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DYNAMIC RESPONSE OF SIMPLE SYSTEMS TO PERIODIC FORCES

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ABSTRACT

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A study of the response of viscously damped single-degree-of-freedom systems to non-harmonic periodic excitations is presented. The objectives have been (1) to assess the effects of the various factors that affect the response of such systems; and (2) to present information and concepts with which the salient features of the response may be identified readily.

The following aspects of the response are examined: (a) the steady-state response, which is the response obtained after the free vibrational component is damped and the resulting motion repeats itself; (b) the absolute maximum response, which is generally obtained prior to the attainment of the steady-state response; (c) the rate of "build-up" of the response; and (d) the effects of possible cessation of the excitation.

The factors investigated include the characteristics of the structure and the excitation. Special attention is paid to the behavior of low-frequency systems. For a number of excitations, closed-form expressions are also presented for the steady-state response of undamped systems.

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I. INTRODUCTION

The response of structures to periodic forces is of interest in a variety of applications. Periodic excitations may be induced by imbalances in rotating or reciprocating machinery, or may be associated with the vortices induced in structures subjected to steady winds. Additionally, the forces induced by waves on offshore structures are often assumed to be periodic. The periodic response of structures also is of interest as a step in evaluating their transient response by Fourier transform techniques.

The following aspects of the response of structures to periodic excitations are of interest: (1) the steady-state response, which is the response obtained after a sufficiently long time such that the free vibrational component is damped and the resulting motion repeats itself; (2) the absolute maximum response, which is generally obtained prior to the attainment of the steady-state response; (3) the rate of "build-up" of the response; and (4) the effects of possible cessation of the excitation before attainment of the steady-state condition.

The methods for evaluating the response of linear structures to periodic excitation are well established. There is, however, a paucity of information regarding the response characteristics of such systems. Even for the simplest possible single-degree-of-freedom system, the available information refers almost exclusively to harmonic excitations or only to limited aspects of the response. The response characteristics of systems subjected to nonharmonic periodic excitations cannot, in general, be predicted without detailed analyses.

The objectives of this study are: (1) to assess the effects of the various factors that affect the response of viscously damped single-degree-of-freedom systems to periodic forces; and (2) to present information and concepts with which the salient features of the response of such systems may be identified readily. Due consideration is given to each of the four aspects of the response enumerated above. Both harmonic and nonharmonic periodic excitations are considered. The factors investigated include the form and duration of the excitation, and the natural frequency and damping of the system. The results are displayed in the form of response histories and response spectra, using tripartite logarithmic plots to emphasize limiting trends. The sensitivity of the response to variations in the characteristics of the structure and excitation are examined through analytical and numerical means, and simple expressions are presented for the limiting behavior of very flexible and very stiff systems. For a number of excitations, closed-form expressions are also presented for the steady-state response and the associated maximum response of undamped systems.

Special attention is given to the behavior of low-frequency, compliant systems which are becoming of increasing importance in offshore construction. It is shown that the maximum steady-state response of such systems can be significantly different from the corresponding maximum transient response. The reason for this difference is identified, and simple practical procedures are presented for estimating the two responses.

II. SYSTEMS, EXCITATIONS AND METHOD OF SOLUTION

The systems investigated are viscously damped, single-degree-of-freedom linear systems subjected to periodic excitations of relatively simple forms. The motion for the system is defined by

$$\ddot{x} + 2\zeta p \dot{x} + p^2 x = \frac{1}{m} P(t) \tag{2.1}$$

in which P(t) is the exciting force at any time t; x, ζ and p are the displacement, fraction of critical damping and circular natural frequency of the system, respectively; and a dot superscript denotes differentiation with respect to time.

Some of the force histories considered are shown in Fig. 1. They include the alternating step and alternating versine functions shown in parts (a) and (b), the two sequences of half-sine waves shown in parts (c) and (d), and the two sequences of triangular pulses shown in parts (e) and (f). The alternating versine excitation may represent the drag component of the force induced by a harmonic wave.

Special attention is given to the effects of the alternating step function mainly because of its simplicity and because the response to this excitation reflects the salient features of the response to more complex excitations as well.

The transient response of the system was evaluated by a step-by-step integration procedure assuming that the exciting force varies linearly within each integration step, Δt. The integration step was in all cases small compared to both the period of the excitation and the natural period of the system. Specifically, Δt was taken equal to or less than 1/20th the undamped natural period of the system, and less than 1/40th the period of the excitation, whichever was smaller. The first criterion controls the analysis of high-frequency systems, whereas the second controls the analysis of low-frequency systems. In addition to ensuring that the response of the system is evaluated for a reasonable number of points for each cycle of the excitation, the latter criterion ensures that the excitation itself is adequately represented by the piecewise linear approximation employed in the solution.

The steady-state responses were in all cases computed from the corresponding transient responses by application of the technique described in Ref. 3. This technique has as its basis the fact that the difference between the transient and steady-state responses for a typical force cycle arises from differences in the initial states of the two motions. The steady-state response may, therefore, be obtained from the corresponding transient response simply by superimposing a corrective solution which appropriately modifies the initial state of the transient response. The analyses were implemented with the aid of a special purpose computer program.

III. STEADY-STATE RESPONSE

Of the four aspects of the response of periodically excited systems referred to in the Introduction, the steady-state response is probably best understood. Even for this case, however, the steady-state response of systems to nonharmonic periodic excitations cannot be identified readily from available information. The information in this chapter is intended to provide improved insight into the characteristics of the steady-state response and into the parameters that control it.

3.1 Representative Response Histories

The solid lines in Fig. 2 represent the steady-state displacements of systems with 10 percent of critical damping, $\zeta = 0.10$, subjected to the alternating step force shown in part (a) of Fig. 1. The results are normalized with respect to x_{st}, the static displacement induced by the peak value of the applied force. Six different values of the frequency parameter, ft_o, are considered, in which $f = p/2\pi$ is the undamped natural frequency of the system in cycles per unit of time, and t_o is the period of the exciting force.

Also shown in dashed lines are the corresponding transient responses from which the steady-state solutions were obtained. In the development of the transient solutions, the system was presumed to be initially at rest. The initial conditions of the steady-state solution are naturally generally different from zero, and these conditions are the same as those at the end of the force cycle.

These plots reveal that the response histories depend importantly on the value of the frequency parameter, ft_o. The larger this parameter, the greater is the number of oscillations per cycle of the forcing function. This is true of both the transient and steady-state response histories. There is, in fact, a close resemblance between the two sets of curves, a fact which the method of analysis employed makes readily apparent. The similiarity in the two sets of response curves is particularly great for the higher values of ft_o.

The highly irregular nature of the response curves for systems with high values of ft_o makes it clear that had the response been computed by the Fourier series approach, an unusually large number of terms in the series would have been required to achieve good accuracy. The method of analysis employed avoids this complexity.

Although the general shapes of the response histories for the steady state and the associated transient response are similar, the peak values of the two responses may be significantly different at critical values of ft_o. A comparison of these peak values for systems with $\zeta=0.05$ is provided in Fig. 3, in which the amplification factors, AF, are plotted as a function of ft_o. The amplification factor is defined as the absolute value of the maximum response, normalized with respect to x_{st}.

In the low-frequency region of Fig. 3, two sets of results are presented for the maximum transient response. One identifies the maximum response during the period that the exciting force acts on the system, whereas the other identifies the value of the peak response following termination of the excitation.

The peaks of the steady-state response spectrum occur at natural frequency values equal to the frequencies of the harmonic components in a Fourier series expansion of the excitation. As would be expected, the amplifications factors for the maximum steady-state response in the neighborhood of these frequency values are significantly greater than those for the transient response induced by one cycle of the forcing function. By contrast, the transient response values are larger than the corresponding steady-state values over the remainder of the response spectrum. The percentage differences between the two sets of results are particularly large for small values of the frequency parameter. This may be appreciated better from Fig. 4, which displays on logarithmic scales the same information as that presented in Fig. 3.

3.2 Steady-State Response Spectra

The response spectra for steady-state displacement presented in Figs. 3 and 4 were for systems with $\zeta = 0.05$. In Fig. 5 are presented similar spectra for systems having four different damping values in the range of zero and 20 percent of the critical value.

As would be expected, damping decreases the resonant peaks. The reductions corresponding to a given damping value are generally significantly greater for the higher order resonant peaks than the first order peak. For example, the amplification factor for systems with $\zeta = 0.05$ is 12.7 at the first resonant peak, 5.5 at the second resonant peak, and 4.1 at the third resonant peak. It is of interest to note further that, within certain ranges of the frequency parameter, an increase in damping is associated with an increase, rather than with a decrease, in the amplification factor for steady-state response.

In Fig. 6 are presented response spectra for the steady-state displacement of systems subjected to the alternating versine force shown in part (b) of Fig. 1. The general trends of these spectra are the same as those of the spectra presented in Fig. 5, except that the higher order resonant peaks are sharper than those for the step pulse, and the effect of damping on the magnitudes of these peaks is also greater. These differences are consequences of the fact that the higher order terms in a Fourier series expansion of the versine excitation are not as large in comparison to the fundamental term as are those for the step pulse. Accordingly, their contributions to the steady-state response are correspondingly smaller.

Further insight into the characteristics of the steady-state response spectra may be gained from a comparison of the information presented in Figs. 7a through 7c. In part (a) are given the well known response spectra for the steady-state response of systems excited by a sinusoidal force, and in parts (b) and (c) are given the corresponding spectra for systems excited by each of the two sequences of half-sine pulses shown in parts (c) and (d) of Fig. 1. These data are intended to demonstrate the influence that the direction of the excitation pulses has on the ensuing response.

Note that the number, magnitude, and generally the location of the resonant peaks are different in the three cases. The location of the peaks for the discontinuous half-sine pulses can readily be determined from Fourier expansions of the forcing functions. For the 'half-sine' excitation shown in part (c) of Fig. 1, this expansion is

$$P(t) = p\left[\frac{1}{\pi} + \frac{1}{2}\sin\omega t - \frac{2}{\pi}\sum_{k=2,4,6\ldots}^{\infty} \frac{1}{k^2-1}\cos k\omega t \right] \qquad (3.1)$$

and for the 'absolute sine' shown in part (d), it is

$$P(t) = p\left[\frac{2}{\pi} - \frac{4}{\pi}\sum_{k=2,4,6\ldots}^{\infty} \frac{1}{k^2-1}\cos k\omega t \right] \qquad (3.2)$$

in which P represents the peak value of the exciting force. It should be clear that the 'absolute sine' excitation will exhibit resonant response peaks at values of ft_o equal to an even integer, whereas the 'half-sine' pulse will exhibit, in addition, a peak at a value of $ft_o = 1$.

A few comments are also in order concerning the limiting behavior of the spectra as ft_o tends to zero. The peak value of the steady-state displacement at this limit can be shown to equal the static displacement induced by a force equal to the mean value of the excitation. For the sinusoidal excitation this force is zero, whereas for the excitations considered in Figs. 7b and 7c, they are determined from Eqs. 3.1 and 3.2 to be equal to $\frac{1}{\pi}$ P and $\frac{2}{\pi}$ P, respectively.

The response spectra for the triangular sequences of pulses considered in parts (e) and (f) of Fig. 1 are shown in Figs. 8 and 9, respectively. The general trends of these spectra are similar to those of the spectra presented previously. It should be noted, however, that the high-frequency limits of these spectra, and of the spectra for some of the other excitations considered, are different. This matter is examined in detail in a subsequent section.

3.3 One-Term Fourier Series Approximations

As already noted, the steady-state response may also be analyzed by a Fourier series decomposition of the forcing function. It is of interest to examine here how the individual terms in such an approach contribute to the maximum value of the steady-state response of the system.

In Fig. 10 the response spectra for the steady-state response of systems subjected to the alternating step excitation are compared with the corresponding spectra obtained by considering only the effect of the jth ($j = 1,2,3$) term in the Fourier expansion. It can be seen that the results obtained by the exact and the one-term solution are in excellent agreement in the region of the spectrum up to and slightly past the first resonant peak. The agreement deteriorates, however, beyond this region, and the one-term solution leads to significant errors even when the term considered is the dominant contributor to the response.

It can also be shown that use of any of the standard rules for approximating the maximum value of the steady-state response from the corresponding values of the component terms in a Fourier series representation also leads to substantial errors. For example, use of the root mean square procedure can be shown to lead to errors well in excess of 25% for the excitations considered.

3.4 Closed-Form Solutions

The method of analysis employed in this study can also be used to obtain closed-form expressions for the steady-state response of the system in special cases. For example, starting with the following well-known expression for the transient displacement, x(t), of an initially at rest, undamped system subjected to a rectangular step,

$$x(t) = x_{st}(1-\cos pt) \qquad \text{for} \quad 0 \le t \le t_o/2 \tag{3.3}$$

it can be readily shown that the steady-state displacement, y(t), for the alternating step excitation is given by

$$y(t) = x_{st}\left[(1-\cos pt) - \tan\frac{pt_o}{4}\sin pt\right] \quad \text{for} \quad 0 \le t \le t_o/2 \tag{3.4}$$

and

$$y\left(\frac{t_o}{2}+\tau\right) = -y(\tau) \qquad \text{for} \quad 0 \le \tau \le t_o/2 \tag{3.5}$$

The term on the extreme right of Eq. 3.4 represents the corrective solution, $\xi(t)$ in the notation of Ref. 3, which transforms the transient solution to the steady-state solution.

Equation 3.4 can also be obtained by an analysis of the general solution of the governing differential equation of motion. By choosing the constant coefficients of the homogeneous solution such that the motion repeats itself after a time interval equal to the excitation period, t_o, the steady-state response can be found. For the undamped system and the step excitation considered, the solution of Eq. 2.1 for $0 \le t \le t_o/2$ is given by

$$x(t) = c_1 \cos pt + c_2 \sin pt + x_{st} \tag{3.6}$$

For a forcing function that is antisymmetric about the midpoint of the excitation period, the steady-state response also is antisymmetric, and the values of c_1 and c_2 can most effectively be determined from the solution valid for $0 \le t \le t_o/2$ by requiring that

$$x(0) = -x(t_o/2) \tag{3.7}$$

and

$$\dot{x}(0) = -\dot{x}(t_o/2) \tag{3.8}$$

On satisfying these conditions, one obtains a system of two algebraic equations in c_1 and c_2, the solution of which is

$$c_1 = -x_{st} \tag{3.9}$$

$$c_2 = -\tan \frac{pt_o}{4} \, x_{st} \tag{3.10}$$

Finally, on substituting these values of c_1 and c_2 into Eq. 3.6 and replacing x(t) by y(t), one obtains the expression presented in Eq. 3.4.

With the instantaneous value of the steady-state displacement established, the maximum value of this displacement may be determined by differentiation. This leads to the following expressions for the amplification factor, $AF = y_{max}/x_{st}$

$$AF = \sec\left(\frac{\pi}{2} ft_o\right) - 1 \qquad \text{for} \quad ft_o \leq 1 \tag{3.11a}$$

and

$$AF = \left|\sec\left(\frac{\pi}{2} ft_o\right)\right| + 1 \qquad \text{for} \quad ft_o \geq 1 \tag{3.11b}$$

Similar closed-form expressions have also been obtained for damped systems subjected to the alternating step force, as well as for undamped systems subjected to a number of other periodic excitations. These results are summarized in Appendices C and D. The closed-form expressions for damped systems subjected to the alternating step force give results which are naturally the same as those obtained by numerical integration and presented in Figs. 2 and 5.

3.5 Limiting Behavior for Low-Frequency Systems

This section deals with the response of systems for which the value of the frequency parameter ft_o tends to zero. Both the transient and steady-state responses are examined.

The equation of motion of such systems reduces to

$$\ddot{x}(t) = \frac{1}{m} P(t) \tag{3.12}$$

from which it is clear that the structural acceleration in this case is a function only of the exciting force and the system mass. Note further that the equation of motion, and hence the response, are independent of the system damping in this case.

3.5.1 Transient Response. For systems that are initially at rest, integration of Eq. 3.12 leads to the following simple expressions for the velocity and displacement of the system,

$$\dot{x}(t) = \frac{1}{m} I_1(t) \tag{3.13}$$

$$x(t) = \frac{1}{m} I_2(t) \tag{3.14}$$

in which $I_1(t)$ and $I_2(t)$ represent the first and second integrals of P(t), respectively, with the initial values of these integrals taken as zero. The units of $I_1(t)$ are naturally force multiplied by time, and those of $I_2(t)$ are force multiplied by time squared. The maximum values of the responses are then

$$\ddot{x}_{max} = \frac{P}{m} \tag{3.15}$$

$$\dot{x}_{max} = \frac{I_1}{m} \tag{3.16}$$

$$x_{max} = \frac{I_2}{m} \tag{3.17}$$

in which P, I_1, and I_2 represent the peak numerical values without regard to sign of the exciting force, of the first integral of P(t), and of the second integral of P(t), respectively.

3.5.2 Steady-State Response. Making use of the approach described in Ref. 3, the steady-state response, y(t), may be expressed in the form

$$y(t) = x(t) + y(0)g(t) + \dot{y}(0)h(t) \tag{3.18}$$

in which y(0) and $\dot{y}(0)$ are the yet to be determined initial values of the steady-state displacement and velocity of the system; and g(t) and h(t) represent the displacements at time t induced by a unit initial displacement and a unit initial velocity, respectively.

Consider now a periodic exciting force, P(t), with a zero mean. When the systems with negligible stiffness considered herein are subjected to such an excitation, g(t) = 1 and h(t) = t. On substituting these unit response functions into Eq. 3.18, differentiating the latter expression, and making use of Eqs. 3.13 and 3.14, the following expressions are obtained

$$\dot{y}(t) = \frac{1}{m} I_1(t) + \dot{y}(0) \tag{3.19}$$

and

$$y(t) = \frac{1}{m} I_2(t) + \dot{y}(0)t + y(0) \tag{3.20}$$

These results could, of course, also have been obtained by integration of Eq. 3.12, giving due interpretation to the constants of integration.

Since the exciting force considered is periodic with a zero mean, the steady-state response functions, $\ddot{y}(t)$, $\dot{y}(t)$ and y(t), will also be periodic and have a zero mean. Specifically, by ensuring that $y(t_o)$ = y(0), the following expression is obtained from Eq. 3.20 for $\dot{y}(0)$:

$$\dot{y}(0) = -\frac{1}{m} \frac{I_2(t_o)}{t_o} \tag{3.21}$$

Equation 3.19 can then be written as

$$\dot{y}(t) = \frac{1}{m}\left[I_1(t) - \frac{\bar{I}_2(t_o)}{t_o} \right] \tag{3.22}$$

in which the bracketed term represents the first integral of the exciting force, with the initial value of the integral adjusted such that the integral has a zero mean. This balanced or shifted version of $I_1(t)$ will be denoted by $I_1'(t)$. With this notation, Eq. 3.22 may be expressed as

$$\dot{y}(t) = \frac{1}{m} I_1'(t) \tag{3.23}$$

By ensuring that the integral of y(t) also be periodic, it can similarly be shown that

$$y(t) = \frac{1}{m} I_2'(t) \tag{3.24}$$

in which $I_2'(t)$ represents the integral of $I_1'(t)$, i.e. the second integral of the exciting force, with the initial value of $I_2'(t)$ adjusted such that $I_2'(t)$ has a zero mean. It should be clear that $I_2'(t)$ is effectively the balanced or adjusted version of $I_2(t)$. The maximum values of $I_1'(t)$ and $I_2'(t)$ will be denoted by I_1' and $\dot{I}_2'$, respectively.

In addition to the periodic force component with a zero mean, P(t), if the excitation includes a constant component, P_o, the total force, $P_T(t)$, will be of the form

$$P_T(t) = P_o + P(t) \tag{3.25}$$

The total steady-state displacement of the system, $y_T(t)$ is then given by

$$y_T(t) = (x_{st})_o + y(t) \tag{3.26}$$

in which $(x_{st})_o = P_o/k$ is a constant representing the static displacement induced by P_o.

3.5.3 Comparison of Results. From the information presented in the preceding sections it is clear that the transient and steady-state responses of systems having negligibly small resistances are related simply to the integrals of the exciting force.

The interrelationship of the two responses is illustrated in Fig. 11 for systems subjected to the alternating step force pulse. In part (a) of the figure are shown the transient response histories, and in part (b) are shown the corresponding steady-state responses. It should be clear that the maximum value of the steady-state displacement is significantly less than that of the transient displacement. Similar results are presented in Figs. 12 and 13 for a simulated wave loading which has an effective period of slightly in excess of 100 sec.

An indication of the extent to which these limiting responses may approximate the behavior of systems having small but finite stiffnesses is provided in Fig. 14. The limiting responses are compared for the alternating step force with the actual responses computed for a system for which $ft_o = 0.1$. In the development of the transient solution, the system is presumed to be initially at rest, and system damping is presumed to equal 5 percent of the critical value in all cases. It can be seen that the agreement between the limiting and actual responses are excellent, particularly for the steady-state responses. Further insight into the range of values of the frequency parameter for which the actual maximum responses may adequately be approximated by the corresponding limiting response may be obtained from the response spectra presented in subsequent sections.

3.6 Tripartite Response Spectra

It is instructive to display the response spectra presented earlier in Figs. 3 and 4 also in a tripartite logarithmic format analogous to that used to display response spectra for earthquake ground motions. Figure 15 shows results plotted in such a format.

The maximum response of the system in this plot is expressed in terms of the maximum spring force, F. This force may also be interpreted as the equivalent static force which produces the same structural response as the maximum response produced by the actual time-dependent force. The vertical and the two diagonal scales are alternative dimensionless measures of F. The diagonal scale on the right is normalized with respect to the maximum value of the exciting force, P; the vertical scale is normalized with respect to pI_1; and the diagonal scale on the left is normalized with respect to p^2I_2.

An alternate but equivalent means of expressing the scales of the tripartite log plot can be found which uses terms analogous to the pseudo-acceleration and pseudo-velocity concepts used in studies of ground-excited systems. In particular, the diagonal scale on the right can be written as $p^2x_{max}/[P/m]$, where p^2x_{max} is analogous to the pseudo-acceleration of the ground-excited system. The vertical scale can be expressed as $px_{max}/[I_1/m]$, where px_{max} is analogous to the pseudo-velocity of the ground-excited system. Finally, the left diagonal scale may be expressed as $x_{max}/[I_2/m]$. It may further be noted that the normalizing terms P/m, I_1/m, and I_2/m represent the maximum transient values of the acceleration, velocity, and displacement of the system when its natural frequency tends to zero. These values correspond to the maximum values of the ground acceleration, ground velocity, and ground displacement in the corresponding plot for ground-excited systems.

The tripartite logarithmic format used in Fig. 15 best displays the maximum transient displacement. One notes, for instance, that the maximum steady-state displacement for low-frequency systems subjected to the alternating step excitation is one eighth that of the maximum transient displacement. The steady-state response spectrum can, however, be emphasized in a format which uses the corresponding adjusted or balanced force integrals, I_1' and I_2'. In this case the steady-state response spectrum approaches unity on the left diagonal scale for low-frequency systems.

The well-known response spectra for undamped systems subjected to n pulses of the sinusoidal excitation are shown in Fig. 16 on the tripartite log format. One notices that the response spectra for low and high-frequency systems are again emphasized, and that the low-frequency response spectra approach a limit on the vertical scale for odd values of n and a limit on the left diagonal scale for even values of n. The existence of a limit on the vertical scale and left diagonal scale implies, of course, that these response spectra approach zero linearly and parabolically, respectively, on an arithmetic plot.

3.7 Limiting Behavior for High-Frequency Systems

This section deals with the response of systems for which the value of the frequency parameter ft_o tends to infinity. The three most important factors in the determination of this response are 1) the non-harmonic nature of the excitation, 2) the amount of damping in the system, and 3) the locations and magnitudes of discontinuities in the exciting force.

A high-frequency limiting response does not in general exist for undamped systems since the steady-state response becomes unbounded when the natural frequency values equal the frequencies of the harmonic components of the Fourier series expansion of the excitation.

A steady-state limiting response can always be identified for damped systems, however, although the rate of convergence to this limit is highly dependent on the amount of damping in the system and the shape of the forcing function. The response histories shown in Fig. 17, for instance, illustrate that the effects of discontinuities in the forcing function decay for damped systems and that these effects are, furthermore, localized for high-frequency systems.

3.7.1 Continuous Excitations. The steady-state response of high-frequency systems subjected to an excitation whose periodic extension is continuous approaches a "static" condition since the effective period of the excitation becomes very long and the system behaves as if loaded very gradually; i.e., the maximum steady-state response due to continuous excitations is given by

$$y_{max} = x_{st} \tag{3.27}$$

This limiting response is valid for undamped systems as well as damped, provided that the continuous forcing function may be represented by a finite number of harmonic components. Figure 7a shows that this is, of course, true for sinusoidal force.

Figures 6, 7b, 7c and 8, furthermore illustrate that the steady-state response converges fastest to the limit defined in Eq. 3.27 for highly damped systems. One also notes from these figures that the higher order resonant peaks for undamped systems subjected to these excitations are sharper than the resonant peaks of a lesser order. This phenomenon is due to the fact that the higher order terms in the Fourier series expansions of these excitations become very small. The undamped high-frequency response spectra for these excitations do not differ significantly from the damped response spectra except near the resonant peaks.

3.7.2 Discontinuous Excitations. The limiting response of damped high-frequency systems subjected to discontinuous periodic excitations may or may not be given by Eq. 3.27; the locations and magnitudes of the discontinuities may be such that the steady-state limit is given by the response after one of them occurs.

As the frequency parameter ft_o tends to infinity, every discontinuity in the forcing function may be considered a step loading of an infinite duration and the displacement just before the discontinuity occurs may be considered equal to the corresponding "static" displacement. The maximum steady-state response which results after a discontinuity is therefore given by the larger of the relations

$$\frac{1}{K}\left(P_a + P_b e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi}\right) \qquad (3.28)$$

and

$$\frac{1}{K}\left(P_a - P_b e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}2\pi}\right) \qquad (3.29)$$

in which P_a represents the value of the forcing function directly after the discontinuity occurs and P_b is the magnitude of the change of force due to the discontinuity. Both P_a and P_b are signed quantities and are therefore dependent on the orientation of the force axis. As an illustration, P_a and P_b are both positive for the excitation shown in Fig. 17a, whereas P_a is zero and P_b negative for the force shown in Fig. 17b.

