



Oscillation of a perturbed nonlinear third order functional differential equation

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Received 7 September 2017, appeared 1 May 2018

Communicated by Ferenc Hartung

Abstract. In this paper, the authors present some new results on the oscillatory and asymptotic behavior of solutions of the perturbed nonlinear third order functional differential equation

$$\left(b(t) (a(t)(x'(t))^\alpha)'\right)' + p(t)f(x(\tau(t))) = h(t, x(t), x(\tau(t)), x'(t)).$$

In addition to other conditions, the authors assume that $uf(u) > 0$ for $u \neq 0$ and f is increasing. Examples to illustrate the results are included.

Keywords: third order differential equations, perturbed equations, delay argument, oscillation, asymptotic behavior.

2010 Mathematics Subject Classification: 34C10, 34C11, 34K11.

1 Introduction

We consider the third order nonlinear functional differential equation with a perturbation term

$$\left(b(t) (a(t)(x'(t))^\alpha)'\right)' + p(t)f(x(\tau(t))) = h(t, x(t), x(\tau(t)), x'(t)), \quad (1.1)$$

where $\alpha \geq 1$ is the ratio of odd positive integers, and we assume:

(H1) $a, b, p, \tau \in C([t_0, \infty))$ are positive;

(H2) $f : \mathbb{R} \rightarrow \mathbb{R}$ and $h : [t_0, \infty) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous, $uf(u) > 0$ for $u \neq 0$, and f is nondecreasing;

(H3) $f(uv) \geq f(u)f(v)$ for $uv > 0$;

(H4) $\tau(t) \leq t$ and $\lim_{t \rightarrow \infty} \tau(t) = \infty$.

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By a *solution* of (1.1) we mean a function $x(t)$ whose quasi-derivatives $a(t)(x'(t))^\alpha$ and $(a(t)(x'(t))^\alpha)'$ are continuous on $[T_x, \infty)$, $T_x \geq t_0$, and which satisfies Eq. (1.1) on $[T_x, \infty)$. We consider only those solutions $x(t)$ of (1.1) that satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq T_x$. A solution of (1.1) is said to be *oscillatory* if it has arbitrarily large zeros, and *nonoscillatory* otherwise. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

In two very nice papers Baculiková and Džurina studied the oscillatory and asymptotic behavior of solutions of some third order nonlinear delay differential equations. In [1], they considered the equation

$$(b(t)(x''(t))^\alpha)' + p(t)f(x(\tau(t))) = 0$$

under the same covering assumptions as those above and assumed that

$$\int_{t_0}^{\infty} \frac{1}{b(s)^{\frac{1}{\alpha}}} ds = \infty.$$

In [2], they considered the equation

$$\left(b(t) (a(t)(x'(t)))' \right)' + p(t)f(x(\tau(t))) - q(t)h(x(\sigma(t))) = 0$$

under conditions (H1)–(H4),

$$\int_{t_0}^{\infty} \frac{1}{b(s)} ds = \infty, \quad \int_{t_0}^{\infty} \frac{1}{a(s)} ds = \infty, \quad (1.2)$$

and

$$\int_{t_0}^{\infty} \frac{1}{a(t)} \int_t^{\infty} \frac{1}{b(s)} \int_s^{\infty} q(u) du ds < \infty. \quad (1.3)$$

They employed a new technique to obtain some interesting results on the oscillatory and asymptotic behavior of solutions (see [2, Theorem 2.1]). They obtained another oscillation result (see [2, Theorem 2.6]) by replacing condition (1.3) with

$$\int_{t_0}^{\infty} \frac{1}{a(t)} \int_{t_0}^t \frac{1}{b(s)} \int_{t_0}^s q(u) du ds dt < \infty. \quad (1.4)$$

Notice that condition (1.3) implies that q is small in that we must have

$$\int_{t_0}^{\infty} q(u) du < \infty \quad \text{and} \quad \int_{t_0}^{\infty} \frac{1}{b(s)} \int_s^{\infty} q(u) du ds < \infty.$$

Condition (1.4) requires q to be small in some sense relative to b and a .

