On the structure of isentropes of real polynomials

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Abstract

In this paper we will modify the Milnor–Thurston map, which maps a one dimensional mapping to a piece-wise linear of the same entropy, and study its properties. This will allow us to give a simple proof of monotonicity of topological entropy for real polynomials and better understand when a one dimensional map can and cannot be approximated by hyperbolic maps of the same entropy. In particular, we will find maps of particular combinatorics which cannot be approximated by hyperbolic maps of the same entropy.

1 Introduction

In this paper we will study sets of one dimensional real polynomial maps which have the same topological entropy which we will be calling *isentropes*.

First the structure of isentropes was understood for the family of quadratic maps: every isentrope in this case is connected and, since the parameter space of the normalised quadratic maps is one dimensional, is either a point or an interval, see [MT88], [DH84], [Dou95], [Tsu00]. When the dimension of the parameter space increases, the structure of isentropes becomes much more complicated. Even establishing whether isentropes are connected for families of real polynomials with all critical points real took quite an effort: in 1992 Milnor conjectured that isentropes are connected in this case and proved it with Tresser for polynomials of degree three ([MT00]); the general case was proved later by Bruin and van Strien in [BvS15]. However, it is still unknown if isentropes are connected for real polynomial maps when one allows some critical points to be complex (though we have made some progress in this direction and we can prove the connectedness of isentropes for some families (e.g. $x \mapsto x^4 + ax^2 + b$) where complex critical points are allowed, see Section 3).

The main goal of this paper is to develop a set of tools which gives a better understanding of the structure of isentropes and is used to prove monotonicity. The strategy is based on some modifications of the Milnor-Thurston map which maps every one dimensional smooth map to a piece-wise linear map with constant slopes of the same entropy. We will demonstrate how it works on two problems: we will generalise and give a much simpler proof of monotonicity of topological entropy (i.e. we will reprove the main results of [BvS15] in a more general setting), and then we will make some progress in answering one of Thurston's questions, see below.

The proof in [BvS15] is rather complicated and long. Let us review some general ideas used to prove monotonicity of entropy.

We start with defining what we mean by a monotone map.

Definition 1. *Let X and Y be some topological spaces and $F : X \to Y$ be a map. We say that the map F is* monotone *if for any $y \in Y$ the set $F^{-1}(y)$ is connected.*

The following simple fact will be proved in the Appendix:

Fact. *Let X and Y be compact connected topological spaces, and the map $F : X \to Y$ be continuous, surjective and monotone. Let $Z \subset Y$ be a connected subset of Y. Then the preimage of Z under F is connected.*

This statement enables us to use the following strategy for proving monotonicity of entropy. Let X be a connected component of the space of polynomials of given degree with real critical points (actually, this approach would work for any space of maps). Now suppose we can find another space of maps Y which is somewhat "simpler" than X and has the following properties:

- There is a map $F : X \to Y$ which is continuous, surjective and monotone.
- The map F preserves the topological entropy.
- The map $\mathrm{h_{top}} : Y \to \mathbb{R}$ is monotone.

Then, due to Fact above the map $\mathrm{h_{top}} : X \to \mathbb{R}$ which can be seen as the composition $\mathrm{h_{top}}|_Y \circ F$ is monotone.

In [BvS15] the authors use the space of stunted sawtooth maps as the probe space Y. Stunted sawtooth maps were introduced in [MT00]. They are piece-wise linear maps whose branches have slopes ±constant or 0. It is rather easy to show monotonicity of $\mathrm{h_{top}}|_Y$. The map F is defined using the kneading invariants of the maps and, thus, the maps $f \in X$ and $F(f) \in Y$ have the same combinatorial structure. This immediately implies that F preserves the topological entropy. To prove monotonicity of F one should use the rigidity result for real polynomials, see [KSvS07b], [KSvS07a], [CST17].

So far the strategy worked out perfectly, but now some problems arise. It turns out that the map F is neither continuous nor surjective. The authors of [BvS15] had to overcome the lack of these two properties which was not straightforward.

Now let us try a different probe space Y, for example the usual space of piece-wise linear maps with constant slopes. The map F in this case is given by the Milnor-Thurston map [MT88]. However, again the map F is not continuous and not surjective.

In this paper we use a slight modification of the space of piece-wise linear maps of constant slopes. This modification makes the Milnor-Thurston map continuous and surjective and all other required properties we get almost for free.

Another new ingredient we introduce is the notion of multi-interval maps. At first sight one might think that these maps should not be of great use: after all, the dynamics of a multi-interval map can be described in terms of a usual one dimensional interval map. However, such multi-interval maps provide a useful decomposition of iterates of a map and will enable us to formulate certain results in the more general (and useful) settings.

As we have already mentioned one of the aims of this paper is to give a short proof of monotonicity of topological entropy. There is another profound reason for finding different approaches to this problem. The stunted sawtooth maps used in [MT00] and [BvS15] have rather complicated dynamics and though it is easy to prove that in the space of stunted sawtooth maps sets of constant topological entropy are connected, the structure of the isentropes is completely unclear and it is impossible to see what stunted sawtooth maps belong to a given isentrope.

On the other hand, in the space of piece-wise linear maps of constant slopes the isentropes can be easily understood: such an isentrope consists of maps whose slopes are equal to $\pm\exp(h)$ where h is the topological entropy of the given isentrope.