Obviously the steady-state maximum response which occurs after a discontinuity converges fastest to the limit given by Eqs. 3.28 and 3.29 for 1) systems that are highly damped, 2) forcing functions which have only minimal slopes before and after the discontinuities and 3) forcing functions which have discontinuities which are not close together in time.

Equation 3.29 will, of course, never control the maximum steady-state response since this displacement is always less than the "static" displacement induced by the force just before the discontinuity. One can note, for instance, from Fig. 17d that the displacement given by Eq. 3.29 represents the maximum response after the first discontinuity in the forcing function but is still less than the displacement just before this discontinuity occurs.

Using Eqs. 3.27 and 3.28 one can conclude that the steady-state limiting response for damped systems subjected to the periodic excitation shown in Fig. 17f is given by the greater of the relations

$$\frac{1}{K}\left(P_A + P_B e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi}\right) \quad ; \quad \frac{1}{K} P_C$$

$$\frac{1}{K}\left(P_D + P_E e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi}\right) \quad ; \quad \frac{1}{K} P_F$$

For the alternating step and saw-tooth excitations shown in parts (a) and (f) of Fig. 1, respectively, the limiting response for high-frequency systems is given by Eq. 3.28 such that

$$y_{max} = x_{st}\left(1 + 2e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi}\right) \tag{3.30}$$

The results given in App. C for the alternating step excitation can be used to show that Eq. 3.30 is valid.

From Fig. 5 one notes that over the frequency range shown the response spectra for the more highly damped systems approach the steady-state limit given by Eq. 3.30, while the spectra for the slightly damped systems still experience increases at resonance. The same trends are exhibited in Fig. 9 by the steady-state response spectra for the saw-tooth excitation. The high-frequency limiting response is attained more slowly in this case, however, since the slope just before and after the discontinuity in the saw-tooth forcing function becomes negligible only for relatively high-frequency systems.

IV. ABSOLUTE MAXIMUM RESPONSE

The absolute maximum response of a periodically excited system generally occurs prior to the maximum steady-state response. The purpose of this section is to define the interrelationship of the two maxima, paying special attention to the response of low-frequency systems which are finding increased application in offshore construction. Both undamped and damped systems are examined.

4.1 Comparison of Maximum Steady-State and Absolute Maximum Responses

In Figs. 18 and 19 are presented spectra for the maximum steady-state and absolute maximum responses of systems with $\zeta = 0.05$ subjected to the alternating step force. The results are plotted in arithmetic format in Fig. 18. The vertical scale and left-hand diagonal scale in Fig. 19 are normalized with respect to the peak values of the balanced versions of the force integrals, I_1' and I_2', rather than those of the unbalanced versions, I_1 and I_2. As a consequence, the low-frequency limit of the spectrum for steady-state response is unity on the left-hand diagonal scale.

The results for absolute maximum response were evaluated by numerical integration of the equation of motion by carrying out the solution over a sufficiently large number of excitation cycles. The systems were presumed to be initially at rest. Also shown in Fig. 19 in dashed lines is the response spectrum for absolute maximum response for a modified version of the excitation, which is identified in greater detail later.

The resonant peaks for the two sets of spectra in Figs. 18 and 19 coincide because, as it is well known, the response of the system in this case simply builds up with time to the steady-state response. In regions away from the resonant peaks, the absolute maximum response may be substantially greater than the steady-state maximum, the percentage difference between the two maxima becoming greatest in the low frequency region on the system.

The source of the latter difference is identified in Fig. 20, in which the initial transient and the final steady-state responses of systems subjected to the alternating step force are compared for three different damping values. In all cases, the value of the frequency parameter $ft_0 = 0.1$, i.e., the excitation period, t_0, is considered to be one-tenth of the natural period of the system, $T = 1/f$. It can be seen that the initial, transient response in this case is composed of a very small steady-state component superimposed on a much larger free vibrational component. The period of the latter component is equal to the natural period of the systems, and hence the absolute maximum response occurs at a time equal or close to one-quarter the natural period of the system. Furthermore, unlike the limiting value of the steady-state maximum which is independent of the system damping, the absolute maximum is a function of the amount of damping present.

4.2 Limiting Behavior for Low-Frequency Systems

It is desirable to consider the effects of forces with zero and non-zero means separately.

4.2.1 Excitation with Zero Mean. From the information presented in Figs. 18 and 19, it should be clear that the transient response of a low-frequency system is dominated by the free vibrational component, which depends on the natural frequency of the system itself. Accordingly, for the purpose of determining the absolute maximum response of low-frequency systems, the second and third terms on the right-hand side of Eq. 2.1 cannot be neglected, as was done in determining the corresponding steady-state maximum.

The low-frequency limit for the absolute maximum response can be determined from the corresponding steady-state response by application of the approach used in Ref. 3. Specifically, the transient displacement, $x(t)$, may be expressed in terms of the corresponding steady-state displacement, $y(t)$, by the following rearranged version of Eq. 3.18,

$$x(t) = y(t) - y(0)g(t) - \dot{y}(0)h(t) \tag{4.1}$$

and the transient velocity, $\dot{x}(t)$, may be expressed as

$$\dot{x}(t) = \dot{y}(t) - y(0)\dot{g}(t) - \dot{y}(0)\dot{h}(t) \tag{4.2}$$

As the natural frequency of the system tends to zero, the first two terms on the right-hand side of Eq. 4.1 become negligible in comparison to the third provided $\dot{y}(0)$ is not zero, and the second term in Eq. 4.2 becomes negligible in comparison to the other two. Accordingly, Eqs. 4.1 and 4.2 may be simplified to

$$x(t) \simeq -\dot{y}(0)h(t) \tag{4.3}$$

and

$$\dot{x}(t) \simeq \dot{y}(t) - \dot{y}(0)\dot{h}(t) \tag{4.4}$$

Equation 4.3 reveals that the transient displacement of a low-frequency periodically excited system may be approximated by the free vibration induced by an initial velocity change equal in magnitude to the negative of the initial value of the steady-state velocity. The latter value is determined from Eq. 3.23 to be

$$\dot{y}(0) \simeq \frac{1}{m} I_1'(0) \tag{4.5}$$

On substituting Eq. 4.5 into Eqs. 4.3 and 4.4 and Eq. 3.23 into Eq. 4.4, and making use of the exact expression for h(t) and its derivative -- not the limiting expression for h(t) used in the derivation of the corresponding steady-state response --, one obtains

$$x(t) \simeq -\frac{1}{m\bar{p}} I_1'(0)\, e^{-\zeta p t} \sin \bar{p} t \tag{4.6}$$

and

$$\dot{x}(t) \simeq -\frac{1}{m} I_1'(0)\, e^{-\zeta p t}\left[\cos \bar{p} t - \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \bar{p} t\right] + \frac{1}{m} I_1'(t) \tag{4.7}$$

in which $\bar{p} = p\sqrt{1-\zeta^2}$ is the damped circular natural frequency of the system. It should be noted that, because of differences in the order of the approximations involved in the two expressions, Eq. 4.7 is not strictly equal to the derivative of Eq. 4.6 with respect to t.

The time of absolute maximum response, t_{max}, is close to one-quarter of the natural period of the system, and is given by

$$\bar{p} t_{max} = \cos^{-1} \zeta \simeq \frac{\pi}{2} - \zeta \tag{4.8}$$

On substituting this equation into Eq. 4.6, the following expression is obtained for the absolute maximum displacement x_{max}:

$$x_{max} \simeq -\frac{1}{mp} I_1'(0)\, e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}\left(\frac{\pi}{2} - \zeta\right)} \tag{4.9}$$

The maximum value of the spring force, F, is then given by

$$F \simeq -pI_1'(0)\, e^{\frac{-\zeta}{\sqrt{1-\zeta^2}}\left(\frac{\pi}{2}-\zeta\right)} \tag{4.10}$$

When displayed in the tripartite logarithmic format, the spectra for absolute maximum response must therefore approach horizontal limits on the left. For small values of damping, the factor ζ within the parentheses of Eqs. 4.9 and 4.10 may, of course, be neglected.

Equations 4.9 and 4.10 reveal that the absolute maximum response of low-frequency systems is proportional to the initial value of the balanced version of the first integral of the exciting force, $I_1'(t)$. Slight changes in the shape of the exciting force may, therefore, strongly affect the value of $I_1'(0)$ and hence the absolute maximum response of low-frequency systems. Consider, for example, that the origin of the alternating step force has been shifted to $t_0/4$ such that it is symmetric about the midpoint of excitation half-cycle. The dashed line in Fig. 19 represents the spectrum of absolute maximum response for such an excitation while the steady-state response spectrum is still given by the dotted line. Note that the low-frequency limits of the absolute maximum response are significantly different for the original and shifted excitations.

As a further example, in Fig. 21 are presented spectra for the steady-state and absolute maximum responses of systems with $\zeta = 0.05$ subjected to a sinusoidal force. Also shown is the corresponding response spectrum for a cosine force. Note that changing the forcing function from sine to cosine drastically alters the low-frequency limit for absolute maximum response.

System damping affects, of course, not only the low-frequency limits of the spectrum but also the entire spectrum. In Fig. 22 are given spectra for the absolute maximum response of systems subjected to the sinusoidal force for a range of damping values. As would be expected, the effect of damping is greatest for frequencies near the resonant peak. Incidentally, the low-frequency limits of these spectra are in excellent agreement with those predicted by Eq. 4.10.

4.2.2 Excitation with Non-Zero Mean. Let P_o be the constant component of the exciting force and $(x_{st})_o$ be the corresponding static displacement. For a system that is initially at rest, the instantaneous value of the dynamic displacement produced by P_o is given by

$$x(t) = (x_{st})_o \left[1 - e^{-\zeta p t} \left(\cos \bar{p}t + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \bar{p}t \right) \right] \tag{4.11}$$

The maximum value of this displacement is given by

$$x_{max} = (x_{st})_o \left[1 + e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi} \right] \tag{4.12}$$

and the corresponding value of the spring force is given by

$$F = P_o \left[1 + e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi} \right] \tag{4.13}$$

Note that these peak response values defined by Eqs. 4.12 and 4.13 are independent of the natural frequency of the system, and that on the tripartite logarithmic plot they are represented by a 45° diagonal line extending upward from right to left. By contrast, the peak values of the responses induced by the oscillating force with a zero mean are linear functions of the natural frequency of the system. Provided P_o

is different from zero, it follows that as ft_o tends to zero, the maximum response will be defined by Eqs. 4.11 and 4.12. For systems with moderately low-frequency values, the response contributed by the constant or oscillating force component may be the dominant one depending on the relative values of P_o and $I_1'(0)$.

4.3 Limiting Behavior for High Frequency Systems

The differences between the transient and steady-state responses of systems in the high-frequency region of the spectrum are not as great as those in the low-frequency region. In fact, the absolute maximum response for high-frequency systems may be greater than the steady-state response for only one class of excitation; for all others, the two limiting responses are the same.

4.3.1 Similarities of Transient and Steady-State Responses. The absolute maximum response of high-frequency undamped systems, like that of the corresponding steady-state response, can be defined only for continuous excitations which can be expressed by a finite number of Fourier components. Discontinuous periodic excitations lead to unbounded absolute maximum responses. Furthermore, high values of damping accelerate the rate of convergence of the absolute maximum response to its limiting value in essentially the same manner that damping affects the corresponding steady-state response values.

4.3.2 Effect of Initial Discontinuity in Forcing Function. The one class of excitation for which the absolute maximum limiting response may not equal that of the steady-state response is identified in Fig. 17e. The force shown in Fig. 17e has an initial discontinuity which is not present in its periodic extension. The steady-state response of systems subjected to this excitation does not undergo this discontinuity and therefore has a limiting value given by the larger of the "static" conditions

$$\frac{1}{K} P_A \qquad ; \qquad \frac{1}{K} P_B$$

The absolute maximum response of systems subjected to this excitation on the other hand is strongly influenced by this initial discontinuity and its limit is given by the larger of the relations

$$\frac{1}{K} P_A \left(1 + e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi}\right) \qquad ; \qquad \frac{1}{K} P_B$$

in which the first equation represents the maximum response due to a step loading , P_A, of an infinite duration. Obviously, if the first of these relations governs, the absolute maximum and steady-state limiting behavior will not be equal.

The response of high-frequency systems subjected to the cosine excitation results in a difference of this type. Obviously, the steady-state and absolute maximum limiting responses for the sinusoidal excitation are given by the "static" condition of Eq. 3.27. The absolute maximum limiting response for the cosine excitation on the other hand is given by

$$x_{max} = \left(1 + e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi}\right) x_{st} \qquad (4.14)$$

One notes from Fig. 21 that this limit is approached gradually; since the cosine forcing function has a nonzero slope just after the discontinuity, the limiting response for this excitation only approaches that due to a step loading for relatively high-frequency systems.

The shifted alternating step excitation also has an initial discontinuity which is not present in the periodic extension but since subsequent discontinuities induce responses which are greater than that due to the initial one, the absolute maximum response for this case approaches the limit given in Eq. 3.30 as does the steady-state and absolute maximum response for the alternating step excitation shown in part (a) of Fig. 1.

V. BUILDUP OF RESPONSE AND EFFECTS OF CESSATION OF EXCITING FORCE

The previous sections of this paper investigated the response of systems subjected to periodic excitation; i.e., the results presented assumed that the exciting force continues to act indefinitely on the system. For the purpose of derivation of the low-frequency behavior of the steady-state response, however, the transient response of low-frequency systems subjected to one cycle of excitation was also considered. Reference was also made to Fig. 3 in which the steady-state response spectrum and the response spectrum for one cycle only were compared for the alternating step excitation.

This section of the paper will investigate the behavior of systems which are initially at rest and subjected to excitations which are applied for <u>any</u> finite number of cycles and are then removed. The systems considered under this section, therefore, are allowed to undergo a free vibration motion as well as a forced motion.

It is of interest to investigate the maximum possible responses that the system will experience during these two stages and their relationships with the characteristics of the system and excitation. It is also of interest to compare these responses with those given for systems subjected to periodic excitation.

In the following section the maximum forced response will be discussed and in the subsequent section the free vibration motion will be accounted for and an absolute maximum response will be defined for systems subjected to excitations of a finite duration.

5.1 Response Spectra for Maximum Forced Response

One would expect that the maximum forced response is dependent on the natural frequency and damping ratio of the system as well as the shape and duration of the excitation. This is borne out in Figs. 23-26, which depict spectra for the forced response of systems with $\zeta = 0.05$ and subjected to one, two, three, and an infinite number of cycles of two harmonic and two alternating step excitations. The maximum response during an infinite number of cycles, of course, corresponds to the absolute maximum response for the case of periodic excitation.

One can observe three basic characteristics from these response spectra:

(1) The absolute maximum response for periodic excitation is in some regions given during the first cycle of excitation (typically in the high-frequency region, except at higher order resonance);

(2) At resonance, the absolute maximum response for periodic excitation is attained only after an infinite number of cycles of the excitation; and

(3) The low-frequency region of the response spectra may be most sensitive to the number of cycles which the system undergoes, but may also be very sensitive to the shape of the excitation.

One concludes, therefore, that the periodicity of the excitation has its greatest effect in the low-frequency region of the response spectra and at resonance.

5.2 Response Spectra for Absolute Maximum Response

The absolute maximum response of systems which undergo a finite number of cycles must include an account of the free vibration which occurs after cessation of the excitation. By including the free vibration motion one can reasonably expect to obtain in some cases larger results than those obtained by considering the forced motion alone. As an illustration of this behavior, spectra for the absolute maximum response of undamped systems subjected to one, three and five cycles of sinusoidal excitation are shown in Fig. 27. The analytical solution of this problem is well documented (2) but use of the tripartite logarithmic plot again accentuates the limiting characteristics of the low and high-frequency regions of the response spectra.

The same can be noted from Fig. 28 which depicts the response spectra for the absolute maximum response of damped systems subjected to one, three, and five cycles of the alternating step excitation.

A direct comparison of the maximum forced response and absolute maximum response for three cycles of the alternating step excitation are shown in Fig. 29. One notes that inclusion of the free vibration motion increases the response substantially for low-frequency systems but has no effect on the maximum response of high-frequency systems; i.e., for low-frequency systems the maximum response occurs after cessation of the force and for high-frequency systems during the excitation for this case.

5.3 Behavior of Low-Frequency Systems

One would expect from closer examination of Figs. 23-29 that the behavior of low-frequency systems when subjected to a finite number of cycles of excitation may be described in a manner similar to that which was employed for the steady-state and absolute maximum responses for periodic excitation. The limiting response of low-frequency systems subjected to n cycles of excitation is, for instance, assymptotic on the left-hand diagonal scale of the tripartite log plot. It is also apparent that the response spectra may also be bounded on the vertical scale of the tripartite log plot for moderately low-frequency systems. The limiting responses of low-frequency systems must, therefore, be proportional to the first and second integrals of the forcing function.

5.3.1 Limiting Behavior. The limiting response of low-frequency systems which are initially at rest is given by Eq. 3.14 such that the maximum forced displacement for n cycles of excitation may be expressed as

$$x_{max} = \frac{1}{m} (I_2)_n \tag{5.1}$$

in which $(I_2)_n$ represents the peak numerical value of the second integral of P(t), evaluated over the duration of the excitation.

The maximum free vibration response of low-frequency systems which undergo n cycles of excitation is furthermore given by

$$x_{max} = \frac{1}{m} I_2(nt_o) \tag{5.2}$$

in which $I_2(nt_o)$ represents the value of the second integral of P(t), evaluated at the end of the excitation. Equations 5.1 and 5.2 correspond, of course, to limits on the left-hand diagonal scale of the tripartite log plot.

Additionally, Eq. 5.1 corresponds to translation of the system mass during the excitation, while Eq. 5.2 indicates that the system mass remains stationary in its final position after cessation of the excitation. For excitations for which the maximum value of the second force integral occurs at the end of each cycle (e.g., the sinusoidal and alternating step), Eqs. 5.1 and 5.2 reduce to

$$x_{max} = \frac{n}{m} I_2 \tag{5.3}$$

The convergence of these two responses to the limit given in Eq. 5.3 is verified in Fig. 29 for low-frequency systems subjected to three cycles of the alternating step force.

5.3.2 <u>Maximum Forced Response of Moderately Low-Frequency Systems.</u> The behavior of the forced response of moderately low-frequency systems which undergo n cycles of an excitation can also be found, as was the case for the absolute maximum limiting behavior due to periodic excitation, through the use of a corrective displacement, $\xi(t)$, and the low-frequency representation of the steady-state response.

The maximum response of moderately low-frequency systems which occurs during n cycles of a zero-mean excitation, therefore, is simply given by Eq. 4.9 which, of course, corresponds to a limit of the response spectrum on the vertical scale of the tripartite logarithmic plot for such excitations. Since, however, the time which the system has to obtain this maximum is not unlimited, but rather bounded by the cessation of the excitation one must recognize that the duration of the excitation, nt_o, is limited to

$$nt_o \geq \frac{1}{p} \cos^{-1} J \tag{5.4}$$

by Eq. 4.8 ; or in terms of a variable frequency and fixed excitation duration, that

$$ft_o \geq \frac{1}{2\pi n\sqrt{1-\zeta^2}} \cos^{-1}\zeta \tag{5.5}$$

One notices, for instance, that the dashed line in Fig. 29 , which represents the response spectrum for the forced response of systems with $\zeta = 0.05$ subjected to three cycles of the alternating step excitation, is approximately equal to that value on the vertical scale given by Eq. 4.10 for values of $ft_o > 0.08$.

The maximum forced response for moderately low-frequency systems subjected to n cycles of an excitation which does <u>not</u> have a zero mean, on the other hand, is given by Eq. 4.12 provided that 1) the value of the mean of the excitation, P_o, is large and $I_1'(0)$ is small, or that 2) the duration of the excitation is long. The accuracy of Eq. 4.12 for nonzero-mean excitations is, therefore, highly dependent on the magnitudes of $I_1'(0)$, P_o, and nt_o.

5.3.3 <u>Maximum Free Response of Moderately Low-Frequency Systems.</u> The behavior of the free vibration response of moderately low-frequency systems is dependent on the end conditions of the forced response when the excitation ceases. After these end conditions have been identified, the free vibration solution can be found and solved for its maximum value.

Equations 4.1 and 4.2 which relate the forced response to the steady-state response can be used to find the appropriate end conditions. Evaluation of these equations at the time of cessation of the excitation , nt_o, yields the following values of the forced displacement and velocity:

$$x(nt_0) = y(0)\left[1 - e^{-\zeta p n t_0}\left(\cos \bar{p} n t_0 + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \bar{p} n t_0\right)\right] - \frac{\dot{y}(0)}{\bar{p}} e^{-\zeta p n t_0} \sin \bar{p} n t_0 \tag{5.6}$$

and

$$\dot{x}(nt_0) = \dot{y}(0)\left[1 - e^{-\zeta p n t_0}\left(\cos \bar{p} n t_0 - \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \bar{p} n t_0\right)\right] + \frac{y(0) p}{\sqrt{1-\zeta^2}} e^{-\zeta p n t_0} \sin \bar{p} n t_0 \tag{5.7}$$

From closer inspection of Eqs. 5.6 and 5.7 one notices that the resulting end displacement and velocity are equivalent to those responses produced by the superposition of a step displacement of magnitude y(0) over the duration of the excitation and two impulsive velocities; the first, which is of magnitude $-\dot{y}(0)$, is applied at the time that the excitation originates and the second, which is of magnitude $\dot{y}(0)$, is applied at the time the excitation ceases. The two representative impulsive velocities are, therefore, equal but opposite in direction.

For any particular system and excitation, Eqs. 5.6 and 5.7 may be evaluated and the free vibration response obtained from the following equation:

$$x(t) = e^{-\zeta p t}\left[x(nt_0) \cos \bar{p} t + \left(\frac{\dot{x}(nt_0)}{\bar{p}} + \frac{\zeta}{\sqrt{1-\zeta^2}} x(nt_0)\right) \sin \bar{p} t\right] \tag{5.8}$$

in which the time t is now oriented at the beginning of the free vibration motion. Equation 5.8 can be differentiated and the time at which the first maximum occurs can be found. This maximum will obviously be very dependent on the end conditions given in Eqs. 5.6 and 5.7 and therefore on the frequency parameter $\bar{p}nt_o$ and the steady-state initial values, y(0) and $\dot{y}(0)$. One would further expect that there exists a combination of these two end conditions, or in other words a particular value of $\bar{p}nt_o$, which makes the system experience the largest possible displacement during its free vibration motion. It is obvious, however, that without further knowledge of the exact nature of the steady-state initial values, which also vary with respect to $\bar{p}t_o$, that this largest possible displacement cannot be exactly identified.

Evaluation of Eqs. 5.6, 5.7, and 5.8, for instance, reveals that the maximum free vibration response for undamped systems subjected to n cycles of excitation is given by

$$x_{max} = 2\sqrt{y^2(0) + \left(\frac{\dot{y}(0)}{p}\right)^2}\ \sin \pi n f t_o \tag{5.9}$$

Since Eq. 5.9 involves the unknown quantities y(0) and $\frac{\dot{y}(0)}{p}$ the largest possible free vibration displacement cannot generally be found. One can, however, normalize the maximum displacement, x_{max}, by the radical of Eq. 5.9 and effectively remove the steady-state terms. In other words, Eq. 5.9 can be expressed as

$$\frac{x_{max}}{\sqrt{y^2(0) + \left(\frac{\dot{y}(0)}{p}\right)^2}} = 2 \sin \pi n f t_o \tag{5.10}$$

which obviously is a maximum when the duration of the excitation is equal to one-half of the structural period; i.e., when $nt_o = T/2$.

Since it has already been established that the free vibration response is most important for moderately low-frequency systems, it is useful to simplify Eq. 5.10 by substitution of the appropriate force integral relations for the steady-state response, Equation 5.10 can, therefore, be restated as

$$\frac{x_{max}}{\sqrt{\left[\frac{1}{m} I_2'(0) + (x_{st})_o\right]^2 + \left[\frac{I_1'(0)}{mp}\right]^2}} \simeq 2 \sin \pi n f t_o \tag{5.11}$$

The displacement for which Eq. 5.11 is a maximum will henceforth be called $x_{max,max}$ and although it does not define the largest possible free vibration response it is apparent from Eq. 5.11 that it may define a significant reference point on the tripartite log plot.

It is also interesting to note that it is possible to obtain closed form expressions for the maximum free vibration response provided that the initial conditions of the steady-state response are available in equation form. One can easily show, for instance, from Eq. 5.9 that the free vibration maxima of undamped systems subjected to n cycles of sinusoidal excitation are given by

$$x_{max} = \left[\frac{2ft_o}{(ft_o)^2 - 1} \right] x_{st} \sin \pi n f t_o \tag{5.12}$$

Closed form expressions for the free vibration maxima of undamped systems subjected to n cycles of nonharmonic excitations can also be found from Eq. 5.9 in certain cases when the techniques described in Sec. 3.4 are applied. The free vibration maxima for undamped systems subjected to n cycles of the alternating step excitation, for instance, are given by

$$x_{max} = \left[2 \tan \frac{\pi}{2} f t_o \right] x_{st} \sin \pi n f t_o \tag{5.13}$$

The envelope of these free vibration maxima are indicated in Figs. 27 and 28 in the low-frequency region of the response spectra for the sinusoidal and alternating step excitations, respectively. One notes from Fig. 28 that the maximum free vibration response of low-frequency systems may be significantly affected by the amount of damping in the system.