Our goal here is to establish oscillation results for equation (1.1) without imposing a “smallness” condition on the perturbation term. We also present some results on the boundedness and oscillatory behavior of a special case of (1.1), namely,

$$\left(b(t) (a(t)(x'(t))^\alpha) \right)' + p(t)x^\beta(t) = e(t) + q(t)x^\gamma(t), \quad (1.5)$$

where β and γ are the ratios of odd positive integers with $\beta > \gamma$ and $e : [t_0, \infty) \rightarrow \mathbb{R}$ is a continuous function. As was done in [1,2], we will use a comparison approach.

2 Oscillation of equation (1.1)

We assume that there exists a positive continuous function $q : [t_0, \infty) \rightarrow \mathbb{R}^+$ such that

$$(H5) \quad |h(t, u, v, w)| \leq q(t)f(v) \text{ for all } (t, u, v, w) \in [t_0, \infty) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}.$$

For any $t_1 \geq t_0$, we set

$$I(t; t_1) = \int_{t_1}^{\tau(t)} \left(\frac{1}{a(u)} \int_{t_1}^u \frac{1}{b(s)} ds \right)^{1/\alpha} du. \quad (2.1)$$

We also assume that there are functions $\zeta, \eta \in C^1[t_0, \infty)$ satisfying

$$\tau(t) \leq \zeta(t) \leq \eta(t) \leq t \text{ for all large } t, \quad (2.2)$$

and set

$$I^*(t) = \left(\int_{\tau(t)}^{\zeta(t)} \frac{1}{a^{1/\alpha}(s)} ds \right) \left(\int_{\zeta(t)}^{\eta(t)} \frac{1}{b(s)} ds \right)^{1/\alpha}, \quad (2.3)$$

and

$$Q(t) = p(t) - q(t). \quad (2.4)$$

In some of our results we will also ask that

$$\int_{t_0}^{\infty} \frac{1}{b(s)} ds = \infty \quad \text{and} \quad \int_{t_0}^{\infty} \frac{1}{a^{1/\alpha}(s)} ds = \infty. \quad (2.5)$$

Our first oscillation result is contained in the following theorem.

Theorem 2.1. *Let $Q(t) > 0$ for large t , conditions (H1)–(H5), (2.2), and (2.5) hold, and assume that all solutions of the first order delay differential equations*

$$y'(t) + Q(t)f(I(t))f(y^{1/\alpha}(\tau(t))) = 0 \quad (2.6)$$

and

$$z'(t) + Q(t)f(I^*(t))f(z^{1/\alpha}(\eta(t))) = 0 \quad (2.7)$$

are oscillatory. Then equation (1.1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1). Without loss of generality we may assume that $x(t)$ and $x(\tau(t))$ are positive and condition (2.2) holds for $t \geq t_1$ for some $t_1 \geq t_0$. If $x(t)$ is eventually negative, a similar proof holds. From our assumptions and equation (1.1), we see that

$$\left(b(t) (a(t)(x'(t))^\alpha)' \right)' \leq -Q(t)f(x(\tau(t))) < 0, \quad (2.8)$$

for all $t \geq t_1$.

It is easy to see that we need to consider the following two cases:

(I) $a(t)(x'(t))^\alpha > 0$ and $b(t) (a(t)(x'(t))^\alpha)' > 0$, or

(II) $a(t)(x'(t))^\alpha < 0$ and $b(t) (a(t)(x'(t))^\alpha)' > 0$

for $t \geq t_2$ for some $t_2 \geq t_1$. We will first examine Case (I). For $t \geq t_2$, we see that

$$a(t)(x'(t))^\alpha \geq \int_{t_2}^t \frac{1}{b(s)} b(s) (a(s)(x'(s))^\alpha)' ds \geq \left(\int_{t_2}^t \frac{1}{b(s)} ds \right) b(t) (a(t)(x'(t))^\alpha)',$$

or

$$x'(t) \geq \left(\frac{1}{a(t)} \int_{t_2}^t \frac{1}{b(s)} ds \right)^{1/\alpha} \left(b(t) (a(t)(x'(t))^\alpha)' \right)^{1/\alpha}.$$