The following question was asked by W. Thurston:

Question 1. *Consider the space of real polynomials of degree $d > 2$ with all critical points real. Does there exist a dense set $H \subset [0, \log(d)]$ of entropy levels such that the hyperbolic polynomials are dense in the isentrope of entropy h for every $h \in H$?*

As usual we call a polynomial hyperbolic if the iterates of all critical points converge to attracting periodic points and there are no neutral periodic points. It is clear that there are only countably many combinatorially different hyperbolic maps, so there exists at most countably many entropy levels whose isentropes contain hyperbolic maps. In fact, a simple argument (presented in Section 9) will show that the entropy of a hyperbolic map is always the logarithm of an algebraic number. In view of this discussion one might ask questions related to Thurston's one:

Question 2. *Consider the space of real polynomials of degree $d > 2$ with all critical points real. Do there exist isentropes of positive entropy which contain hyperbolic maps of infinitely many different combinatorial types? Is there a dense set of entropy levels with such the property?*

Of course, an affirmative answer on Thurston's question implies the affirmative answer of the above questions, however we conjecture that the answer on Thurston's question is negative. More precisely we conjecture the following:

Conjecture. *In the space of polynomials of degree $d > 2$ with all critical points real there are no isentropes of entropy $h \in (0, \log d)$ where hyperbolic polynomials are dense.*

The results of this paper give some insight on how one can prove the conjecture. In Section 9 we will explain how to reduce this conjecture first to a question about piece-wise linear maps and then to some number theory question. In fact we will find a combinatorial obstruction which prevents a map from being approximable by a hyperbolic map of the same topological entropy. Also, we will demonstrate that in case of cubic polynomials the answer on the first part of question 2 is positive.

The paper is structured as follows. After introducing some necessary notation we state monotonicity of entropy theorems in Section 3. Then we introduce the space of piece-wise linear maps with constant slopes, define the Milnor-Thurston map and prove that after an appropriate modification this map becomes continuous. This will take Sections 4–6. The proof of the monotonicity theorems are in Sections 7 and 8. Then we will study when a map cannot be approximated by hyperbolic maps of the same entropy and discover that under certain (rather non-restrictive) condition a map which has all critical points in basins of periodic attractors except one critical point, cannot be approximated by hyperbolic maps of the same entropy (Section 9). Finally, we study more the mentioned condition, give some examples when it is not satisfied, prove that it is always satisfied if the entropy of the map is larger than $\log 3$ and argue that every isentrope should have such a "codimension one hyperbolic" map (Sections 10 and 11).

There are many more other open questions related to monotonicity of entropy where the approach introduced here can be useful. For example, it is unknown if the isentropes in the space of real polynomials are contractible. We suggest the reader to consult [vS14] and the introduction of [BvS15] where the history and importance of monotonicity of entropy together with remaining open problems are discussed with very fine details.

2 Multi-interval Multi-modal maps

Surprisingly enough to prove monotonicity of entropy for polynomials we will have to consider more general spaces of maps which we will call multi-interval multi-modal, and which are introduced in this section. Because of use of these multi-interval maps our main theorems will apply to the wider class of spaces compared to [BvS15], however our way of proof will require these maps even for the proof of the monotonicity of the entropy just for the space of polynomial maps considered in [BvS15].

Let $I = \cup_{k=1}^{N} I_k$ be a union of disjoint intervals and $f : I \to I$ be a differentiable map which maps the set of boundary points of I to itself. We will call such a map *multi-interval multi-modal*. The domain of definition I of f will be denoted by $\mathrm{Dom}(f)$.

Every interval I_k is mapped by f into another interval which we denote $I_{\sigma(k)}$ where $\sigma : \{1, \dots, N\} \to \{1, \dots, N\}$. So $f(I_k) \subset I_{\sigma(k)}$ and $f(\partial I_k) \subset \partial I_{\sigma(k)}$. Note that we do not assume that σ is a permutation.

We also define two more function associated to f: $l(k)$ will denote the number of critical points of the map $f|_{I_k}$ counting with their multiplicities; $s(k)$ is defined to be $+1$ if f is mapping the left boundary point of I_k onto the left boundary point of $I_\sigma(k)$ and -1 otherwise. The total number of critical points will be denoted by $|l| := \sum_{k=1}^{N} l(k)$.

The space of C^b multi-interval multi-modal maps has the topology induced by the C^b norm. $\mathcal{M}^b_{N,\sigma,l,s}$ will denote all multi-interval multi-modal C^b maps with the prescribed combinatorial data N, σ, l, and s. Notice that $\mathcal{M}^b_{N,\sigma,l,s}$ is a connected set.

We will need to consider subsets of $\mathcal{M}^b_{N,\sigma,l,s}$ defined as follows. The set of boundary points of intervals I_k is mapped to itself by f, and this map depends only on the combinatorial information N, σ, l, s. Let P be the set of orbits of periodic boundary points and $\mathsf{b} : \mathsf{P} \to \{0, 1\}$ be a function which assumes only two values $\{0, 1\}$. Then $\mathcal{M}^b_{N,\sigma,l,s,\mathsf{b}}$ defined as a subset of $\mathcal{M}^b_{N,\sigma,l,s}$ such that $|Df^n(x)| \geq \mathsf{b}(p)$ if x is a periodic boundary point of period n and p is the periodic orbit corresponding to x. Here Df denotes the derivative of f. Clearly, if b is a zero function, then $\mathcal{M}^b_{N,\sigma,l,s} = \mathcal{M}^b_{N,\sigma,l,s,\mathsf{b}}$.