For damped systems, it can be argued that the displacement $x_{max,max}$ will be given when the duration of the excitation equals one-half of the damped structural period; i.e., when $nt_o = \bar{T}/2$. Equations 5.6 and 5.7 indicate that the end conditions of the forced response for this excitation duration are given by

$$x(nt_o) = y(0)\left[1 + e^{\frac{-\zeta}{\sqrt{1-\zeta^2}}\pi}\right] \tag{5.14}$$

and

$$\dot{x}(nt_o) = \dot{y}(0)\left[1 + e^{\frac{-\zeta}{\sqrt{1-\zeta^2}}\pi}\right] \tag{5.15}$$

Substitution of these values into Eq. 5.8 reveals that the free vibration response for systems subjected to an excitation of this duration is given by

$$x(t) = \left[1 + e^{\frac{-\zeta}{\sqrt{1-\zeta^2}}\pi}\right] e^{-\zeta pt}\left\{y(0)\cos\bar{p}t + \left(\frac{\dot{y}(0)}{\bar{p}} + \frac{\zeta}{\sqrt{1-\zeta^2}}y(0)\right)\sin\bar{p}t\right\} \tag{5.16}$$

Differentiation of Eq. 5.16 enables one to find the time, t_{max}, at which this response is first a maximum:

$$t_{max} = \frac{nt_o}{\pi}\tan^{-1}\left[\frac{\dot{y}(0)}{\bar{p}} \Big/ \left(\frac{y(0)}{\sqrt{1-\zeta^2}} + \zeta\frac{\dot{y}(0)}{\bar{p}}\right)\right] \tag{5.17}$$

Finally, substitution of Eq. 5.17 into Eq. 5.16 indicates that the displacement $x_{max,max}$ is given by

$$x_{max,max} = \left(1 + e^{\frac{-\zeta}{\sqrt{1-\zeta^2}}\pi}\right) e^{\frac{-\zeta}{\sqrt{1-\zeta^2}}\frac{\pi t_{max}}{nt_o}} \sqrt{y^2(0) + 2\zeta y(0)\frac{\dot{y}(0)}{\bar{p}} + \left(\frac{\dot{y}(0)}{\bar{p}}\right)^2} \tag{5.18}$$

In order to locate this reference point on the tripartite logarithmic plot one must use the force integral formulation for the initial values of the steady-state response. One must also make an additional approximation regarding the relative magnitudes of the quantities $y(0)$ and $\frac{\dot{y}(0)}{p}$. This second approximation will differ for zero-mean and nonzero-mean excitations.

5.3.3.1 Excitation with Zero Mean. For excitations which have a zero mean the force integral expression for the initial steady-state displacement is given by

$$y(0) \simeq \frac{1}{m} I_2'(0) = \frac{p^2}{k} I_2'(0) \tag{5.19}$$

The steady-state initial velocity divided by the circular natural frequency is similarly given by

$$\frac{\dot{y}(0)}{p} \simeq \frac{1}{p}\left[\frac{1}{m} I_1'(0)\right] = \frac{p}{k} I_1'(0) \tag{5.20}$$

For moderately low-frequency systems, the value of p is small such that the initial steady-state displacement as expressed in Eq. 5.19 becomes negligibly small with respect to the value given in Eq. 5.20. The accuracy of this approximation, of course, also depends on the magnitudes of $I_2'(0)$ and $I_1'(0)$. This approximation is namely valid if 1) $I_1'(0)$ is large, or 2) $I_2'(0)$ is small, or 3) the excitation duration is long.

Using this approximation, the value of $x_{max,max}$ as given in Eq. 5.18 may be expressed as

$$x_{max,max} \simeq \frac{1}{mp} I_1'(0)\left(1 + e^{\frac{-\zeta}{\sqrt{1-\zeta^2}}\pi}\right) e^{\frac{-\zeta}{\sqrt{1-\zeta^2}}\left(\frac{\pi}{2}-\zeta\right)} \tag{5.21}$$

For reasonable amounts of damping, $x_{max,max}$ is approximated well by

$$x_{max,max} \simeq \frac{1}{mp} I_1'(0)\left(1 + e^{-\pi\zeta}\right) e^{-\frac{\pi\zeta}{2}} \tag{5.22}$$

The corresponding equivalent static force, F, is similarly given by

$$F \simeq p I_1'(0)\left(1 + e^{\frac{-\zeta}{\sqrt{1-\zeta^2}}\pi}\right) e^{\frac{-\zeta}{\sqrt{1-\zeta^2}}\left(\frac{\pi}{2}-\zeta\right)} \tag{5.23}$$

One also notes that the ratio on the vertical scale of the tripartite log plot is given by

$$\frac{F}{pI_1'} \simeq \frac{I_1'(0)}{I_1'}\left\{\left(1+e^{\frac{-\zeta}{\sqrt{1-\zeta^2}}\pi}\right)e^{\frac{-\zeta}{\sqrt{1-\zeta^2}}\left(\frac{\pi}{2}-\zeta\right)}\right\} \tag{5.24}$$

As was evident for the absolute maximum response for periodic excitation the initial value of the force integral $I_1'(t)$ and hence the shape of the excitation has an important effect on the response in the moderately low-frequency region.

The accuracy of Eq. 5.24 can be noted from Fig. 28 for the absolute maximum response spectra for systems with ζ = 0.05 and subjected to one, three, and five cycles of the alternating step excitation. The initial value of the force integral $I_1'(t)$ is, of course, equal to the maximum value of that integral so that the bracketed term in Eq. 5.24 gives the value of the pertinent point on the vertical scale of the tripartite logarithmic plot in this case. For five cycles of excitation, for instance, the reference point occurs at $ft_o \simeq 0.1$ as predicted. Equation 5.24, furthermore, gives the value 1.72 for this value of damping whereas the value of this ratio for the actual response is 1.73. The agreement is excellent due in part to the fact that the assumption that the initial steady-state displacement approximately equals zero is very good for this excitation.

One can further note that when it is questionable whether an excitation which has a zero mean is to be considered periodic or if the excitation may indeed cease after any cycle, Eq. 4.10 and 5.23 may be used as a direct evaluation of the difference involved in the assumption of a periodic or nonperiodic excitation for moderately low-frequency systems. In such cases, cessation of the excitation after any cycle implies that the envelope of the absolute maximum response spectra governs and that the maximum equivalent static force may be up to two times greater for nonperiodic excitation than it is for periodic excitation.

5.3.3.2 Excitation with Nonzero Mean. For excitations which have a nonzero mean, the initial value of the steady-state displacement is given by

$$y(0) \simeq \frac{1}{m} I_2'(0) + (x_{st})_0 = \frac{1}{k}\left[p^2 I_2'(0) + P_0 \right] \qquad (5.25)$$

The value of the initial steady-state velocity divided by p is, of course, still given by Eq. 5.20.

For moderately low-frequency systems, the value of p is small such that the force integral dependent part of the initial steady-state displacement becomes negligibly small with respect to the value of the static displacement induced by the mean of the excitation. The ratio expressed in Eq. 5.20 also becomes negligibly small, so that the value of $x_{max,max}$ is dependent on the mean of the excitation only. These assumptions are valid if 1) P_o is large or 2) the duration of the excitation is long.

From Eq. 5.18, the resulting $x_{max,max}$ considering these approximations is given by

$$x_{max,max} \simeq (x_{st})_0 \left[1 + e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi} \right] \qquad (5.26)$$

and is equal to the largest possible free vibration response since the value $(x_{st})_o$ is not frequency dependent.

It may also be seen that under such conditions the maximum value of the absolute maximum response spectrum for moderately low-frequency systems subjected to n cycles of excitation equals that of the absolute maximum response for periodic excitation for one value of the frequency parameter ft_o but is less than the absolute maximum response for periodic excitation for all other values. For this limiting case, therefore, one notes that cessation of excitations which have a nonzero mean does not increase the low-frequency limiting response over that which is indicated for a periodic version of the excitation.

5.4 Resonant Response

It was previously stated that for forcing functions which have a predominant Fourier series component the steady-state maxima at and near the resonant peak corresponding to this term are approximated well by consideration of this Fourier component alone. One would expect that this notion would also hold for the buildup of the response at resonance of systems which are initially at rest and subjected to n periods of excitations which have a predominant Fourier series term.

5.4.1 One-Term Sinusoidal Fourier Series Approximations. The response at the resonance corresponding to a predominant sinusoidal Fourier series component is approximately given then by

$$x(t) \simeq x_{st} \frac{a_r}{2\zeta} \left[e^{-\zeta r \omega t} - 1 \right] \cos r\omega t \tag{5.27}$$

and

$$x(t) \simeq x_{st} \frac{a_r}{2} \left[\sin r\omega t - r\omega t \cos r\omega t \right] \tag{5.28}$$

for damped and undamped systems, respectively, in which ω is the circular frequency of the excitation, a_r is the predominant Fourier coefficient, and r is the value of the Fourier index for this predominant term. The general form of these equations is well known (1); Eq. 5.28 is exactly correct for simple sinusoidal excitation whereas

Eq. 5.27 assumes that the systems at resonance are only moderately damped.

Both of these equations are a maximum at the end of the n^{th} period, such that the amplification factor at this resonant buildup is approximately given by

$$AF \simeq \frac{a_r}{2\zeta}\left[1 - e^{-2\pi\zeta r n}\right] \tag{5.29}$$

and

$$AF \simeq a_r\left[\pi r n\right] \tag{5.30}$$

for damped and undamped systems, respectively.

5.4.2 <u>One-Term Cosine Fourier Series Approximations.</u> The response at the resonance corresponding to a predominant cosine Fourier series component is approximately given by

$$x(t) \simeq x_{st}\frac{a_r}{2\zeta}\left[1 - e^{-\zeta r\omega t}\right] \sin r\omega t \tag{5.31}$$

and

$$x(t) \simeq x_{st}\frac{a_r}{2}\left[r\omega t\right] \sin r\omega t \tag{5.32}$$

for damped and undamped systems, respectively, which are initially at rest. Equation 5.32 is exact for simple cosine excitation whereas Eq. 5.31 assumes that the systems at resonance are only moderately damped.

Equations 5.31 and 5.32 are both approximately a maximum at the time given by

$$t = \left[n - \frac{1}{4r}\right] t_o \tag{5.33}$$

The amplification factor at this resonant buildup is, therefore, approximately given by

$$AF \simeq \frac{a_r}{2\zeta}\left[1 - e^{-2\pi\zeta(rn - \frac{1}{4})}\right] \tag{5.34}$$

and

$$AF \simeq a_r \pi \left[rn - \frac{1}{4}\right] \tag{5.35}$$

for damped and undamped systems, respectively.

The alternating step and shifted alternating step excitations represent forcing functions which have a predominant sinusoidal and cosine Fourier series component, respectively. Use of Eqs. 5.29 and 5.34 for these excitations, in fact, predicts the buildup of the first resonant peak (r = 1) indicated in Figs. 25 and 26 within 3% for all values of n.

The amplification factor for the undamped resonant response of systems subjected to the alternating step excitation, furthermore, is exactly given by Eq. 5.30 for the first resonant peak (2); i.e.,

$$AF \equiv \frac{4}{\pi}\left[\pi \times 1 \times n\right] = 4n \tag{5.36}$$

Consideration of the first Fourier term only yields the exact solution since the maximum response for the system subjected to the actual excitation does indeed occur at the end of the n^{th} period and the remaining sinusoidal Fourier series terms do not contribute in this case. One can, in fact, state that Eq. 5.30 will yield the exact amplification factor for the resonant buildup of undamped systems subjected to excitations which are defined by a sinusoidal Fourier series provided that the actual maximum undergone by the system occurs at the end of the n^{th} period.

5.5 Limiting Behavior of High-Frequency Systems

The limiting behavior of the response of high-frequency systems subjected to a finite number of cycles of an excitation may be determined using the same concepts developed earlier for the limiting behavior of systems subjected to periodic excitations. A limiting response can, however, also be found for the response of undamped systems due to a finite number of cycles of a nonharmonic excitation whereas this was, in general, not the case for the response due to periodic excitation.

5.5.1 Damped Systems. The limiting response equations developed for periodic excitation of damped high-frequency systems assume that the effects of discontinuities are such that the displacements induced by them quickly reduce to the value given by a "static" condition and that in the limit the effects of discontinuities on other portions of the response are negligible. Using this idea, formulae were presented which compared the effects of each discontinuity and force maximum separately. It was also stated that convergence to this limiting condition could be expected to be slower for less highly damped systems. These concepts naturally also apply to the case of damped systems subjected to a finite number of cycles of an excitation.

One can further note that the absolute maximum response occurs in the first cycle of excitation for high-frequency systems, except when the excitation has an initial discontinuity which is present in its periodic extension. For instance, the maximum response during the first cycle of the excitation shown in Fig. 17f is given by the larger of the following relations:

$$\frac{P_A}{k}\left[1+e^{\frac{-\beta}{\sqrt{1-\beta^2}}\pi}\right] ; \quad \frac{P_C}{k} ; \quad \frac{1}{k}\left[P_D + P_E e^{\frac{-\beta}{\sqrt{1-\beta^2}}\pi}\right] ; \quad \frac{P_F}{k}$$

for systems which are initially at rest. The first equation is not equal to that given for the steady-state and absolute maximum responses. If this relation governs, the limiting behavior of the absolute maximum response will first be given in the second cycle of the excitation.

The free vibration response is also seen to never govern the maximum response of high-frequency damped systems since cessation of the excitation results in a maximum response equal only to a fraction of the "static" value just before termination. For the excitation shown in Fig. 17f, for instance, the maximum free vibration response is given by

$$\frac{P_G}{k} e^{\frac{-\beta}{\sqrt{1-\beta^2}}\pi} < \frac{P_G}{k}$$

5.5.2 Undamped Systems. The maximum response of undamped high-frequency systems subjected to a finite number of cycles on the other hand may be many times greater than that for damped systems since the response due to a discontinuity may not for this case be considered localized at that one point in the excitation. The free vibration response is also a factor for undamped systems since the response just before termination of the excitation may not equal the "static" value that was assumed for the response of damped systems.

Figure 30 depicts the responses of high-frequency undamped systems subjected to one cycle of various excitations. The maximum response is seen to be strongly dependent on the interaction of the effects due to the discontinuities. Additionally, the response histories drawn in Fig. 30 assume that the natural period of the system is small compared to the smallest time interval between consecutive discontinuities.

This behavior may be proved from closer inspection of Eqs. 3.27, 3.28, and 3.29. From these equations one can conclude that the maximum response of undamped systems after a discontinuity is given by the relation:

$$\frac{1}{K} [P_a \pm P_b] \tag{5.37}$$

in which P_b is the magnitude of the change of force due to the discontinuity and P_a represents the maximum value of the forcing function after the discontinuity; P_a and P_b are signed quantities and are therefore dependent on the orientation of the force axis. The value of P_a is no longer restricted to that value of the forcing function directly after the discontinuity since the effect of the discontinuity on the response does not dampen and the maximum response necessarily occurs where the maximum value of the forcing function occurs.

These conclusions may be verified by consideration of the analogous ground excited system. Reference 4, for instance, considers the high-frequency limiting behavior of undamped systems subjected to ground shaking and concludes that 1) the effect of a continuous input acceleration is a "static" condition such that the pseudoacceleration equals that of the maximum input acceleration, and that 2) the effect of a discontinuity in the input acceleration is to make the amplitude of the periodic component of the system acceleration equal to the magnitude of the discontinuity.

Using these concepts, one can conclude that the maximum displacement for one cycle of the excitation shown in Fig. 30f is given by the larger of the relations:

$$\frac{1}{K}\left[P_A + P_C\right] \; ; \; \frac{1}{K}\left[-P_A + P_E + P_F\right] \; ; \; \frac{1}{K}\left[P_A - P_E + P_G\right]$$

Damping is, therefore, seen to have an immense influence on the limiting response of high-frequency systems subjected to discontinuous excitations. The limiting response of undamped high-frequency systems subjected to n cycles of the alternating step excitation, for example, is given by $4n(x_{st})$ whereas the limit for damped systems is given by Eq. 3.30 which is always less than $3(x_{st})$.

VI. CONCLUSIONS

The following aspects of the response of structures to periodic excitation have been analyzed: 1) the steady-state response; 2) the absolute maximum response; 3) the rate of "build-up" of the response; and 4) the effects of possible cessation of the excitation. Emphasis has been placed on the response at resonance and the limiting behavior of the response of low and high-frequency systems for these four areas of interest; general equations have been presented which identify the limiting behavior.

It has further been shown that the response of low-frequency systems is strongly influenced by the percentage of critical damping, the shapes and maxima of the appropriate integrals of the forcing function, and by the mean or constant force component of the excitation. The absolute maximum response and the response during the buildup to periodic excitation were shown to be significantly different from the steady-state response for low-frequency systems.

The response of high-frequency systems has been shown to be most sensitive to the percentage of critical damping, nonharmonic nature of the forcing function, and the presence, magnitudes, and locations of discontinuities in the excitation and its periodic extension. The limiting response of high-frequency systems was found to be the same for the four areas mentioned above except for certain cases which were specifically identified.

These concepts were illustrated through response histories and response spectra for several simple excitations. It was also shown that a tripartite log plot which is analogous to that used for the response spectra of ground excited systems is very useful in identifying the limiting behavior.

Additionally, the method of analysis used (3), which interrelates the steady-state and the transient responses, was found to be particularly efficient in obtaining the steady-state response and useful in deriving the equations for the limiting behavior of low-frequency systems. It was further shown that this procedure even enables one to obtain closed form solutions for the steady-state response in certain special cases.

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APPENDIX A - REFERENCES

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APPENDIX B - NOTATION

The following symbols are used in this paper;

AF = dynamic amplification factor;

f = natural frequency of the system;

F = maximum equivalent static force;

$g(t)$ = displacement produced by a unit initial displacement;

$h(t)$ = displacement produced by a unit initial velocity;

$I_1(t)$ = instantaneous value of the first integral of $P(t)$ where the initial value of the integral is taken equal to zero;

I_1 = peak numerical value of $I_1(t)$;

$I_1'(t)$ = instantaneous value of the integral of $P(t)$ where the initial value is taken such that the integral has a zero mean;

I_1' = peak numerical value of $I_1'(t)$;

$I_2(t)$ = instantaneous value of the second integral of $P(t)$ where the initial value of the integral is taken equal to zero;

I_2 = peak numerical value of $I_2(t)$;

$I_2'(t)$ = instantaneous value of the integral of $I_1'(t)$ where the initial value is taken such that the integral has a zero mean;

I_2' = peak numerical value of $I_2'(t)$;

k = stiffness coefficient of the system;

m = mass of the system;

$p, \bar{p}$ = circular natural frequency of the system without and with damping, respectively;

$P(t)$ = instantaneous value of the exciting force;

P = peak numerical value of $P(t)$;

P_o = constant component of $P_T(t)$;

$P_T(t)$ = instantaneous value of nonzero-mean exciting force;

P_a = numerical value of the forcing function after a discontinuity;

P_b = signed magnitude of a discontinuity in the forcing function;

t = time;

t_o = period of the excitation;

$T, \bar{T}$ = period of the system without and with damping, respectively;

$x(t)$ = transient displacement of the system;

x_{st} = static displacement of the system produced by P;

$(x_{st})_o$ = static displacement of the system produced by P_o;

x_{max} = maximum transient displacement;

$x_{max,max}$ = reference displacement for the free vibration response spectrum on the tripartite log plot;

$y(t)$ = steady state displacement of the system;

y_{max} = maximum steady state response;

ζ = fraction of critical damping;

$\xi(t)$ = corrective displacement;

ϕ = ratio of the circular frequency of excitation to the circular natural frequency of the system;

ω = circular frequency of the excitation.

APPENDIX C - CLOSED-FORM EXPRESSIONS FOR STEADY-STATE RESPONSE OF DAMPED SYSTEMS SUBJECTED TO ALTERNATING STEP FORCE

Using either of the two procedures discussed in Sec. 3.4, the steady-state response of damped systems subjected to the alternating step force can be shown to be given by

$$y(t) = x_{st}\left[1 - \frac{2}{\lambda\sqrt{1-\zeta^2}}\, e^{-\zeta p t} \sin(\bar{p}t + \alpha)\right] \quad \text{for } 0 \le t \le \frac{t_o}{2} \tag{C1}$$

and

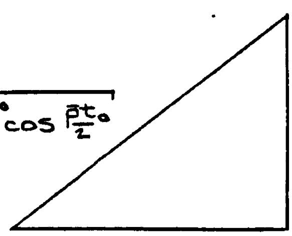
$$y\left(\frac{t_o}{2} + \tau\right) = -y(\tau) \quad \text{for } 0 \le \tau \le \frac{t_o}{2} \tag{C2}$$

in which $\bar{p} = p\sqrt{1-\zeta^2}$ is the damped circular natural frequency of the system; x_{st} is the static displacement of the system produced by the peak value of the applied force; α is defined by

$$\tan\alpha = \frac{\delta - \frac{\zeta}{\sqrt{1-\zeta^2}}\eta}{\frac{\zeta}{\sqrt{1-\zeta^2}}\delta + \eta} \quad \text{and} \quad 0 \le \alpha \le \pi \;;$$

and λ, η, and δ are given by the sides of the triangle shown below.

$$\lambda = \sqrt{1 + e^{-\zeta p t_o} + 2e^{-\frac{\zeta p t_o}{2}}\cos\frac{\bar{p}t_o}{2}}$$



$$\eta = e^{-\frac{\zeta p t_o}{2}}\sin\frac{\bar{p}t_o}{2}$$

$$\delta = 1 + e^{-\frac{\zeta p t_o}{2}}\cos\frac{\bar{p}t_o}{2}$$

Differentiation of Eq. C1 reveals that the maxima and minima of the steady-state response during the first half of the excitation period occur at times given by

$$t = \frac{1}{\bar{p}}(\theta + \pi n) \tag{C3}$$

in which n is a nonnegative integer but also complies with the restriction

$$n \leq \frac{1}{\pi}\left(\frac{\bar{p}t_0}{2} - \theta\right) \quad ; \tag{C4}$$

and $\theta = \tan^{-1}\frac{\eta}{\delta}$. The restriction on the value of n given in Eq. C4 arises from the fact that Eq. C1 is only valid over the first half of the excitation period.

The maxima and minima of the steady-state response during the first half of the excitation period are given then by

$$y_{max,min} = x_{st}\left[1 + \frac{2(-1)^{n+1}}{\lambda} e^{\frac{-\zeta}{\sqrt{1-\zeta^2}}(\theta+\pi n)}\right] \tag{C5}$$

It can further be shown that, for reasonable amounts of damping, the maximum value of the steady-state response for systems subjected to the alternating step force corresponds to the case of

$n = 0$ when $ft_o < 1$

and

$n = 1$ when $ft_o > 1$

The steady-state amplification factor, $AF = y_{max}/x_{st}$, can therefore be expressed in closed form as

$$AF = \frac{2}{\lambda} e^{\frac{-\zeta}{\sqrt{1-\zeta^2}}\theta} - 1 \qquad \text{for } ft_o < 1 \tag{C6}$$

and

$$AF = \frac{2}{\lambda} e^{\frac{-\zeta}{\sqrt{1-\zeta^2}}(\theta+\pi)} + 1 \qquad \text{for } ft_o > 1 \tag{C7}$$

APPENDIX D - MISCELLANEOUS CLOSED-FORM EXPRESSIONS FOR STEADY-STATE RESPONSE

The equations given below were obtained by application of the methods described in Sec. 3.4. The excitations considered are shown in Fig. 1; the orientation of the time axis for each excitation should be carefully noted.

The steady-state displacement, y(t), of undamped systems subjected to the "alternating versine" excitation is given by

$$y(t) = x_{st} \frac{2\phi^2}{4\phi^2 - 1} \left[1 - \cos pt - \tan \frac{pt_o}{4} \sin pt - \frac{1}{2\phi^2} \sin^2 \omega t \right] \tag{D1}$$

for $0 \leq t \leq \frac{t_o}{2}$ and by

$$y\left(\frac{t_o}{2} + \tau\right) = -y(\tau) \qquad \text{for} \quad 0 \leq \tau \leq \frac{t_o}{2} \tag{D2}$$

The value ϕ is the ratio of the circular frequency of the excitation, ω, and the undamped circular frequency of the system, p; t_o represents the period of the excitation.

It can be noted that Eq. D1 is indeterminate for systems whose structural period, T, is exactly one half the duration of the period of the excitation. Applying L'Hospital's rule to Eq. D1, the steady-state response for this special system can be shown to be given by

$$y(t) = x_{st} \left[\sin^2\left(2\pi \frac{t}{t_o}\right) + \pi \left(\frac{1}{4} - \frac{t}{t_o}\right) \sin\left(4\pi \frac{t}{t_o}\right)\right] \tag{D3}$$

for $0 \leq t \leq \frac{t_o}{2}$ and by Eq. D2 for the second half of the excitation period.

The steady-state response of undamped systems subjected to the "absolute sine" excitation is given by

$$y(t) = x_{st} \frac{1}{1-\phi^2} \left[\sin \omega t - \phi \left(\sin pt + \cot \frac{pt_o}{4} \cos pt \right) \right] \tag{D4}$$

for $0 \leq t \leq \frac{t_o}{2}$ and by

$$y\left(\frac{t_o}{2} + \tau\right) = y(\tau) \qquad \text{for} \qquad 0 \leq \tau \leq \frac{t_o}{2} \tag{D5}$$

Equation D4 results in an indeterminate condition when the structural period is exactly equal to the period of the excitation. Use of L'Hospital's rule in this case indicates that the steady-state response is given by

$$y(t) = x_{st} \left[\frac{1}{2} \sin\left(2\pi \frac{t}{t_o}\right) + \pi \left(\frac{1}{4} - \frac{t}{t_o}\right) \cos\left(2\pi \frac{t}{t_o}\right) \right] \tag{D6}$$

for $0 \leq t \leq \frac{t_o}{2}$ and by Eq. D5 for the second half of the excitation period.

The steady-state response of undamped systems subjected to the "half-sine" excitation is given by

$$y(t) = x_{st} \frac{1}{1-\phi^2} \left[-\frac{\phi}{2} \left(\sin pt + \cot \frac{pt_o}{4} \cos pt \right) + \sin \omega t \right] \tag{D7}$$

for $0 \leq t \leq \frac{t_o}{2}$ and by

$$y(t) = x_{st} \frac{1}{1-\phi^2} \left[-\frac{\phi}{2} \left(\sin p\left(t - \frac{t_o}{2}\right) + \cot \frac{pt_o}{4} \cos p\left(t - \frac{t_o}{2}\right) \right) \right] \tag{D8}$$

for $\frac{t_o}{2} \leq t \leq t_o$.