Integrating this inequality from t_2 to $\tau(t) \geq t_2$, we have

$$\begin{aligned} x(\tau(t)) &\geq \left(\int_{t_2}^{\tau(t)} \left(\frac{1}{a(u)} \int_{t_2}^u \frac{1}{b(s)} ds \right)^{1/\alpha} du \right) \left(b(t) (a(t)(x'(t))^\alpha)' \right)^{1/\alpha} \\ &= I(t; t_2) y^{1/\alpha}(t), \end{aligned} \quad (2.9)$$

where $y(t) = b(t) (a(t)(x'(t))^\alpha)'$. Using (2.9) in (2.8) and applying (H3), we obtain

$$y'(t) + Q(t)f(I(t))f(y^{1/\alpha}(\tau(t))) \leq 0 \quad \text{for } t \geq t_2.$$

It follows from [11, Corollary 1] that the corresponding differential equation (2.6) also has a positive solution. This contradiction completes the proof for Case (I).

For Case (II), it is easy to see that

$$\begin{aligned} -a(\zeta(t))(x'(\zeta(t)))^\alpha &\geq \left(\int_{\zeta(t)}^{\eta(t)} \frac{1}{b(s)} ds \right) b(\eta(t)) (a(\eta(t))(x'(\eta(t)))^\alpha)' \\ &= \left(\int_{\zeta(t)}^{\eta(t)} \frac{1}{b(s)} ds \right) z(\eta(t)), \end{aligned} \quad (2.10)$$

for $t \geq t_2$, where $z(t) = b(t) (a(t)(x'(t))^\alpha)'$.

Now $x' = \left[\frac{a(x')^\alpha}{a} \right]^{1/\alpha} = \frac{1}{a^{1/\alpha}} [a(x')^\alpha]^{1/\alpha}$ so integrating for $v \geq u \geq t_2$, we have

$$x(v) - x(u) = \int_u^v \frac{1}{a^{1/\alpha}(s)} [a(s)(x'(s))^\alpha]^{1/\alpha} ds,$$

or

$$x(u) - x(v) = \int_u^v \frac{1}{a^{1/\alpha}(s)} [-a(s)(x'(s))^\alpha]^{1/\alpha} ds \geq \int_u^v \frac{1}{a^{1/\alpha}(s)} ds [-a(v)(x'(v))^\alpha]^{1/\alpha} ds.$$

Hence,

$$x(u) \geq \left(\int_u^v \frac{1}{a^{1/\alpha}(s)} ds \right) \left(-a(v)(x'(v))^\alpha \right)^{1/\alpha}.$$

Setting $u = \tau(t)$ and $v = \zeta(t)$ in the above inequality, we obtain

$$x(\tau(t)) \geq \left(\int_{\tau(t)}^{\zeta(t)} \frac{1}{a^{1/\alpha}(s)} ds \right) \left(-a(\zeta(t))(x'(\zeta(t)))^\alpha \right)^{1/\alpha}. \quad (2.11)$$

From (2.10) and (2.11) we see that

$$x(\tau(t)) \geq \left(\int_{\tau(t)}^{\zeta(t)} \frac{1}{a^{1/\alpha}(s)} ds \right) \left(\int_{\zeta(t)}^{\eta(t)} \frac{1}{b(s)} ds \right)^{1/\alpha} z^{1/\alpha}(\eta(t)) =: I^*(t) z^{1/\alpha}(\eta(t)). \quad (2.12)$$

Using (2.12) in equation (2.8), we have

$$z'(t) + Q(t)f(I^*(t))f(z^{1/\alpha}(\eta(t))) \leq 0.$$

It follows from [11, Corollary 1] that the corresponding differential equation (2.7) also has a positive solution, which is a contradiction. This completes the proof of the theorem. \square

The next two corollaries follow immediately from known oscillation criteria for first order delay differential equations; for example, see [9, Theorem 2].

Corollary 2.2. *Let $f(x) = x^\alpha$, $Q(t) > 0$ for all large t , and conditions (H1), (H4), (2.2), and (2.5) hold. If*

$$\liminf_{t \rightarrow \infty} \int_{\tau(t)}^t Q(s) I^\alpha(s) ds > \frac{1}{e} \quad \text{and} \quad \liminf_{t \rightarrow \infty} \int_{\eta(t)}^t Q(s) (I^*(s))^\alpha ds > \frac{1}{e}, \quad (2.13)$$

then equation (1.1) is oscillatory.

