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Uniqueness theorem of differential system with coupled integral boundary conditions

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Abstract. The paper is devoted to study the uniqueness of solutions for a differential system with coupled integral boundary conditions under a Lipschitz condition. Our approach is based on the Banach's contraction principle. The interesting point is that the Lipschitz constant is related to the spectral radius corresponding to the related linear operators.

Keywords: differential system, coupled integral boundary conditions, spectral radius, Banach's contraction principle.

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1 Introduction

In this paper, we consider the uniqueness of solutions for the following differential system with coupled integral boundary conditions

$$\begin{cases} -x''(t) = f(t, x(t), y(t)), & t \in (0,1), \\ -y''(t) = g(t, x(t), y(t)), & t \in (0,1), \\ x(0) = y(0) = 0, \quad x(1) = \alpha[y], & y(1) = \beta[x] \end{cases} \tag{1.1}$$

where $\alpha[x], \beta[x]$ are bounded linear functionals on $C[0,1]$ given by

$$\alpha[x] = \int_0^1 x(t)dA(t), \qquad \beta[x] = \int_0^1 x(t)dB(t)$$

involving Riemann–Stieltjes integrals, in particular, A, B are non-decreasing functions, so dA, dB are positive Stieltjes measures.

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Differential system with coupled boundary conditions arise from the study of reaction-diffusion equations and Sturm–Liouville problems, and have extensive applications in various fields of sciences and engineering such as the heat equation and mathematical biology.

The existence of solutions or positive solutions of differential system with coupled boundary conditions has been studied by many researchers, see [1–4, 6–10, 13] for some recent work. For example, by using the Guo–Krasnosel'skii fixed-point theorem, the existence of positive solution of the following singular system with coupled four-point boundary value conditions are obtained [1]

$$\begin{cases} -x''(t) = f_1(t, x(t), y(t)), & t \in (0,1), \\ -y''(t) = f_2(t, x(t), y(t)), & t \in (0,1), \\ x(0) = y(0) = 0, \quad x(1) = \alpha y(\xi), \quad y(1) = \beta x(\eta). \end{cases}$$

In [8], Infante, Minhós and Pietramala, by means of classical fixed point index theory, provided a general theory for existence of positive solutions for coupled systems.

The uniqueness of solutions can be an important problem for boundary value problems of differential equation or differential system. This problem has been investigated by many authors by use of techniques of nonlinear analysis. We refer the reader to [3, 4] for some recent uniqueness results for differential system, to [5, 12, 14] for differential equation. In [3], by means of the Guo–Krasnosel'skii fixed-point theorem and mixed monotone method, Cui, Liu and Zhang investigated the uniqueness of positive solutions of singular system (1.1) in the case that the nonlinearities f and g may be singular at $t = 0, 1$.

However, to our best knowledge, there are fewer results concerned the uniqueness of solutions for differential systems with coupled integral boundary conditions. So, we consider the uniqueness of solutions for differential system (1.1) under a Lipschitz condition on f and g. By using Banach's contraction principle, a new result on the uniqueness of solutions for differential system (1.1) is obtained. It is worthwhile to mention that the Lipschitz constant is related to the spectral radius corresponding to the related linear operators.

Throughout the paper, we assume that the following conditions hold.

(H_1) $\alpha[t] = \int_0^1 t dA(t) > 0$, $\beta[t] = \int_0^1 t dB(t) > 0$, $\kappa = 1 - \alpha[t]\beta[t] > 0$.

(H_2) $f, g : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous.

2 Preliminaries

Let $C[0,1]$ be the Banach space of continuous functions endowed with the norm $\|x\| = \max_{t\in[0,1]} |x(t)|$ and let P_1 be the cone of nonnegative functions in $C[0,1]$ given by

$$P_1 = \{x \in C[0,1] : x(t) \geq 0, \forall\, t \in [0,1]\}.$$

Thus $E = C[0,1] \times C[0,1]$ is a Banach space with the norm defined by $\|(x,y)\|_E = \max\{\|x\|, \|y\|\}$, and $P = P_1 \times P_1$ is a cone in E.