The steady-state response of undamped systems subjected to the "alternating triangle" excitation is given by

$$y(t) = x_{st} \left[1 - 4\frac{t}{t_o} + \frac{4}{pt_o} \left(\sin pt - \tan \frac{pt_o}{4} \cos pt \right) \right] \tag{D9}$$

for $0 \leq t \leq \frac{t_o}{2}$ and by

$$y\left(\frac{t_o}{2}+\tau\right) = -y(\tau) \qquad \text{for} \qquad 0 \le \tau \le \frac{t_o}{2} \tag{D10}$$

The steady-state amplification factor, $AF = y_{max}/x_{st}$, can also be found for this case and may be expressed as

$$AF = \left| 1 - \frac{4n}{ft_o} - \frac{2}{\pi ft_o}\tan\left(\frac{\pi}{2}ft_o\right) \right| \tag{D11}$$

in which the value of n is given by zero or the largest integer which is less than or equal to $\frac{1}{2}ft_o$; the largest amplification factor calculated for these two cases governs.

The steady-state response of undamped systems subjected to the <u>"saw-tooth" excitation</u> is given by

$$y(t) = x_{st}\left[1 - \cos pt - 2\frac{t}{t_o} + \cot\frac{pt_o}{2}\sin pt\right] \tag{D12}$$

for $0 \le t \le t_o$.

The steady-state amplification factor can also be found for this case and may be expressed as

$$AF = \Bigg| 1 - \frac{2n}{ft_o} - \sqrt{\csc^2\pi ft_o - \left(\frac{1}{\pi ft_o}\right)^2} + \frac{1}{\pi ft_o} \times$$

$$\left[-\sin^{-1}\left(\frac{1}{\pi ft_o}|\sin\pi ft_o|\right) + \tan^{-1}(\cot\pi ft_o)\right] \Bigg| \tag{D13}$$

in which the value of n is given by the largest integer which is less than or equal to ft_o.

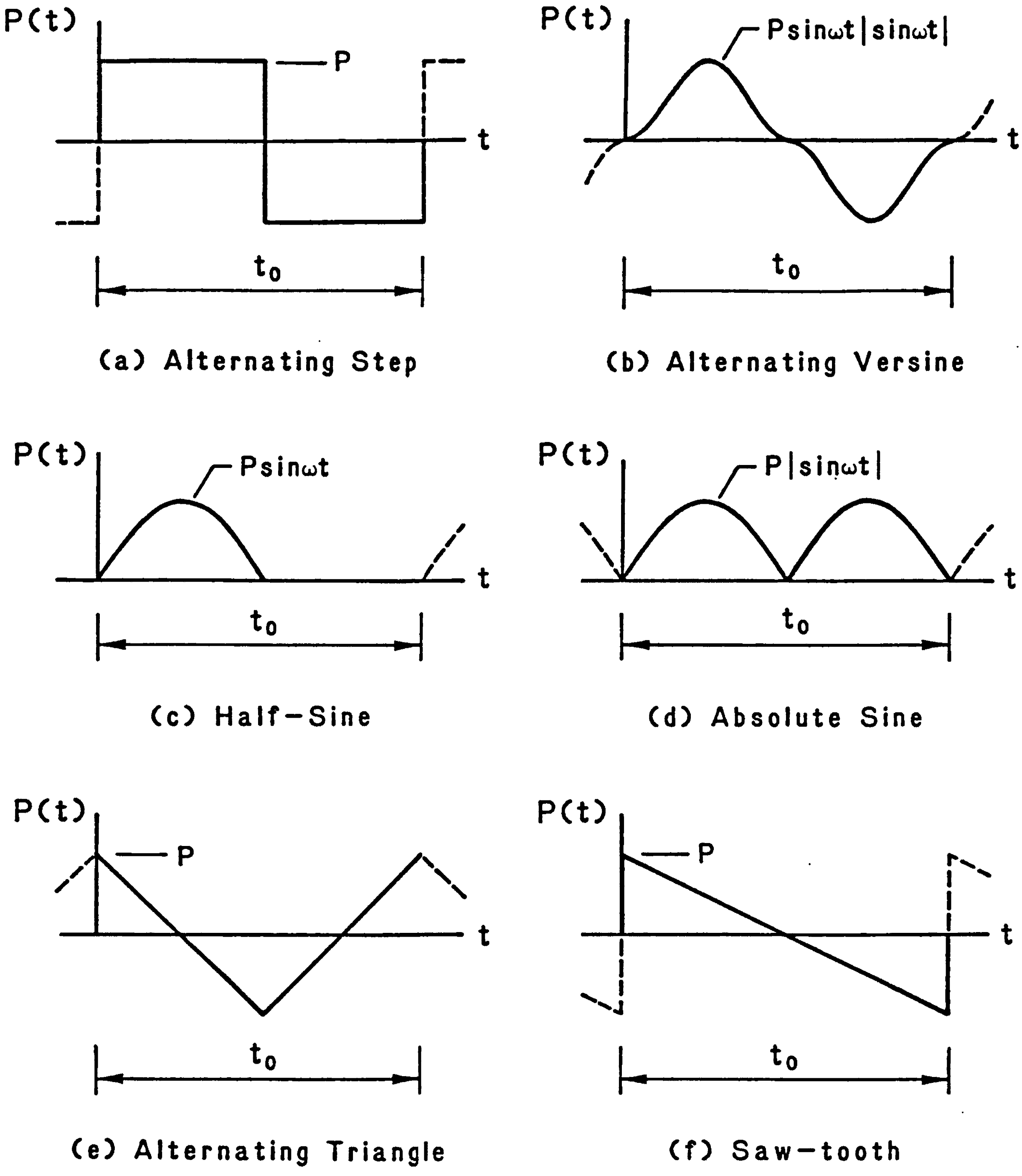


FIG. 1 Definition of Periodic Excitations Considered

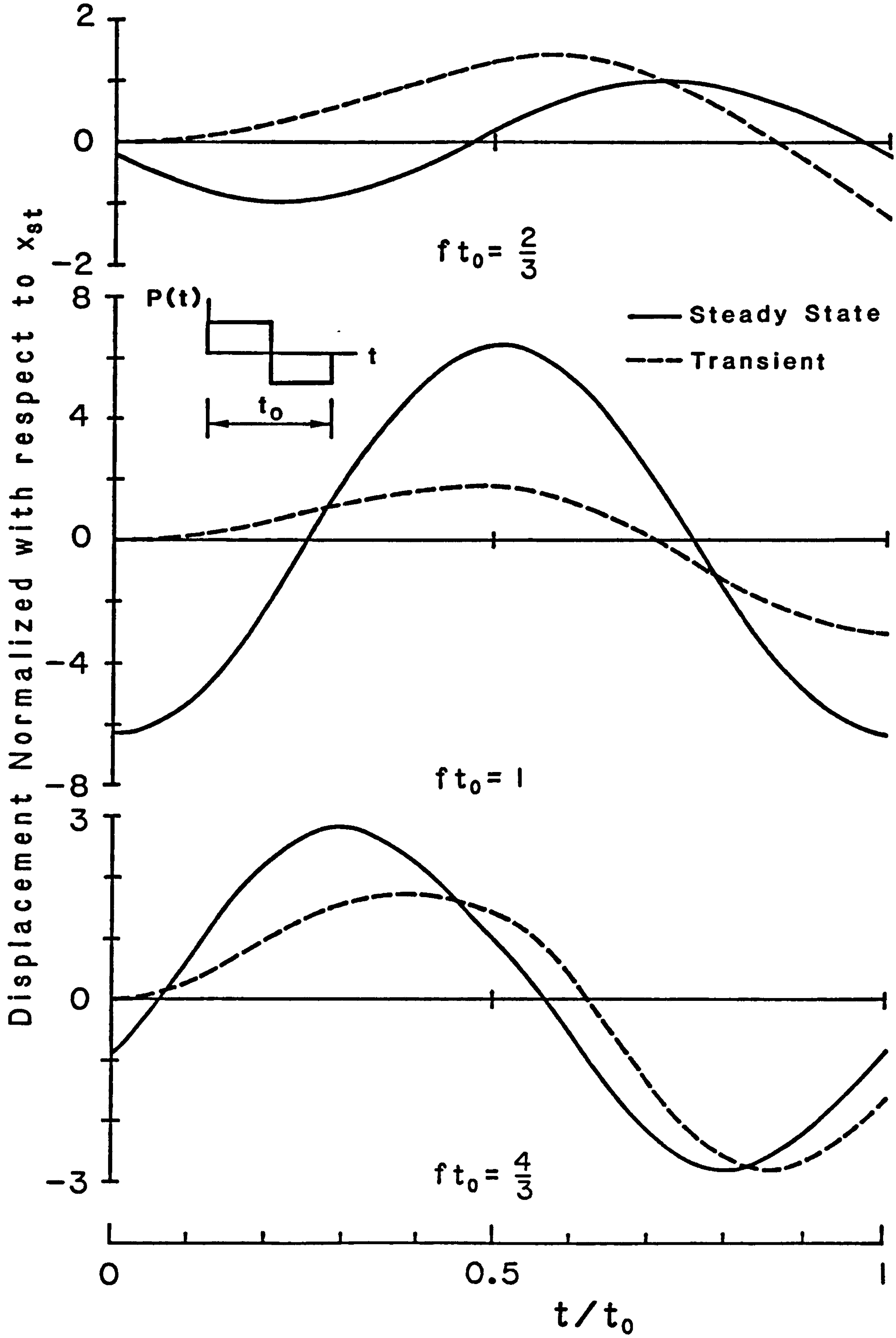


FIG. 2a Comparison of Transient and Steady-State Responses for Systems with $\zeta = 0.10$ Subjected to the Alternating Step Force

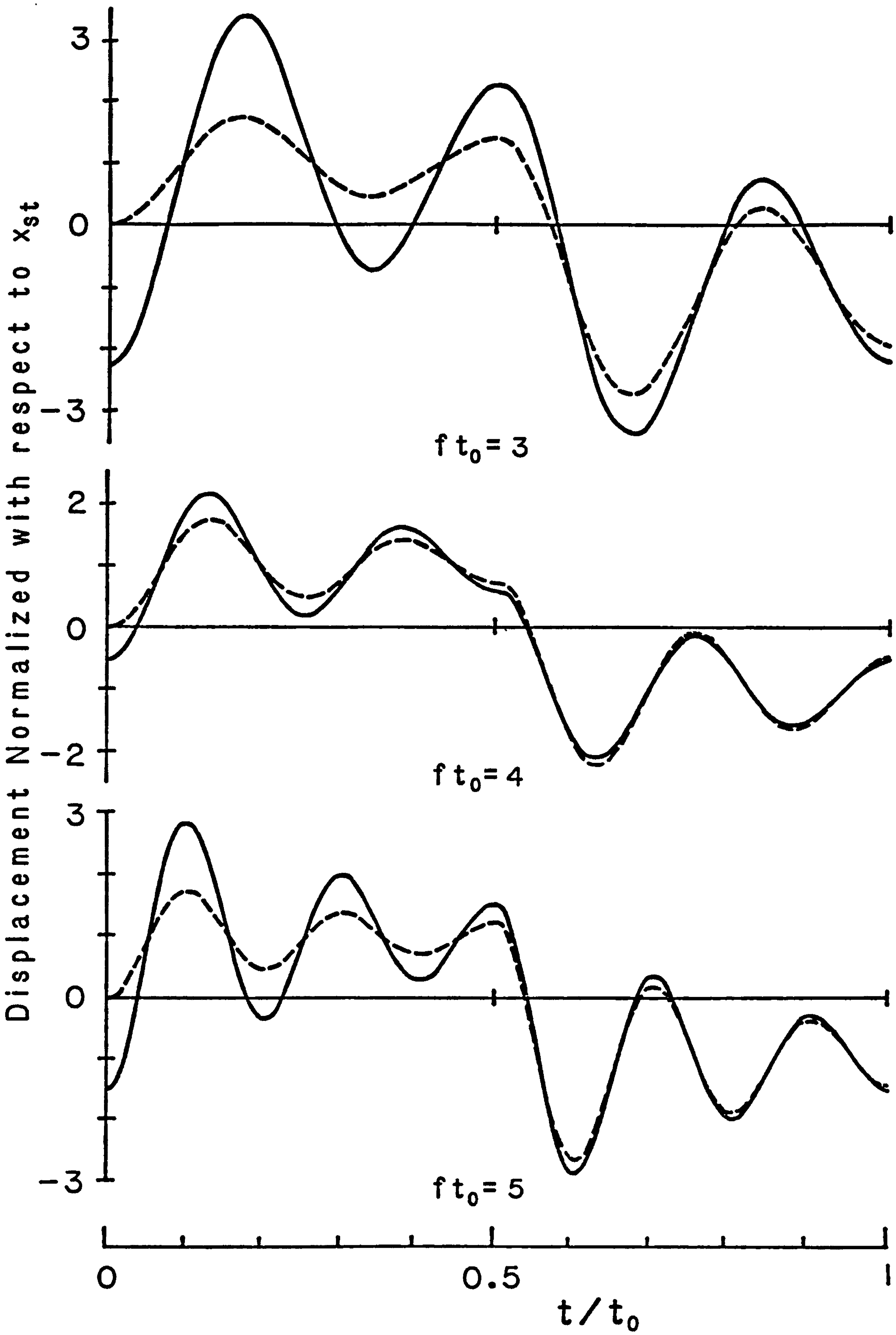


FIG. 2b Comparison of Transient and Steady-State Responses for Systems with $\zeta = 0.10$ Subjected to the Alternating Step Force

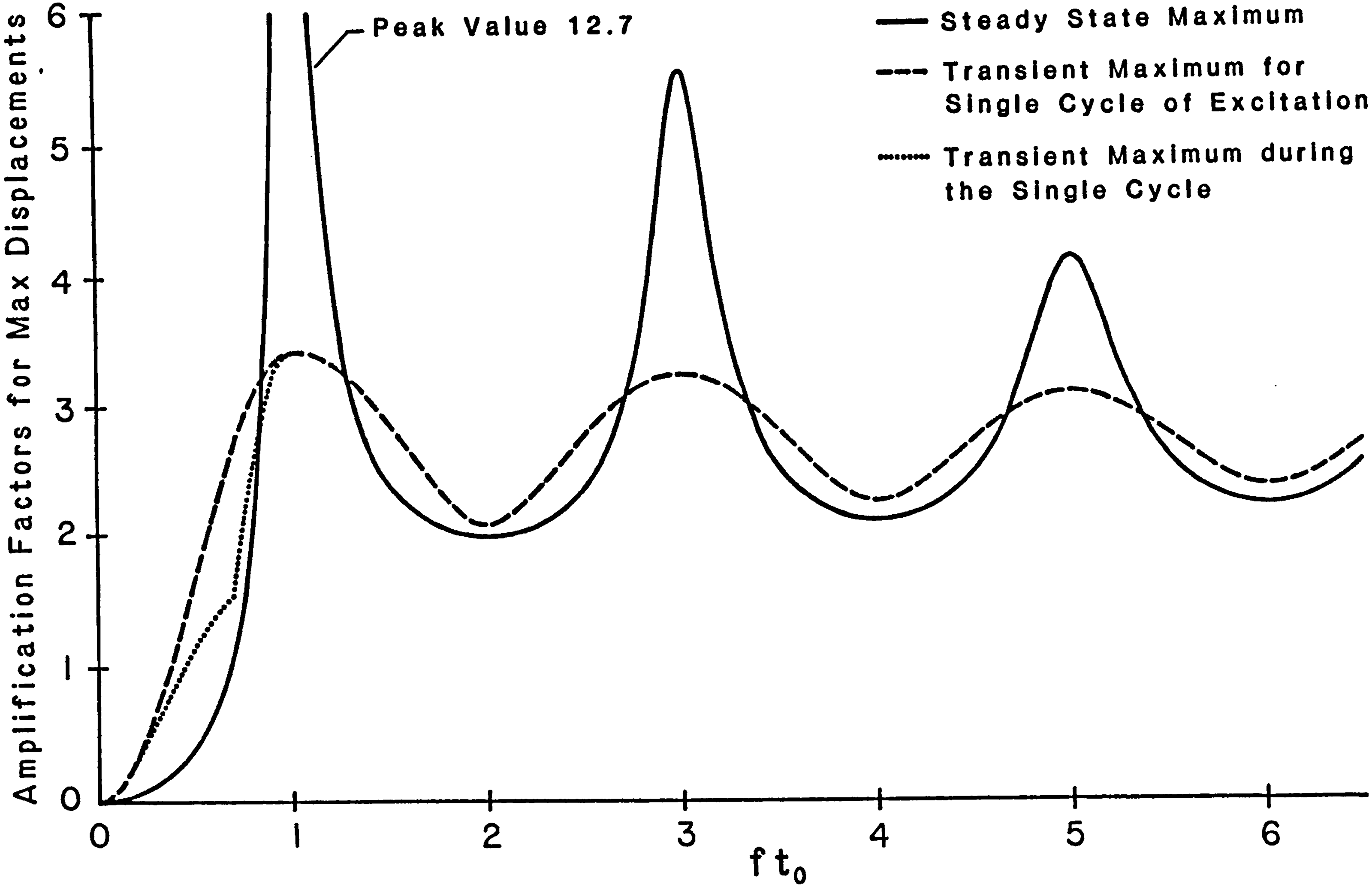


FIG. 3 Comparison of Response Spectra for Steady-State and Transient Displacements of Systems with $\zeta = 0.05$ Subjected to Alternating Step Force

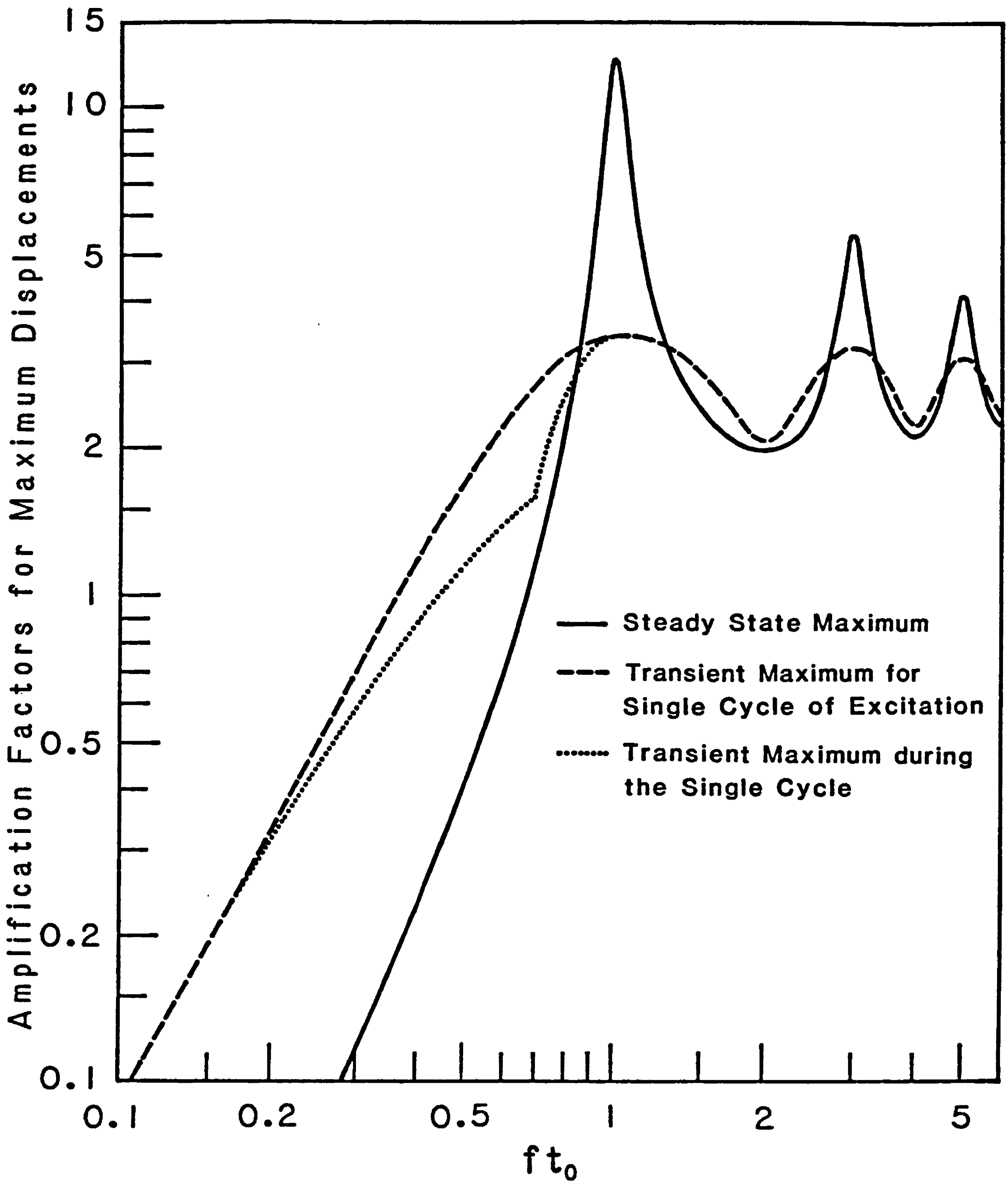


FIG. 4 Logarithmi('lots of Response Spectra for Steady-State and Transient isplacements of Systems with $\zeta = 0.05$ Subjected to Alternati .g Step For.-

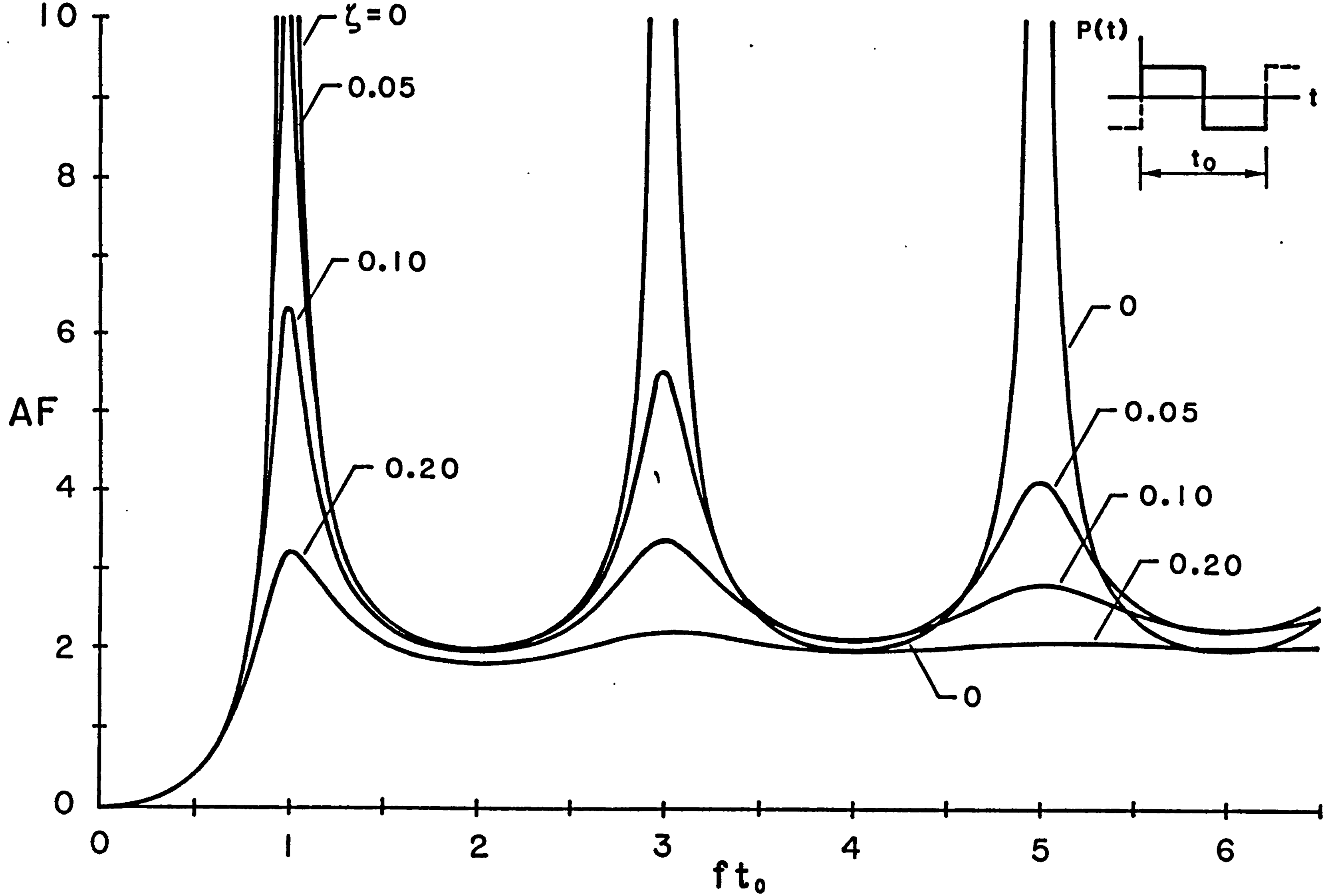


FIG. 5 Response Spectra for Steady-State Displacement of Systems Subjected to Alternating Step Force

