View metadata, citation and similar papers at core.ac.uk

brought to you by CORE

provided by Repository of the Academy's Library

Electronic Journal of Qualitative Theory of Differential Equations
2018, No. **28**, 1–14; https://doi.org/10.14232/ejqtde.2018.1.28 www.math.u-szeged.hu/ejqtde/

Solutions to nonlocal Neumann boundary value problems

Katarzyna Szymańska-Dębowska ✉

Institute of Mathematics, Lodz University of Technology, 90-924 Łódź, ul. Wólczańska 215, Poland

Received 29 October 2017, appeared 18 May 2018
Communicated by Gennaro Infante

Abstract. In this paper we study the nonlocal Neumann boundary value problem of the following form

$$u'' = f(t, u, u'), \quad u'(0) = 0, \quad u'(1) = \int_0^1 u'(s)dg(s),$$

where $f : [0, 1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $g = \operatorname{diag}(g_1, \dots, g_n)$ with $g_i : [0, 1] \to \mathbb{R}$, $i = 1, \dots, n$. The case when the function f does not depend on u' is also considered. The existence of solutions is obtained by means of the generalized Miranda theorem. The main results can be applied to many problems of this type depending on which conditions will be imposed upon the function f.

Keywords: nonlocal boundary value problem, boundary value problem at resonance, Neumann problem, the Miranda theorem, R_δ-sets.

2010 Mathematics Subject Classification: 34B10, 34B15, 34K10.

1 Introduction

We consider the following nonlocal boundary value problem

$$u'' = f(t, u, u'), \quad u'(0) = 0, \quad u'(1) = \int_0^1 u'(s)dg(s), \tag{1.1}$$

where $f : [0, 1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $g : [0, 1] \to \mathbb{R}^n$ with $g = \operatorname{diag}(g_1, \dots, g_n)$. Observe that the problem (1.1) is always resonant, since the functions $u(t) \equiv b \in \mathbb{R}^n$ are solutions to the corresponding homogenous linear problem

$$u'' = 0, \quad u'(0) = 0, \quad u'(1) = \int_0^1 u'(s)dg(s).$$

When $g \equiv 0$, the problem (1.1) reduces to the classical Neumann boundary value problem, which has been extensively studied (see, for instance, [7,8,12,16,21] and the references therein).

✉ Email: katarzyna.szymanska-debowska@p.lodz.pl

Other types of generalizations of Neumann boundary value problems involving Riemann–Stieltjes integrals than those discussed in this paper can be found, for instance, in [9,19].

Recently, the following problem

$$x'' = f(t,x), \quad x'(0) = 0, \quad x'(1) = \int_0^1 x'(s)\,dg(s), \tag{1.2}$$

was considered in [17], where was shown that, under some Landesman–Lazer–Nirenberg-type asymptotic condition, the problem (1.2) has at least one solution. In [13], one can find a number of existence results of the problem (1.1), for instance: existence theorem in the case when the function f satisfies some Villari-type conditions, existence conditions in terms of the non-vanishing of the Brouwer degree of some mapping in $\mathbb{R}^n$ depending upon f and g (the conditions for the problem (1.2) made in [17] follow from this result).

So far as we aware, the problems (1.1) and (1.2) were studied only in [13] and [17]. However, in both papers the function f is considered to be *bounded*. In this paper, using the generalized Miranda theorem, we shall weaken the assumptions imposed upon the function f in [13] and [17].

First, for the convenience of the reader, let us recall some notation and terminology needed later on. Let X, Y be nonempty metric spaces. We say that a space X is *contractible*, if there exist $x_0 \in X$ and a homotopy $M : X \times [0,1] \to X$ such that $M(x,0) = x$ and $M(x,1) = x_0$ for every $x \in X$. A compact space X is an *R_δ-set* (we write $X \in R_\delta$) if there is a decreasing sequence X_n of compact contractible spaces such that $X = \bigcap_{n\geq 1} X_n$. A set-valued map $H : X \multimap Y$ is *upper semicontinuous* (written USC) if, given an open $V \subset Y$, the set $\{x \in X \mid H(x) \subset V\}$ is open. We say that $H : X \multimap Y$ is an *R_δ-map* if it is USC and, for each $x \in X$, $H(x) \in R_\delta$. The key tool in our approach is the following generalization of the Miranda theorem:

Theorem 1.1 ([18]). *Let $A_i > 0$, $i = 1, \ldots, n$, and F be an admissible map from $\prod_{i=1}^n [-A_i, A_i]$ to $\mathbb{R}^n$, i.e. there exist a Banach space E, $\dim E \geq n$, a linear, bounded and surjective map $h : E \to \mathbb{R}^n$ and an R_δ–map H from $\prod_{i=1}^n [-A_i, A_i]$ to E such that $F = h \circ H$. If for any $i = 1, \ldots, n$ and every $y \in F(a)$, where $|a_i| = A_i$, we have*

$$a_i \cdot y_i \geq 0 \tag{1.3}$$

or

$$a_i \cdot y_i \leq 0, \tag{1.4}$$

then there exists $a \in \prod_{i=1}^n [-A_i, A_i]$ such that $0 \in F(a)$.