Lemma 2.1 ([2]). *Let $u, v \in C[0,1]$, then the system of BVPs*

$$\begin{cases} -x''(t) = u(t), \quad -y''(t) = v(t), & t \in [0,1], \\ x(0) = y(0) = 0, \quad x(1) = \alpha[y], \quad y(1) = \beta[x] \end{cases}$$

has integral representation

$$\begin{cases} x(t) = \int_0^1 G_1(t,s)u(s)ds + \int_0^1 H_1(t,s)v(s)ds, \\ y(t) = \int_0^1 G_2(t,s)v(s)ds + \int_0^1 H_2(t,s)u(s)ds, \end{cases}$$

where

$$G_1(t,s) = \frac{\alpha[t]t}{\kappa}\int_0^1 k(s,\tau)dB(\tau) + k(t,s),\ H_1(t,s) = \frac{t}{\kappa}\int_0^1 k(s,\tau)dA(\tau),$$
$$G_2(t,s) = \frac{\beta[t]t}{\kappa}\int_0^1 k(s,\tau)dA(\tau) + k(t,s),\ H_2(t,s) = \frac{t}{\kappa}\int_0^1 k(s,\tau)dB(\tau),$$
$$k(t,s) = \begin{cases} t(1-s), & 0 \le t \le s \le 1, \\ s(1-t), & 0 \le s \le t \le 1. \end{cases}$$

Employing Lemma 2.1, we can reformulate BVP (1.1) as a fixed point for the following integral equations:

$$\begin{cases} x(t) = \int_0^1 G_1(t,s)f(s,x(s),y(s))ds + \int_0^1 H_1(t,s)g(s,x(s),y(s))ds, \\ y(t) = \int_0^1 G_2(t,s)g(s,x(s),y(s))ds + \int_0^1 H_2(t,s)f(s,x(s),y(s))ds. \end{cases}$$

Define an operator S by

$$S(x,y) = (S_1(x,y), S_2(x,y)), \qquad (x,y) \in E, \tag{2.1}$$

where operators $S_1, S_2 : E \to C[0,1]$ are defined by

$$\begin{cases} S_1(x,y)(t) = \int_0^1 G_1(t,s)f(s,x(s),y(s))ds + \int_0^1 H_1(t,s)g(s,x(s),y(s))ds, & t \in [0,1], \\ S_2(x,y)(t) = \int_0^1 G_2(t,s)g(s,x(s),y(s))ds + \int_0^1 H_2(t,s)f(s,x(s),y(s))ds, & t \in [0,1]. \end{cases}$$

Then the existence of a solution of differential system (1.1) is equivalent to the existence of a fixed point of S on E.

It is well known that the function $k(t,s)$ has the following properties:

$$t(1-t)s(1-s) \le k(t,s) \le t(1-t), \qquad \forall\ t,s \in [0,1].$$

From this and (H_1), for $t,s \in [0,1]$, we have

$$G_1(t,s) \le t + \frac{\alpha[t]t}{\kappa}\int_0^1 dB(\tau), \qquad H_1(t,s) \le \frac{t}{\kappa}\int_0^1 dA(\tau),$$
$$G_2(t,s) = t + \frac{\beta[t]t}{\kappa}\int_0^1 dA(\tau), \qquad H_2(t,s) = \frac{t}{\kappa}\int_0^1 dB(\tau),$$

and

$$G_1(t,s) \ge \frac{\alpha[t]t}{\kappa}\int_0^1 k(s,\tau)dB(\tau) \ge \frac{\alpha[t]s(1-s)}{\kappa}\int_0^1 \tau(1-\tau)dB(\tau)\cdot t,$$
$$G_2(t,s) \ge \frac{\beta[t]t}{\kappa}\int_0^1 k(s,\tau)dA(\tau) \ge \frac{\beta[t]s(1-s)}{\kappa}\int_0^1 \tau(1-\tau)dA(\tau)\cdot t,$$

$$H_1(t,s) \geq \frac{s(1-s)}{\kappa} \int_0^1 \tau(1-\tau)dA(\tau) \cdot t,$$
$$H_2(t,s) \geq H_2(t,s) = \frac{s(1-s)}{\kappa} \int_0^1 \tau(1-\tau)dB(\tau) \cdot t.$$