To simplify notation we set $\mathcal{X} = \{N, \sigma, l, s, \mathsf{b}\}$ and will write $\mathcal{M}^b_{\mathcal{X}}$ instead of $\mathcal{M}^b_{N,\sigma,l,s,\mathsf{b}}$ when it does not create a confusion. We will not distinguish maps in $\mathcal{M}^b_{\mathcal{X}}$ which can be obtained from each other by a linear rescaling of intervals I_k, so we can assume that all intervals I_k are of the unit length. Also, notice that for any combinatorial information $\mathcal{X}$ the space $\mathcal{M}_{\mathcal{X}}$ is connected.

If σ is a cyclic permutation, we will call the space $\mathcal{M}^b_{\mathcal{X}}$ *cyclic*. If there exists k_0 such that for any $k \in \{1, \dots, N\}$ there is $n \in \mathbb{N}$ such that $\sigma^n(k) = k_0$, then the corresponding space $\mathcal{M}^b_{\mathcal{X}}$ is called *primitive*. For every primitive space $\mathcal{M}^b_{\mathcal{X}}$ there exists a unique set of subintervals $I_{k_1}, \dots, I_{k_{N'}}$ such that the restriction of the maps in $\mathcal{M}^b_{\mathcal{X}}$ to the union of these subintervals forms a cyclic space $\mathcal{M}^b_{\mathcal{X}'}$ with an appropriate $\mathcal{X}'$. The number N' will be called the *period* of the primitive space $\mathcal{M}^b_{\mathcal{X}}$.

Every space $\mathcal{M}^b_{\mathcal{X}}$ can be decomposed into a Cartesian product of primitive spaces:

$$\mathcal{M}^b_{\mathcal{X}} = \mathcal{M}^b_{\mathcal{X}_1} \times \cdots \times \mathcal{M}^b_{\mathcal{X}_m},$$

where all $\mathcal{M}^b_{\mathcal{X}_i}$ are primitive. The minimum of all periods of $\mathcal{M}^b_{\mathcal{X}_i}$ will be called the *minimal period* of $\mathcal{M}^b_{\mathcal{X}}$ and will be denoted by $\mathrm{P}_{\min}$.

Given two data sets $\mathcal{X}$ and $\mathcal{X}'$ we will say that $\mathcal{X}'$ is *subordinate* to $\mathcal{X}$ if either $|l| > |l'|$ or $|l| = |l'|$ and $\mathrm{P}_{\min}(\mathcal{X}) < \mathrm{P}_{\min}(\mathcal{X}')$.

Finally, if $l(k) > 0$ for any $k \leq N$ such that $k \notin \text{Image}(\sigma)$, the corresponding space $\mathcal{M}^b_{\mathcal{X}}$ will be called *essential.* In other words, an essential space cannot have an interval without critical points which does not contain an image of another interval.

The multi-interval multi-modal maps are not much different from just the multi-modal maps and the combinatorial theory of one dimensional maps can be applied to them.

The *basin of attraction* of a non-repelling periodic point x of the map f is defined as the interior of all points whose trajectories converge to the orbit of x under iterates of f and denoted by $B(f, x)$. The intervals of $B(f, x)$ which contain points of orbit of x is called the *immediate basin of attraction.* Basins of attraction whose immediate basins of attraction contain critical points of f are called *essential.* Finally, the *basin of attraction* of the map f is the union of basins of attraction of all non-repelling periodic points of f and denoted by $B(f)$.

Two maps $f_1, f_2 \in \mathcal{M}^1_{\mathcal{X}}$ are called *semi-conjugate* if there exists a continuous monotone map $H : I \to I$ such that $H(I_k) = I_k$ for all k, the map H maps the critical points of f_1 onto the critical points of f_2 of the same order and $H \circ f_1 = f_2 \circ H$.

Two maps $f_1, f_2 \in \mathcal{M}^1_{\mathcal{X}}$ are called *partially conjugate* if there exists a homeomorphism $H : I \to I$ such that $H(I_k) = I_k$ for all k, the map H maps the critical points of f_1 onto the critical points of f_2 of the same order, H maps the basins of attraction $B(f_1)$ onto the basins of attraction $B(f_2)$, i.e. $H(B(f_1)) = B(f_2)$, and $H \circ f_1|_{I \setminus B(f_1)} = f_2 \circ H|_{I \setminus B(f_1)}$.

3 Polynomial model

In the space $\mathcal{M}^\infty_{\mathcal{X}}$ consider maps p such that the restriction of p to any interval I_k is a polynomial of degree $l(k)+1$. Notice that this implies that all critical points of the polynomial $p|_{I_k}$ belong to the interval I_k and, therefore, p_{I_k} has non-positive Schwarzian derivative. We denote the space of such maps by $\mathcal{P}_{\mathcal{X}}$.

To state the main result of this paper we will use the following notation: if X is a space of maps (e.g. $\mathcal{M}^1_{\mathcal{X}}$ or $\mathcal{P}_{\mathcal{X}}$), then for any $h \geq 0$ we define

$$
\begin{aligned}
X(= h) &:= \{f \in X : \mathrm{h_{top}}(f) = h\} \\
X(\leq h) &:= \{f \in X : \mathrm{h_{top}}(f) \leq h\}.
\end{aligned}
$$

Theorem A. *The isentrope $\mathcal{P}_{\mathcal{X}}(= h)$ is connected for any $\mathcal{X}$ and $h \geq 0$, in other words the map $\mathrm{h_{top}}|_{\mathcal{P}_{\mathcal{X}}}$ is monotone.*

Notice that the space $\mathcal{M}^b_{\mathcal{X}}$ as well as $\mathcal{P}_{\mathcal{X}}$ contains maps with degenerate critical points. Let us remove these maps and denote by $\mathcal{M}^{0,b}_{\mathcal{X}} \subset \mathcal{M}^b_{\mathcal{X}}$ the set of maps which have only quadratic critical points, and set $\mathcal{P}^0_{\mathcal{X}} := \mathcal{P}_{\mathcal{X}} \cap \mathcal{M}^{0,b}_{\mathcal{X}}$. The topological entropy function is also monotone on this space:

Theorem B. *The isentrope $\mathcal{P}^0_{\mathcal{X}}(= h)$ is connected for any $\mathcal{X}$ and $h \geq 0$.*

We will see that the sets $\mathcal{P}_{\mathcal{X}}(\leq h)$ and $\mathcal{P}^0_{\mathcal{X}}(\leq h)$ are connected as well.