Remark 1.2. Theorem 1.1 differs from Theorem 5 proved in [18] with the condition (1.4). To show that Theorem 1.1 holds true with the condition (1.4), in the proof given in [18] it is sufficient to consider the following set-valued map $G : \prod_{i=1}^n [-A_i, A_i] \multimap \prod_{i=1}^n [-A_i, A_i]$ and the diagram

$$D : \prod_{i=1}^n [-A_i, A_i] \overset{\Phi_0}{\multimap} E \overset{h}{\to} \prod_{i=1}^n [-A_i, A_i],$$

where $G = h \circ \Phi_0$ and $\Phi_0(q) := \{x \in E \mid x = j(q) + \varepsilon z,\ z \in H(q)\}$ with $\varepsilon > 0$ small enough and $j : \mathbb{R}^n \to E$ given by

$$j(q) = \sum_{i=1}^n q_i e_i,$$

where e_i is the element of the space E such that $h_j(e_i) = \delta_{ij}$, $i, j = 1, \dots, n$. Note that in order to prove Theorem 1.1 with the condition (1.4), it is sufficient to change only the definition of the map Φ_0, the rest of the proof remains the same.

In this paper, using the generalized Miranda theorem (Theorem 1.1), general theorems of the existence of solutions to the problems (1.1) and (1.2) are proved (Theorems 2.1–2.5). On the one hand, this approach allows to consider functions f which can be unbounded, on the other hand, as opposed to the previous results in which g was of bounded variation ([13, 17]), we need to make some additional assumptions upon the function g. Special cases and examples, for which the assumptions of Theorems 2.1–2.5 are satisfied, are given in Section 3.

The generalized Miranda theorem (Theorem 1.1) can be applied to systems of ordinary differential equations, to both nonresonant and resonant cases. In [18] some examples of using this method for systems of n differential equations of the form $u'' = f(t, u, u')$ subject to various local boundary conditions posed on an interval or on a half-line are given. Recently, this approach has been applied to the following nonlocal resonant problem

$$x'' = f(t, x, x'), \quad x'(0) = 0, \quad x(1) = \int_0^1 x(s)\, dg(s).$$

Under standard growth and sign conditions imposed on the function f and assuming that for each $i = 1, \dots, k$ functions g_i are nondecreasing, it was showed that the above problem has at least one solution (see [10] for more details).

2 Existence results

Denote by $C^1([0, 1], \mathbb{R}^n)$ the space of once continuously differentiable functions with the usual norm.

Let us consider the following family of initial value problems

$$u'' = f(t, u, u'), \quad u(0) = a, \quad u'(0) = 0, \tag{2.1}$$

where $a \in \mathbb{R}^n$ and $f : [0, 1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$.

Define an operator $T : \mathbb{R}^n \times C^1([0, 1], \mathbb{R}^n) \to C^1([0, 1], \mathbb{R}^n)$ by

$$T_a(u)(t) = a + \int_0^t (t - s) f(s, u(s), u'(s)) ds. \tag{2.2}$$

Let $a \in \mathbb{R}^n$ be fixed and set

$$\operatorname{Fix} T_a := \{u \in C^1([0, 1], \mathbb{R}^n) \mid T_a u = u\}.$$

Observe, that $u \in \operatorname{Fix} T_a$ if and only if u is a solution to the problem (2.1).

Let us consider a map $H : \mathbb{R}^n \multimap C^1([0, 1], \mathbb{R}^n)$ such that

$$H(a) = \operatorname{Fix} T_a \tag{2.3}$$

and define a map $h : C^1([0, 1], \mathbb{R}^n) \to \mathbb{R}^n$ by

$$h(u) = u'(1) - \int_0^1 u'(s) dg(s). \tag{2.4}$$

Now, let a multifunction $F : \mathbb{R}^n \multimap \mathbb{R}^n$ be such that $F = h \circ H$, i.e.,

$$F(a) = \left\{ u'(1) - \int_0^1 u'(s) dg(s) \,\middle|\, u \in \operatorname{Fix} T_a \right\}. \tag{2.5}$$

Theorem 2.1. *Let the following assumptions be fulfilled:*

(F1) $f : [0,1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ *is continuous;*

(T1) *(a priori estimate) for each* $a \in \mathbb{R}^n$ *the possible solutions to the problem* (2.1) *are equibounded in the space* $C^1([0,1], \mathbb{R}^n)$*;*

(G) *for each* $i = 1, \dots, k$, g_i *is nondecreasing,* $0 \le \int_0^1 dg_i(s) \le 1$ *and, if* $\int_0^1 dg_i(s) = 1$*, then* g_i *is not constant on* $[0,1)$.

Moreover, assume that there are constants $A_i > 0$, $i = 1, \dots, n$, *such that*

(A1) *if* $u \in \operatorname{Fix} T_a$ *with* $a_i := A_i$*, then the functions* u_i' *are nondecreasing on* $[0,1]$, $i = 1, \dots, n$*;*

(A2) *if* $u \in \operatorname{Fix} T_a$ *with* $a_i := -A_i$*, then the functions* u_i' *are nonincreasing on* $[0,1]$, $i = 1, \dots, n$.