Therefore we have

$$G_i(t,s) \leq \rho t, \qquad H_i(t,s) \leq \rho t, \ i = 1,2, \tag{2.2}$$

and

$$G_i(t,s) \geq \nu t s(1-s), \quad H_i(t,s) \geq \nu t s(1-s), \qquad i = 1,2, \tag{2.3}$$

where

$$\rho = \max\left\{\frac{\alpha[t]}{\kappa}\beta[1]+1,\ \frac{\beta[t]}{\kappa}\alpha[1]+1, \frac{1}{\kappa}\beta[1], \frac{1}{\kappa}\alpha[1]\right\},$$
$$\nu = \min\left\{\frac{\alpha[t]}{\kappa}\beta[t(1-t)],\ \frac{\beta[t]}{\kappa}\alpha[t(1-t)],\ \frac{1}{\kappa}\beta[t(1-t)],\ \frac{1}{\kappa}\alpha[t(1-t)]\right\}.$$

Let $\mathbb{R}_+ = [0,+\infty)$. For $\mathbf{a} = (a,b,c,d) \in \mathbb{R}_+^4$ with $a^2+b^2+c^2+d^2 \neq 0$, define an operator $T : E \to E$ by

$$T_{\mathbf{a}}(x,y) = (T_{\mathbf{a},1}(x,y), T_{\mathbf{a},2}(x,y)), \tag{2.4}$$

where operators $T_{\mathbf{a},1}, T_{\mathbf{a},2} : E \to C[0,1]$ are defined by

$$T_{\mathbf{a},1}(x,y)(t) = \int_0^1 G_1(t,s)(ax(s)+by(s))ds + \int_0^1 H_1(t,s)(cx(s)+dy(s))ds, \qquad t \in [0,1],$$
$$T_{\mathbf{a},2}(x,y)(t) = \int_0^1 G_2(t,s)(cx(s)+dy(s))ds + \int_0^1 H_2(t,s)(ax(s)+by(s))ds, \qquad t \in [0,1].$$

It is not difficult to verify that $T_{\mathbf{a}} : E \to E$ is a completely continuous linear operator.

Definition 2.2 ([11]). Let E be a Banach space, $P \subset E$ be a cone in E. Let $e \in P\backslash\{\theta\}$, a mapping $T : P \to P$ is called $e-$positive if for every nonzero $x \in P$ a natural number $n = n(x)$ and two positive number c_x, d_x can be found such that

$$c_x e \leq T^n x \leq d_x e.$$

Recall that a real number λ is an eigenvalue of the operator T if there exists a non-zero element $x \in E$ such that $Tx = \lambda x$.

Lemma 2.3 ([11, Theorem 2.5, Lemma 2.1, Theorem 2.10]). *Suppose that $T : E \to E$ is a $e-$positive, completely continuous linear operator. If there exist $\psi \in E\backslash(-P)$ and a constant $c > 0$ such that $cT\psi \geq \psi$, then the spectral radius $r(T) \neq 0$, and $r(T)$ is the unique positive eigenvalue with its eigenfunction in P.*

Lemma 2.4. *Suppose that (H_1) holds. Then for the operator $T_{\mathbf{a}}$ defined by (2.4), there is a unique positive eigenvalue $r(T_{\mathbf{a}})$ with its eigenfunction in P.*

Proof. First, we show that $T_{\mathbf{a}}$ is $e-$positive with $e(t) = (t,t)$, that is, for any $(x,y) \in P\backslash\{\theta\}$, there exist $c_{x,y}, d_{x,y} > 0$ such that

$$c_{x,y} \cdot e \leq T_{\mathbf{a}}(x,y) \leq d_{x,y} \cdot e. \tag{2.5}$$

Let $d_{x,y} = \rho(a+c)\int_0^1 x(s)ds + \rho(b+d)\int_0^1 y(s)ds$. By (2.2), we can derive $T_{\mathbf{a}}(x,y)(t) \leq d_{x,y} \cdot (t,t) = d_{x,y} \cdot e(t)$. Let $c_{x,y} = \nu(a+c)\int_0^1 s(1-s)x(s)ds + \nu(b+d)\int_0^1 s(1-s)y(s)ds$. By (2.3), $T_{\mathbf{a}}(x,y)(t) \geq c_{x,y} \cdot e(t)$ holds, in particular, we have $T_{\mathbf{a}}e(t) \geq c_{e(t)} \cdot e(t)$. So (2.5) is proved and Lemma 2.4 holds follows from Lemma 2.3. This completes the proof. □