Interestingly enough the use of multi-interval spaces enable us to prove the connectedness of isentropes for some families. For example, the family $p_4 : x \mapsto x^4 + ax^2 + b$ can be seen as a composition of two quadratic maps: $p_4(x) = (x^2 + \frac{1}{2}a)^2 + b - \frac{1}{4}a^2$. Then the Theorem A applied in the case $\mathcal{X} = \{2, (1 \to 2 \to 1), (1, 1), (-1, -1)\}$ implies that the isentropes in the family p_4 are connected. Notice that for some values of parameters (a, b) the map p_4 is a real unicritical map of positive entropy having complex critical points. In general, the following corollary holds. Let $\mathcal{Q}_{d,s}$ denote the set of all real polynomials of degree d which satisfy the following conditions: all critical points of these polynomials are real and in the unit interval; such polynomials define proper maps of the unit interval into itself; the leading coefficients of the polynomials have the same sign s.

Corollary. *Consider a family of real polynomial maps obtained as a composition of polynomials $p_n \circ \cdots \circ p_1$, where $p_1 \in \mathcal{Q}_{d_1,s_1}, \dots, p_n \in \mathcal{Q}_{d_n,s_n}$ for some $s_1, \dots, s_n \in \{+,-\}$ and natural $d_1, \dots, d_n$. Then the isentropes in such a family are connected. Moreover, if maps with degenerate critical points are removed from the family, the isentropes remain connected.*

One of the ingredients of the proof is based on the Rigidity Theorems [KSvS07b], [CST17] and can be proved for multi-interval maps exactly in the same way as Lemma 3.12 in [BvS15]. Later this lemma will enable us to prove monotonicity of a certain map.

Lemma 3.1. *Let f be in $\mathcal{P}_\mathcal{X}$ and let $\mathcal{PH}_\mathcal{X}(f) \subset \mathcal{P}_\mathcal{X}$ denote the set of maps partially conjugate to f. Then the set $\mathcal{PH}_\mathcal{X}(f)$ is connected.*

$\tilde{\mathcal{P}}_\mathcal{X}$ will denote the quotient space of $\mathcal{P}_\mathcal{X}$ with respect to the partial conjugacy. For any map $f \in \mathcal{M}^1_\mathcal{X}$ there exists a map $p \in \mathcal{P}_\mathcal{X}$ which is semi-conjugate to f. Moreover, this semi-conjugacy collapses only intervals which are in the non-essential basins of attraction and wandering intervals, see Theorem 6.4, page 156 in [dMS93]. If there are two maps $p_1, p_2 \in \mathcal{P}_\mathcal{X}$ which are both semi-conjugate to f, then p_1 and p_2 are partially conjugate. Thus we can define the map $\Upsilon : \mathcal{M}^1_\mathcal{X} \to \tilde{\mathcal{P}}_\mathcal{X}$ so $\Upsilon(f)$ is a set of partially conjugate polynomial maps which contains a map semi-conjugate to f. Obviously, Υ is surjective, it is also easy to see that it is continuous.

4 Piece-Wise Linear model

Fix $h \geq 0$ and let us consider a space of piece-wise linear maps whose slopes are $\pm e^h$ and which satisfy the same combinatorial properties as $\mathcal{M}_\mathcal{X}$. More precisely, for $\mathcal{X} = \{N, \sigma, l, s, \mathsf{b}\}$ as before we will study the space of piece-wise linear maps $q : I \to I$, where $I = \cup_{k=1}^N I_k$, q maps boundary of I to itself, for any $k \leq N$ one has $q(I_k) \subset I_{\sigma(k)}$, there are precisely $l(k)$ turning points of q in the interval I_k (though some of them we allow to collide), and $s(k)$ tells us if q is decreasing or increasing at the left boundary point of the interval I_k. The function b does not play any role here.

To normalise the settings and slightly abusing the notation we consider the points $a_0 = 0 \leq a_1 \leq \cdots \leq a_N = 1$ and set $I_k = [a_{k-1}, a_k]$. Then the map q is discontinuous at points a_k. To distinguish the different values of the map q on different sides of the points a_k we introduce the following notation: $q(a_k^+) = \lim_{x \searrow a_k} q(x)$ and $q(a_k^-) = \lim_{x \nearrow a_k} q(x)$.

Given a map q described above for any branch of q there exists b such that for that branch we have $q(x) = \pm e^h x + b$. So, any map as above can be described by the following data: the combinatorial data $\mathcal{X}$, the points a_k for $k = 0, \dots, N$, the coefficients b_k^i for $k = 1, \dots, N$, $i = 0, \dots, l(k)$. The i^{th} branch of q on I_k is then given by the formula $q(x) = (-1)^i s(k) e^h x + b_k^i$.