Then the problem (1.1) *has at least one solution.*

Proof. First, observe that h defined in (2.4) is a linear and continuous map. We shall show that h is surjective. Let $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. By the assumption (G), one can always find a function $\overline{u} \in C^1([0,1], \mathbb{R}^n)$ such that $\overline{u}_i'(1) = 0$ and $\int_0^1 \overline{u}_i'(s)\, dg_i(s) \neq 0$, $i = 1, \dots, n$. Let $d_i := \int_0^1 \overline{u}_i'(s)\, dg_i(s)$ and set

$$u_i(t) := -\frac{x_i}{d_i}\, \overline{u}_i(t).$$

Thus, for each x there is an u such that $h(u) = x$ and h is surjective.

From (F1), using the theorem on existence, the problem (2.1) has local solutions. Because of local existence, it is enough to know that the possible global solutions are bounded to conclude that they exist. By the assumption (T1), for every $a \in \mathbb{R}^n$ there is a constant $R_a > 0$ such that $|u(t)| \le R_a$ and $|u'(t)| \le R_a$ for all $t \in [0,1]$. Now, using the theorem on a priori bounds (cf. [15, p. 146]), for each fixed $a \in \mathbb{R}^n$ there exists at least one global solution u to the problem (2.1), i.e. $u \in C^1([0,1], \mathbb{R}^n)$. Hence, the map H is well defined.

It is standard, using the Arzelà–Ascoli theorem, to show that under assumptions (F1) the operator T is completely continuous. Now, the set-valued map

$$H : \prod_{i=1}^{n} [-A_i, A_i] \multimap C^1([0,1], \mathbb{R}^n)$$

is USC with compact values (cf. [18, Lemma 2]).

Set

$$f_0(t,u,v) = \begin{cases} f(t,u,v), & \text{for } t \in [0,1] \wedge |u| \le R_a \wedge |v| \le R_a, \\ f(t, R_a \frac{u}{|u|}, v), & \text{for } t \in [0,1] \wedge |u| \ge R_a \wedge |v| \le R_a, \\ f(t, u, R_a \frac{v}{|v|}), & \text{for } t \in [0,1] \wedge |u| \le R_a \wedge |v| \ge R_a, \\ f(t, R_a \frac{u}{|u|}, R_a \frac{v}{|v|}), & \text{for } t \in [0,1] \wedge |u| \ge R_a \wedge |v| \ge R_a. \end{cases}$$

One can check that f_0 is continuous and bounded. It is easy to see that the problem (2.1) is equivalent to the following one

$$u'' = f_0(t,u,u'), \quad u(0) = a, \quad u'(0) = 0. \tag{2.6}$$

Hence, the set of solutions to the problem (2.1) is equal to the set of solutions to the problem (2.6). Consequently, since f_0 is integrably bounded, for each $a \in \prod_{i=1}^{n}[-A_i, A_i]$, $H(a) \in R_\delta$ (see [4], [5] p. 162 or [6] p. 352 for more details). Hence, the set of all solutions to the problem (2.1) is an R_δ-set.

Consequently, by the assumption (F1), (T1) and (G), F defined in (2.5) is the admissible map in the sense of Theorem 1.1.

Now, applying Theorem 1.1, we shall show that there is an $\bar{a} \in \prod_{i=1}^{n}[-A_i, A_i]$ such that $0 \in F(\bar{a})$, which means that there is an $\bar{a}$ for which the solution u to the problem (2.1) is also a solution to the problem (1.1), i.e. u satisfies the second nonlocal boundary condition of the problem (1.1).

Let us consider the initial value problem (2.1). If the assumption (A1) holds, then each solution $u \in C^1([0,1], \mathbb{R}^n)$ to the problem (2.1) with $a_i = A_i$, $i = 1, \ldots, n$, is such that u_i' is nondecreasing on $[0,1]$. In the case when $a_i = -A_i$, from the assumption (A2), every solution $u \in C^1([0,1], \mathbb{R}^n)$ to the initial value problem (2.1) is such that u_i' is nonincreasing on $[0,1]$.

Thus, from the assumption (A1), for $a_i = A_i$, we get

$$u_i'(1) \geq 0, \tag{2.7}$$

since $u_i'(0) = 0$. Moreover, we have

$$u_i'(t) \leq u_i'(1), \tag{2.8}$$

for all $t \in [0,1]$. Integrating both sides of (2.8) over $[0,1]$ with respect to the measure dg_i and using (2.7) and the assumption (G), one gets

$$\int_0^1 u_i'(s)dg_i(s) \leq u_i'(1) \int_0^1 dg_i(s) \leq u_i'(1).$$

If $a_i = -A_i$, then, from the assumption (A2),

$$u_i'(1) \leq 0, \tag{2.9}$$

and

$$u_i'(t) \geq u_i'(1), \tag{2.10}$$

for each $t \in [0,1]$. Integrating both sides of (2.10), from (2.9) and the assumption (G), we obtain

$$\int_0^1 u_i'(s)dg_i(s) \geq u_i'(1) \int_0^1 dg_i(s) \geq u_i'(1).$$

Consequently, for each $u_i'(1) - \int_0^1 u_i'(s)dg_i(s) \in F(a)$ with $|a_i| = A_i$, one has

$$a_i \left(u_i'(1) - \int_0^1 u_i'(s)dg_i(s) \right) \geq 0 \tag{2.11}$$

and the condition (1.3) of Theorem 1.1 is satisfied, what ends the proof. □

Remark 2.2. In the assumption (G), the condition that g_i is not constant on $[0,1)$, $i = 1, \ldots, n$, also guarantees that the second boundary conditions of the problems (1.1) and (1.2) are well posed.