Remark 2.5. Let (φ, ψ) be the positive eigenfunction of $T_{\mathbf{a}}$ corresponding to $r(T_{\mathbf{a}})$, thus

$$T_{\mathbf{a}}(\varphi, \psi) = r(T_{\mathbf{a}})(\varphi, \psi). \tag{2.6}$$

Then by the proof of Lemma 2.4 and Definition 2.2, there exist $c_{\varphi,\psi} > 0$ such that

$$c_{\varphi,\psi} \cdot (t, t) = c_{\varphi,\psi} \cdot e(t) \le T_{\mathbf{a}}(\varphi, \psi)(t) = r(T_{\mathbf{a}}) \cdot (\varphi(t), \psi(t)),$$

i.e.,

$$t \le \frac{r(T_{\mathbf{a}})}{c_{\varphi,\psi}} \varphi(t), \qquad t \le \frac{r(T_{\mathbf{a}})}{c_{\varphi,\psi}} \psi(t), \qquad t \in [0,1]. \tag{2.7}$$

3 Main results

Theorem 3.1. *Suppose that there exists $\mathbf{a} = (a, b, c, d) \in \mathbb{R}^4_+$ with $a^2 + b^2 + c^2 + d^2 \neq 0$ such that*

$$|f(t, u_1, v_1) - f(t, u_2, v_2)| \le a|u_1 - u_2| + b|v_1 - v_2|, \qquad \forall\, t \in [0,1],\ u_1, u_2, v_1, v_2 \in \mathbb{R}, \tag{3.1}$$

and

$$|g(t, u_1, v_1) - g(t, u_2, v_2)| \le c|u_1 - u_2| + d|v_1 - v_2|, \qquad \forall\, t \in [0,1],\ u_1, u_2, v_1, v_2 \in \mathbb{R}. \tag{3.2}$$

If $r(T_{\mathbf{a}}) < 1$, then differential system (1.1) has a unique solution in E.

Proof. It is clear that the fixed points of operator S coincide with the solutions to differential system (1.1).

For $(x, y) \in E$, by (2.2), (2.7), (3.1) and(3.2) we have

$$\begin{aligned}
&|S_1(x,y)(t)| \\
&\quad \le \left| \int_0^1 G_1(t,s) f(s, x(s), y(s)) ds - \int_0^1 G_1(t,s) f(s,0,0) ds \right| + \left| \int_0^1 G_1(t,s) f(s,0,0) ds \right| \\
&\qquad + \left| \int_0^1 H_1(t,s) g(s, x(s), y(s)) ds - \int_0^1 H_1(t,s) g(s,0,0) ds \right| + \left| \int_0^1 H_1(t,s) g(s,0,0) ds \right| \\
&\quad \le \int_0^1 G_1(t,s) |f(s, x(s), y(s)) - f(s,0,0)| ds + \int_0^1 G_1(t,s) |f(s,0,0)| ds \\
&\qquad + \int_0^1 H_1(t,s) |g(s, x(s), y(s)) - g(s,0,0)| ds + \int_0^1 H_1(t,s) |g(s,0,0)| ds \\
&\quad \le \rho t \left((a+c) \int_0^1 |x(s)| ds + (b+d) \int_0^1 |y(s)| ds + \int_0^1 |f(s,0,0)| ds + \int_0^1 |g(s,0,0)| ds \right) \\
&\quad \le \frac{r(T_{\mathbf{a}})\rho}{c_{\varphi,\psi}} \left((a+c) \int_0^1 |x(s)| ds + (b+d) \int_0^1 |y(s)| ds \right. \\
&\qquad\qquad \left. + \int_0^1 |f(s,0,0)| ds + \int_0^1 |g(s,0,0)| ds \right) \cdot \varphi(t), \qquad t \in [0,1].
\end{aligned}$$