Corollary 2.3. *Let $f(x) = x^\beta$, $\beta/\alpha \in (0, 1)$, $Q(t) > 0$ for all large t , and conditions (H1), (H4), (2.2), and (2.5) hold. If*

$$\limsup_{t \rightarrow \infty} \int_{\tau(t)}^t Q(s) I^\beta(s) ds > 0 \quad \text{and} \quad \limsup_{t \rightarrow \infty} \int_{\eta(t)}^t Q(s) (I^*(s))^\beta ds > 0, \quad (2.14)$$

then equation (1.1) is oscillatory.

The following example illustrates the above results.

Example 2.4. Consider the equation

$$\left(t (t^3 (x'(t))^3) \right)' + p(t) x^3(\lambda_1 t) = q(t) \frac{x^3(\lambda_1 t) x'(t)}{(1 + x^2(\lambda_1 t))(1 + |x'(t)|)}, \quad t \geq 1. \quad (2.15)$$

Here $a(t) = t^3$, $b(t) = t$, $\tau(t) = \lambda_1 t$, $\xi(t) = \lambda_2 t$ and $\eta(t) = \lambda_3 t$, where $0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq 1$, $h(t, x(t), x(\tau(t)), x'(t)) = q(t) \frac{x^3(\tau(t)) x'(t)}{(1 + x^2(\tau(t)))(1 + |x'(t)|)}$, $f(x) = x^3$, and $\alpha = 3$. Let $p(t)$ and $q(t)$ be positive continuous functions with $Q(t) = p(t) - q(t)$ positive for all large t . Now,

$$I(t; 1) = \int_1^{\lambda_1 t} \left(\frac{1}{u^3} \int_1^u \frac{1}{s} ds \right)^{1/3} du = \frac{3}{4} (\ln \lambda_1 t)^{4/3}$$

and

$$I^*(t) = \left(\int_{\lambda_1 t}^{\lambda_2 t} \frac{1}{s} ds \right) \left(\int_{\lambda_2 t}^{\lambda_3 t} \frac{1}{s} ds \right)^{1/3} = \left(\ln \frac{\lambda_2}{\lambda_1} \right) \left(\ln \frac{\lambda_3}{\lambda_2} \right)^{1/3}.$$

If both of the equations

$$y'(t) + Q(t) I^3(t) y(\lambda_1 t) = 0$$

and

$$z'(t) + Q(t) (I^*(t))^3 z(\lambda_3 t) = 0$$

are oscillatory, then equation (2.15) is oscillatory.

Instead of condition (2.2), we assume that there exists a function $\rho(t) \in C^1([t_0, \infty))$ satisfying

$$\rho'(t) > 0, \quad \rho(t) > t, \quad \omega(t) = \rho(\rho(\tau(t))) < t, \quad (2.16)$$

and we set

$$I^{**}(t) = \int_{\tau(t)}^{\rho(\tau(t))} \left(\frac{1}{a(u)} \int_u^{\rho(u)} \frac{1}{b(s)} ds \right)^{1/\alpha} du. \quad (2.17)$$

We can then obtain the following theorem.

Theorem 2.5. Let $Q(t) > 0$ for large t , conditions (H1)–(H5), (2.5), and (2.16) hold. If all solutions of equations (2.6) and

$$z'(t) + Q(t)f(I^{**}(t))f(z^{1/\alpha}(\omega(t))) = 0 \quad (2.18)$$

are oscillatory, then equation (1.1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1) such that $x(t)$ and $x(\tau(t))$ are positive and condition (2.16) holds for $t \geq t_1$ for some $t_1 \geq t_0$. Proceeding as in the proof of Theorem 2.1 we again obtain (2.8). The proof for Case (I) holding is similar to that of Theorem 2.1 and hence is omitted.

