gamma}\right)^2 \overline{m}^{\frac{1}{1-p}}.$$

Then

(A) the system (3.15)–(3.16) has a unique positive solution (x^*, y^*) if and only if one of the following conditions is satisfied:

(A1) $0 < \overline{m} \le \left(\frac{p-1}{\gamma}e\right)^{p-1}$;

(A2) $\overline{m} > \left(\frac{p-1}{\gamma}e\right)^{p-1}$ and $\overline{m}^{\frac{1}{p-1}}e^{-\frac{\gamma}{p-1}x_1} + \frac{p-1}{\gamma}\log x_1 < \frac{1}{\gamma}\log \underline{m}$;

(A3) $\overline{m} > \left(\frac{p-1}{\gamma}e\right)^{p-1}$ and $\overline{m}^{\frac{1}{p-1}}e^{-\frac{\gamma}{p-1}x_2} + \frac{p-1}{\gamma}\log x_2 > \frac{1}{\gamma}\log \underline{m}$.

(B) The system (3.15)–(3.16) has exactly two positive solutions if and only if one of the following conditions is satisfied:

(B1) $\overline{m} > \left(\frac{p-1}{\gamma}e\right)^{p-1}$ and $\overline{m}^{\frac{1}{p-1}}e^{-\frac{\gamma}{p-1}x_1} + \frac{p-1}{\gamma}\log x_1 = \frac{1}{\gamma}\log \underline{m}$;

(B2) $\overline{m} > \left(\frac{p-1}{\gamma}e\right)^{p-1}$ and $\overline{m}^{\frac{1}{p-1}}e^{-\frac{\gamma}{p-1}x_2} + \frac{p-1}{\gamma}\log x_2 = \frac{1}{\gamma}\log \underline{m}$.

(C) The system (3.15)–(3.16) has exactly three positive solutions if and only if

$$\overline{m}^{\frac{1}{p-1}}e^{-\frac{\gamma}{p-1}x_2} + \frac{p-1}{\gamma}\log x_2 < \frac{1}{\gamma}\log \underline{m} < \overline{m}^{\frac{1}{p-1}}e^{-\frac{\gamma}{p-1}x_1} + \frac{p-1}{\gamma}\log x_1.$$

Proof. Since we are looking for positive solutions, system (3.15)–(3.16) is equivalent to

$$x^{p-1} = \underline{m}e^{-\gamma y}, \tag{3.17}$$
$$y^{p-1} = \overline{m}e^{-\gamma x}. \tag{3.18}$$

Equation (3.17) yields

$$y = \frac{1}{\gamma}\log \underline{m} - \frac{p-1}{\gamma}\log x. \tag{3.19}$$

The vector $(x, y) = (x^*, y^*)$ is a positive solution of the system (3.17)–(3.18) if and only if $x = x^*$ is a positive solution of the equation

$$\overline{m}^{\frac{1}{p-1}}e^{-\frac{\gamma}{p-1}x} + \frac{p-1}{\gamma}\log x - \frac{1}{\gamma}\log \underline{m} = 0, \tag{3.20}$$

and $y = y^*$ is defined by (3.19). We define the function $\theta : \mathbb{R}_+ \to \mathbb{R}$ by

$$\theta(u) := \overline{m}^{\frac{1}{p-1}}e^{-\frac{\gamma}{p-1}u} + \frac{p-1}{\gamma}\log u.$$

Then (3.20) can be written in the form

$$\theta(x) = \frac{1}{\gamma}\log \underline{m}. \tag{3.21}$$

Since $\lim_{u\to 0+}\theta(u) = -\infty$ and $\lim_{u\to\infty}\theta(u) = \infty$, it follows that equation (3.21) has at least one positive solution. We have

$$\theta'(x) = -\frac{\gamma}{p-1}\overline{m}^{\frac{1}{p-1}}e^{-\frac{\gamma}{p-1}u} + \frac{p-1}{\gamma}\frac{1}{u} = \frac{\gamma}{p-1}\overline{m}^{\frac{1}{p-1}}\frac{1}{u}\left(\left(\frac{p-1}{\gamma}\right)^2\overline{m}^{\frac{1}{1-p}} - \eta(u)\right),$$

where

$$\eta(u) := ue^{-\frac{\gamma}{p-1}u}.$$

It is easy to check that η is unimodal with the properties that $\eta(0) = 0$, $\lim_{u\to\infty}\eta(u) = 0$ and its maximum is

$$\eta_{\max} := \frac{p-1}{\gamma e},$$

which is taken at the point $u_{\max} = \frac{p-1}{\gamma}$. Therefore, if

$$\left(\frac{p-1}{\gamma}\right)^2\overline{m}^{\frac{1}{1-p}} \geq \eta_{\max}$$

(or equivalently, (A1) holds), then $\theta'(u) > 0$ for all $u > 0$ (except possibly at $u = u_{\max}$), and hence equation (3.21) has exactly one positive solution.