The case when the measure dg_i, $i = 1, \dots, n$, is negative provides the following existence result

Theorem 2.3. *Let the assumptions* (F1), (T1), (A1) *and* (A2) *hold. Moreover, let the following assumption be satisfied*

(G′) *for each $i = 1, \dots, k$, g_i is nonincreasing, $-1 \le \int_0^1 dg_i(s) < 0$.*

Then there is at least one solution to the problem (1.1).

Proof. The beginning of the proof is analogous to the previous one. Observe that, if $a_i = A_i$, $i = 1, \dots, n$, then, from (2.7) and (2.8), we obtain

$$u_i'(t) \ge -u_i'(1),$$

for every $t \in [0,1]$. From (2.7) and the assumption (G′), one has

$$\int_0^1 u_i'(s) dg_i(s) \le u_i'(1) \left[- \int_0^1 dg_i(s) \right] \le u_i'(1).$$

In the case when $a_i = -A_i$, from (2.9) and (2.10), we get

$$u_i'(t) \le -u_i'(1),$$

for each $t \in [0,1]$. Consequently, from (2.9) and the assumption (G′), we obtain

$$\int_0^1 u_i'(s) dg_i(s) \ge [-u_i'(1)] \int_0^1 dg_i(s) \ge u_i'(1). \qquad \square$$

In the case when the function f does not depend upon u', we consider the following family of initial value problems

$$u'' = f(t,u), \quad u(0) = a, \quad u'(0) = 0, \tag{2.12}$$

where $a \in \mathbb{R}^n$ and $f : [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$. Then the operator $T : \mathbb{R}^n \times C^1([0,1], \mathbb{R}^n) \to C^1([0,1], \mathbb{R}^n)$ is given by

$$T_a(u)(t) = a + \int_0^t (t-s) f(s, u(s)) ds. \tag{2.13}$$

The above immediately leads to the following results:

Theorem 2.4. *Let the assumption* (G) *be satisfied and the assumptions* (A1) *and* (A2) *hold for the operator T defined in* (2.13)*. Moreover, let the following assumptions be satisfied:*

(F2) *$f : [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous;*

(T2) *for each $a \in \mathbb{R}^n$ the possible solutions to the problem* (2.12) *are equibounded in the space $C^1([0,1], \mathbb{R}^n)$.*

Then the problem (1.2) *has at least one solution.*

Theorem 2.5. *If the assumptions* (G′), (F2) *and* (T2) *are fulfilled and the assumptions* (A1) *and* (A2) *hold for the operator T defined in* (2.13)*, then the problem* (1.2) *has at least one solution.*

Remark 2.6. One can generalize the assumptions (F1) and (F2) and assume that f is an Carathéodory mapping. In this case one can use the Aronszajn characterization of the set $\mathrm{Fix}\, T_a$ (see [1] or [6, p. 351]).

3 Examples of the applications of the main theorems

Known results on the problems (1.1) and (1.2) refer to the case in which the function f is bounded (cf. [13,17]). The assumptions made in Section 2 and the use of the generalization of the Miranda theorem allow us to obtain the existence of solutions to the problems (1.1) and (1.2) with much weaker assumptions upon the function f.

In this section we shall consider some conditions for which the assumptions of Theorems 2.1–2.5 are fulfilled. One can see that the assumptions (T1) and (T2) are crucial here. From this assumptions we obtain that the solutions to the problems (2.1) and (2.12) are global and also we know something about the topological structure of the set of solutions to this problems. Instead of the two examples of well known growth conditions given below, one can impose on f any conditions that will guarantee *equiboundedness* of solutions to the problems (2.1) and (2.12) (see the assumption (F9)). Some Gronwall's type inequalities might be useful here (cf. [3,14]).

First, the following assumption upon f will be needed:

(F3) there are $c_1, c_2, c_3 \in \mathbb{R}_+$ such that $|f(t,u,v)| \leq c_1|u| + c_2|v| + c_3$ for all $(t,u,v) \in [0,1] \times \mathbb{R}^n \times \mathbb{R}^n$;

(F4) for each $i = 1,\dots,k$ there is $M_i > 0$ such that $u_i f_i(t,u,v) \geq 0$ for all $t \in [0,1]$, $v \in \mathbb{R}^k$, $u \in \mathbb{R}^k$ with $|u_i| \geq M_i$.

Let us consider the problem (2.1). For any solution one has

$$u'(t) = \int_0^t f\left(s, u(s), u'(s)\right) ds$$

and

$$u(t) = a + \int_0^t u'(s) ds. \tag{3.1}$$

Hence, in the case when f has linear growth (the assumption (F3)), using Gronwall's Lemma, one can observe that if u is a solution to the problem (2.1), then there are constants $U_a, V_a > 0$ such that $|u(t)| \leq U_a$ and $|u'(t)| \leq V_a$, for all $t \in [0,1]$ (cf. [18, Example 1]). Consequently, under the assumptions (F1) and (F3) the assumption (T1) is satisfied.