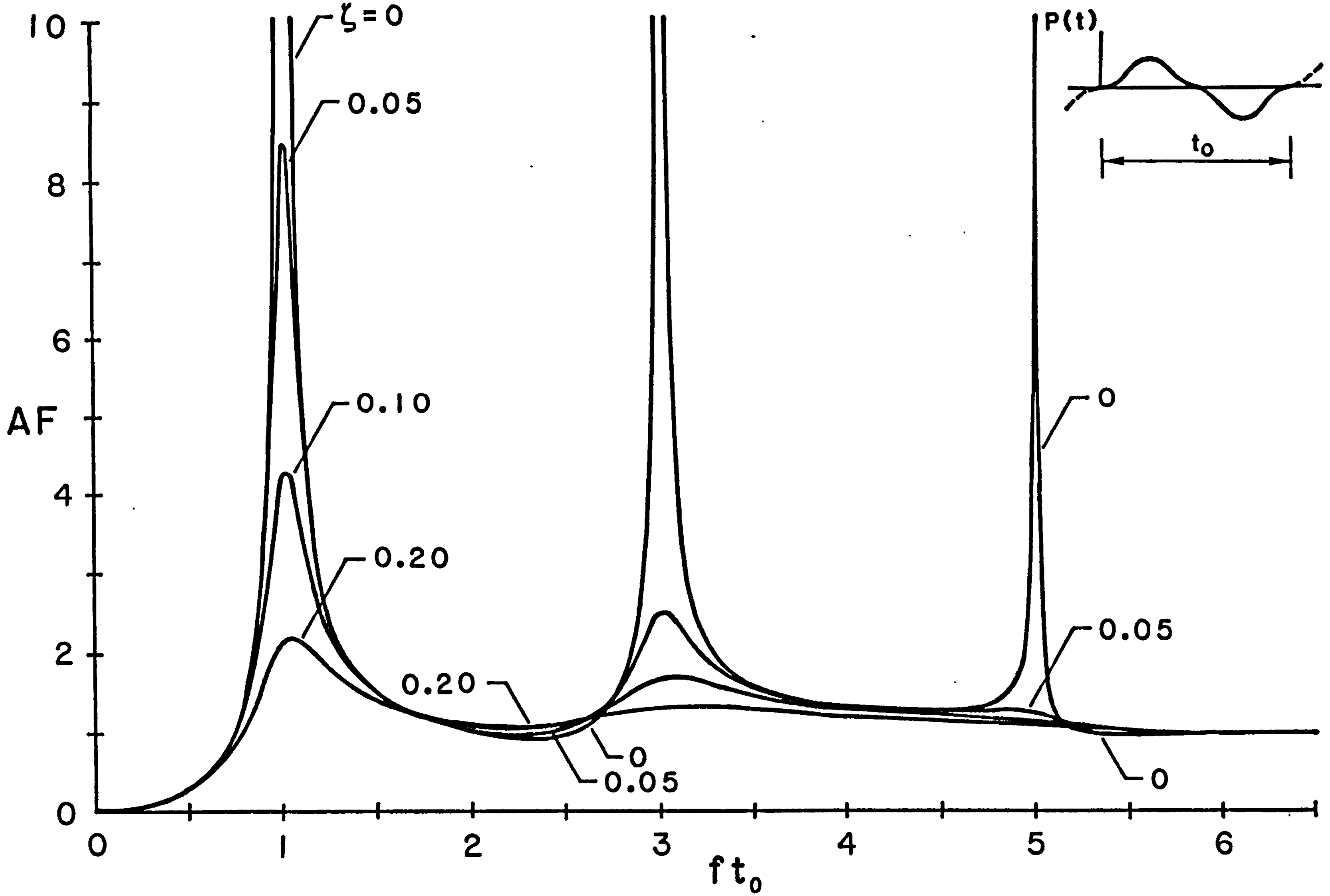


FIG. 6 Response Spectra for Steady-State Displacement of Systems Subjected to Alternating Versine Force

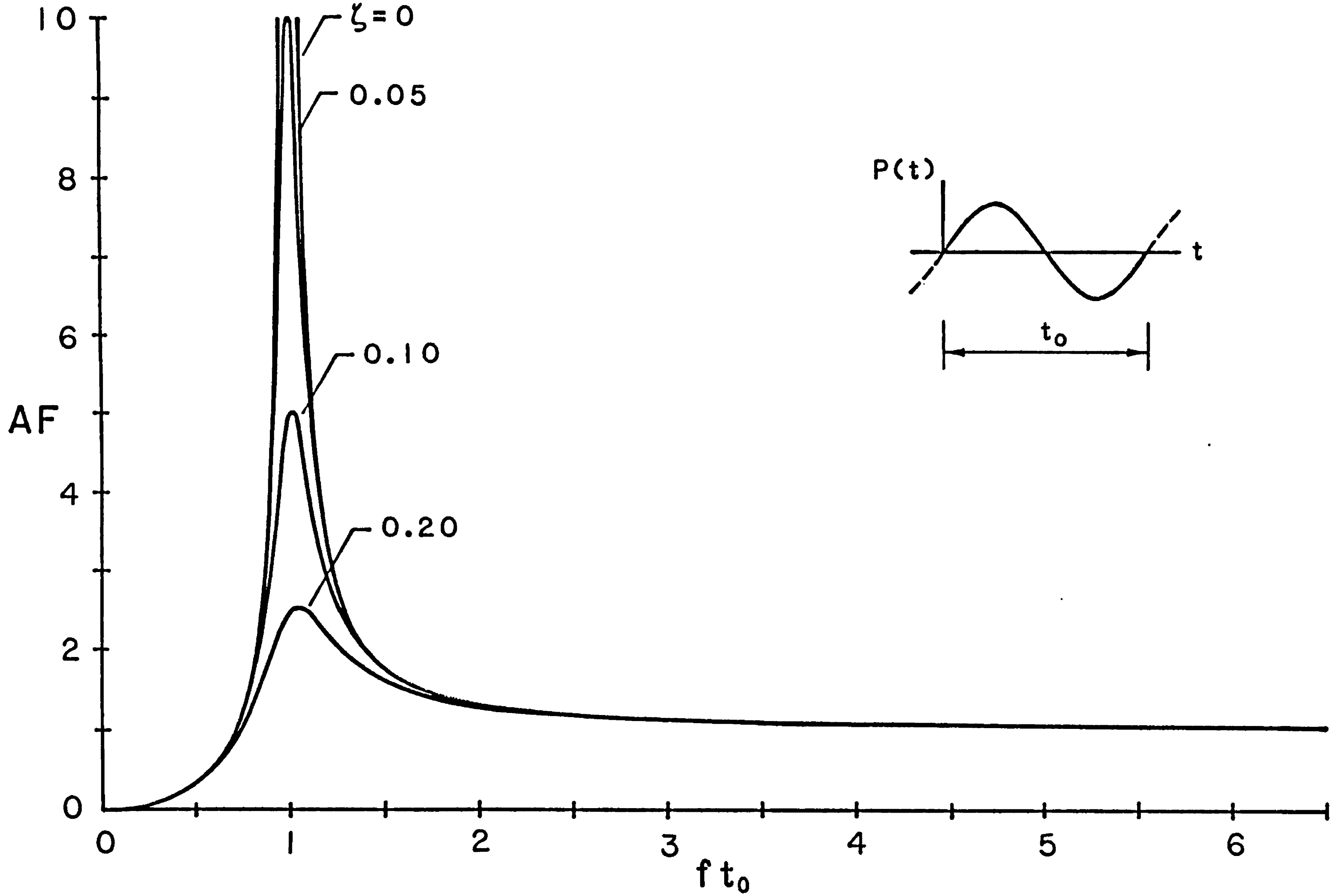


FIG. 7a Response Spectra for Steady-State Displacement of Systems Subjected to Three Different Arrangements of Sinusoidal Force Pulses

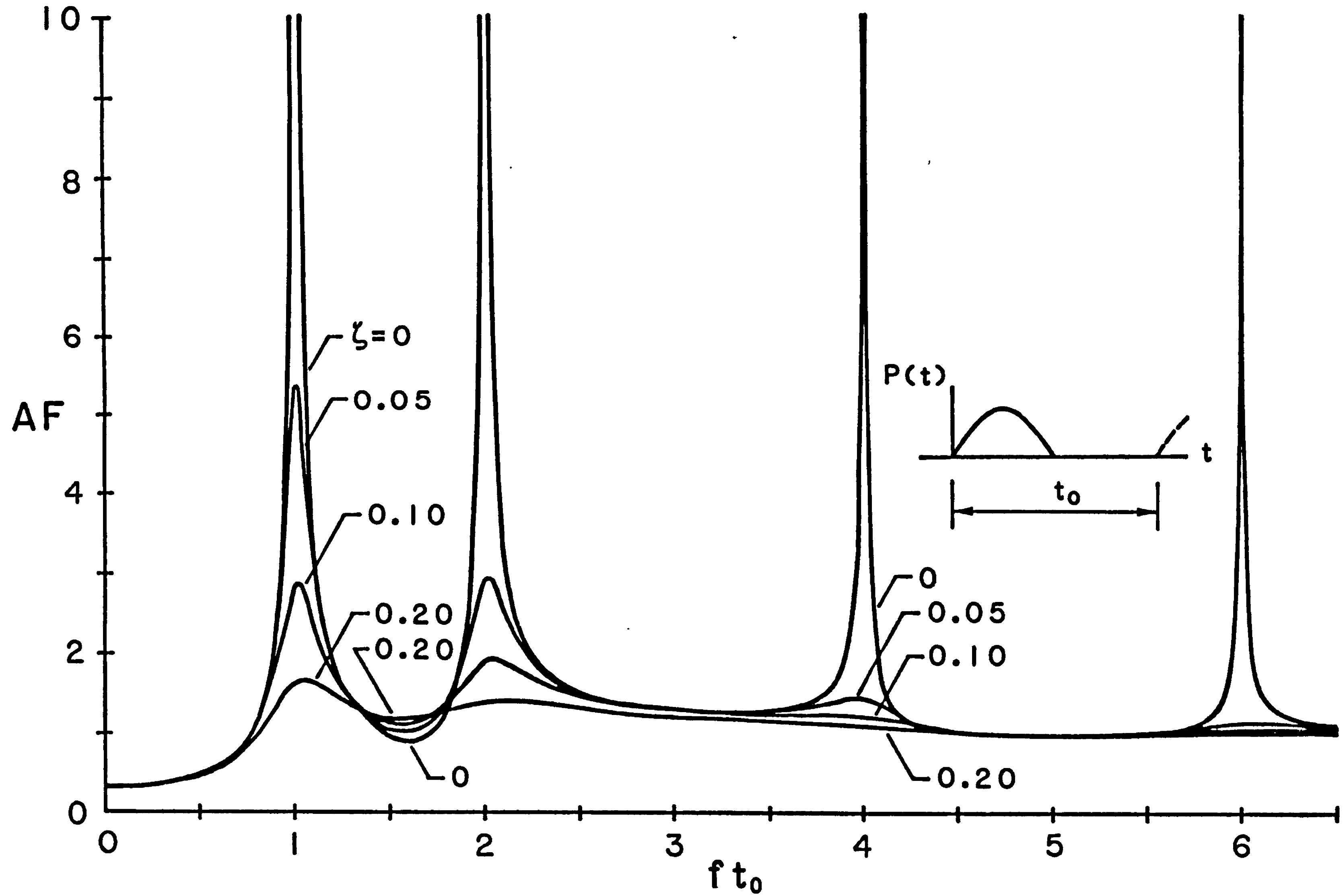


FIG. 7b Response Spectra for Steady-State Displacement of Systems Subjected to Three Different Arrangements of Sinusoidal Force Pulses

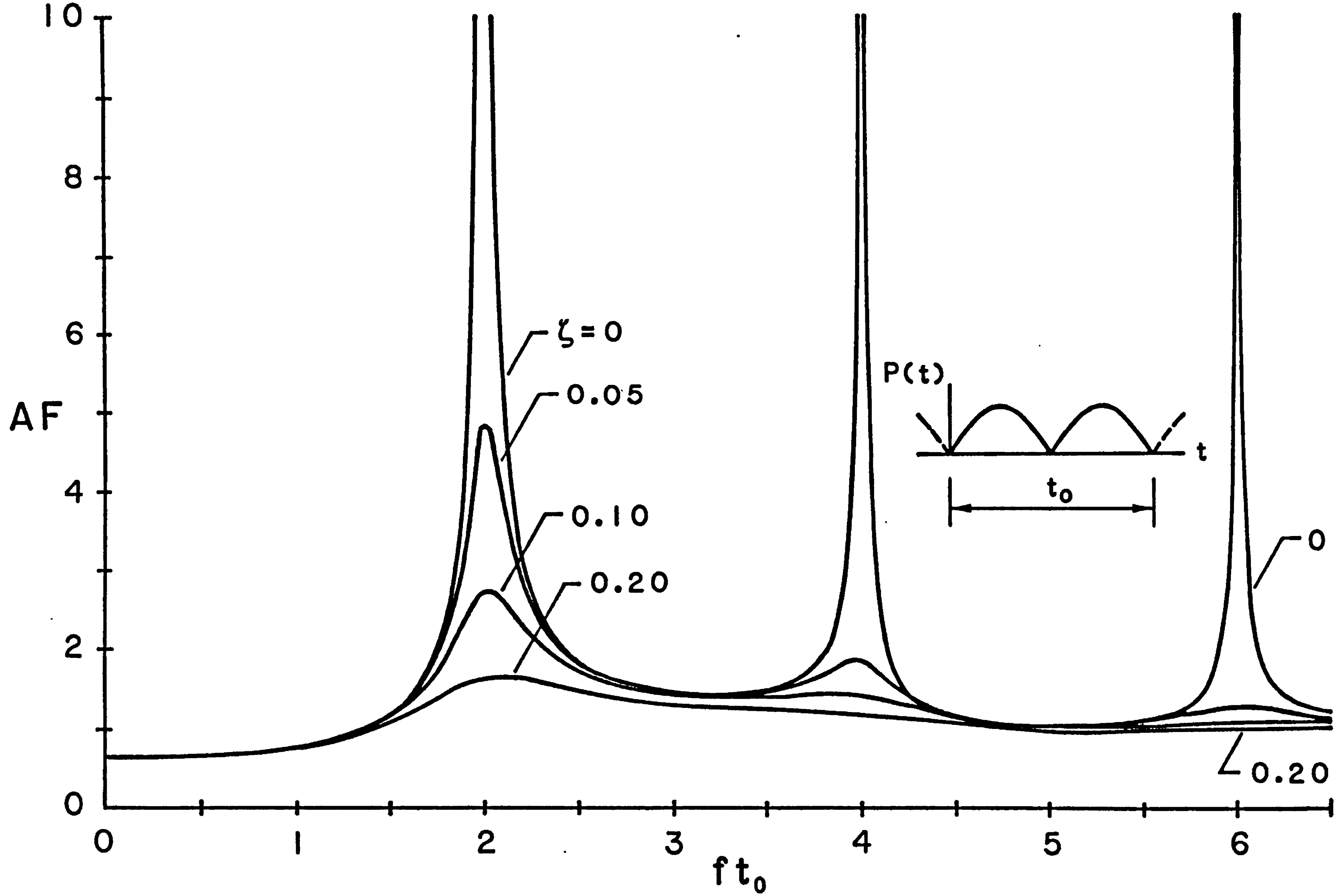


FIG. 7c Response Spectra for Steady-State Displacement of Systems Subjected to Three Different Arrangements of Sinusoidal Force Pulses

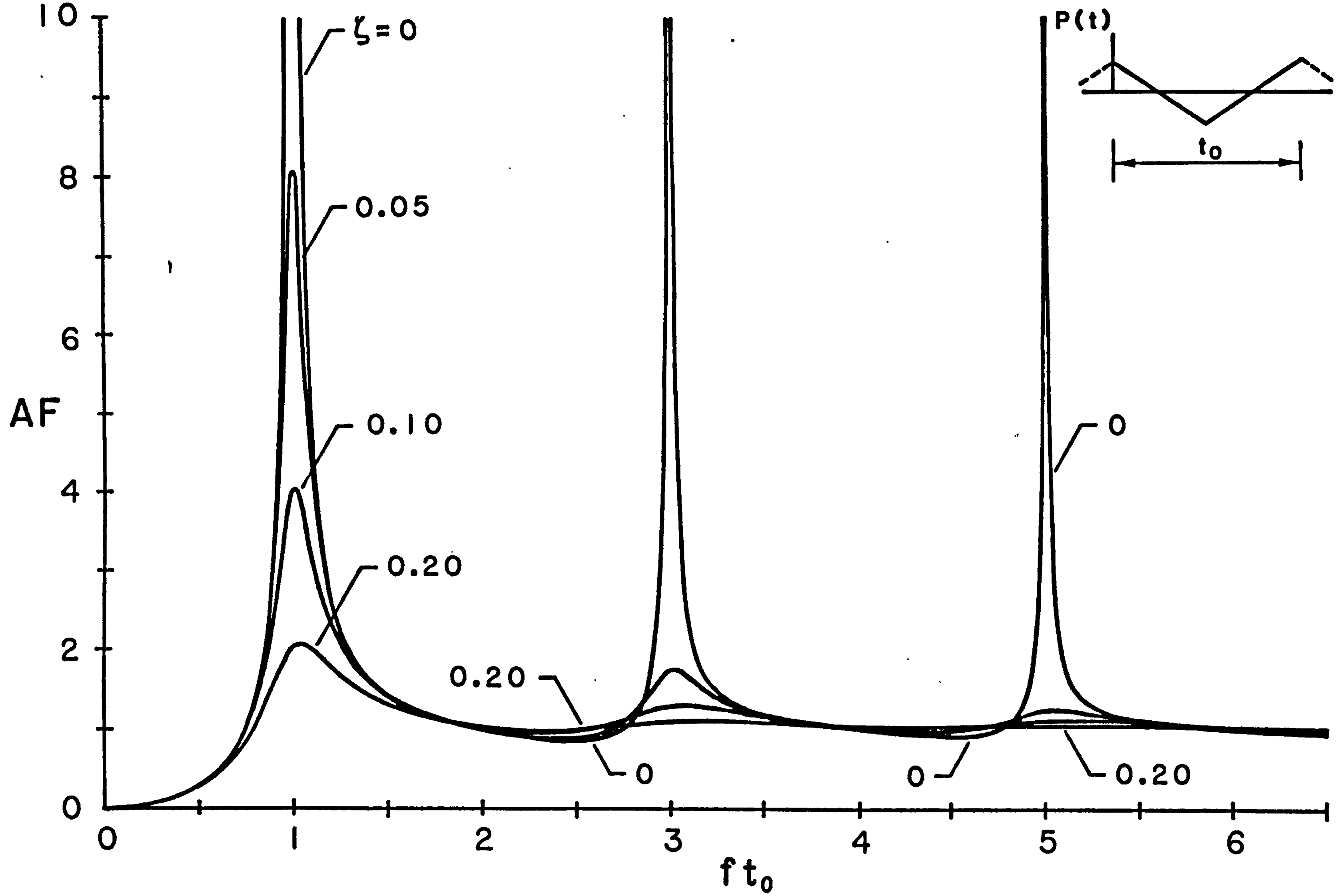


FIG. 8 Response Spectra for Steady-State Displacement of Systems Subjected to Alternating Triangular Force

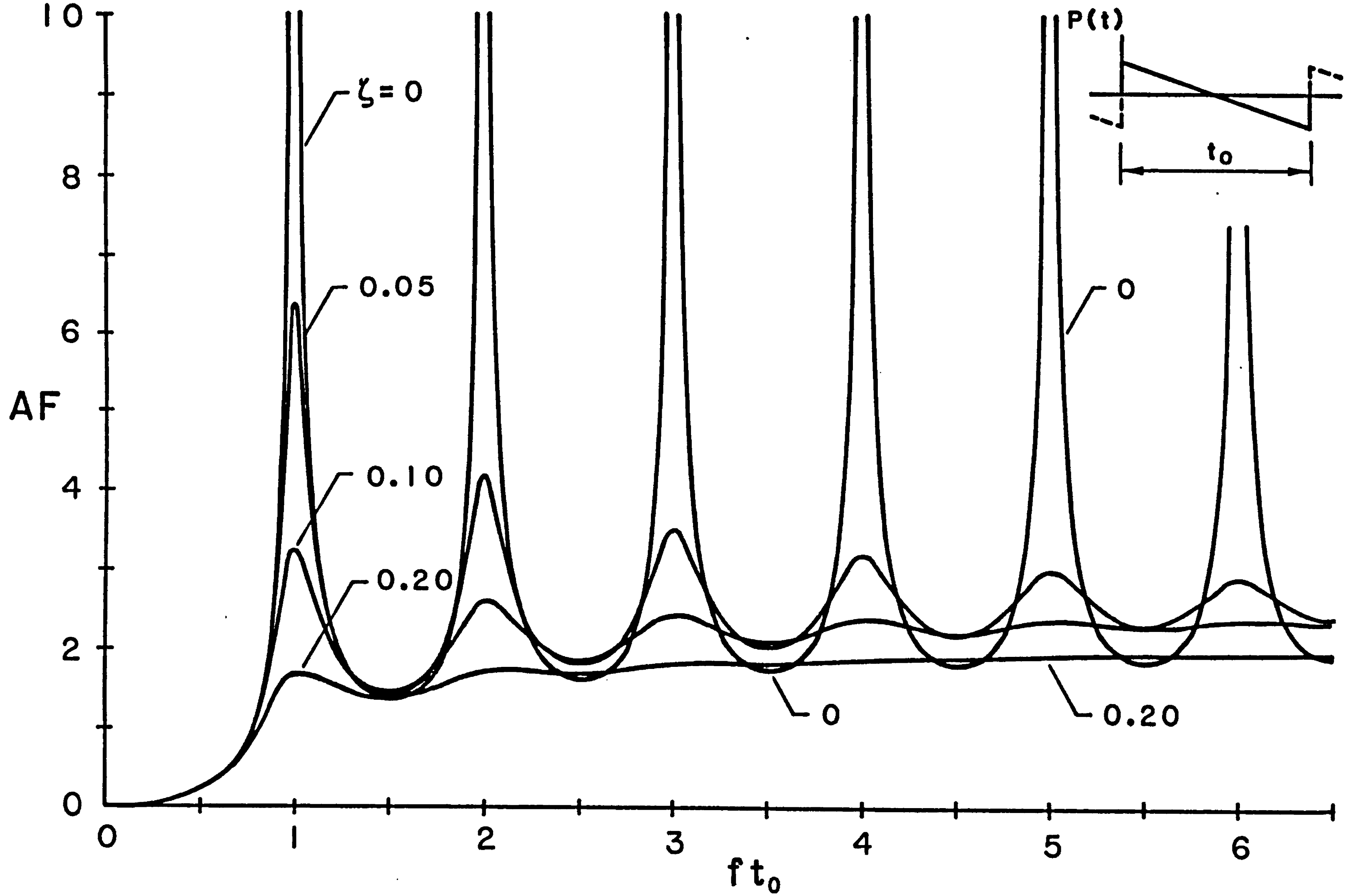


FIG. 9 Response Spectra for Steady-State Displacement of Systems Subjected to Saw-Tooth Force

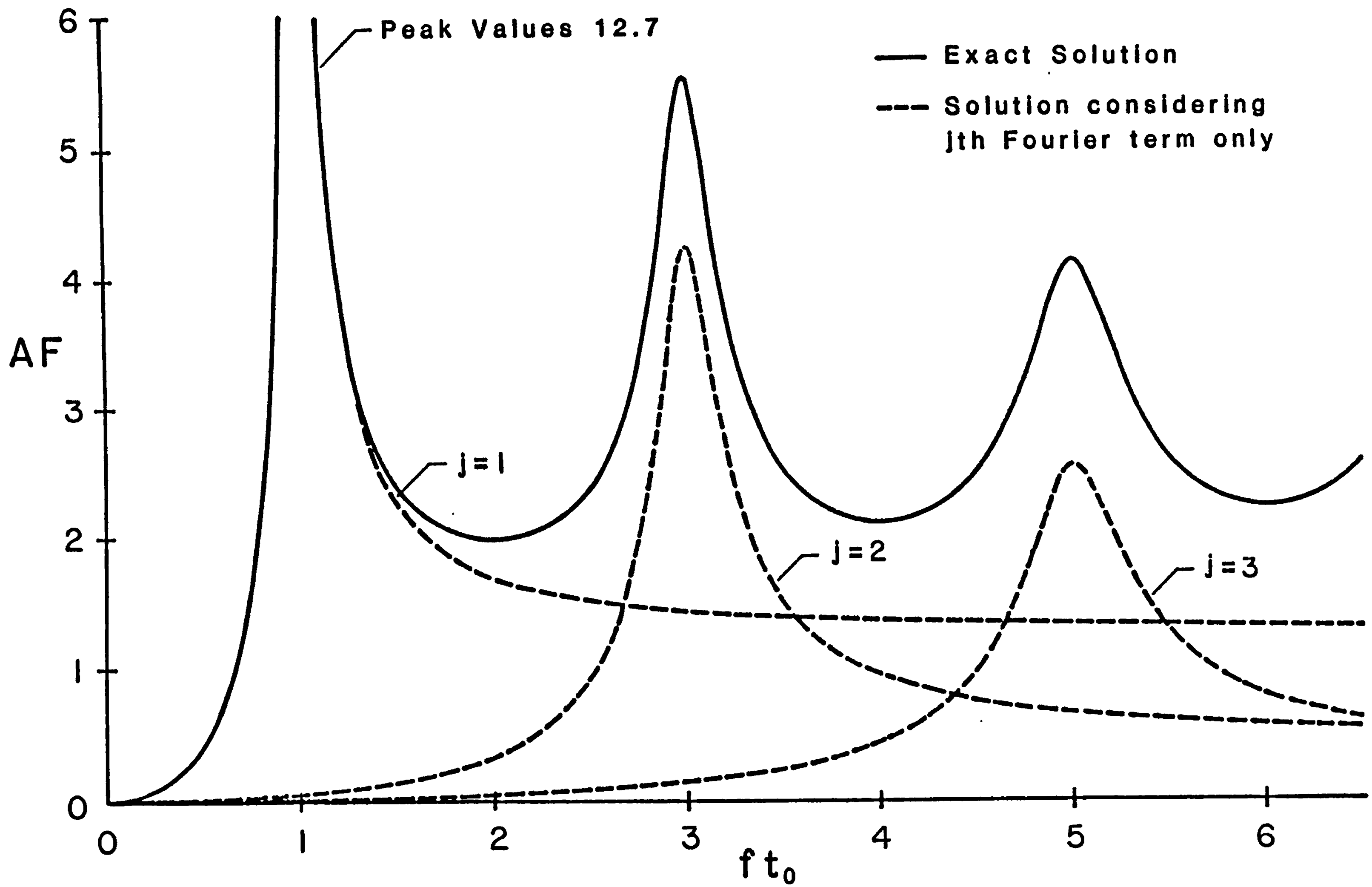


FIG. 10 Comparison of Exact with Approximate Response Spectra for Systems with $\zeta = 0.05$ Subjected to Alternating Step Force