Suppose for the rest of the proof that $\left(\frac{p-1}{\gamma}\right)^2\overline{m}^{\frac{1}{1-p}} < \eta_{max}$, i.e., $\overline{m} > \left(\frac{p-1}{\gamma}e\right)^{p-1}$. Then the graph of η has two intersections $0 < x_1 < x_2$ with the graph of the constant function $\left(\frac{p-1}{\gamma}\right)^2\overline{m}^{\frac{1}{1-p}}$, and hence θ is monotone increasing on the intervals $(-\infty, x_1]$ and $[x_2,\infty)$, and it is decreasing on $[x_1, x_2]$. Hence equation (3.21) has also one positive solution if $\theta(x_1) < \frac{1}{\gamma}\log\underline{m}$ (condition (A2)) or $\theta(x_2) > \frac{1}{\gamma}\log\underline{m}$ (condition (A3)). Similarly, if $\theta(x_1) = \frac{1}{\gamma}\log\underline{m}$ (condition (B1)) or $\theta(x_2) = \frac{1}{\gamma}\log\underline{m}$ (condition (B2)), then equation (3.21) has two positive solutions. Finally, if $\theta(x_2) < \frac{1}{\gamma}\log\underline{m} < \theta(x_1)$ (condition (C)), then equation (3.21) has three positive solutions. □

Remark 3.5. The proof of the previous lemma yields that in the cases when the system (3.15)–(3.16) has more than one solution, its dominant positive solution (x^*, y^*) satisfies the inequality $x^* \leq x_1$, and for other solutions (x, y) of the system (3.15)–(3.16) it follows $x > x_1$.

We have therefore the following corollary of our main Theorem 2.4.

Corollary 3.6. *Assume that $p > 1$, $\gamma > 0$ and α, β satisfy (H1). Then, for any initial function $\varphi \in C_+$, the solution $x(\varphi)(t)$ of the IVP (3.14) and (3.2) satisfies*

$$\underline{x}^* \leq \liminf_{t\to\infty} x(\varphi)(t) \leq \limsup_{t\to\infty} x(\varphi)(t) \leq \overline{x}^*,$$

where $(\underline{x}^*, \overline{x}^*)$ is the dominant positive solution of the algebraic system (3.15)–(3.16) with $\underline{m} := \liminf_{t\to\infty}\frac{\alpha(t)}{\beta(t)}$ and $\overline{m} := \limsup_{t\to\infty}\frac{\alpha(t)}{\beta(t)}$.

Example 3.7. Consider the following nonlinear differential equation,

$$\dot{x}(t) = \sqrt[4]{t}e^{0.25\sin t}x(t-e)e^{-x(t-1)} - \sqrt[4]{t}x^2(t), \qquad t \geq 0. \tag{3.22}$$

Note that the conditions of Corollary 3.6 are satisfied for (3.22). We have $\underline{m} := \liminf_{t\to\infty} \frac{\alpha(t)}{\beta(t)} = e^{-0.25} \approx 0.7788$ and $\overline{m} := \limsup_{t\to\infty} \frac{\alpha(t)}{\beta(t)} = e^{0.25} \approx 1.2840 < e$, so condition (A1) of Lemma 3.4 holds. Therefore, the algebraic system

$$x^2 = \underline{m}xe^{-y} \tag{3.23}$$
$$y^2 = \overline{m}ye^{-x} \tag{3.24}$$

has a unique positive solution $(\underline{x}^*, \overline{x}^*)$. Then Corollary 3.6 implies

$$\underline{x}^* \le \liminf_{t\to\infty} x(\varphi)(t) \le \limsup_{t\to\infty} x(\varphi)(t) \le \overline{x}^* \tag{3.25}$$

for any $\varphi \in C_+$.

We solve the system (3.23)–(3.24) numerically by the fixed point iteration

$$x_{k+1} = e^{-0.25}e^{-y_k} \tag{3.26}$$
$$y_{k+1} = e^{0.25}e^{-x_k}. \tag{3.27}$$

We computed the sequence defined by the iteration (3.26)–(3.27) starting from the initial value $(x_0, y_0) = (0.001, 0.002)$. The first 23 terms of this sequence are displayed in Table 3.2. We can observe that the sequence is convergent, and its limit is $(\underline{x}^*, \overline{x}^*) \approx (0.30115, 0.95013)$. Therefore Corollary 3.6 yields (3.25) with $\underline{x}^* \approx 0.30115$ and $\overline{x}^* \approx 0.95013$. We plotted the numerical solution of equation (3.22) in Figure 3.2 corresponding to the constant initial functions $\varphi(t) = 0.2$, $\varphi(t) = 0.7$ and $\varphi(t) = 1.4$. The horizontal lines in Figure 3.2 correspond to the lower and upper bounds 0.30115 and 0.95013, respectively. The numerical results demonstrate the theoretical bounds (3.25). □