The assumptions made upon f also imply the following results:

Lemma 3.1. *Let the assumptions* (F1), (F3) *and* (F4) *be fulfilled and let* $a_i := M_i + 1$, $i = 1,\dots,n$. *Then the assumption* (A1) *holds.*

Proof. Let $a_i = M_i + 1 =: A_i$ and let $u \in C^1([0,1], \mathbb{R}^n)$ be a global solution to the problem (2.1).

First, we will show that $u_i'(t) \geq 0$, $t \in [0,1]$. Note that $u_i'(0) = 0$ and assume that for some t we have $u_i'(t) < 0$. Then there exists $t_0 := \inf\{t \mid x_i'(t) < 0\}$ such that, $u_i'(t_0) = 0$ and $u_i'(t) \geq 0$ for $t < t_0$. Consequently, since u_i' is continuous, there is $t_1 > t_0$ such that $\int_{t_0}^{t_1} |u_i'(t)| dt \leq 1$. By (3.1), we have

$$u_i(t) = M_i + 1 + \int_{t_0}^{t} u_i'(s) ds \geq M_i, \tag{3.2}$$

for all $t \in [t_0, t_1]$. Now, since $u_i'(t_0) = 0$, we reach a contradiction. Indeed, by (3.2) and (F4), one gets

$$u_i(t) f_i\left(t, u(t), u'(t)\right) = u_i(t) u_i''(t) \geq 0. \tag{3.3}$$

Hence, $u_i''(t) \geq 0$ for $t \in [t_0, t_1]$, which means that $u_i'(t)$ is nondecreasing on $[t_0, t_1]$.

Now, since $u_i'(t) \geq 0$ on $[0,1]$ and $u_i(0) = M_i + 1$, one get $u_i(t) \geq M_i + 1$, $t \in [0,1]$. Consequently, for all $t \in [0,1]$, by (F4) and (3.3), we have : $u_i''(t) \geq 0$. Hence, u_i' is nondecreasing on $[0,1]$. □

Lemma 3.2. *Let the assumptions* (F1), (F3) *and* (F4) *hold and let* $a_i := -M_i - 1$, $i = 1, \ldots, n$. *Then the assumption* (A2) *is satisfied.*

Proof. To the proof it is sufficient to follow in the same way as in the proof of Lemma 3.1, showing first that $u_i'(t) \leq 0$ for $t \in [0,1]$. □

From Lemmas 3.1–3.2, Theorems 2.1 and 2.3, it immediately follows:

Corollary 3.3. *Let the assumptions* (F1), (F3), (F4) *and* (G) *be fulfilled. Then the problem* (1.1) *has at least one solution.*

Corollary 3.4. *Under the assumptions* (F1), (F3), (F4) *and* (G′), *there is at least one solution to the problem* (1.1).

Example 3.5. Let $g_i(s) = s$, $i = 1, 2$, and

$$f_1(t, u, v) = a_1(t, u, v)\left(u_1 + \arctan u_2 + 1\right),$$
$$f_2(t, u, v) = a_2(t, u, v)\left(u_2 + \arctan v_2 - 2\right),$$

where a_i are continuous and there are constants $0 < l_i \leq L_i$ such that

$$l_i := \inf_{t \in [0,1], u \in \mathbb{R}^n, v \in \mathbb{R}^n} a_i(t, u, v)$$

and

$$\sup_{t \in [0,1], u \in \mathbb{R}^n, v \in \mathbb{R}^n} a_i(t, u, v) =: L_i,$$

$i = 1, 2$. It is easy to check that in this case the assumptions (F1), (F3), (F4) and (G) are satisfied. Consequently, from Corollary 3.3, the problem (1.1) with the functions f and g defined above has at least one nontrivial solution.

For the special case when the function f does not depend on u', let us make the following additional assumptions:

(F5) there are $c_1, c_2 \in \mathbb{R}_+$ such that $|f(t, u)| \leq c_1|u| + c_2$ for all $(t, u) \in [0,1] \times \mathbb{R}^n$;

(F6) for each $i = 1, \ldots, k$ there is $M_i > 0$ such that $u_i f_i(t, u) \geq 0$ for all $t \in [0,1]$, $u \in \mathbb{R}^k$ with $|u_i| \geq M_i$.

It is easy to observe that in this case Lemmas 3.1 and 3.2 hold as well. Now, Theorems 2.4 and 2.5 imply the following existence results:

Corollary 3.6. *Let the assumptions* (F2), (F5), (F6) *and* (G) *be fulfilled. Then the problem* (1.2) *has at least one solution.*

Corollary 3.7. *Under the assumptions* (F2), (F5), (F6) *and* (G'), *the problem* (1.2) *has at least one solution.*

Now, we shall consider some generalization of the sublinear case. Let $n = 1$. The following assumptions are made:

(F7) $|f(t,u,v)| \leq c_1\omega(|v|) + c_2$, $t \in [0,1]$, $u,v \in \mathbb{R}$, where $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous nondecreasing function;

(F8) there exists $M > 0$ such that $uf(t,u,v) \geq 0$ for all $t \in [0,1]$, $v \in \mathbb{R}$, $u \in \mathbb{R}$ with $|u| \geq M$.