Of course, not for all possible choices of a_k and b_k^i there is a map which has this prescribed data. The following conditions should be satisfied:

- The i^{th} turning point c_k^i of $q|_{I_k}$ must belong to I_k. The value of c_k^i can be found from

$$-(-1)^i s(k) e^h c_k^i + b_k^{i-1} = (-1)^i s(k) e^h c_k^i + b_k^i,$$

so $c_k^i = \frac{1}{2}(-1)^i s(k) e^{-h}(b_k^{i-1} - b_k^i)$. All turning points should be ordered correctly, thus the following inequalities must hold:

$$0 = a_0 \leq c_1^1 \leq c_1^2 \leq \cdots \leq c_1^{l(1)} \leq a_1 \leq c_2^1 \leq \cdots \leq a_N = 1 \tag{1}$$

- The turning values should belong to the corresponding interval as well. The turning value $q(c_k^i)$ is $\frac{1}{2}(b_k^{i-1} + b_k^i)$, therefore

$$a_{\sigma(k)-1} \leq \frac{1}{2}(b_k^{i-1} + b_k^i) \leq a_{\sigma(k)} \tag{2}$$

should be satisfied for all $k = 1, \ldots, N$ and $i = 1, \ldots, l(k)$.

- Finally, the map q must have the prescribed values at the boundary points of the intervals I_k. We know that $q(I_k) \subset I_{\sigma(k)}$ and the boundary points of I_k are mapped to the boundary points of $I_{\sigma(k)}$. Let $q(a_{k-1}^+) = a_{\sigma_l(k)}$ and $q(a_k^-) = a_{\sigma_r(k)}$, where the functions σ_l and σ_r are completely defined by the combinatorial data $\mathcal{X}$ and $\sigma_l(k)$ and $\sigma_r(k)$ can assume one of the two values : $\sigma(k)$ or $\sigma(k) - 1$ depending on $s(k)$ and $l(k)$. Therefore,

$$s(k) e^h a_{k-1} + b_k^0 = a_{\sigma_l(k)}, \tag{3}$$

$$(-1)^{l(k)} s(k) e^h a_k + b_k^{l(k)} = a_{\sigma_r(k)}. \tag{4}$$

For given h and $\mathcal{X}$ if a_k and b_k^i satisfy the inequalities and equalities above, then the corresponding piece-wise linear map described by these data exists. The set of these maps we will denote by $\mathcal{L}_\mathcal{X}(= h)$. Obviously, $\mathcal{L}_\mathcal{X}(= h)$ is a compact subset of $\mathbb{R}^D$ for some D depending on $\mathcal{X}$. Moreover, since $\mathcal{L}_\mathcal{X}(= h)$ is described by linear inequalities and equalities in $\mathbb{R}^D$, it is connected as an intersection of finitely many connected convex subsets of $\mathbb{R}^D$. So, we have proved

Lemma 4.1. *The set $\mathcal{L}_\mathcal{X}(= h)$ is connected.*

Finally, the space $\mathcal{L}_\mathcal{X}$ we define as $\mathcal{L}_\mathcal{X} := \cup_{h>0} \mathcal{L}_\mathcal{X}(= h)$.

Let us repeat that we allow maps in $\mathcal{L}_\mathcal{X}$ to have colliding turning points. For example, if two turning points c_k^i and c_k^{i+1} of the map q collide, i.e. $c_k^i = c_k^{i+1}$, then the graph of q will have not $|l| + N$ branches as a generic map in $\mathcal{L}_\mathcal{X}$ but only $|l| + N - 1$ branches and the point c_k^i might not be a turning point on the graph. However, we will keep track of such collided points and we will still call them turning. Other (i.e. non-collided) turning points of q will be called *simple*.

5 A link between $\mathcal{M}_\mathcal{X}$ and $\mathcal{L}_\mathcal{X}$

Milnor and Thurston [MT88] (see also [Par66]) defined the function $\Lambda : \mathcal{M}_\mathcal{X}^1 \to \mathcal{L}_\mathcal{X}$ such that the maps $f \in \mathcal{M}_\mathcal{X}^1$ and $\Lambda(f)$ are semi-conjugate and of the same topological entropy (they did it for the maps of an interval, but their construction can be applied to our case with no alterations). The particular definition of Λ is of no importance for us, the only thing we are going to use is the fact that for any function f there exists $q \in \mathcal{L}_{\mathcal{X}}$ semi-conjugate to f and of the same topological entropy.

For a map $q \in \mathcal{L}_{\mathcal{X}}$ let us define a set of all maps in $\mathcal{M}^b_{\mathcal{X}}$ which are semi-conjugate to q and denote it by $\mathcal{SH}^b_{\mathcal{X}}(q)$. Notice that maps in $\mathcal{SH}^b_{\mathcal{X}}(q)$ can have topological entropy different from the entropy of q, and $\mathrm{h_{top}}(q) \le \mathrm{h_{top}}(f)$ for any $f \in \mathcal{SH}^b_{\mathcal{X}}(q)$.

The set $\mathcal{SH}^b_{\mathcal{X}}(q)$ is closely related to the notion of a restrictive interval. An interval $J \subset I$ is called a *restrictive* interval of a map $f \in \mathcal{M}^1_{\mathcal{X}}$ if there exists $n \in \mathbb{N}$ such that $f^n(J) \subset J$ and $f^n(\partial J) \subset \partial J$. A connected component of a preimage of a restrictive interval we will also call a restrictive interval.