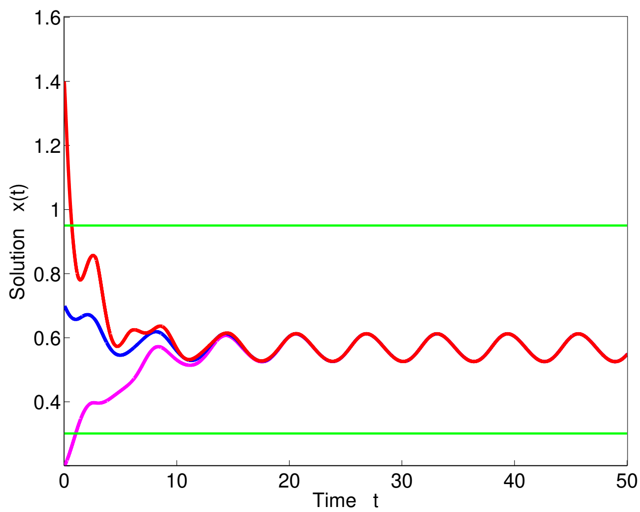

Figure 3.2: Numerical solution of equation (3.22).

4 Proofs

In this section we present the proofs of our main results. First we recall the next lemma from [13], which is needed in our proofs later.

k	x_k	y_k
0	0.00100	0.00200
1	0.77724	1.28274
2	0.21594	0.59023
3	0.43161	1.03464
4	0.27675	0.83393
5	0.33827	0.97361
6	0.29417	0.91552
7	0.31176	0.95679
8	0.29915	0.94011
9	0.30419	0.95203
10	0.30058	0.94725
11	0.30202	0.95068
12	0.30099	0.94931
13	0.30140	0.95029
14	0.30111	0.94990
15	0.30123	0.95018
16	0.30114	0.95006
17	0.30117	0.95014
18	0.30115	0.95011
19	0.30116	0.95014
20	0.30115	0.95013
21	0.30116	0.95013
22	0.30115	0.95013
23	0.30115	0.95013

Table 3.2: Fixed point iteration (3.26)–(3.27).

Lemma 4.1. *Consider the ordinary differential equation*

$$\dot{y}(t) = \beta(t)\Big(c - f(y(t))\Big), \qquad t \geq T \geq 0 \tag{4.1}$$

with the initial condition

$$y(T) = y^*, \tag{4.2}$$

where $c \geq 0$, *and* $\beta \in C(\mathbb{R}_+, \mathbb{R}_+)$ *with* $\beta(t) > 0$ *for* $t > 0$, $\int_0^\infty \beta(s)ds = \infty$ *and* $f \in C(\mathbb{R}, \mathbb{R}_+)$ *satisfies* $0 = f(0) < f(x_1) < f(x_2)$ *for* $0 < x_1 < x_2$. *Then for any solution* $y(T, y^*, c)(t)$ *of the IVP* (4.1)–(4.2) *we have*

(i) $c > 0$ *and* $0 < y^* < f^{-1}(c)$ *yield that*

$$0 < y(T, y^*, c)(t) < f^{-1}(c), \qquad \dot{y}(T, y^*, c)(t) > 0, \qquad t \geq T$$

and

$$\lim_{t\to\infty} y(T, y^*, c)(t) = f^{-1}(c);$$

(ii) $y^* = f^{-1}(c)$ *yields that* $y(T, y^*, c)(t) = f^{-1}(c), \;\; t \geq T;$

(iii) $c \geq 0$ *and* $y^* > f^{-1}(c)$ *yield that*

$$y(T, y^*, c)(t) > f^{-1}(c), \qquad \dot{y}(T, y^*, c)(t) < 0, \quad t \geq T$$

and

$$\lim_{t\to\infty} y(T, y^*, c)(t) = f^{-1}(c).$$

Proof of Lemma 2.1. Let $x(t) = x(\varphi)(t)$ be any solution of the IVP (2.1)–(2.2). Since $x(0) = \varphi(0) > 0$, there exists a $\xi > 0$ such that $x(t) > 0$ for $0 \leq t < \xi$. If $\xi = \infty$, then the proof is completed. Otherwise, there exists a $t_1 \in (0, \infty)$ such that $x(t) > 0$ for $0 \leq t < t_1$ and $x(t_1) = 0$. Since $\alpha(t) \geq 0$ for $t \geq 0$ and $h(u, v) \geq 0$ for any $(u, v) \in \mathbb{R}_+ \times \mathbb{R}_+$, we have from (2.1) that

$$\dot{x}(t) \geq -\beta(t) f(x(t)), \qquad 0 \leq t \leq t_1. \tag{4.3}$$

But from the comparison theorem of the differential equations (see, e.g., [6]), we have

$$x(t) \geq y(t), \quad 0 \leq t \leq t_1,$$

where $y(t) = y(0, \varphi(0), 0)(t)$ is the positive solution of (4.1), with $c = 0$, $T = 0$ and with the initial condition

$$y(0) = x(0) = \varphi(0) > 0.$$

Then at $t = t_1$ we get $x(t_1) \geq y(t_1) > 0$, which is a contradiction with our assumption that $x(t_1) = 0$. Hence $x(t) > 0$ for $t \in \mathbb{R}_+$. □