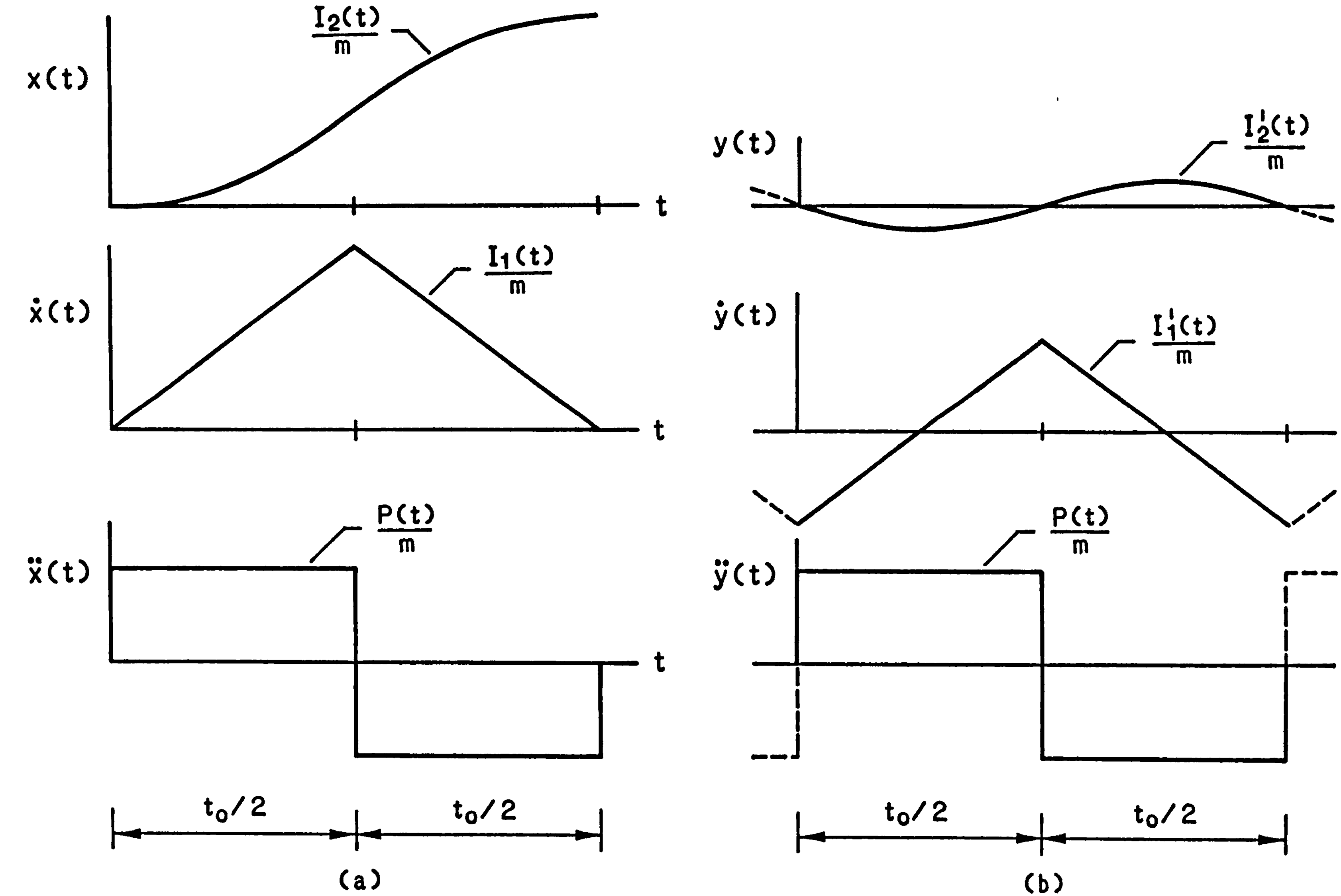


FIG. 11 Limiting Transient and Steady-State Responses of Low-Frequency Systems

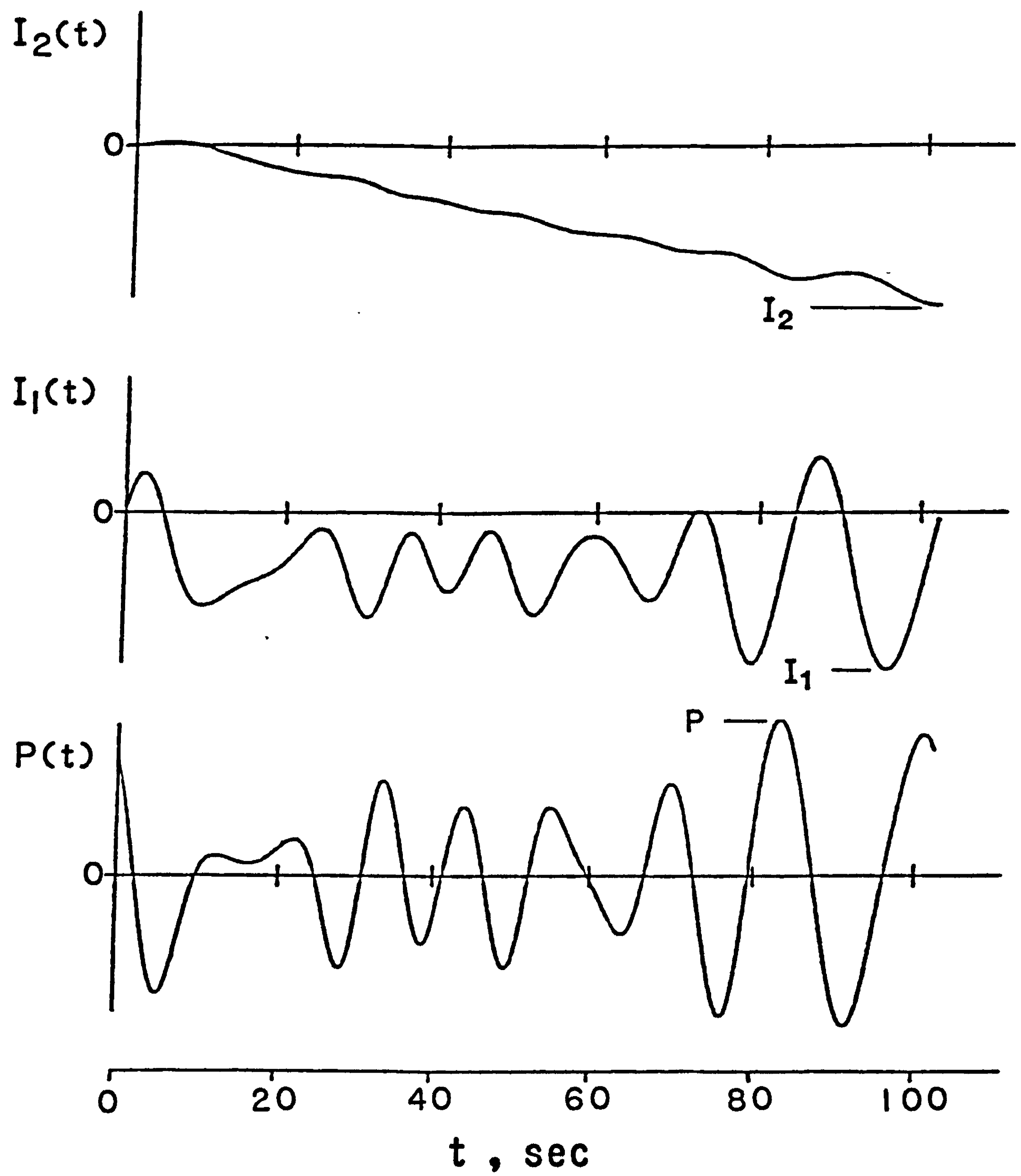


FIG. 12 Representative Wave Force History and its First Two Integrals with Zero Initial Values

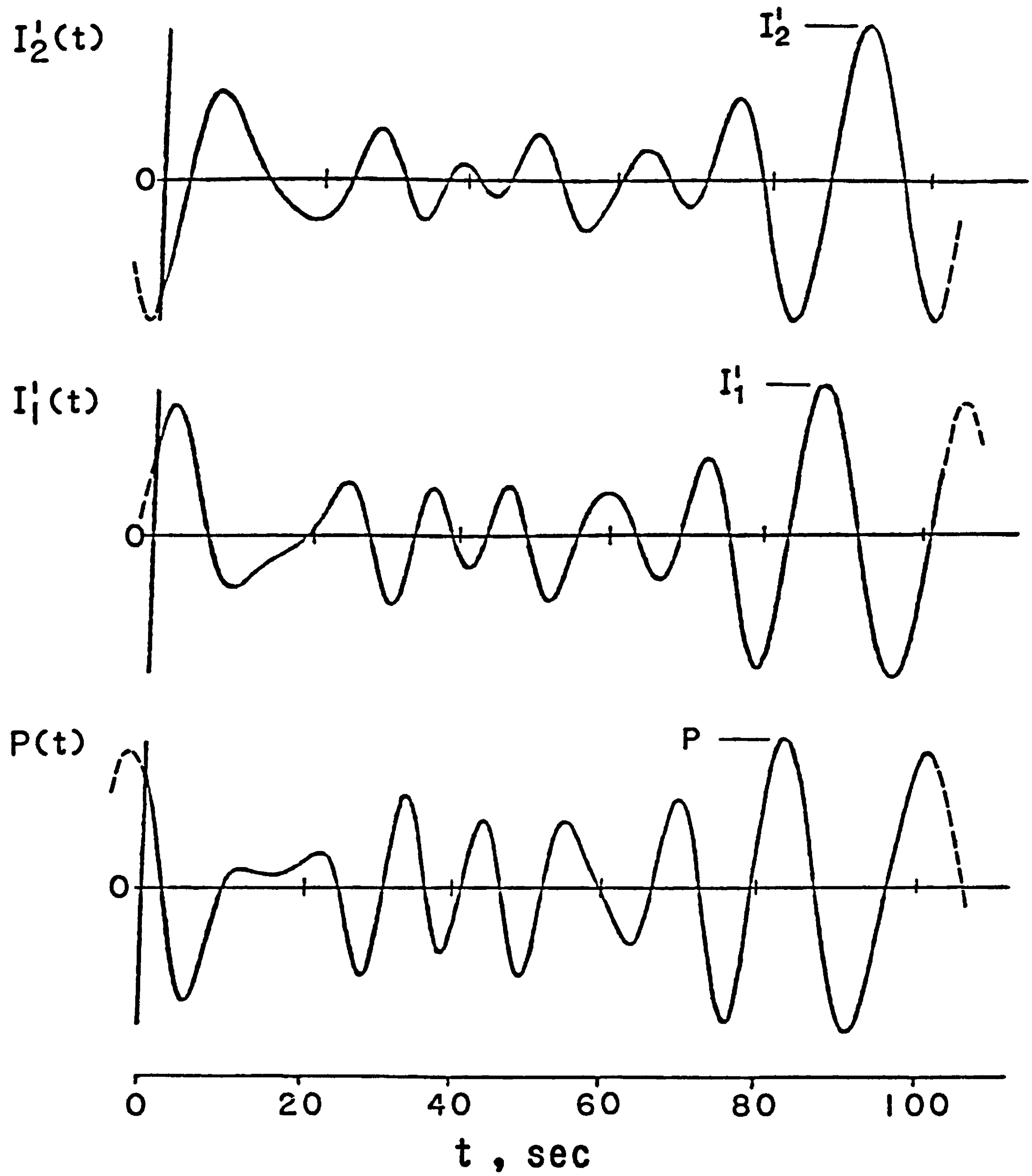


FIG. 13 Representative Wave Force History and its First Two Integrals Adjusted to Have Zero Means

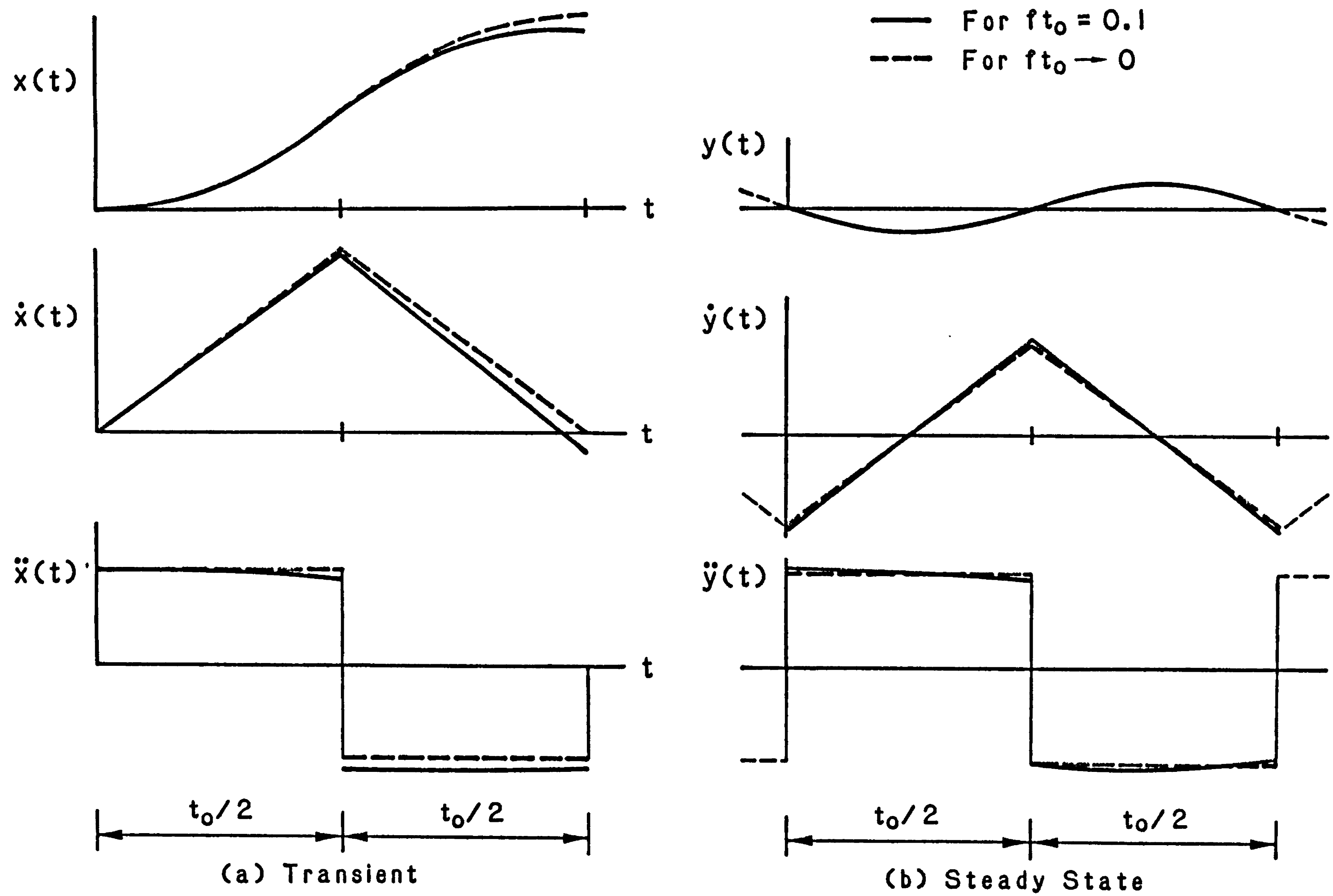


FIG. 14 Comparison of Limiting and Actual Responses of Low-Frequency Systems with $\zeta = 0.05$ Subjected to Alternating Step Force

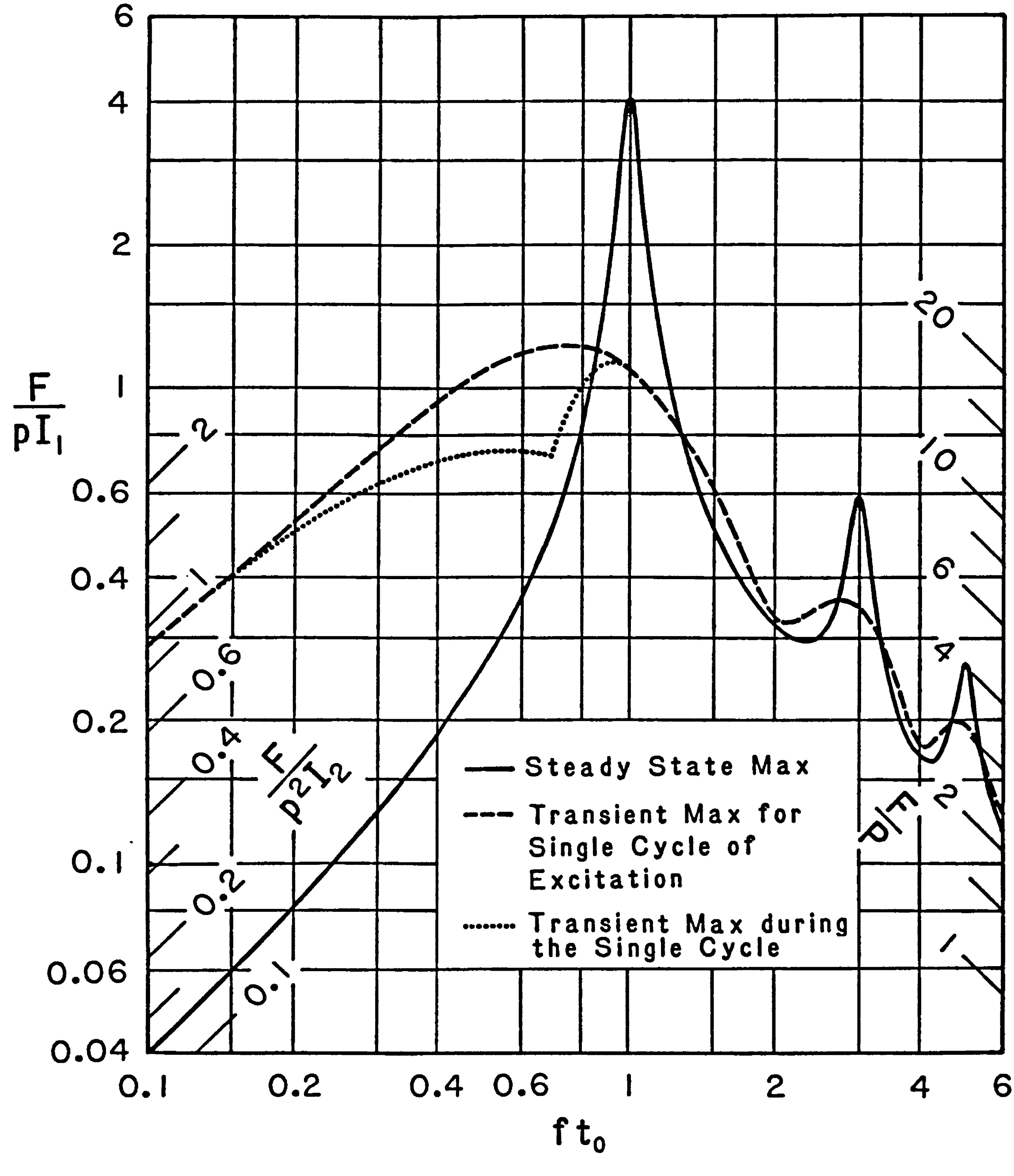


FIG. 15 Tripartite Logarithmic Plots of Response Spectra for Steady-State and Transient Spring Forces of Systems with $\zeta = 0.05$ Subjected to Alternating Step Force

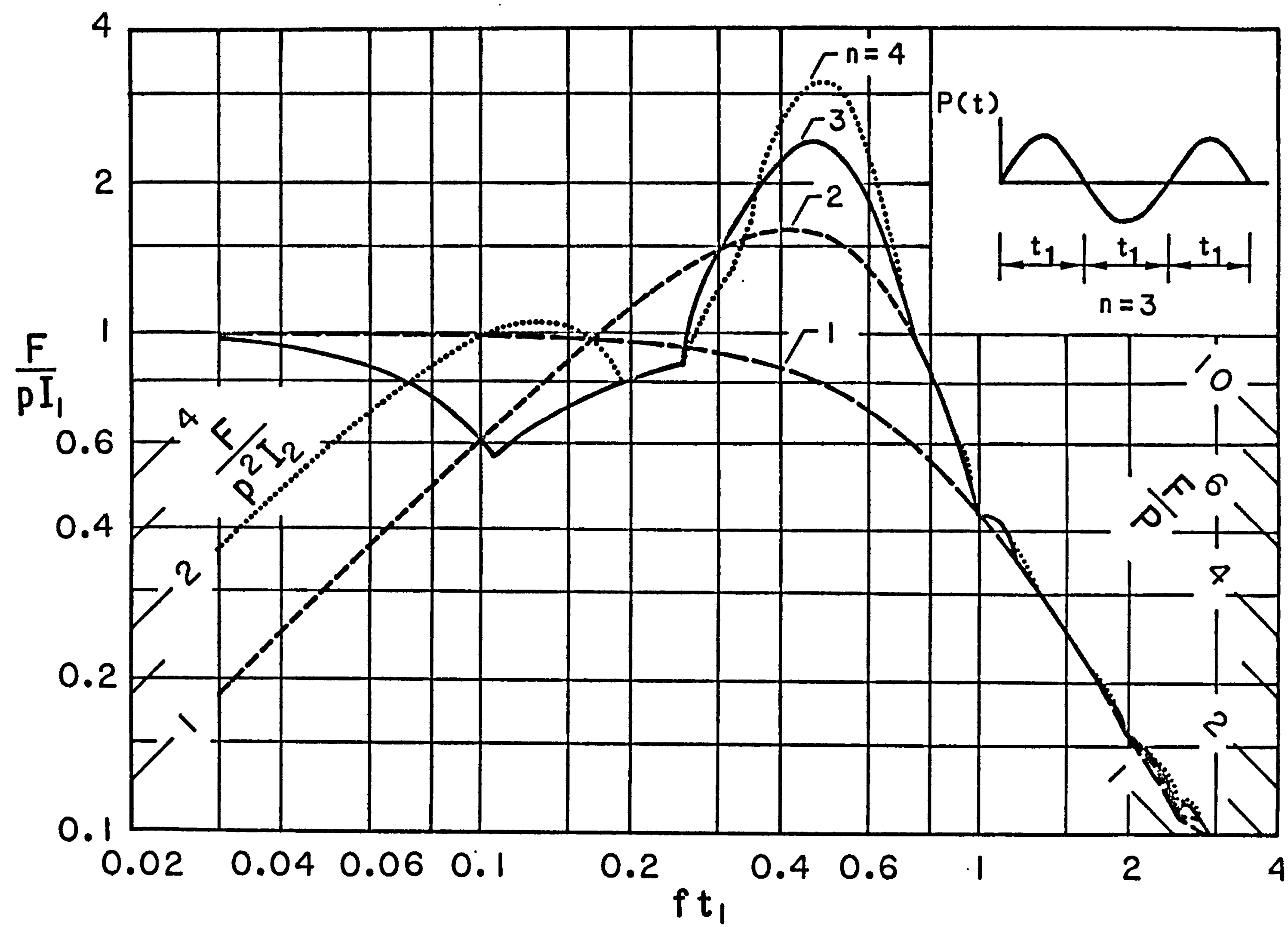


FIG. 16 Spectra for Maximum Response of Undamped Systems Subjected to n Half-Sine Force Pulses

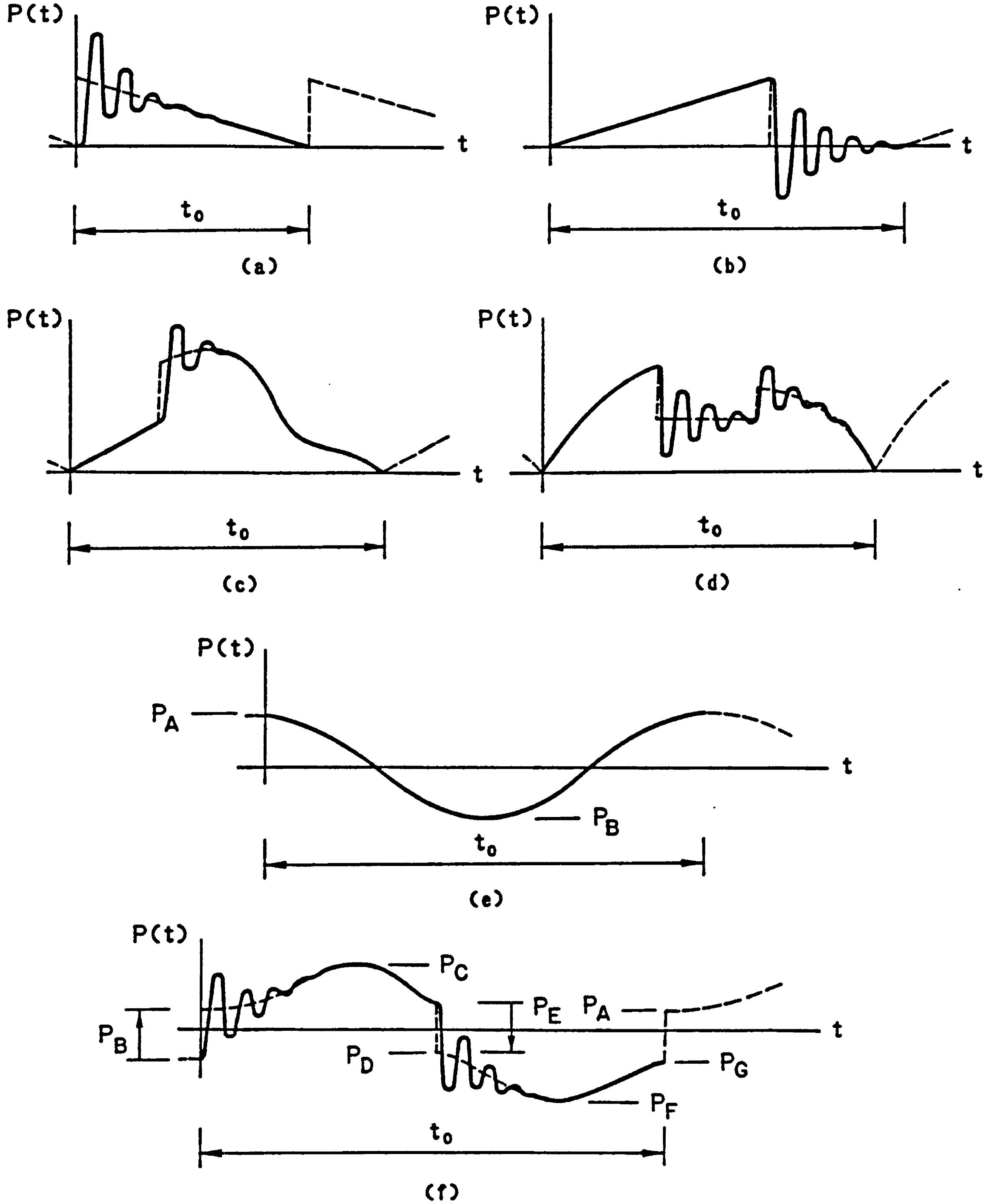


FIG. 17 Steady-State Response Histories of High-Frequency Damped Systems Subjected to Various Periodic Excitations