Fix maps $q \in \mathcal{L}_{\mathcal{X}}$, $f \in \mathcal{SH}^1_{\mathcal{X}}(q)$ and let H be the semi-conjugacy between f and q. Suppose that one of turning points c_q of the map q is periodic of period n. The set $H^{-1}(c_q)$ cannot be just a point. Indeed, if $H^{-1}(c_q)$ is a point, then it would be a critical point of f and, therefore, $c_f := H^{-1}(c_q)$ would be a superattractor. Iterates of all points in a neighbourhood of c_f would converge to the orbit of c_f, which is impossible if H is not locally constant near c_f.

Thus $H^{-1}(c_q)$ is an interval, and let us define $J_k := H^{-1}(q^k(c_q))$ for $k = 0, \dots, n-1$. It is easy to see that J_k are restrictive intervals, $f(J_k) \subset J_{k+1(\mathrm{mod}\, n)}$, $f(\partial J_k) \subset \partial J_{k+1(\mathrm{mod}\, n)}$. The map f restricted to $\cup_{k=0}^{n-1} J_k$ belongs to the cyclic space $\mathcal{M}_{n,\sigma',l',s',\mathsf{b}'}$, where σ', l', s' are defined in an obvious way. The definition of the function b' is more subtle and is done as follows. One or both boundary points of J_0 are periodic. Let x be a periodic boundary point of J_0 of period n_0 (where n_0 is either n or $2n$). If x is an interior point of $\mathrm{Dom}(f)$, then x cannot be a hyperbolic attractor. Indeed, otherwise it would attract trajectories of points on both sides of x, so H must be locally constant around x and then x cannot be a boundary point of J_0. Thus, $|Df^{n_0}(x)| \ge 1$ and, in this case, we set $\mathsf{b}'(x) = 1$. If the point x is a boundary point of $\mathrm{Dom}(f)$, then we set $\mathsf{b}'(x) = \mathsf{b}(x)$.

If there exists another turning point c^2_q of q so that $q^m(c^2_q) = c_q$ where m is minimal with this property, we can do a similar construction: define $J^2_k := H^{-1}(q^k(c^2_q))$ for $k = 0, \dots, m-1$. Then again the map f restricted to $\left(\cup_{k=1}^{n-1} J_k\right) \bigcup \left(\cup_{k=1}^{m-1} J^2_k\right)$ is a essential multi-interval multi-modal map.

We can repeat this construction for all periodic turning points of q and for all turning points of q one of whose iterates is mapped onto a periodic turning point. In this way to any map $f \in \mathcal{SH}^1_{\mathcal{X}}(q)$ we will associate another multi-interval multi-modal map (which is a restriction of f to the union of the restrictive intervals as above) in $\mathcal{M}^1_{\mathcal{X}_q}$ for an appropriate $\mathcal{X}_q := \{N_q, \sigma_q, l_q, s_q, \mathsf{b}_q\}$. Notice that $\mathcal{X}_q$ depends only on q and is independent of f. Also, from the construction it follows that the space $\mathcal{M}^b_{\mathcal{X}_q}$ is essential.

The union of all restrictive intervals used in this construction we will denote by $\mathrm{RDom}(f,q)$. From the definition of Λ it follows that if $q = \Lambda(f)$, then $\mathrm{h_{top}}(f|_{\mathrm{Dom}(f)\setminus \mathrm{RDom}(f,q)}) = \mathrm{h_{top}}(f)$ and $\mathrm{h_{top}}(f|_{\mathrm{RDom}(f,q)}) \le \mathrm{h_{top}}(f)$.

Thus for any map $q \in \mathcal{L}_{\mathcal{X}}$ there exists a map from $\mathcal{SH}^1_{\mathcal{X}}(q)$ to $\mathcal{M}^1_{\mathcal{X}_q}$ defined as above. Notice that because of the way we have constructed the function b_q this map is surjective. We will be more interested in the restriction of this map to the space $\mathcal{P}_{\mathcal{X}}$ and denote this map by $\Gamma_q : \mathcal{SH}^{\mathcal{P}}_{\mathcal{X}}(q) \to \mathcal{M}^\infty_{\mathcal{X}_q}$, where $\mathcal{SH}^{\mathcal{P}}_{\mathcal{X}}(q)$ denotes $\mathcal{SH}^1_{\mathcal{X}}(q) \cap \mathcal{P}_{\mathcal{X}}$. If q does not have periodic turning points, we set $N_q = 0$ and the map Γ_q is trivial.

Let us list a few properties of $\mathcal{X}_q$ and Γ_q. In what follows we denote the map $\Upsilon \circ \Gamma_q$ by $\tilde{\Gamma}_q$.

Lemma 5.1. *For any $q \in \mathcal{L}_{\mathcal{X}}$*

1. *Γ_q and $\tilde\Gamma_q$ are continuous;*

2. *the map $\tilde\Gamma_q : \mathcal{SH}^{\mathcal{P}}_{\mathcal{X}}(q) \to \tilde{\mathcal{P}}_{\mathcal{X}_q}$ is surjective;*

3. *if for $p_1, p_2 \in \mathcal{SH}^{\mathcal{P}}_{\mathcal{X}}(q)$ one has $\tilde\Gamma_q(p_1) = \tilde\Gamma_q(p_2)$, then $\Upsilon(p_1) = \Upsilon(p_2)$ (i.e. p_1 and p_2 are partially conjugate);*

4. *the map $\tilde\Gamma_q : \mathcal{SH}^{\mathcal{P}}_{\mathcal{X}}(q) \to \tilde{\mathcal{P}}_{\mathcal{X}_q}$ is monotone, i.e. for any $\upsilon \in \tilde{\mathcal{P}}_{\mathcal{X}_q}$ the set $\tilde\Gamma_q^{-1}(\upsilon)$ is connected;*

5. *if $\mathcal{X}$ is cyclic and $\mathrm{h_{top}}(q) > 0$, then $\mathcal{X}_q$ is subordinate to $\mathcal{X}$.*

Proof. The continuity of Γ_q is obvious and the map $\tilde\Gamma_q$ is a composition of two continuous maps.