If Case (II) holds, it is easy to see that

$$\begin{aligned} -a(t)(x'(t))^\alpha &\geq \int_t^{\rho(t)} \frac{1}{b(s)} \left(b(s) (a(s)(x'(s))^\alpha)' \right) ds \\ &\geq \left(\int_t^{\rho(t)} \frac{1}{b(s)} ds \right) \left(b(\rho(t)) (a(\rho(t))(x'(\rho(t)))^\alpha)' \right) \\ &= \left(\int_t^{\rho(t)} \frac{1}{b(s)} ds \right) z(\rho(t)), \end{aligned} \quad (2.19)$$

where $z(t) = b(t) (a(t)(x'(t))^\alpha)'$. Dividing by $a(t)$ and integrating from $\tau(t)$ to $\rho(\tau(t))$, we obtain

$$\begin{aligned} x(\tau(t)) &\geq \int_{\tau(t)}^{\rho(\tau(t))} \left(\frac{z(\rho(u))}{a(u)} \right)^{1/\alpha} \left(\int_u^{\rho(u)} \frac{1}{b(s)} ds \right)^{1/\alpha} du \\ &\geq I^{**}(t) z^{1/\alpha}(\omega(t)), \end{aligned} \quad (2.20)$$

for all large t . Using (2.20) in (2.8) and proceeding as in the proof of Case (II) in Theorem 2.1, we arrive at the desired contradiction. This completes the proof of the theorem. \square

To illustrate this result we have the following example.

Example 2.6. Consider Example 2.4 with $\rho(t) = \theta t$ and $\theta > 1$. Now $\omega(t) = \theta^2 \lambda_1 t$ and

$$I^{**}(t) = \int_{\lambda_1 t}^{\theta \lambda_1 t} \left(\frac{1}{u^3} \int_u^{\theta u} \frac{1}{s} ds \right)^{1/3} du = \int_{\lambda_1 t}^{\theta \lambda_1 t} \frac{1}{u} \left(\ln \frac{\theta u}{u} \right)^{1/3} du = (\ln \theta)^{4/3}.$$

If $\theta^2 \lambda_1 t \leq 1$ and the equations

$$y'(t) + Q(t)I^3(t)y(\lambda_1 t) = 0$$

and

$$z'(t) + Q(t) (\ln \theta)^4 z(\omega(t)) = 0$$

are oscillatory, then equation (2.15) is oscillatory by Theorem 2.5.

3 Boundedness and oscillation of equation (1.5)

In order to obtain our results in this section, we need the following lemma.

Lemma 3.1 (Young's inequality). *Let X and Y be nonnegative, $n > 1$, and $1/n + 1/m = 1$. Then*

$$XY \leq \frac{1}{n}X^n + \frac{1}{m}Y^m, \quad (3.1)$$

and equality holds if and only if $Y = X^{n-1}$.

Theorem 3.2. *In addition to condition (H1), assume that*

$$\int_{t_0}^{\infty} \frac{1}{b(s)} ds < \infty, \quad \int_{t_0}^{\infty} \frac{1}{a^{1/\alpha}(s)} ds < \infty, \quad \int_{t_0}^{\infty} \frac{1}{b(u)} \int_{t_0}^u q^{\frac{\beta}{\beta-\gamma}}(s) p^{\frac{\gamma}{\gamma-\beta}}(s) ds du < \infty, \quad (3.2)$$

and

$$\int_{t_0}^{\infty} \frac{1}{b(u)} \int_{t_0}^u |e(s)| ds du < \infty. \quad (3.3)$$

Then every nonoscillatory solution of equation (1.5) is bounded.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.5) such that $x(t) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. Applying (3.1) to $[q(s)x^\gamma(s) - p(s)x^\beta(s)]$ with

$$n = \frac{\beta}{\gamma} > 1, \quad X(s) = x^\gamma(s), \quad Y = \frac{\gamma}{\beta} \left(\frac{q(s)}{p(s)} \right), \quad \text{and} \quad m = \frac{\beta}{\beta - \gamma},$$

we obtain

$$\begin{aligned} q(s)x^\gamma(s) - p(s)x^\beta(s) &= \frac{\beta}{\gamma} p(s) \left[x^\gamma(s) \frac{\gamma}{\beta} \frac{q(s)}{p(s)} - \frac{\gamma}{\beta} (x^\gamma(s))^{\beta/\gamma} \right] \\ &= \frac{\beta}{\gamma} p(s) \left[XY - \frac{1}{n} X^n \right] \\ &\leq \frac{\beta}{\gamma} p(s) \left(\frac{1}{m} Y^m \right) \\ &= \left(\frac{\beta - \gamma}{\gamma} \right) \left[\frac{\gamma}{\beta} q(s) \right]^{\frac{\beta}{\beta-\gamma}} (p(s))^{\frac{\gamma}{\gamma-\beta}}. \end{aligned} \quad (3.4)$$