Proof of Lemma 2.2. First, we prove part (i). Consider first equation (2.8) and fix an $x \geq 0$. Since $g(0, x) = 0$, $\lim_{y\to\infty} g(y, x) = \infty$, and $g(\cdot, x)$ is a strictly monotone increasing continuous function, there exists a unique $y > 0$ such that $g(y, x) = \overline{m}$. Thus there exists a function $y = s(x)$ such that $s : \mathbb{R}_+ \to (0, \infty)$ satisfies

$$g(s(x), x) = \overline{m}, \qquad x \geq 0. \tag{4.4}$$

We claim that s satisfies the following properties:

(i) s is strictly monotone decreasing,

(ii) s is continuous on $\mathbb{R}_+$,

(iii) $\lim_{x\to\infty} s(x) = 0$.

To prove (i), let $0 \leq x_1 < x_2$, then we get

$$\overline{m} = g(s(x_1), x_1) < g(s(x_1), x_2).$$

But

$$\overline{m} = g(s(x_2), x_2) < g(s(x_1), x_2),$$

thus the strict monotonicity of g in its first variable yields $s(x_2) < s(x_1)$, and hence $s(x)$ is strictly monotone decreasing. To show (ii), let x_n be a strictly monotone increasing sequence of nonnegative numbers such that $\lim_{n\to\infty} x_n = \overline{x}$. Then $s(x_n)$ is a monotone decreasing sequence by part (i), so it has a limit, say $\lim_{n\to\infty} s(x_n) = A$. Since we have $x_n < \overline{x}$, then $s(x_n) > s(\overline{x})$, and so $A \geq s(\overline{x})$. Suppose $s(\overline{x}) < A$, we have $\overline{m} = g(s(x_n), x_n)$ for all n. Taking the limit as $n \to \infty$ gives

$$\overline{m} = g(A, \overline{x}) > g(s(\overline{x}), \overline{x}) = \overline{m}.$$

This contradiction yields that $\lim_{n\to\infty} s(x_n) = s(\overline{x})$. Similarly for any monotone decreasing sequence x_n converging to $\overline{x}$, we can show that $\lim_{n\to\infty} s(x_n) = s(\overline{x})$. Hence $s(x)$ is continuous on $\mathbb{R}_+$. To prove (iii), suppose that $s(x) \geq B > 0$ for $x > 0$. Then, the monotonicity of g and (2.11) yield

$$\overline{m} = g(s(x), x) \geq g(B, x) \to \infty \quad \text{as } x \to \infty,$$

which is a contradiction, and hence $\lim_{x\to\infty} s(x) = 0$.

Now, consider equation (2.7) and fix an $x \geq 0$. Then $g(x, \cdot)$ is a strictly monotone increasing continuous function which tends to ∞ at ∞ by (2.11). Thus equation (2.7) has a unique solution y if and only if $g(x, 0) \leq \underline{m}$. But $g(\cdot, 0)$ is a strictly monotone increasing function which tends to ∞ at ∞ and $g(0, 0) = 0$. Therefore there exists a positive constant $x_{\underline{m}}$ such that $g(x, 0) \leq \underline{m}$ for $x \in (0, x_{\underline{m}}]$. Hence for any $x \in (0, x_{\underline{m}}]$ there exists a unique $y \geq 0$ such that (2.7) holds. This implies that there exists a function $y = r(x)$ satisfying

$$g(x, r(x)) = \underline{m}, \qquad x \in (0, x_{\underline{m}}]. \tag{4.5}$$

We claim that r satisfies the following properties:

(i) r is strictly monotone decreasing,

(ii) r is continuous on $(0, x_{\underline{m}}]$,

(iii) $r(x_{\underline{m}}) = 0$,

(iv) $\lim_{x\to 0^+} r(x) = \infty$.

The proofs of (i) and (ii) are similar to that for the function s, so they are omitted here. Part (iii) is clear from the above definitions of r. To prove (iv), suppose that $r(x) \leq C$, where $C > 0$. Then, using (2.9) and (4.5),

$$\underline{m} = g(x, r(x)) \leq g(x, C) \to 0 \quad \text{as } x \to 0,$$

which is a contradiction, since $\underline{m} > 0$, and hence (iv) is proved.

The above properties of the functions r and s imply that their graphs have at least one intersection, i.e., there exists an x such that $r(x) = s(x)$. See Figure 4.1 for a possible situation of the graphs. Hence the system (2.7) and (2.8) has at least one positive solution. Let $x^* = \inf\{u : (u, v) \text{ is a positive solution of (2.7) and (2.8)}\}$, and let $y^* = s(x^*)$. Clearly, (x^*, y^*) is a positive solution of (2.7) and (2.8). The monotonicity of r and s imply that $y^* \geq v$ for any solution (u, v) of the system (2.7) and (2.8), so (x^*, y^*) is the dominant solution of (2.7) and (2.8).