The surjectivity is also easy to see: fix any $f \in \mathcal{SH}^1_{\mathcal{X}}(q)$ and $\upsilon \in \tilde{\mathcal{P}}_{\mathcal{X}_q}$, and take $g \in \Upsilon^{-1}(\upsilon)$ which has matching derivatives as f at boundary points of I_q. Then one can glue g into corresponding restrictive intervals of f and obtain a map which is still semi-conjugate to q and has a prescribed image under $\tilde\Gamma_q$. Then take $p \in \mathcal{P}_{\mathcal{X}}$ semi-conjugate to f given by aforementioned Theorem 6.4, [dMS93]. It is easy to see that $\tilde\Gamma_q(p) = \upsilon$.

Claim 3 is straightforward: in the set $\mathrm{Dom}(p_i) \setminus \mathrm{RDom}(p_i, q)$ the partial conjugacy is given by the semi-conjugacies between p_i and q, and inside of $\mathrm{RDom}(p_i, q)$ it is defined by $\tilde\Gamma_q(p_i)$.

Claim 4 follows from Claim 3 and Lemma 3.1.

For the last claim of the lemma consider $\mathcal{X}_q = \{N_q, \sigma_q, l_q, s_q, \mathsf{b}_q\}$ and take some $f \in \mathcal{SH}^1_{\mathcal{X}}(q)$. Clearly, $|l_q| \le |l|$ and $\mathrm{P_{min}}(\mathcal{X}_q) \ge N$. Suppose that $|l_q| = |l|$ and $\mathrm{P_{min}}(\mathcal{X}_q) = N$. In this case each connected component of $\mathrm{Dom}(f)$ contains one (and only one) of connected components of $\mathrm{RDom}(f, q)$ and all branches of $f|_{\mathrm{Dom}(f)\setminus \mathrm{RDom}(f,q)}$ are monotone (as $|l_q| = |l|$). Then $\mathrm{h_{top}}(q) \le \mathrm{h_{top}}(f|_{\mathrm{Dom}(f)\setminus \mathrm{RDom}(f,q)}) = 0$, and we get a contradiction. Thus either $|l_q| < |l|$ or $\mathrm{P_{min}}(\mathcal{X}_q) > N$, and, therefore, $\mathcal{X}_q$ is subordinate to $\mathcal{X}$. □

6 On the continuity of Λ

In the previous section we have defined the map $\Lambda : \mathcal{M}^b_{\mathcal{X}} \to \mathcal{L}_{\mathcal{X}}$. This map is neither continuous nor surjective. We will modify the space $\mathcal{L}_{\mathcal{X}}$ to fix this.

Two maps q_1 and q_2 in $\mathcal{L}_{\mathcal{X}}$ are called *similar* if their topological entropies are the same and there exists a map $f \in \mathcal{M}^1_{\mathcal{X}}$ with $\mathrm{h_{top}}(f) = \mathrm{h_{top}}(q_1) = \mathrm{h_{top}}(q_2)$ and which is semi-conjugate to both q_1 and q_2. We will denote this by $q_1 \approx q_2$.

For every map $f \in \mathcal{M}^1_{\mathcal{X}}$ there exists a map $p \in \mathcal{P}_{\mathcal{X}}$ semi-conjugate to f which just collapses the possible wandering intervals and non-essential basins of attraction. Hence, if $q_1 \approx q_2$, then there exists $p \in \mathcal{P}_{\mathcal{X}}$ such that $\mathrm{h_{top}}(p) = \mathrm{h_{top}}(q_1) = \mathrm{h_{top}}(q_2)$ and p is semi-conjugate to both q_1 and q_2.

The relation $\approx$ is reflexive and symmetric, but not necessary transitive. The relation we are about to introduce will generalise $\approx$ and will be transitive, thus, it will be an equivalence relation. Two maps q_1 and q_2 in $\mathcal{L}_{\mathcal{X}}$ are called *related* if there exist finitely many maps $q'_1, \dots, q'_m \in \mathcal{L}_{\mathcal{X}}$ such that

$$q_1 \approx q'_1 \approx \cdots \approx q'_m \approx q_2.$$

In this case we will write $q_1 \sim q_2$.

The quotient space of $\mathcal{L}_{\mathcal{X}}$ with respect to $\sim$ will be denoted by $\tilde{\mathcal{L}}_{\mathcal{X}}$ and let $\Psi : \mathcal{L}_{\mathcal{X}} \to \tilde{\mathcal{L}}_{\mathcal{X}}$ be the corresponding projection. Define $\tilde{\Lambda} := \Psi \circ \Lambda$.

Theorem C. *The map $\tilde{\Lambda} : \mathcal{P}_{\mathcal{X}} \to \tilde{\mathcal{L}}_{\mathcal{X}}$ is surjective and continuous.*

To proof this theorem we need the following lemma first.