Applying the assumption (F7) to the problem (2.1), one gets

$$|u'(t)| \leq \int_0^t \left| f\left(s, a + \int_0^s u'(z)\,dz, u'(s)\right)\,ds \right| \leq c_2 + c_1 \int_0^t \omega(|u'(s)|)ds.$$

Consequently, using the generalization of Gronwall's inequality due to Bihari (cf. [2,3]), we obtain

$$|u'(t)| \leq W^{-1}\Big(W(c_2) + c_1\Big), \tag{3.4}$$

where $t \in [0,1]$ and $W(\xi) = \int_{\xi_0}^{\xi} \frac{ds}{\omega(s)}$ with $\xi_0 > 0, \xi \geq 0$, provided that $W(c_2) + c_1 \in \operatorname{dom}(W^{-1})$. Thus, by (3.1),

$$|u(t)| \leq a + W^{-1}\Big(W(c_2) + c_1\Big), \tag{3.5}$$

Consequently, the assumption (T1) is satisfied.

Now, Theorems 2.1 and 2.3 can be written as follows:

Corollary 3.8. *Let the assumptions* (F1), (F7), (F8) *and* (G) *be fulfilled. Then the problem* (1.1) *has at least one solution.*

Corollary 3.9. *Under the assumptions* (F1), (F7), (F8) *and* (G'), *the problem* (1.1) *has at least one solution.*

Example 3.10. Let $n = 1$, $g(s) = s$ and

$$f(t,u,v) = b_1(t,u,v)(v^2+1)k_1(u) + b_2(t,u,v)k_2(u),$$

where the functions b_i and k_i are continuous, there are $M_i > 0$ such that

$$uk_i(u) \geq 0,$$

for every $|u| \geq M_i$ and at least one k_i is such that $k_i(0) \neq 0$, $i = 1,2$. Moreover, there exist constants $0 < l_i \leq L_i$, $i = 1,2$, such that

$$l_i := \inf_{t\in[0,1],u\in\mathbb{R},v\in\mathbb{R}} b_i(t,u,v),$$

$$\sup_{t\in[0,1],u\in\mathbb{R},v\in\mathbb{R}} b_1(t,u,v)k_1(u) =: L_1 \le \frac{1}{2},$$

and

$$\sup_{t\in[0,1],u\in\mathbb{R},v\in\mathbb{R}} b_2(t,u,v)k_2(u) =: L_2 \le 1.$$

Obviously the condition (F1) holds. Observe that for all $|u| \ge M := \max\{M_1, M_2\}$ the condition (F8) is satisfied. Moreover, one has

$$|f(t,u,v)| \le \frac{1}{2}(v^2+1)+1.$$

Hence, setting $\omega(|v|) = v^2+1$, the assumption (F7) is also fulfilled.

In this case $W(\xi) = \arctan(\xi)$, $\xi \in \mathbb{R}$, and $W^{-1}(\xi) = \tan(\xi)$ with $\xi \in \big(-\frac{\pi}{2}, \frac{\pi}{2}\big)$. Since $c_1 = \frac{1}{2}$ and $c_2 = 1$ in the assumption (F7), we obtain : $W(c_2)+c_1 = \frac{\pi}{4}+\frac{1}{2}$. Consequently, $W^{-1}\big(W(c_2)+c_1\big)$ exists and the estimations (3.4) and (3.5) hold.

Using Corollary 3.8, we obtain that the problem (1.1) with the functions f and g defined above has at least one nontrivial solution.

Now, we shall give the last example of conditions for which the assumptions of Theorem 2.1 are fulfilled. The following assumption upon f will be needed

(F9) $|f(t,u,v)| \le c_1 + c_2|v| + c_3|v|^p$, where $t \in [0,1]$, $u, v \in \mathbb{R}$, $p \ge 0$, $c_1 > 0$ and $c_2, c_3 \ge 0$.

Let the assumptions (F1) and (F9) hold. We have

$$|u'(t)| \le c_1 + c_2\int_0^t |u'(s)|ds + c_3\int_0^t |u'(s)|^p ds,$$

for each $t \in [0,1]$. Now, using the approach due to Lakshmikantham (cf. [11, Theorem 1]), we know that, for any solution u to the problem (2.1), $|u'(t)|$ is bounded by the solution $x(t)$ to the following problem

$$x' = c_2x + c_3x^p, \quad x(0) = c_1, \tag{3.6}$$

which can be solved explicitly as a Bernoulli equation.

Note that if $p \in \{0,1\}$ or $c_3 = 0$ then one gets the assumption (F3) and can use Gronwall's lemma to estimate $|u'(t)|$.