From equation (1.5) we then have

$$\left(b(t) (a(t)(x'(t))^\alpha) \right)' \leq |e(t)| + \left(\frac{\beta - \gamma}{\gamma} \right) \left[\frac{\gamma}{\beta} q(t) \right]^{\frac{\beta}{\beta-\gamma}} (p(t))^{\frac{\gamma}{\gamma-\beta}} = |e(t)| + cq^{\frac{\beta}{\beta-\gamma}}(t)p^{\frac{\gamma}{\gamma-\beta}}(t),$$

where $c = \left(\frac{\beta-\gamma}{\gamma} \right) \left(\frac{\gamma}{\beta} \right)^{\frac{\beta}{\beta-\gamma}}$. Integrating this inequality from t_1 to t gives

$$(a(t)(x'(t))^\alpha)' \leq \frac{c_1}{b(t)} + \frac{1}{b(t)} \int_{t_1}^t |e(s)| ds + \frac{c}{b(t)} \int_{t_1}^t q^{\frac{\beta}{\beta-\gamma}}(s) p^{\frac{\gamma}{\gamma-\beta}}(s) ds,$$

where $c_1 = b(t_1) (a(t_1)(x'(t_1))^\alpha)'$. Another integration yields

$$\begin{aligned} a(t)(x'(t))^\alpha &\leq c_2 + c_1 \int_{t_1}^t \frac{1}{b(s)} ds + c \int_{t_1}^t \frac{1}{b(u)} \int_{t_1}^u q^{\frac{\beta}{\beta-\gamma}}(s) p^{\frac{\gamma}{\gamma-\beta}}(s) ds du \\ &\quad + \int_{t_1}^t \frac{1}{b(u)} \int_{t_1}^u |e(s)| ds du, \end{aligned}$$

where $c_2 = a(t_1)(x'(t_1))^\alpha$. From condition (3.2) and (3.3), there exists a constant C such that

$$x'(t) \leq \left(\frac{C}{a(t)} \right)^{1/\alpha}.$$

Integrating this inequality from t_1 to t and using condition (3.2), we arrive at the desired conclusion. \square

The following result is concerned with the oscillation of equation (1.5).

Theorem 3.3. *If*

$$\int_{t_0}^{\infty} \frac{1}{a^{1/\alpha}(s)} ds < \infty, \quad \int_{t_0}^{\infty} \left[\frac{1}{a(u)} \int_{t_0}^u \frac{1}{b(s)} ds \right]^{1/\alpha} du < \infty, \quad (3.5)$$

$$\int_{t_0}^{\infty} \left[\frac{1}{a(v)} \int_{t_0}^v \frac{1}{b(u)} \int_{t_0}^u q^{\beta/\alpha-\gamma}(s) p^{\gamma/\alpha-\beta}(s) ds du \right]^{1/\alpha} dv < \infty, \quad (3.6)$$

$$\liminf_{t \rightarrow \infty} \int_{t_0}^t \left[\frac{1}{a(v)} \int_{t_0}^v \frac{1}{b(u)} \int_{t_0}^u e(s) ds du \right]^{1/\alpha} dv = -\infty, \quad (3.7)$$

and

$$\limsup_{t \rightarrow \infty} \int_{t_0}^t \left[\frac{1}{a(v)} \int_{t_0}^v \frac{1}{b(u)} \int_{t_0}^u e(s) ds du \right]^{1/\alpha} dv = +\infty, \quad (3.8)$$

then equation (1.5) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.5), say $x(t) > 0$ for $t \geq t_1 \geq t_0$. Proceeding as in the proof of Theorem 3.2, we obtain

$$\begin{aligned} (b(t)(a(t)(x'(t))^\alpha)')' &\leq e(t) + \left(\frac{\beta-\gamma}{\gamma} \right) \left[\frac{\gamma}{\beta} q(t) \right]^{\beta/\alpha-\gamma} (p(t))^{\gamma/\alpha-\beta} \\ &= e(t) + cq^{\beta/\alpha-\gamma}(t)p^{\gamma/\alpha-\beta}(t), \end{aligned}$$