Next, we prove part (ii). Consider first inequality (2.13) and we claim that (2.13) is satisfied if and only if $y \leq s(x)$. To prove this claim, suppose that $y \leq s(x)$. Then

$$g(y, x) \leq g(s(x), x) = \overline{m}.$$

On the other hand if $y > s(x)$, then

$$g(y, x) > g(s(x), x) = \overline{m}.$$

Thus our claim is proved.

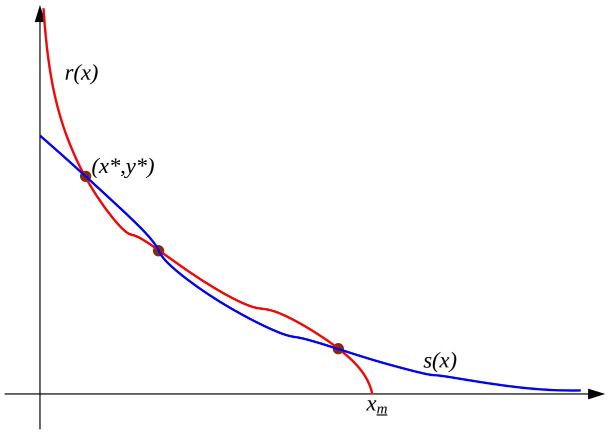

Figure 4.1: A possible graphs of $s(x)$ and $r(x)$.

Now, we consider inequality (2.12) and we claim that that (x, y) is a positive solution of (2.12) if and only if $[y \geq r(x)$ and $x \in (0, x_{\underline{m}}]]$ or $[x > x_{\underline{m}}$ and $y > 0]$. To prove the claim, suppose that $y \geq r(x)$ and $x \in (0, x_{\underline{m}}]$, then

$$g(x, y) \geq g(x, r(x)) = \underline{m}.$$

On the other hand if $y < r(x)$, then

$$g(x, y) < g(x, r(x)) = \underline{m}.$$

Suppose $x > x_{\underline{m}}$ and $y > 0$. Then, using the strict monotonicity of g in both variables, we get

$$g(x, y) > g(x_{\underline{m}}, y) > g(x_{\underline{m}}, 0) = g(x_{\underline{m}}, r(x_{\underline{m}})) = \underline{m}.$$

Thus our claim is proved.

Clearly, all points of the region $A = \{(x, y) : x > x_{\underline{m}}$ and $0 < y < s(x)\}$ give us a positive solution of the system (2.12) and (2.13), and the property $\lim_{x\to\infty} s(x) = 0$ yields that for any $M > 0$ and $\varepsilon > 0$ there exists $(x, y) \in A$ such that $x > M$ and $y < \varepsilon$. Hence, the definition of the dominant solution completes the proof of part (ii). □

Proof of Lemma 2.3. Let $\varphi \in C_+$ be an arbitrary fixed initial function and $x(t) = x(\varphi)(t)$ be any solution of the IVP (2.1)–(2.2). Then, by Lemma 2.1, we have $x(t) > 0$ for $t \geq 0$. We claim that there exist positive constants $d > 0$ and $d^* > 0$ such that the following inequalities are satisfied,

$$\begin{aligned} &\min_{0\leq t\leq\delta} x(t) > d, \qquad \max_{0\leq t\leq\delta} x(t) < d^*, \\ &\frac{\alpha(t)}{\beta(t)} h(d, d^*) > f(d) \quad \text{and} \quad \frac{\alpha(t)}{\beta(t)} h(d^*, d) < f(d^*), \qquad t \geq \delta. \end{aligned} \tag{4.6}$$

The last two inequalities in (4.6) follow if

$$\sup_{t\geq\delta} \frac{\alpha(t)}{\beta(t)} < \frac{f(d^*)}{h(d^*, d)} \tag{4.7}$$

and

$$\inf_{t\geq\delta} \frac{\alpha(t)}{\beta(t)} > \frac{f(d)}{h(d, d^*)}. \tag{4.8}$$

We define the function

$$g(x,y)\begin{cases}\frac{f(x)}{h(x,y)}, & x>0,\ y\geq 0,\\ 0, & x=0,\ y\geq 0.\end{cases} \tag{4.9}$$

The assumed monotonicity of $\frac{f(x)}{h(x,y)}$ in its both variables implies easily that g is continuous on $\mathbb{R}_+\times\mathbb{R}_+$. Then Lemma 2.2 with $\underline{m}=\sup_{t\geq\delta}\frac{\alpha(t)}{\beta(t)}$ and $\overline{m}=\inf_{t\geq\delta}\frac{\alpha(t)}{\beta(t)}$ yields that there exist positive numbers d and d^* satisfying the system of inequalities (4.7)–(4.8), $\max_{0\leq t\leq\delta}x(t)<d^*$ and $\min_{0\leq t\leq\delta}x(t)>d$.