In the same way, we can prove that

$$\begin{aligned}
|S_2(x,y)(t)| \le \frac{r(T_{\mathbf{a}})\rho}{c_{\varphi,\psi}} &\left((a+c) \int_0^1 |x(s)| ds + (b+d) \int_0^1 |y(s)| ds \right. \\
&\left. + \int_0^1 |f(s,0,0)| ds + \int_0^1 |g(s,0,0)| ds \right) \cdot \psi(t), \qquad t \in [0,1].
\end{aligned}$$

Therefore, S maps all of E into the following vector subspace

$$E_1 = \left\{ (x,y) \in E : \frac{|x(t)|}{\varphi(t)},\ \frac{|y(t)|}{\psi(t)} \text{ are bounded for } t \in [0,1] \right\}.$$

Evidently, E_1 is a subspace of E and E_1 is an Banach space with the norm

$$\|(x,y)\|_1 = \max \left\{ \sup_{t\in[0,1]} \frac{|x(t)|}{\varphi(t)},\ \sup_{t\in[0,1]} \frac{|y(t)|}{\psi(t)} \right\}.$$

So it suffices to consider the fixed point of S in E_1. Note that

$$T_{\mathbf{a}}(\varphi,\psi) = r(T_{\mathbf{a}})(\varphi,\psi)$$

means

$$r(T_{\mathbf{a}})\varphi(t) = \int_0^1 G_1(t,s)(a\varphi(s)+b\psi(s))ds + \int_0^1 H_1(t,s)(c\varphi(s)+d\psi(s))ds$$

and

$$r(T_{\mathbf{a}})\psi(t) = \int_0^1 G_2(t,s)(c\varphi(s)+d\psi(s))ds + \int_0^1 H_2(t,s)(a\varphi(s)+b\psi(s))ds.$$

Let $(x_1,y_1),(x_2,y_2) \in E_1$. Then

$$\begin{aligned}
&|S_1(x_1,y_1)(t) - S_1(x_2,y_2)(t)| \\
&\quad \le \left| \int_0^1 G_1(t,s) f(s,x_1(s),y_1(s))ds - \int_0^1 G_1(t,s) f(s,x_2(s),y_2(s))ds \right| \\
&\qquad + \left| \int_0^1 H_1(t,s) g(s,x_1(s),y_1(s))ds - \int_0^1 H_1(t,s) g(s,x_2(s),y_2(s))ds \right| \\
&\quad \le a\int_0^1 G_1(t,s)|x_1(s)-x_2(s)|ds + b\int_0^1 G_1(t,s)|y_1(s)-y_2(s)|ds \\
&\qquad + c\int_0^1 H_1(t,s)|x_1(s)-x_2(s)|ds + d\int_0^1 H_1(t,s)|y_1(s)-y_2(s)|ds \\
&\quad \le a\int_0^1 G_1(t,s)\|(x_1,y_1)-(x_2,y_2)\|_1\varphi(s)ds + b\int_0^1 G_1(t,s)\|(x_1,y_1)-(x_2,y_2)\|_1\psi(s)ds \\
&\qquad + c\int_0^1 H_1(t,s)\|(x_1,y_1)-(x_2,y_2)\|_1\varphi(s)ds + d\int_0^1 H_1(t,s)\|(x_1,y_1)-(x_2,y_2)\|_1\psi(s)ds \\
&\quad = \|(x_1,y_1)-(x_2,y_2)\|_1 \cdot T_{\mathbf{a},1}(\varphi,\psi)(t) = r(T_{\mathbf{a}})\|(x_1,y_1)-(x_2,y_2)\|_1 \cdot \varphi(t).
\end{aligned}$$

In the same way, we can prove that

$$|S_2(x_1,y_1)(t) - S_2(x_2,y_2)(t)| \le r(T_{\mathbf{a}})\|(x_1,y_1)-(x_2,y_2)\|_1 \cdot \psi(t), \qquad t\in[0,1].$$