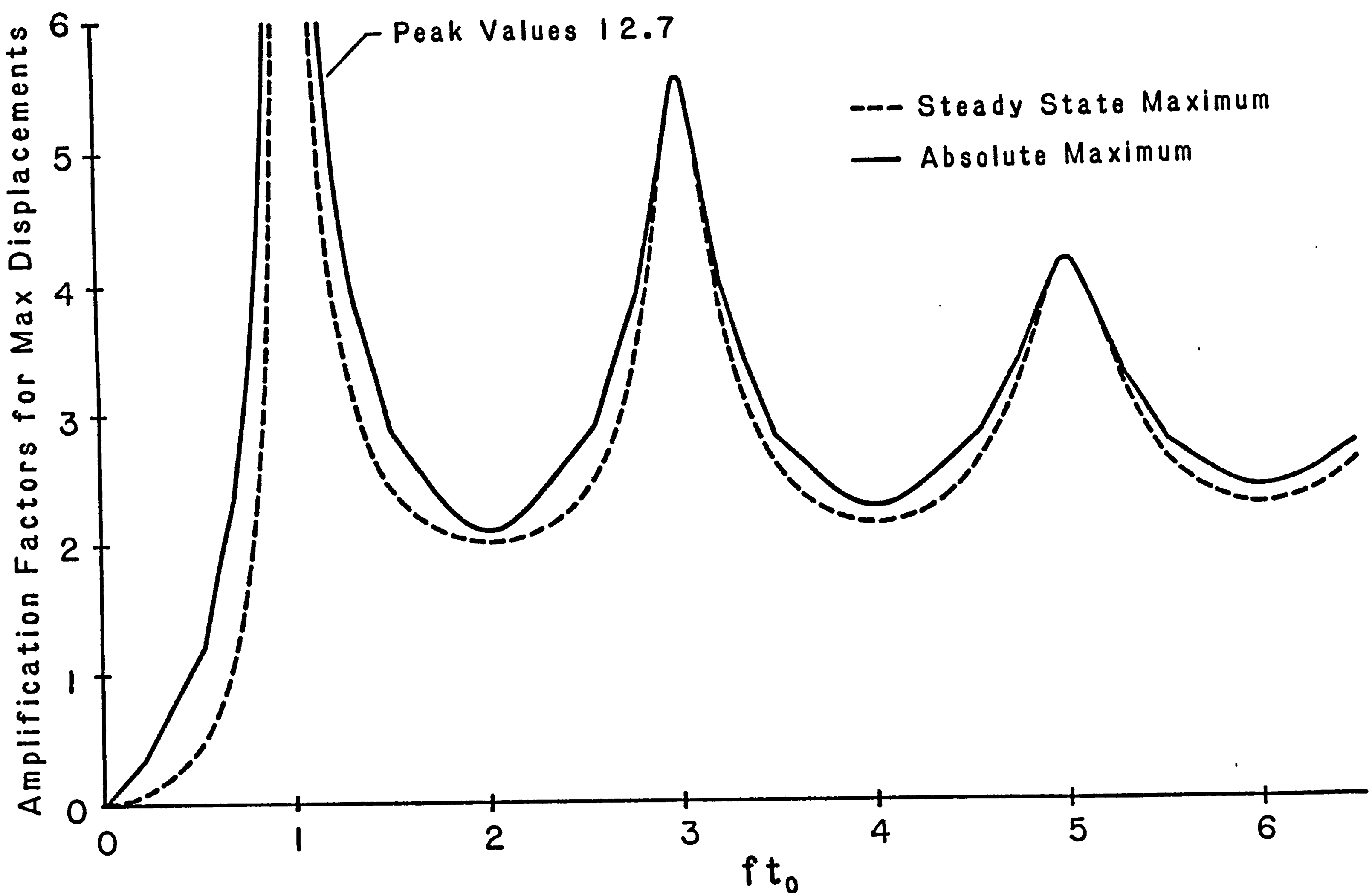


FIG. 18 Comparison of Response Spectra for Steady-State and Absolute Maximum Displacements of Systems with $\zeta = 0.05$ Subjected to Alternating Step Force

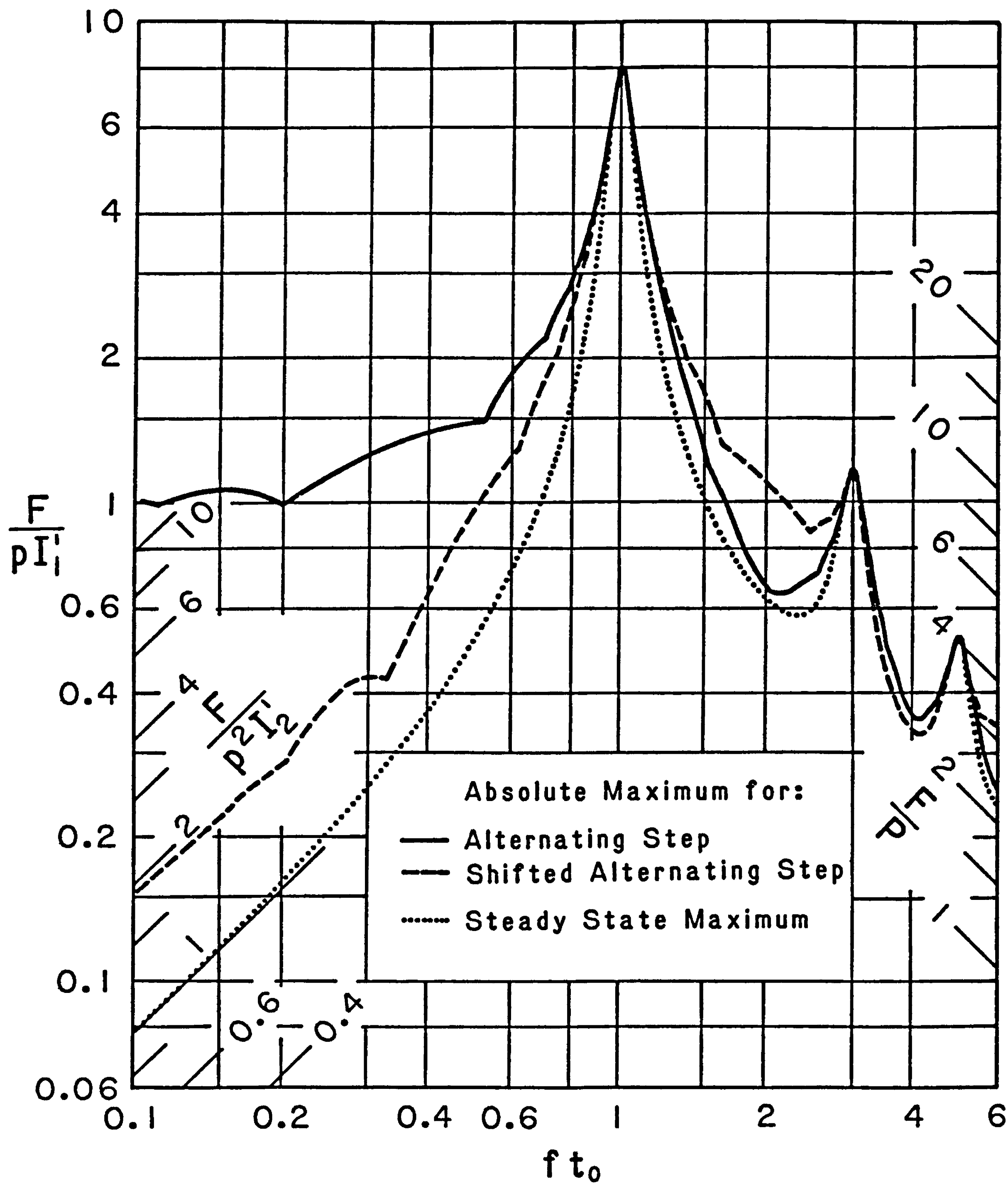


FIG. 19 Comparison of Spectra for Steady-State and Absolute Maximum Response of Systems with $\zeta = 0.05$ Subjected to Alternating Step Forces

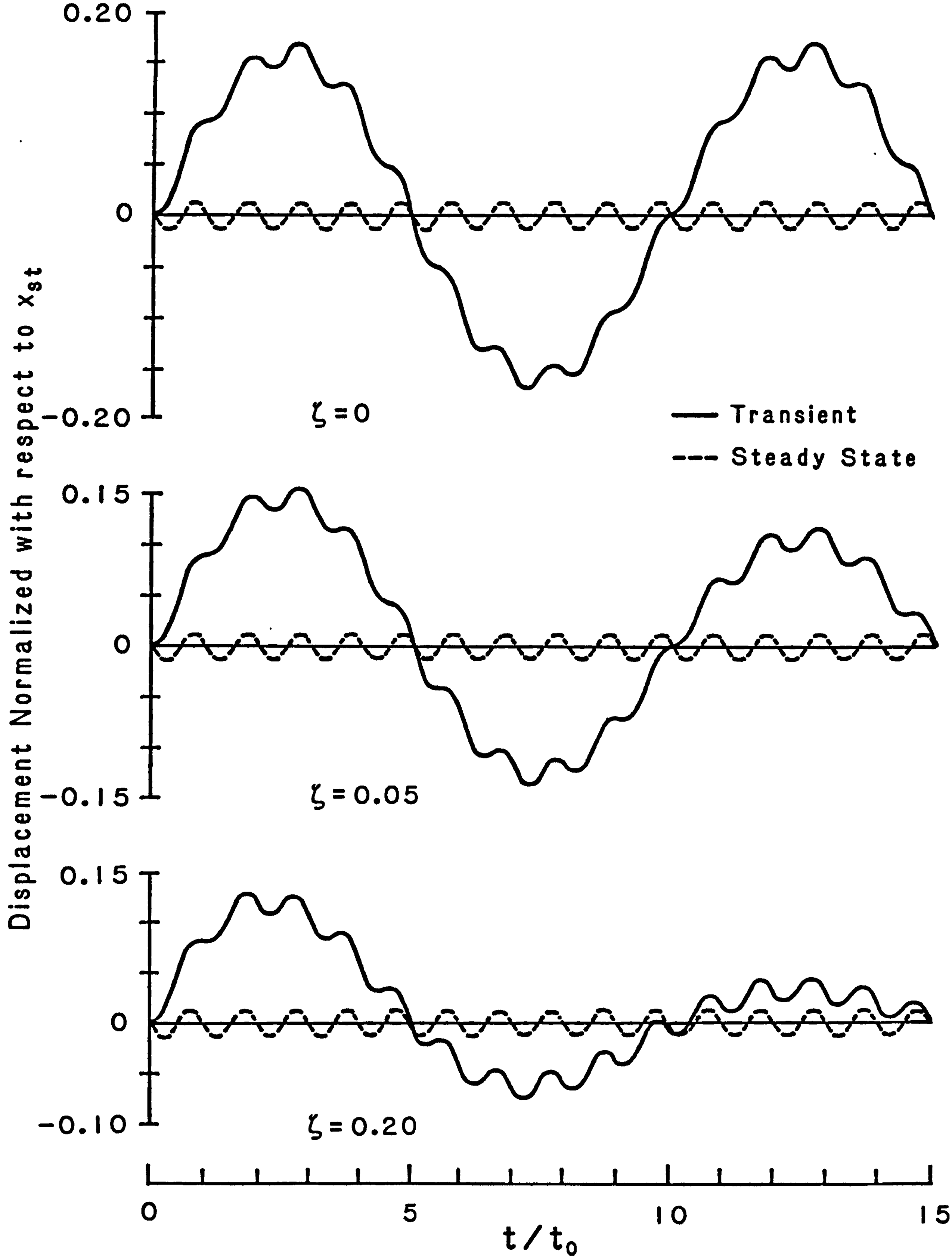


FIG. 20 Comparison of Transient and Steady-State Response Histories for Systems with $ft_o = 0.10$ Subjected to Alternating Step Force

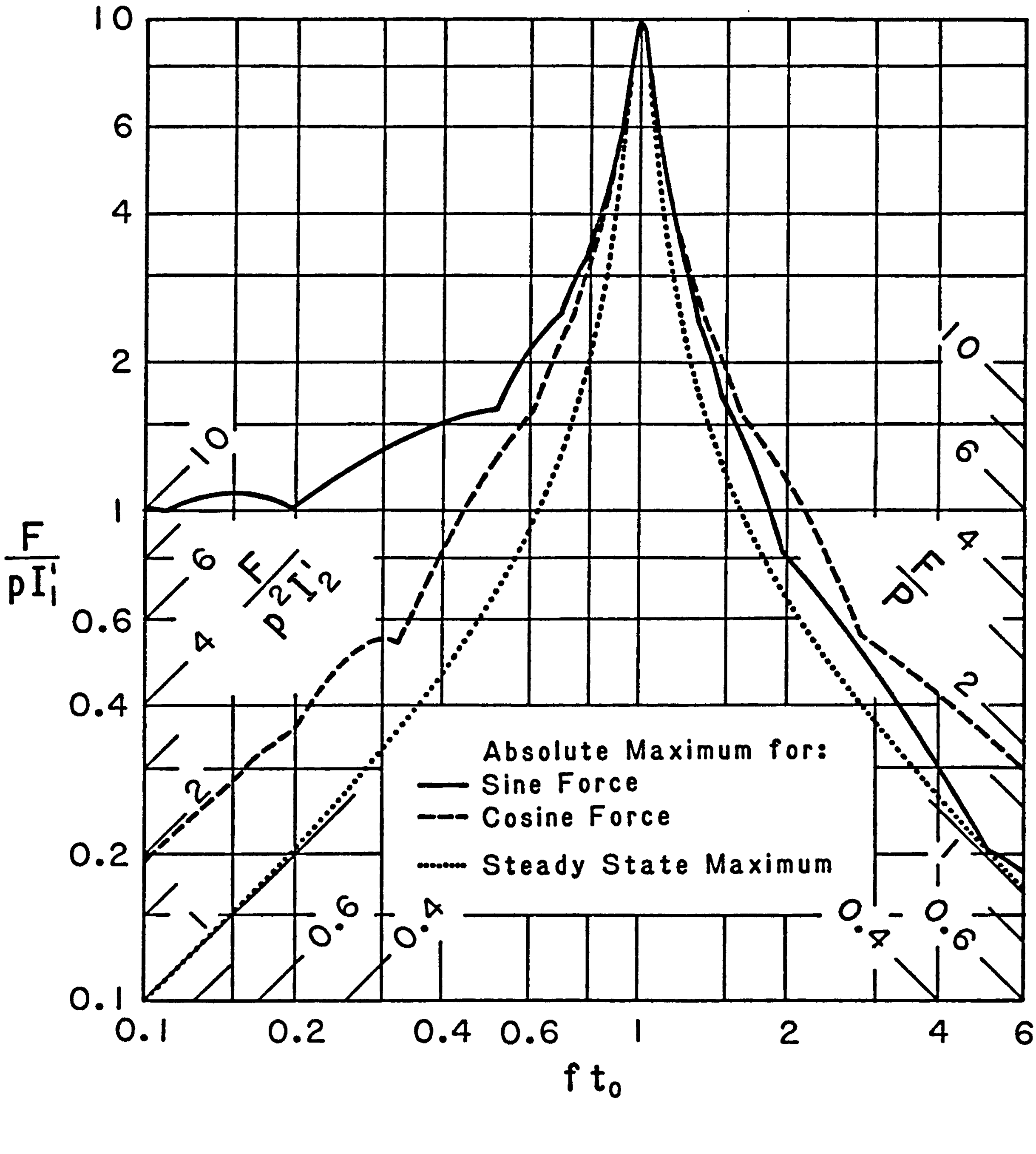


FIG. 21 Comparison of Spectra for Steady-State and Absolute Maximum Response of Systems with $\zeta = 0.05$ Subjected to Harmonic Forces

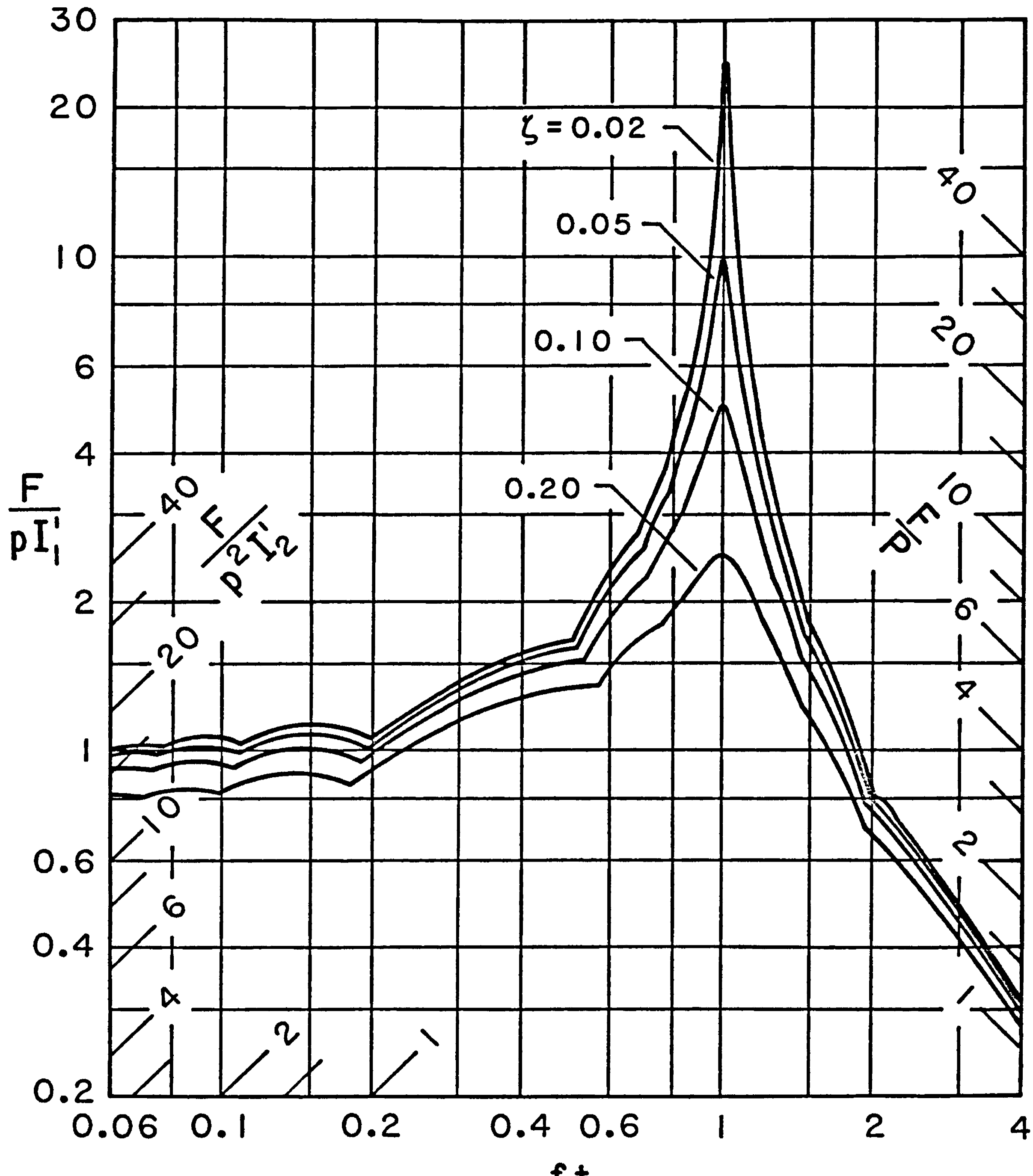


FIG. 22 Spectra for Absolute Maximum Response of Systems Subjected to Sinusoidal Force

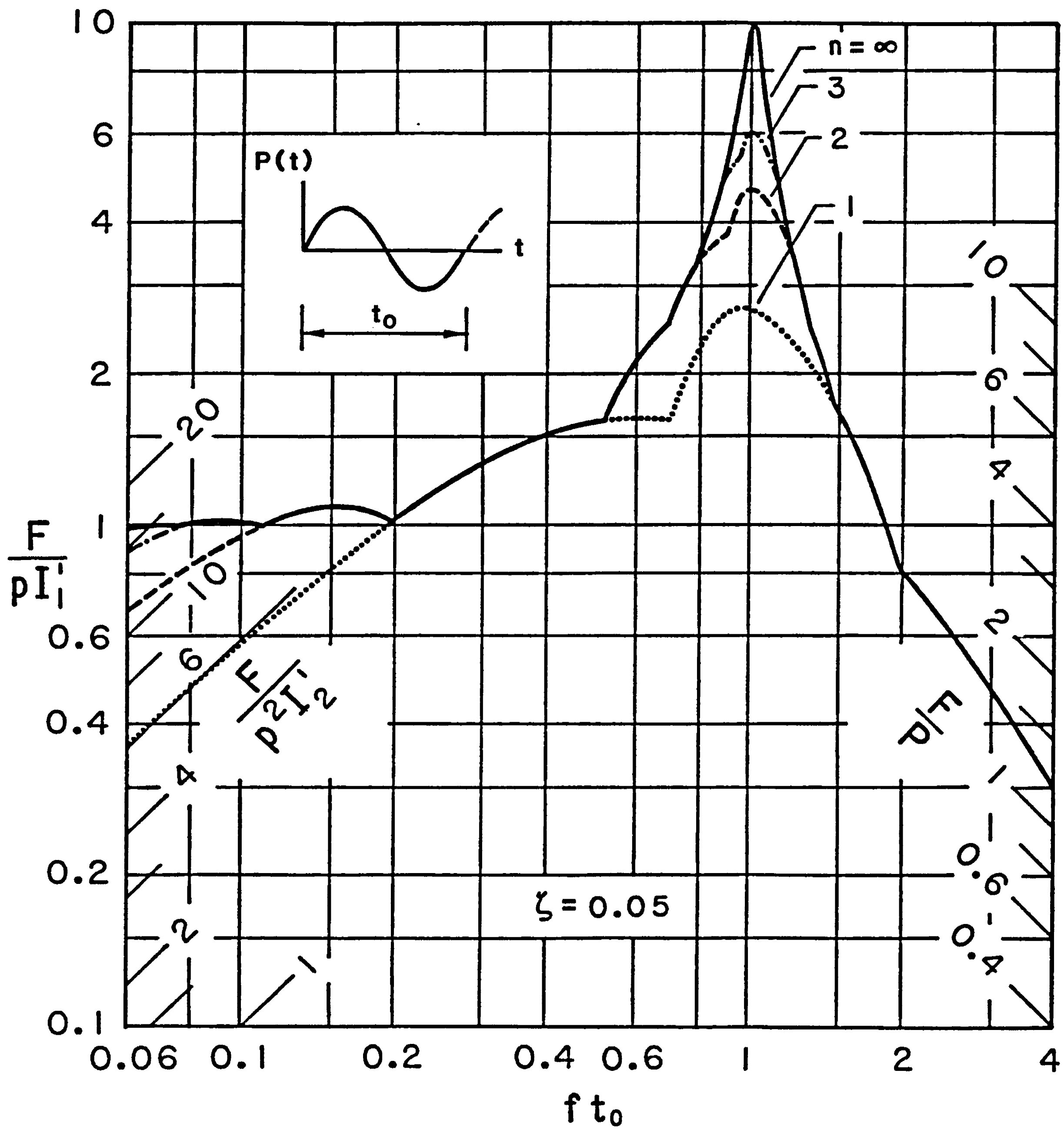


FIG. 23 Spectra for Maximum Forced Response of Systems with $\zeta = 0.05$ Subjected to n Cycles of a Sinusoidal Force