We show that $d<x(t)<d^*$ for all $t\geq 0$. Suppose in contrary that there exists $t_2\in(\delta,\infty)$ such that $d<x(t)<d^*$ for $t\in[0,t_2)$ and either

(i) $x(t_2)=d$ or

(ii) $x(t_2)=d^*$.

First, consider case (i). Then $\dot{x}(t_2)\leq 0$. On the other hand, the mixed monotonicity of h and (4.6) yield that

$$\begin{aligned}\dot{x}(t_2)&=\beta(t_2)\left[\frac{\alpha(t_2)}{\beta(t_2)}h(x(t_2-\tau),x(t_2-\sigma))-f(x(t_2))\right]\\ &\geq\beta(t_2)\left[\frac{\alpha(t_2)}{\beta(t_2)}h(d,d^*)-f(d)\right]\\ &>0,\end{aligned}$$

which is a contradiction, since $\dot{x}(t_2)\leq 0$. Therefore $x(t)>d$ for all $t\geq 0$, and hence (2.14) holds.

Next, consider case (ii). Then $\dot{x}(t_2)\geq 0$. On the other hand, the mixed monotonicity of h and (4.6) yield that

$$\begin{aligned}\dot{x}(t_2)&=\beta(t_2)\left[\frac{\alpha(t_2)}{\beta(t_2)}h(x(t_2-\tau),x(t_2-\sigma))-f(x(t_2))\right]\\ &\leq\beta(t_2)\left[\frac{\alpha(t_2)}{\beta(t_2)}h(d^*,d)-f(d^*)\right]\\ &<0,\end{aligned}$$

which is a contradiction, since $\dot{x}(t_2)\geq 0$. Therefore $x(t)<d^*$ for all $t\geq 0$, and hence (2.15) holds. □

Proof of Theorem 2.4. Let $x(t)$ be any fixed solution of the IVP (2.1)–(2.2), and introduce the short notations

$$\underline{x}(\infty):=\liminf_{t\to\infty}x(t)\quad\text{and}\quad\overline{x}(\infty):=\limsup_{t\to\infty}x(t).$$

Lemma 2.3 implies that $0<\underline{x}(\infty)\leq\overline{x}(\infty)<\infty$. Moreover, for any $T\geq\delta$, the constants defined by

$$a_T:=\inf_{t\geq T}x(t)\leq\sup_{t\geq T}x(t)=:A_T \tag{4.10}$$

and

$$m_T:=\inf_{t\geq T}\frac{\alpha(t)}{\beta(t)}\leq\sup_{t\geq T}\frac{\alpha(t)}{\beta(t)}=:M_T \tag{4.11}$$

are positive and finite. By (4.10), (4.11) and the mixed monotonicity of h, we get from (2.1) that

$$\dot{x}(t) \geq \beta(t)[m_T h(a_T, A_T) - f(x(t))], \qquad t \geq T. \tag{4.12}$$

From (4.12) and the comparison theorem of differential equations we see that

$$x(t) \geq y(t) \quad \text{for } t \geq T,$$

where $y(t) = y(T, x(T), m_T h(a_T, A_T))(t)$ is the solution of equation (4.1) with $c = m_T h(a_T, A_T)$ and with the initial condition

$$y(T) = x(T).$$

From Lemma 4.1, we see that

$$y(\infty) := \lim_{t\to\infty} y(t) = f^{-1}(m_T h(a_T, A_T)).$$

Thus

$$f^{-1}(m_T h(a_T, A_T)) = y(\infty) \leq \underline{x}(\infty), \qquad T \geq \delta,$$

and from the last inequality, we have

$$\lim_{T\to\infty} f^{-1}(m_T h(a_T, A_T)) \leq \underline{x}(\infty).$$

Using relations

$$\lim_{T\to\infty} a_T = \underline{x}(\infty), \qquad \lim_{T\to\infty} A_T = \overline{x}(\infty) \quad \text{and} \quad \lim_{T\to\infty} m_T = \underline{m}$$

and the continuity of f^{-1} and h, we obtain

$$\lim_{T\to\infty} f^{-1}(m_T h(a_T, A_T)) = f^{-1}(\lim_{T\to\infty} m_T h(a_T, A_T)) = f^{-1}(\underline{m} h(\underline{x}(\infty), \overline{x}(\infty))).$$

Therefore we get

$$f^{-1}(\underline{m} h(\underline{x}(\infty), \overline{x}(\infty))) \leq \underline{x}(\infty),$$

and hence

$$\underline{m} h(\underline{x}(\infty), \overline{x}(\infty)) \leq f(\underline{x}(\infty)). \tag{4.13}$$

In a similar way we can obtain the relation

$$\overline{m} h(\overline{x}(\infty), \underline{x}(\infty)) \geq f(\overline{x}(\infty)). \tag{4.14}$$