Let $p > 1$. In the case when $p > 1$ and $c_2 \ne 0$, we obtain

$$[x(t)]^{1-p} = \left(c_1^{1-p} + \frac{c_3}{c_2}\right)\exp\{c_2(1-p)t\} - \frac{c_3}{c_2}. \tag{3.7}$$

Observe that the solution x to the problem (3.6) will be finite for $t \in [0,1]$, if

$$\left(c_1^{1-p} + \frac{c_3}{c_2}\right)\exp\{c_2(1-p)t\} - \frac{c_3}{c_2} > 0$$

for all $t \in [0,1]$, i.e.

$$t < \frac{1}{c_2(p-1)}\ln\left(\frac{c_2}{c_3}c_1^{1-p} + 1\right)$$

for each $t \in [0,1]$. Note that the above holds, if

$$\frac{1}{c_2(p-1)} \ln\left(\frac{c_2}{c_3} c_1^{1-p} + 1\right) > 1.$$

Consequently, it is sufficient to assume that

$$c_1 < \left\{\frac{c_3}{c_2}\left[\exp\left\{c_2(p-1)\right\} - 1\right]\right\}^{\frac{1}{1-p}}. \tag{3.8}$$

Hence, if (3.8) holds, from (3.7) one has

$$\begin{aligned} |u'(t)| &\le \left\{\left(c_1^{1-p} + \frac{c_3}{c_2}\right)\exp\left\{c_2(1-p)t\right\} - \frac{c_3}{c_2}\right\}^{\frac{1}{1-p}} \\ &\le \left\{\left(c_1^{1-p} + \frac{c_3}{c_2}\right)\exp\left\{c_2(1-p)\right\} - \frac{c_3}{c_2}\right\}^{\frac{1}{1-p}}, \end{aligned}$$

and

$$|u(t)| \le a + \left\{\left(c_1^{1-p} + \frac{c_3}{c_2}\right)\exp\left\{c_2(1-p)\right\} - \frac{c_3}{c_2}\right\}^{\frac{1}{1-p}},$$

for $t \in [0,1]$, and the assumption (T1) is satisfied.

If $p > 1$ and $c_2 = 0$, then, from (3.6), we obtain

$$[x(t)]^{1-p} = c_1^{1-p} + c_3(1-p)t. \tag{3.9}$$

In this case x will be finite on $[0,1]$, if for all $t \in [0,1]$

$$t < \frac{c_1^{1-p}}{c_3(p-1)},$$

which leads to the following extra condition

$$c_1 < \left\{c_3(p-1)\right\}^{\frac{1}{1-p}}. \tag{3.10}$$

Consequently, if (3.10) is fulfilled, then, from (3.9), we obtain

$$|u'(t)| < \left[c_1^{1-p} + c_3(1-p)t\right]^{\frac{1}{1-p}} \le \left[c_1^{1-p} + c_3(1-p)\right]^{\frac{1}{1-p}}$$

and

$$|u(t)| \le a + \left[c_1^{1-p} + c_3(1-p)\right]^{\frac{1}{1-p}},$$

for $t \in [0,1]$, and the assumption (T1) holds.

By proceeding in the same way as above, one can obtain an estimation for $|u'(t)|$ in the case when $p \in (0,1)$.

Remark 3.11. The above estimation is a special case of the generalization of Gronwall's inequality due to Perov (cf. [3, p. 11], [14, p. 360]) or Willett and Wong (cf. [20, Theorem 2]).

Now, Theorems 2.1 and 2.3 imply the following corollaries.

Corollary 3.12. *Let the assumptions* (F1), (F8), (F9) *and* (G) *be fulfilled. Then the problem* (1.1) *has at least one solution.*

Corollary 3.13. *Under the assumptions* (F1), (F8), (F9) *and* (G'), *the problem* (1.1) *has at least one solution.*

Example 3.14. Set $n = 1$, $g(s) = -s$ and

$$f(t,u,v) = b_1(t,u,v)k_1(u) + b_2(t,u,v)k_2(u)m_2(|v|) + b_3(t,u,v)k_3(u)m_3(|v|),$$

where the functions b_i and k_i are continuous, there are $M_i > 0$ such that

$$uk_i(u) \geq 0,$$

for every $|u| \geq M_i$ and at least one k_i is such that $k_i(0) \neq 0$, $i = 1,2,3$. Moreover, the functions m_i, $i = 2,3$, are continuous and

$$0 \leq m_2(|v|) \leq d_2|v|, \qquad 0 \leq m_3(|v|) \leq d_3|v|^{77},$$

for $v \in \mathbb{R}$ and there exist constants $l_i > 0$ and $c_i > 0$, $i \in \{1,2,3\}$, such that

$$l_i := \inf_{t\in[0,1],u\in\mathbb{R},v\in\mathbb{R}} b_i(t,u,v),$$

$$\sup_{t\in[0,1],u\in\mathbb{R},v\in\mathbb{R}} d_i b_i(t,u,v)k_i(u) =: c_i, \qquad i \in \{2,3\}$$

and

$$\sup_{t\in[0,1],u\in\mathbb{R},v\in\mathbb{R}} b_1(t,u,v)k_1(u) =: c_1 < \left\{\frac{c_3}{c_2}\left[\exp(76c_2) - 1\right]\right\}^{-\frac{1}{76}}.$$

One can easily check that the assumptions (F1), (F8), (F9) and (G') are satisfied. Consequently, from Corollary 3.13, the problem (1.1) with the functions f and g given above has at least one nontrivial solution.

Acknowledgement

The author would like to thank the anonymous referees for their valuable comments.