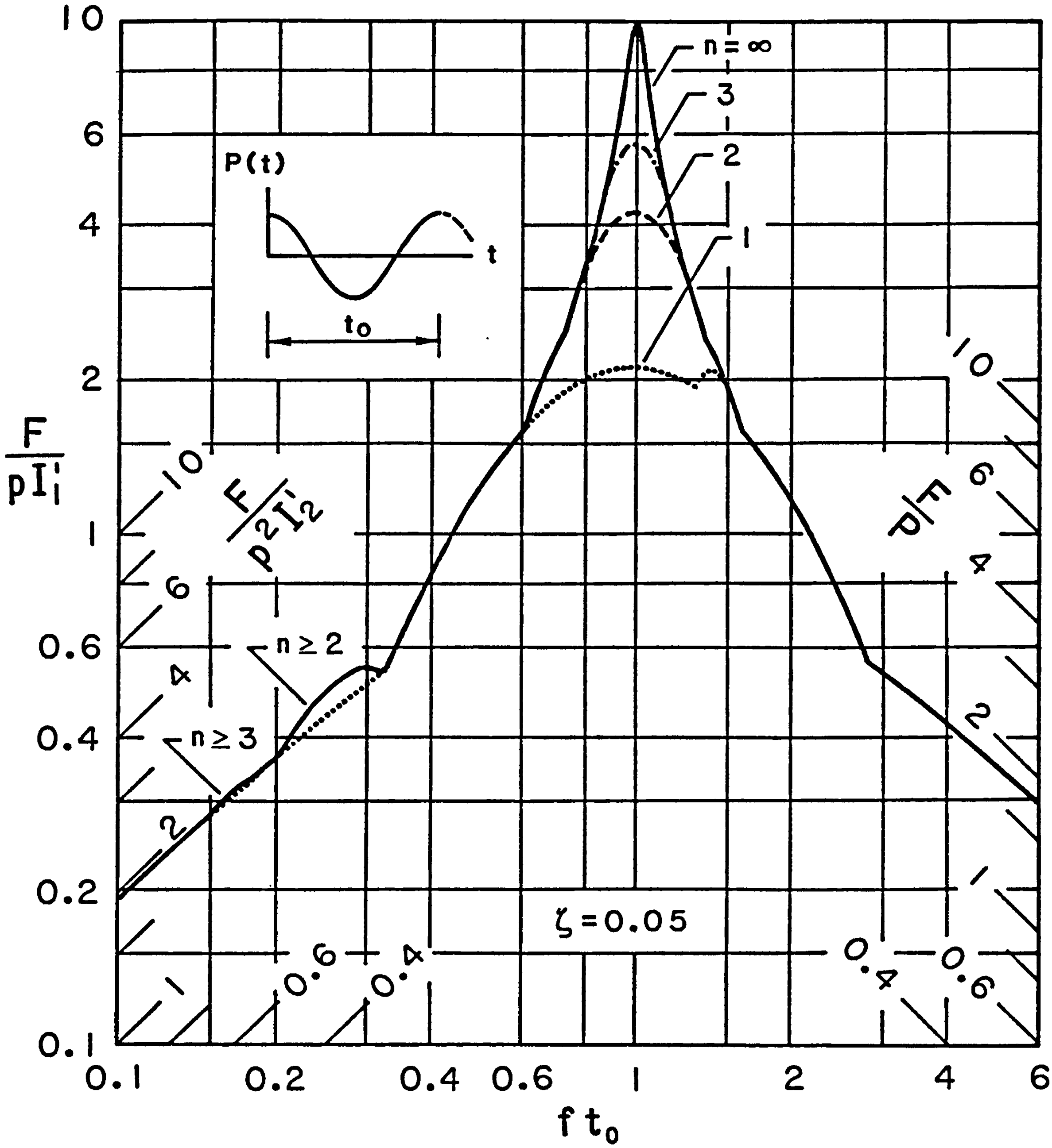


FIG. 24 Spectra for Maximum Forced Response of Systems with $\zeta = 0.05$ Subjected to n Cycles of a Cosine Force

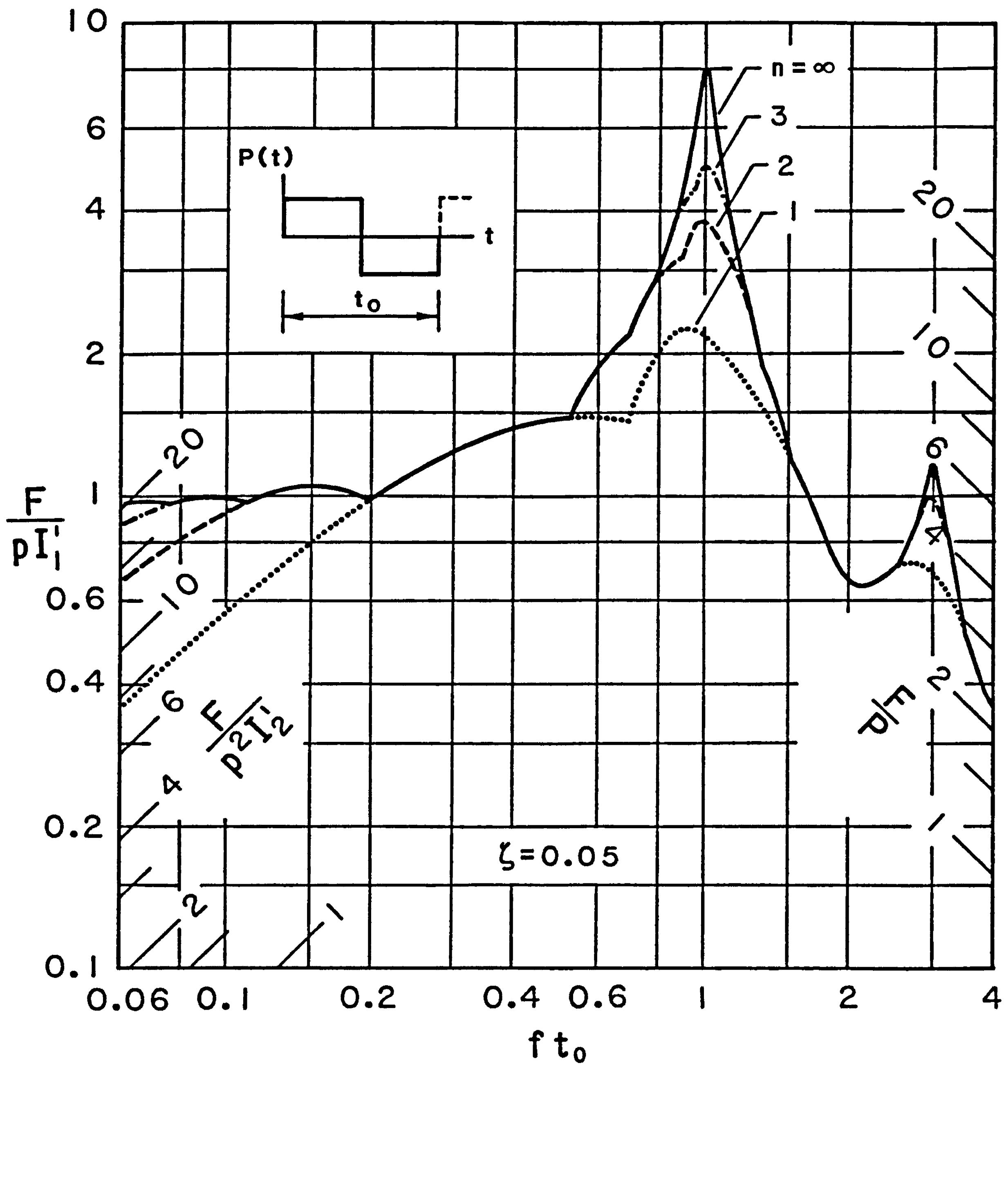


FIG. 25 Spectra for Maximum Forced Response of Systems with $\zeta = 0.05$ Subjected to n Cycles of Alternating Step Force

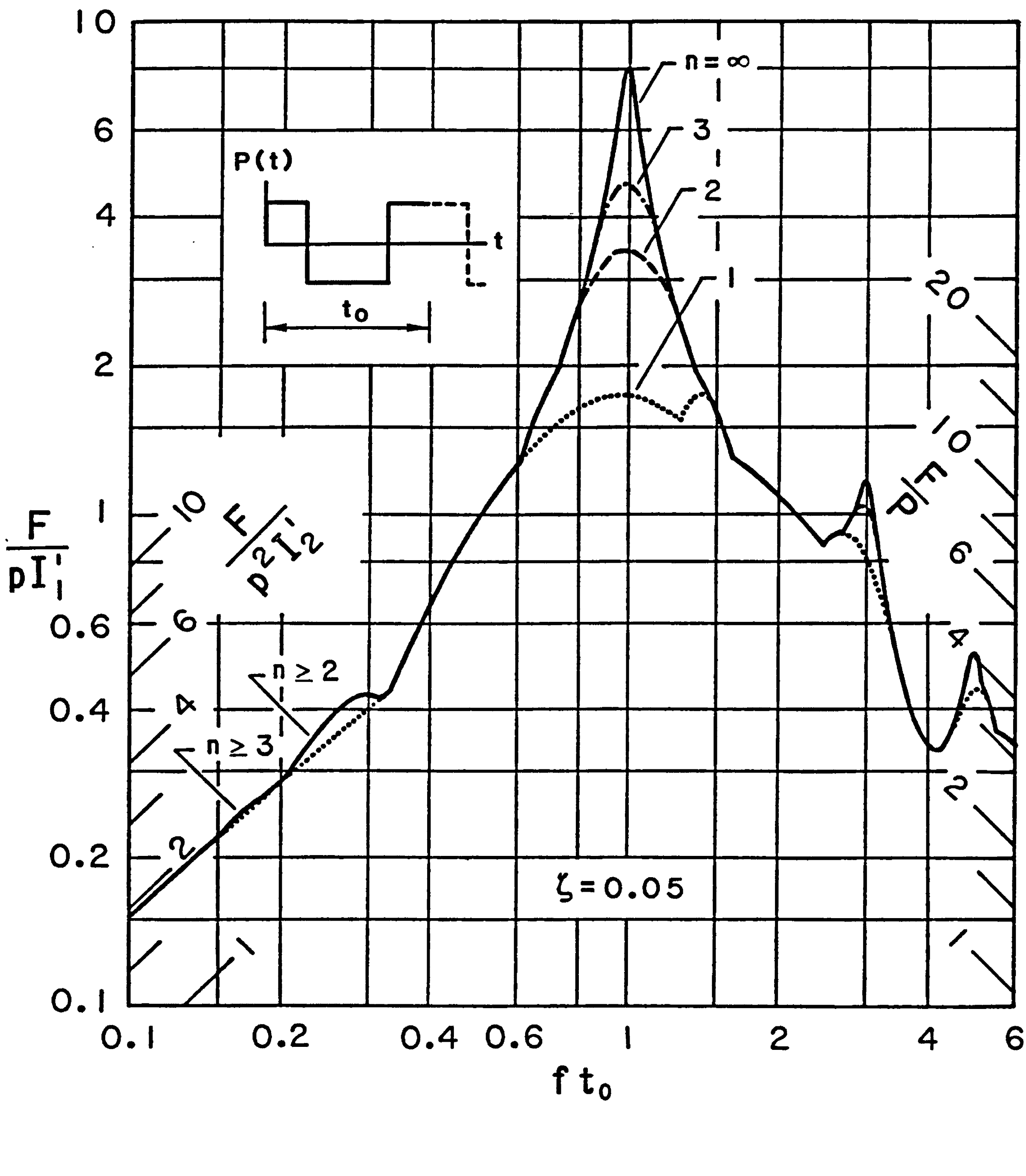


FIG. 26 Spectra for Maximum Forced Response of Systems with $\zeta = 0.05$ Subjected to n Cycles of Shifted Alternating Step Force

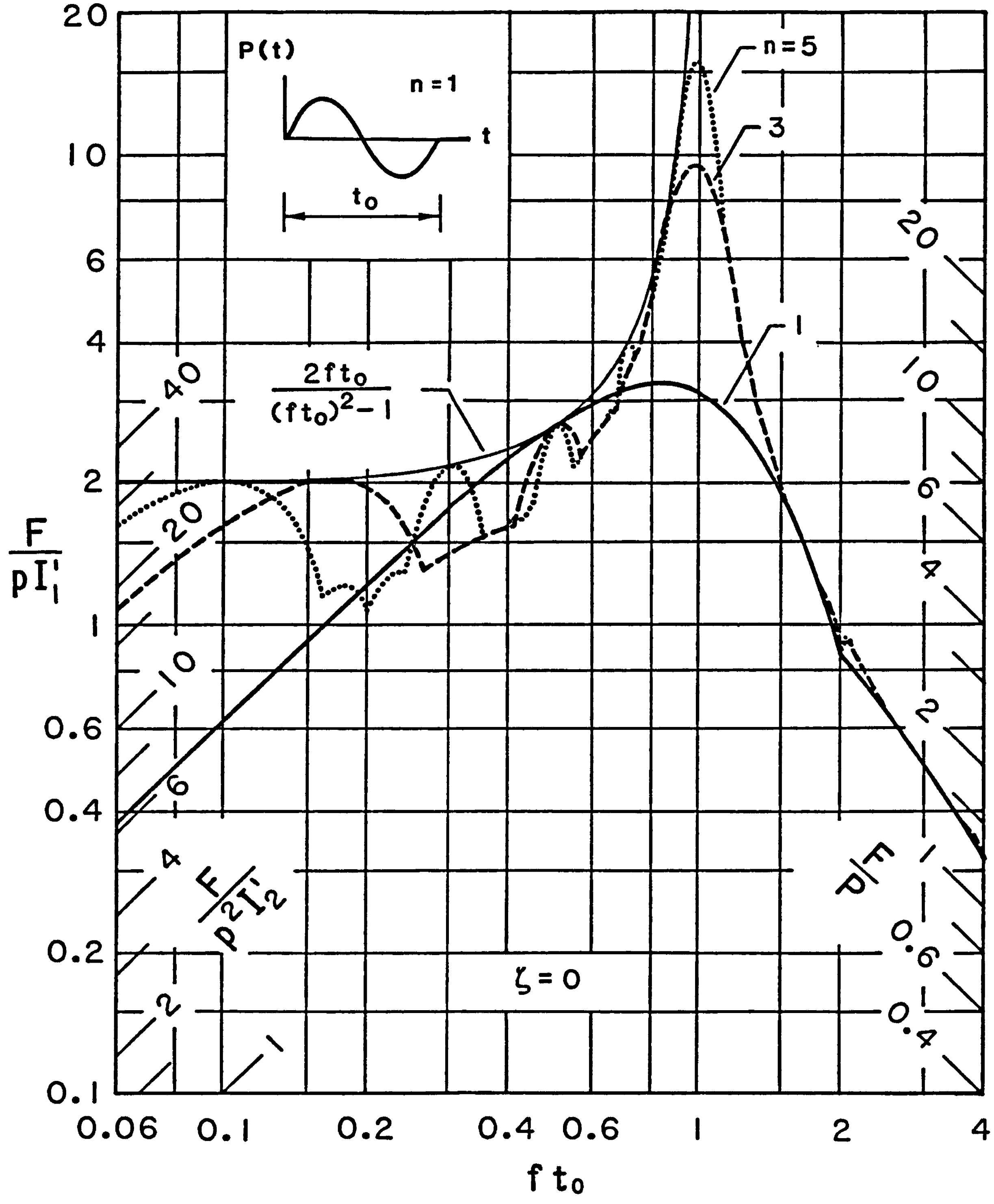


FIG. 27 Spectra for Absolute Maximum Response of Undamped Systems Subjected to n Cycles of Sinusoidal Force

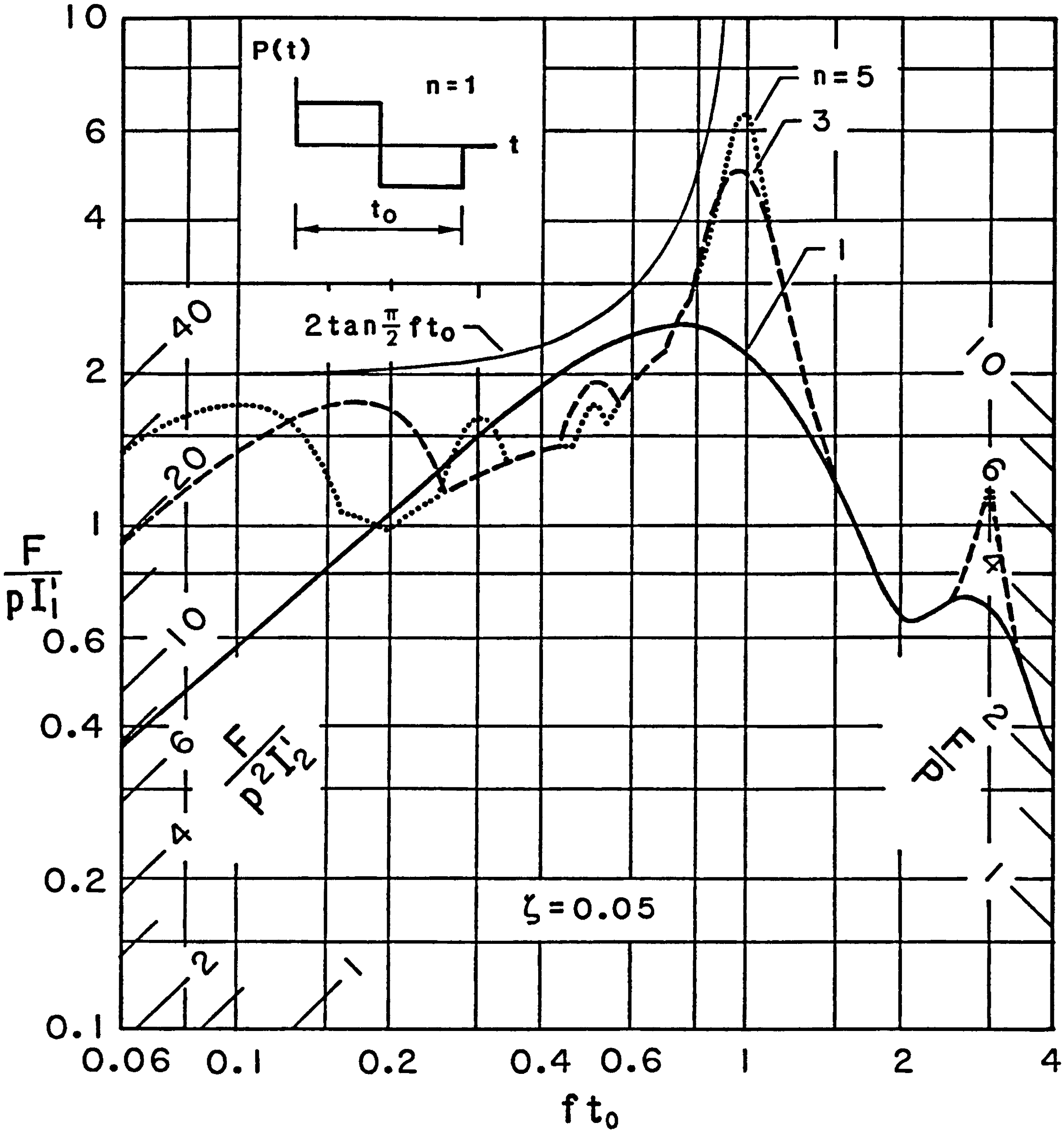


FIG. 28 Spectra for Absolute Maximum Response of Systems with $\zeta = 0.05$ Subjected to n Cycles of Alternating Step Force

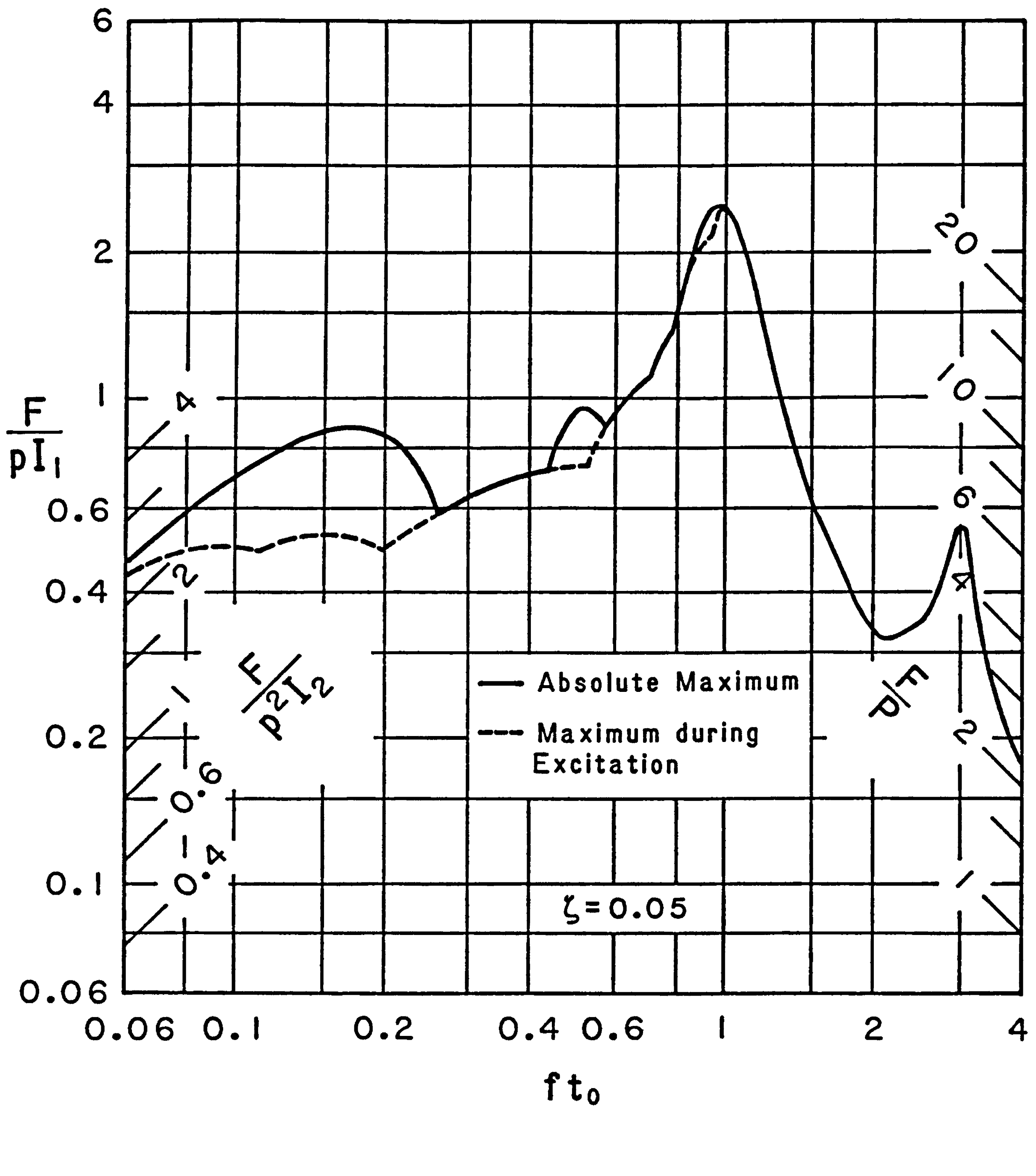


FIG. 29 Spectrum for Absolute Maximum Response of Systems with $\zeta = 0.05$ Subjected to Three Cycles of Alternating Step Force

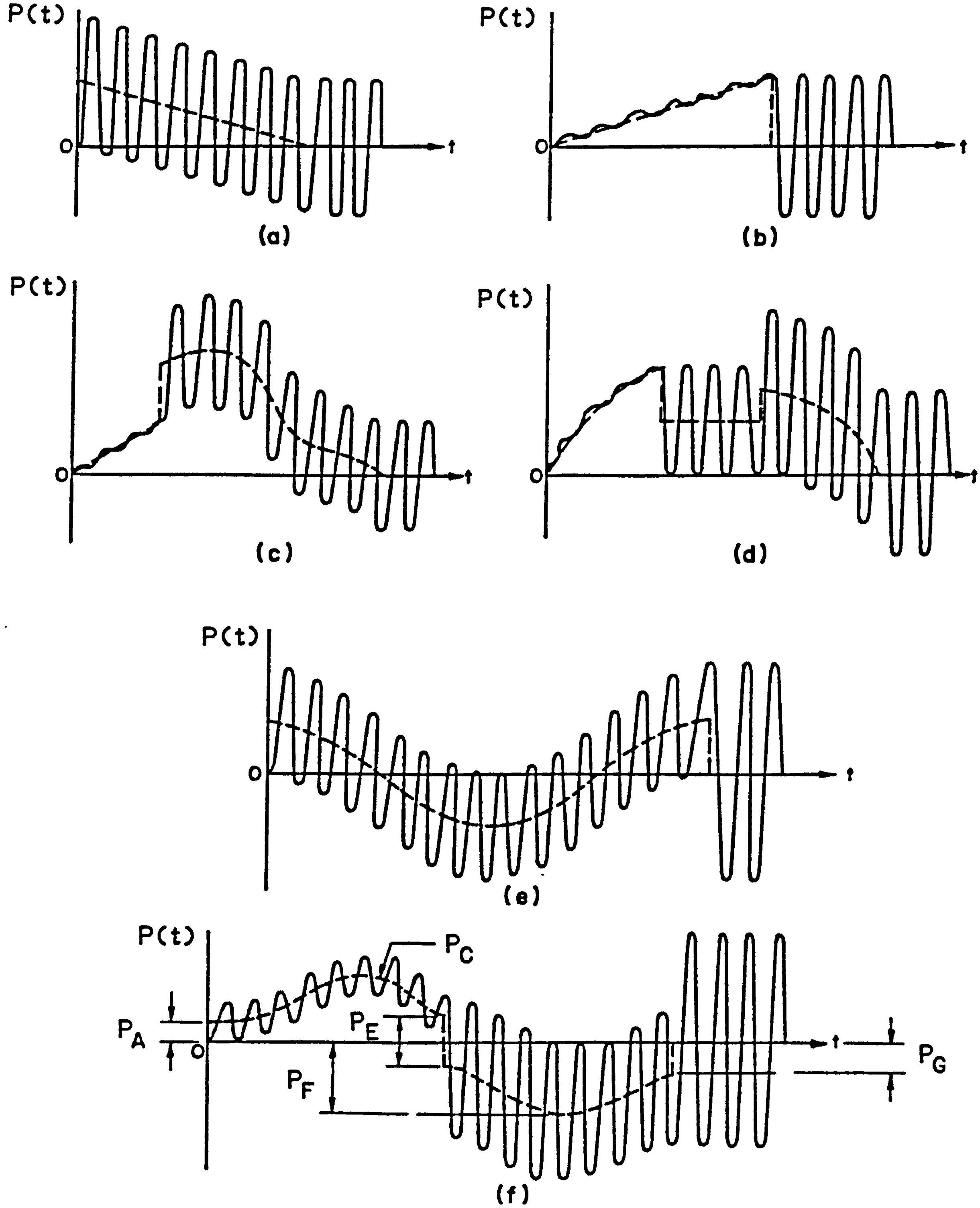


FIG. 30 Response Histories of High-Frequency Undamped Systems Subjected to Various Discontinuous Excitations