where $c = \left(\frac{\beta-\gamma}{\gamma} \right) \left(\frac{\gamma}{\beta} \right)^{\beta/\alpha-\gamma}$. Integrating this inequality from t_1 to t , we obtain

$$(a(t)(x'(t))^\alpha)' \leq \frac{c_1}{b(t)} + \frac{1}{b(t)} \int_{t_1}^t e(s) ds + \frac{c}{b(t)} \int_{t_1}^t q^{\beta/\alpha-\gamma}(s) p^{\gamma/\alpha-\beta}(s) ds,$$

where $c_1 = b(t_1)(a(t_1)(x'(t_1))^\alpha)'$. Integrating one more time, we have

$$a(t)(x'(t))^\alpha \leq c_2 + c_1 \int_{t_1}^t \frac{1}{b(s)} ds + c \int_{t_1}^t \frac{1}{b(u)} \int_{t_1}^u q^{\beta/\alpha-\gamma}(s) p^{\gamma/\alpha-\beta}(s) ds du + \int_{t_1}^t \frac{1}{b(u)} \int_{t_1}^u e(s) ds du,$$

where $c_2 = a(t_1)(x'(t_1))^\alpha$.

Dividing by $a(t)$ gives

$$\begin{aligned} x'(t) \leq \left\{ \frac{c_2}{a(t)} + \frac{c_1}{a(t)} \int_{t_1}^t \frac{1}{b(s)} ds + \frac{c}{a(t)} \int_{t_1}^t \frac{1}{b(u)} \int_{t_1}^u q^{\beta/\alpha-\gamma}(s) p^{\gamma/\alpha-\beta}(s) ds du \right. \\ \left. + \frac{1}{a(t)} \int_{t_1}^t \frac{1}{b(u)} \int_{t_1}^u e(s) ds du \right\}^{1/\alpha}. \end{aligned}$$

Using the fact that $(u + v)^\gamma \leq u^\gamma + v^\gamma$ for $u, v \geq 0$ and $0 < \gamma < 1$, and integrating again, we have

$$x(t) \leq x(t_1) + \int_{t_1}^t \left[\frac{c_2}{a(s)} \right]^{\frac{1}{\alpha}} ds + \int_{t_1}^t \left[\frac{c_1}{a(u)} \int_{t_1}^u \frac{1}{b(s)} ds \right]^{\frac{1}{\alpha}} du \\ + \int_{t_1}^t \left[\frac{c}{a(v)} \int_{t_1}^v \frac{1}{b(u)} \int_{t_1}^u q^{\frac{\beta}{\beta-\gamma}}(s) p^{\frac{\gamma}{\gamma-\beta}}(s) ds du \right]^{\frac{1}{\alpha}} dv + \int_{t_1}^t \left[\frac{1}{a(v)} \int_{t_1}^v \frac{1}{b(u)} \int_{t_1}^u e(s) ds du \right]^{\frac{1}{\alpha}} dv,$$

Taking \liminf of both sides of the above inequality as $t \rightarrow \infty$ and applying conditions (3.5)–(3.7), we obtain a contradiction to $x(t)$ being a positive solution. The proof in case $x(t)$ is eventually negative is similar. \square

The following examples illustrate the above results.

Example 3.4. Consider the equation

$$\left(t^6 (t^6 (x'(t))^3) \right)' + \frac{1}{t^4} x^5(t) = \frac{1}{t^6} x^3(t), \quad t \geq 1. \tag{3.9}$$

All conditions of Theorem 3.2 are satisfied with $e(t) = 0$ and so every nonoscillatory solution of (3.9) is bounded. One such solution is $x(t) = 1/t$.

Example 3.5. Consider the equation

$$\left(t^3 (t^4 (x'(t))^3) \right)' + \frac{1}{t^4} x^5(t) = \frac{1}{t^6} x^3(t) + t^6 \sin t. \tag{3.10}$$

It is easy to check that all the hypotheses of Theorem 3.3 are satisfied and hence all solutions of (3.10) are oscillatory.

Concluding Remarks. With suitable care, the nonlinearity $(x')^\alpha$ in equation (1.1) can be replaced with $|x'|^\alpha \operatorname{sgn}(x')$. There do not appear to be any criteria to ensure the nonoscillation of all solutions of equation (1.1). This would be an interesting topic to explore.

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