Since $\underline{x}(\infty)$ and $\overline{x}(\infty)$ are positive, (H2) yields $h(\underline{x}(\infty), \overline{x}(\infty)) > 0$ and $h(\overline{x}(\infty), \underline{x}(\infty)) > 0$. Hence (4.13) and (4.14) are equivalent to the system of inequalities

$$\frac{f(\underline{x}(\infty))}{h(\underline{x}(\infty), \overline{x}(\infty))} \geq \underline{m} \tag{4.15}$$

$$\frac{f(\overline{x}(\infty))}{h(\overline{x}(\infty), \underline{x}(\infty))} \leq \overline{m}. \tag{4.16}$$

Define the function g by (4.9). Then g is continuous on $\mathbb{R}_+ \times \mathbb{R}_+$, and $(\underline{x}(\infty), \overline{x}(\infty))$ is a solution of the corresponding system of inequalities (2.12)–(2.13). Therefore Lemma 2.2 implies (2.17), where $(\underline{x}^*, \overline{x}^*)$ is the dominant positive solution of (2.7)–(2.8), or equivalently, the dominant positive solution of the system (2.18)–(2.19). □

Proof of Corollary 2.5. We show that the system (2.21)–(2.22) has a solution of the form (x^*, x^*). But, clearly, the system (2.21)–(2.22) has a solution of the form (x^*, x^*) if and only if x^* is a solution of (2.24).

Assumptions (H2) and (H4) imply that the function $\frac{f(x)}{h(x,x)}$ is strictly increasing, since for $0 < x_1 < x_2$ it follows

$$\frac{f(x_1)}{h(x_1, x_1)} < \frac{f(x_1)}{h(x_1, x_2)} < \frac{f(x_2)}{h(x_2, x_2)}.$$

Next, let x_n be a sequence of nonnegative reals with $x_n \to 0$ as $n \to \infty$, and let A be such that $x_n < A$ for all n. Then, using (H2) and (H3), we get

$$0 \le \frac{f(x_n)}{h(x_n, x_n)} < \frac{f(x_n)}{h(x_n, A)} \to 0, \quad \text{as } n \to \infty,$$

so $\lim_{x\to 0+} \frac{f(x)}{h(x,x)} = 0$. Similarly, it can be shown that $\lim_{x\to\infty} \frac{f(x)}{h(x,x)} = \infty$. Therefore, equation (2.24) has a unique solution for any $m > 0$.

Hence (x^*, x^*) is the unique solution of the system (2.21)–(2.22), hence Theorem 2.4 proves the statement of the corollary. $\square$

Proof of Theorem 2.6. First we comment that Lemma 2.3 can be easily extended to equation (2.25), hence the limit inferior and limit superior of the solutions are positive and finite. We use the notations introduced in the proof of Theorem 2.4, and define

$$m_T := \inf_{t\ge T} \frac{\alpha_1(t)}{\beta_2(t)} \le \sup_{t\ge T} \frac{\alpha_2(t)}{\beta_1(t)} =: M_T$$

Then, similarly to the proof of Theorem 2.4, we get

$$\begin{aligned}\dot{x}(t) &\ge \beta_2(t) \left(\frac{\alpha_1(t)}{\beta_2(t)} H_0(t, x(t-\tau_1), \dots, x(t-\tau_k), x(t-\sigma_1), \dots, x(t-\sigma_\ell)) - f(x(t)) \right) \\ &\ge \beta_2(t) \left(m_T H_0(t, a_T, \dots, a_T, A_T, \dots, A_T) - f(x(t)) \right) \\ &= \beta_2(t) [m_T h(a_T, A_T) - f(x(t))], \qquad t \ge T.\end{aligned}$$

This relation implies the inequality (4.13). Similar argument shows (4.14), and that completes the proof. $\square$

Acknowledgements

This research was partially supported by the Hungarian National Foundation for Scientific Research Grant No. K120186. We acknowledge the financial support of Széchenyi 2020 under the EFOP-3.6.1-16-2016-00015. The authors would like to thank the anonymous referee for helpful comments and suggestions.

References

[1] J. Baštinec, L. Berezansky, J. Diblik, Z. Šmarda, On a delay population model with quadratic nonlinearity, *Adv. Difference Equ.* **2012**, 2012:230, 9 pp. https://doi.org/10.1186/1687-1847-2012-230; MR3017366

[2] J. Baštinec, L. Berezansky, J. Diblik, Z. Šmarda, On a delay population model with a quadratic nonlinearity without positive steady state, *Appl. Math. Comput.* **227**(2014), 622–629. https://doi.org/10.1016/j.amc.2013.11.061; MR3146346

[3] L. Berezansky, E. Braverman, Boundedness and persistence of delay differential equations with mixed nonlinearity, *Appl. Math. Comput.* **279**(2016), 154–169. https://doi.org/10.1016/j.amc.2016.01.015; MR3458012