References

[1] N. Aronszajn, Le correspondant topologique de l'unicité dans la théorie des équations différentielles (in French), *Ann. of Math. (2)* **43**(1942), 730–738. https://doi.org/10.2307/1968963; MR0007195

[2] I. Bihari, A generalization of a lemma of Bellman and its application to uniqueness problems of differential equations, *Acta Math. Acad. Sci. Hungar.* **7**(1956), 81–94. https://doi.org/10.1007/BF02022967; MR0079154

[3] S. S. Dragomir, *Some Gronwall type inequalities and applications*, Nova Science Publishers, Inc., Hauppauge, NY, 2003. MR2016992

[4] G. Gabor, Some results on existence and structure of solution sets to differential inclusions on the halfline, *Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8)* **5**(2002), No. 2, 431–446. MR1911199

[5] L. Górniewicz, Topological approach to differential inclusions, in: *Topological methods in differential equations and inclusions (Montreal, PQ, 1994)*, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., Vol. 472, Kluwer Acad. Publ., Dordrecht, 1995, pp. 129–190. https://doi.org/10.1007/978-94-011-0339-8_4; MR1368672

[6] L. Górniewicz, Topological structure of solution sets: current results, in: *CDDE 2000 Proceedings (Brno), Arch. Math. (Brno)* **36**(2000), 343–382. MR1822805

[7] A. Granas, R. Guenther, J. Lee, Nonlinear boundary value problems for ordinary differential equations, *Dissertationes Math. (Rozprawy Mat.)* **244**(1985), 128 pp. MR0808227

[8] G. A. Harris, On multiple solutions of a nonlinear Neumann problem, *J. Differential Equations* **95**(1992), No. 1, 75–104. https://doi.org/10.1016/0022-0396(92)90043-M; MR1142277

[9] G. Infante, P. Pietramala, F. A. F. Tojo, Non-trivial solutions of local and non-local Neumann boundary-value problems, *Proc. Roy. Soc. Edinburgh Sect. A* **146**(2016), No. 2, 337–369. https://doi.org/10.1017/S0308210515000499; MR3475301

[10] I. Kossowski, K. Szymańska-Dębowska, Solutions to resonant boundary value problem with boundary conditions involving Riemann–Stieltjes integrals, *Discrete Contin. Dyn. Syst. Ser. B* **23**(2018), No. 1, 275–282. https://doi.org/10.3934/dcdsb.2018019; MR3721843

[11] V. Lakshmikantham, Upper and lower bounds of the norm of solutions of differential equations, *Proc. Amer. Math. Soc.* **13**(1962), 615–616. https://doi.org/10.2307/2034836; MR0137878

[12] J. Mawhin, Problèmes aux limites du type de Neumann pour certaines équations différentielles ou aux dérivées partielles non linéaires, in: *Équations différentielles et fonctionnelles non linéaires , (Actes Conf. Internat. "Equa-Diff 73"*, Brussels/Louvain-la-Neuve, 1973), Hermann, Paris, 1973, pp. 123–134. MR0405183

[13] J. Mawhin, K. Szymańska-Dębowska, Second-order ordinary differential systems with nonlocal Neumann conditions at resonance, *Ann. Mat. Pura Appl. (4)* **195**(2016), No. 5, 1605–1617. https://doi.org/10.1007/s10231-015-0532-9; MR3537964

[14] D.S. Mitrinović, J. E. Pečarić, A. M. Fink, *Inequalities involving functions and their integrals and derivatives*, Mathematics and its Applications (East European Series), Vol. 53, Kluwer Academic Publishers, Dordrecht, 1991. https://doi.org/10.1007/978-94-011-3562-7; MR1190927

[15] L. C. Piccinnini, G. Stampacchia, G. Vidossich, *Ordinary differential equations in* $\mathbb{R}^n$, Applied Mathematical Sciences, Vol. 39, Springer-Verlag, New York, 1984. https://doi.org/10.1007/978-1-4612-5188-0; MR740539

[16] Y. Sun, Y. J. Cho, D. O'Regan, Positive solutions for singular second order Neumann boundary value problems via a cone fixed point theorem, *Appl. Math. Comput.* **210**(2009), No. 1, 80–86. https://doi.org/10.1016/j.amc.2008.11.025; MR2504122

[17] K. Szymańska-Dębowska, k-dimensional nonlocal boundary-value problems at resonance, *Electron. J. Diff. Equ.* **2016**, No. 148, 1–8. MR3358520

[18] K. Szymańska-Dębowska, On a generalization of the Miranda Theorem and its application to boundary value problems, *J. Differential Equations* **258**(2015), 2686–2700. https://doi.org/10.1016/j.jde.2014.12.022; MR3312640

[19] K. Szymańska-Dębowska, On second order nonlocal boundary value problem at resonance, *Fasc. Math.* **56**(2016), 143–153. https://doi.org/10.1515/fascmath-2016-0010; MR3541578

[20] D. Willett, J. S. W. Wong, On the discrete analogues of some generalizations of Gronwall's inequality, *Monatsh. Math.* **69**(1965), 362–367. https://doi.org/10.1007/BF01297622; MR0185175

[21] F. Wang, F. Zhang, Existence of positive solutions of Neumann boundary value problem via a cone compression-expansion fixed point theorem of functional type, *J. Appl. Math. Comput.* **35**(2011), No. 1–2, 341–349. https://doi.org/10.1007/s12190-009-0360-4; MR2748368