The above two inequalities imply that

$$\|S(x_1,y_1) - S(x_2,y_2)\|_1 \le r(T_{\mathbf{a}})\|(x_1,y_1)-(x_2,y_2)\|_1, \qquad \forall\ (x_1,y_1),(x_2,y_2) \in E_1.$$

Notice that $r(T_{\mathbf{a}}) < 1$, the operator S is a contraction. Hence, it follows from the well known Banach's contraction principle that S has a unique fixed point $(x,y) \in E_1$, which is obviously a unique solution of differential system (1.1). It ends the proof. $\square$

From the above argument, we know that the basic space used in the proof of Theorem 3.1 is E_1, not in E. If we consider differential system (1.1) in E by use of Banach's contraction principle, the result of Theorem 3.1 remains true except that the condition $r(T_{\mathbf{a}}) < 1$ is replaced by $\|T_{\mathbf{a}}\| < 1$, where

$$\|T_{\mathbf{a}}\| = \sup_{(x,y)\in E} \frac{\|T_{\mathbf{a}}(x,y)\|_E}{\|(x,y)\|_E}.$$

It follows from the well-known Gelfand's Formula that

$$r(T_{\mathbf{a}}) = \lim_{n\to\infty} \sqrt[n]{\|T_{\mathbf{a}}^n\|} \le \|T_{\mathbf{a}}\|$$

which concludes that it may be favorable to consider the uniqueness of differential system (1.1) in E_1.

In the following, we give two examples to illustrate our main result. Obviously, it is rather difficult to determine the value of $r(T_{\mathbf{a}})$ in general. In the two examples, we determine the spectral radius $r(T_{\mathbf{a}})$ for certain four-point coupled boundary conditions which can be seen as a special cases of coupled integral boundary conditions.

Example 3.2. Consider the system

$$\begin{cases} -x''(t) = a\sin x(t) + h_1(t), \quad t\in(0,1), \\ -y''(t) = a\sqrt{y^2(t)+1} + h_2(t), \quad t\in(0,1), \\ x(0) = y(0) = 0, \quad x(1) = y(\frac{1}{3}), \quad y(1) = 3x(\frac{1}{4}), \end{cases} \tag{3.3}$$

where $a\in\mathbb{R}$, $h_1, h_2 \in C[0,1]$. In this case the integral boundary conditions are given by the functionals $\alpha[y] = y(\frac{1}{3})$ and $\beta[x] = 3x(\frac{1}{4})$.

Let

$$f(t,x,y) = a\sin x + h_1(t), \qquad g(t,x,y) = a\sqrt{y^2+1} + h_2(t),$$

then

$$|f(t,u_1,v_1) - f(t,u_2,v_2)| \le |a||u_1-u_2|, \qquad \forall\, t\in[0,1], \ \ u_1,u_2,v_1,v_2\in\mathbb{R}$$

and

$$|g(t,u_1,v_1) - g(t,u_2,v_2)| \le |a||v_1-v_2|, \qquad \forall\, t\in[0,1], \ \ u_1,u_2,v_1,v_2\in\mathbb{R}.$$

Thus we have $b = c = 0$, $\kappa = 1 - \alpha[t]\beta[t] = \frac{3}{4}$.

Take $\mathbf{a} = (|a|, 0, 0, |a|)$. Let (φ,ψ) be the positive eigenfunction of $T_{\mathbf{a}}$ corresponding to $r(T_{\mathbf{a}})$, thus

$$T_{\mathbf{a},1}(\varphi,\psi) = r(T_{\mathbf{a}})\varphi, \ \ T_{\mathbf{a},2}(\varphi,\psi) = r(T_{\mathbf{a}})\psi. \tag{3.4}$$

Let $\lambda = \frac{|a|}{r(T_{\mathbf{a}})}$. It follows from (3.4) that

$$\begin{cases} -\varphi''(t) = \lambda\varphi, \ -\psi''(t) = \lambda\psi(t), \ t\in(0,1), \\ \varphi(0) = \psi(0) = 0, \ \varphi(1) = \psi(\frac{1}{3}), \ \psi(1) = 3\varphi(\frac{1}{4}). \end{cases}$$