[4] L. Berezansky, E. Braverman, A note on stability of Mackey–Glass equations with two delays, *J. Math. Anal. Appl.* **450**(2017), 1208–1228. https://doi.org/10.1016/j.jmaa.2017.01.050; MR3639098

[5] S. P. Blythe, R. M. Nisbet, W. S. C. Gurney, Instability and complex dynamic behaviour in population models with long time delays, *Theoret. Population Biol.* **22**(1982), 147–176. https://doi.org/10.1016/0040-5809(82)90040-5; MR679302

[6] M. Budincevic, A comparison theorem of differential equations, *Novi Sad J. Math.* **40**(2010), No. 1, 55–56. MR2816312

[7] K. Cooke, P. van den Driessche, X. Zou, Interaction of maturation delay and nonlinear birth in population and epidemic models, *J. Math. Biol.* **39**(1999), 332–352. https://doi.org/10.1007/s002850050194; MR1727839

[8] T. Faria, A note on permanence of nonautonomous cooperative scalar population models with delays, *Appl. Math. Comput.* **240**(2014), 82–90. https://doi.org/10.1016/j.amc.2014.04.040; MR3213674

[9] T. Faria, R. Obaya, A. M. Sanz, Asymptotic behaviour for a class of non-monotone delay differential systems with applications, *J. Dynam. Differential Equations*, appeared online, 2017. https://doi.org/10.1007/s10884-017-9572-8;

[10] T. Faria, G. Röst, Persistence, permanence and global stability for an n-dimensional Nicholson system, *J. Dynam. Differential Equations* **26**(2014), 723–744. https://doi.org/10.1007/s10884-014-9381-2; MR3274439

[11] K. Gopalsamy, *Stability and oscillations in delay differential equations of population dynamics*, Mathematics and its Applications, Vol. 74, Kluwer Academic Publishers Group, Dordrecht, 1992. https://doi.org/10.1007/978-94-015-7920-9; MR1163190

[12] I. Győri, Global attractivity in delay differential equations using a mixed monotone technique, *J. Math. Anal. Appl.* **152**(1990), No. 1, 131–155. https://doi.org/10.1016/0022-247X(90)90096-X; MR1072931

[13] I. Győri, F. Hartung, N. A. Mohamady, On a nonlinear delay population model, *Appl. Math. Comput.* **270**(2015), 909–925. https://doi.org/10.1016/j.amc.2015.08.090; MR3406944

[14] I. Győri, F. Hartung, N. A. Mohamady, Existence and uniqueness of positive solutions of a system of nonlinear algebraic equations, *Period. Math. Hung.* **75**(2017), No. 1, 114–127. https://doi.org/10.1007/s10998-016-0179-3; MR3686020

[15] I. Győri, F. Hartung, N. A. Mohamady, Boundedness of positive solutions of a system of nonlinear delay differential equations, *Discrete Contin. Dyn. Syst. Ser. B* **23**(2018), No. 2, 809–836. https://doi.org/10.3934/dcdsb.2018044

[16] I. Győri, N. V. Pertsev, On the stability of equilibrium states of functional-differential equations of retarded type possessing a mixed monotone property, *Doklady Akademii Nauk SSSR* **297**(1987), No. 1, 23–25.

[17] I. Győri, N. V. Pertsev, Stability of steady-states of functional-differential equations in the case of mixed monotonocity, Applications to models of some biological process, Department of Numerical Mathematics Preprint No. 126, p. 23, USSR Academy of Science, Moscow, 1986.

[18] D. J. Guo, Existence and uniqueness of positive fixed points for mixed monotone operators and applications, *Appl. Anal.* **46**(1992), No. 1–2, 91–100. https://doi.org/10.1080/00036819208840113; MR1230718

[19] K. P. Hadeler, Delay equations in biology, in: H.-O. Peitgen, H.-O. Walther, Eds., *Functional differential equations and approximation of fixed points (Proc. Summer School and Conf., Univ. Bonn, Bonn, 1978)*, Lecture Notes in Math., Vol. 730, Springer, Berlin, 1979, pp. 136–163. MR547986

[20] K. P. Hadeler, G. Bocharov, Where to put delays in population models, in particular in the neutral case, *Can. Appl. Math. Q.* **11**(2003), No. 2, 159–173. MR2132252

[21] Y. Kuang, *Delay differential equations with applications in population dynamics*, Academic Press, Boston, MA, 1993. MR1218880

[22] R. M. Nisbet, W. S. C. Gurney, *Modelling fluctuating populations*, Wiley, Chichester, 1982.

[23] S. Ruan, Delay differential equations in single species dynamics, in: O. Arino, H. L. Hbid, E. Ait Dads (eds.), *Delay differential equations and applications*, NATO Sci. Ser. II Math. Phys. Chem., Vol. 205, Springer, Berlin, 2006, pp. 477–517. https://doi.org/10.1007/1-4020-3647-7_11; MR2337823