By ordinary method, we conclude that $(\varphi(t),\psi(t)) = (c_1,c_2)\sin\sqrt{\lambda}t$ for some $c_1, c_2\in\mathbb{R}$. This together with the four-point coupled boundary conditions yields

$$c_1\sin\sqrt{\lambda} = c_2\sin\frac{\sqrt{\lambda}}{3}, \qquad c_2\sin\sqrt{\lambda} = 3c_1\sin\frac{\sqrt{\lambda}}{4}.$$

So, λ is the unique positive solution of the equation

$$\sin^2\sqrt{\lambda} = 3\sin\frac{\sqrt{\lambda}}{3}\sin\frac{\sqrt{\lambda}}{4}, \qquad \lambda \in (0, \pi^2).$$

We can obtain $\lambda \approx 1.9585^2 \approx 3.83584$ by MATLAB. Therefore, if $|a| < 3.83584$, the problems (3.3) has a unique solution.

Example 3.3. Consider the differential system

$$\begin{cases} -x''(t) = a\cos x(t) - a\ln(1+y^2(t)) + h_1(t), \quad t \in (0,1), \\ -y''(t) = a\arctan x(t) - ay(t) + h_2(t), \quad t \in (0,1), \\ x(0) = y(0) = 0, \quad x(1) = y(\frac{1}{2}), \quad y(1) = 2x(\frac{1}{4}), \end{cases} \tag{3.5}$$

where $a \in \mathbb{R}$, $h_1, h_2 \in C[0,1]$. Let

$$f(t,x,y) = a\cos x - a\ln(1+y^2) + h_1(t), \qquad g(t,x,y) = a\arctan x - ay + h_2(t),$$

then

$$|f(t,u_1,v_1) - f(t,u_2,v_2)| \le |a||u_1 - u_2| + |a||v_1 - v_2|,$$

and

$$|g(t,u_1,v_1) - g(t,u_2,v_2)| \le |a||u_1 - u_2| + |a||v_1 - v_2|,$$

where $t \in [0,1], u_1, u_2, v_1, v_2 \in \mathbb{R}$.

Take $\mathbf{a} = (|a|, |a|, |a|, |a|)$. Let (φ, ψ) be the positive eigenfunction of $T_{\mathbf{a}}$ corresponding to $r(T_{\mathbf{a}})$, thus

$$T_{\mathbf{a},1}(\varphi,\psi) = r(T_{\mathbf{a}})\varphi, \qquad T_{\mathbf{a},2}(\varphi,\psi) = r(T_{\mathbf{a}})\psi. \tag{3.6}$$

Let $\lambda = \frac{|a|}{r(T_{\mathbf{a}})}$. It follows from (3.6) that

$$\begin{cases} -\varphi''(t) = \lambda\varphi(t) + \lambda\psi(t),\ -\psi''(t) = \lambda\varphi(t) + \lambda\psi(t),\ t \in (0,1), \\ \varphi(0) = \psi(0) = 0,\ \varphi(1) = \psi(\frac{1}{2}),\ \psi(1) = 2\varphi(\frac{1}{4}). \end{cases}$$

By ordinary method, we deduce that $\varphi(t) = \frac{c_1}{2}\sin\sqrt{2\lambda}t + \frac{c_2}{2}t, \psi(t) = \frac{c_1}{2}\sin\sqrt{2\lambda}t - \frac{c_2}{2}t$ for some $c_1, c_2 \in \mathbb{R}$. Clearly, $c_1 \neq 0$ holds from the non-negativity of functions φ and ψ. Without loss of generality, we assume that $c_1 = 2$. Considering the boundary conditions, we have

$$\sin\sqrt{2\lambda} + \frac{c_2}{2} = \sin\frac{\sqrt{2\lambda}}{2} - \frac{c_2}{4},$$

and

$$\sin\sqrt{2\lambda} - \frac{c_2}{2} = 2\left(\sin\frac{\sqrt{2\lambda}}{4} + \frac{c_2}{8}\right).$$

Therefore, λ is the smallest positive solution of the equation

$$2\sin\sqrt{2\lambda} - \sin\frac{\sqrt{2\lambda}}{2} - 2\sin\frac{\sqrt{2\lambda}}{4} = 0, \qquad \lambda \in \left(0, \frac{\pi^2}{2}\right).$$

With the help of MATLAB, we have $\lambda \approx 2.0236421$ which implies that the problems (3.5) has a unique solution if $|a| \le 2.02364$.

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References

[1] N. A. Asif, R. A. Khan, Positive solutions to singular system with four-point coupled boundary conditions, *J. Math. Anal. Appl.* **386**(2012), No. 2, 848–861. MR2834792; https://doi.org/10.1016/j.jmaa.2011.08.039

[2] Y. Cui, J. Sun, On existence of positive solutions of coupled integral boundary value problems for a nonlinear singular superlinear differential system, *Electron. J. Qual. Theory Differ. Equ.* **2012**, No. 41, 1–13. MR2920964; https://doi.org/10.14232/ejqtde.2012.1.41

[3] Y. Cui, L. Liu, X. Zhang, Uniqueness and existence of positive solutions for singular differential systems with coupled integral boundary value problems, *Abstr. Appl. Anal.* **2013**, Art. ID 340487, 9 pp. MR3132530

[4] Y. Cui, Y. Zou, An existence and uniqueness theorem for a second order nonlinear system with coupled integral boundary value conditions, *Appl. Math. Comput.* **256**(2015), 438–444. MR3316082; https://doi.org/10.1016/j.amc.2015.01.068

[5] Y. Cui, Uniqueness of solution for boundary value problems for fractional differential equations, *Appl. Math. Lett.* **51**(2016), 48–54. MR3396346; https://doi.org/10.1016/j.aml.2015.07.002

[6] C. S. Goodrich, Coupled systems of boundary value problems with nonlocal boundary conditions, *Appl. Math. Lett.* **41**(2015), 17–22. MR3282393; https://doi.org/10.1016/j.aml.2014.10.010

[7] J. Henderson, R. Luca, A. Tudorache, On a system of fractional differential equations with coupled integral boundary conditions, *Fract. Calc. Appl. Anal.* **18**(2015), No. 2, 361–386. MR3323907; https://doi.org/10.1515/fca-2015-0024

[8] G. Infante, F. Minhós, P. Pietramala, Non-negative solutions of systems of ODEs with coupled boundary conditions, *Nonlinear Sci. Numer. Simul.* **17**(2012), No. 12, 4952–4960. MR2960289; https://doi.org/10.1016/j.cnsns.2012.05.025

[9] J. Jiang, L. Liu, Y. Wu, Symmetric positive solutions to singular system with multi-point coupled boundary conditions, *Appl. Math. Comput.* **220**(2013), 536–548. MR3091878; https://doi.org/10.1016/j.amc.2013.06.038

[10] J. Jiang, L. Liu, Y. Wu, Positive solutions to singular fractional differential system with coupled boundary conditions, *Commun. Nonlinear Sci.* **18**(2013), No. 11, 3061–3074. MR3072528; https://doi.org/10.1016/j.cnsns.2013.04.009

[11] M. A. Krasnosel'skii, *Positive solutions of operator equations*, P. Noordhoff, Groningen, The Netherlands, 1964. MR0181881

[12] X. Lin, D. Jiang, X. Li, Existence and uniqueness of solutions for singular fourth-order boundary value problems, *J. Comput. Appl. Math.* **196**(2006), No. 1, 155–161. MR2241581; https://doi.org/10.1016/j.cam.2005.08.016

[13] C. Yuan, D. Jiang, D. O'Regan, R. P. Agarwal, Multiple positive solutions to systems of nonlinear semipositone fractional differential equations with coupled boundary conditions, *Electron. J. Qual. Theory Differ. Equ.* **2012**, No. 13, 1–17. MR2889755; https://doi.org/10.14232/ejqtde.2012.1.13

[14] Y. Zou, G. He, On the uniqueness of solutions for a class of fractional differential equations, *Appl. Math. Lett.* **74**(2017), 68–73. MR3677843; https://doi.org/10.1016/j.aml.2017.05.011