Lemma 6.1. *Let $f_i \in \mathcal{P}_{\mathcal{X}}$ be a sequence converging to $f_0 \in \mathcal{P}_{\mathcal{X}}$, the sequence $q_i \in \mathcal{L}_{\mathcal{X}}$ converge to $q_0 \in \mathcal{L}_{\mathcal{X}}$ such that f_i is semi-conjugate to q_i for all i. Then f_0 is semi-conjugate to q_0.*

Proof. Let H_i denote the semi-conjugacy between f_i and q_i. Let us define two function H_0^- and H_0^+ by

$$\begin{aligned} H_0^-(x) &= \inf_{\{x_i\}:\, x_i \to x} \liminf H_i(x_i), \\ H_0^+(x) &= \sup_{\{x_i\}:\, x_i \to x} \limsup H_i(x_i) \end{aligned}$$

for $x \in I$. In other words, $[H_0^-(x), H_0^+(x)]$ is the minimal interval containing all limit points of $H_i(x_i)$ for all sequences $x_i \to x$. From the definition it is clear that $H_0^-(x) \le H_0^+(x)$ and since the maps H_i are non-strictly monotone increasing, for all $x_1 < x_2$ we have $H_0^+(x_1) \le H_0^-(x_2)$. In particular, $H_0^\pm$ are non-strictly monotone increasing too.

It is easy to see that from the definition of $H_0^\pm$ it follows that $\liminf H_0^-(x_i) \ge H_0^-(x)$ and $\limsup H_0^+(x_i) \le H_0^+(x)$ when $x_i \to x$. Indeed, given x for any $\epsilon > 0$ there exists $\delta > 0$ and N such that for all $y \in (x - \delta, x + \delta)$ and all $i > N$ one has

$$H_i(y) > H_0^-(x) - \epsilon. \tag{5}$$

If this were not true, then there would exist $\epsilon > 0$ and a sequences $y_k \to x$ and $i_k \to \infty$ such that $H_{i_k}(y_k) \le H_0^-(x) - \epsilon$ and taking the limit we would obtain a contradiction with the definition of $H_0^-(x)$. Then inequality (5) implies that $H_0^-(y) > H_0^-(x) - \epsilon$ for all $y \in (x - \delta, x + \delta)$ and we are done.

Notice that the last property of $H_0^\pm$ implies that if $H_0^-(x) = H_0^+(x)$, then the functions $H_0^\pm$ are continuous at x.

Define $A(x) := [H_0^-(x), H_0^+(x)]$. We claim that $q_0(A(x)) \subset A(f_0(x))$. Indeed, fix $x_0 \in I$ and $y_0 \in A(x_0)$ and find x_i such that $H_i(x_i) = y_0$ and $x_i \to x_0$. Then, since H_i is a semi-conjugacy we have $q_i(y_0) = H_i(f_i(x_i))$. Clearly, $q_i(y_0)$ converges to $q(y_0)$, $f_i(x_i)$ converges to $f_0(x_0)$, and the set of limit points of the sequence $H_i(f_i(x_i))$ belongs to $A(f_0(x_0))$ because of the definition of $H_0^\pm$.

Suppose that $A(x_0)$ is a non-degenerate interval for some x_0. The orbit of the interval $A(x_0)$ under the map q_0 cannot be disconnected because q_0 is expanding and has only finitely many turning points. So, without loss of generality we can assume that $A(x_0)$ contains a turning point and there exists $n > 0$ such that $q_0^n(A(x_0)) \subset A(x_0)$. This implies that x_0 is a periodic critical point of f_0 and, therefore, it is a superattractor. The corresponding critical points of maps f_i, where i is sufficiently large will be contained in a basin of attraction of a periodic attractor and this basin will contain a definite neighbourhood U of x_0 which does not depend on i when i is sufficiently large. Every semi-conjugacy between a C^1 map and piece-wise linear expanding maps must collapse basins of attraction, so all maps H_i are constants on U for i sufficiently large. This implies that $H_0^\pm$ are also constant on U, so $H_0^-(x_0) = H_0^+(x_0)$ which contradicts the fact that $A(x_0)$ is non-degenerate.

Therefore, we have proved that $H_0^-(x) = H_0^+(x)$ for all x and the sequence H_i converges to a continuous non-strictly monotone increasing map H_0 which is a semi-conjugacy between f_0 and q_0. □

We can proceed with the proof of the theorem now.

Proof of Theorem C. The surjectivity of $\tilde{\Lambda}$ follows from the fact that every combinatorics of a piece-wise linear map can be realised by a polynomial and the fact that the combinatorially equivalent maps in $\mathcal{L}_{\mathcal{X}}$ are similar.

The continuity of $\tilde{\Lambda}$ is a consequence of the above lemma. Indeed, take a sequence $f_i \in \mathcal{P}_{\mathcal{X}}$ converging to f_0 as in the lemma and let $q_i = \Lambda(f_i)$. Assume q_i converges to q_0. From Lemma 6.1 we know that f_0 is semi-conjugate to q_0. By continuity of the topological entropy we know that $\mathrm{h_{top}}(f_0) = \lim \mathrm{h_{top}}(f_i) = \lim \mathrm{h_{top}}(q_i) = \mathrm{h_{top}}(q_0)$. The map $\Lambda(f_0)$ is semi-conjugate to f_0 and has the same topological entropy, hence $q_0 \approx \Lambda(f_0)$. Thus, for any sequence f_i converging to f_0 we have that $\tilde{\Lambda}(f_i)$ converges to $\tilde{\Lambda}(f_0)$. □

Now consider some $f_0 \in \mathcal{P}_{\mathcal{X}}$ and let $q = \Lambda(f_0)$. As we already know (by Lemma 5.1(3)) for every $\upsilon \in \tilde{\mathcal{P}}_{\mathcal{X}_q}$ there exists a map $f \in \mathcal{P}_{\mathcal{X}}$ which is semi-conjugate to q and such that $\tilde{\Gamma}_q(f) = \upsilon$. If $\mathrm{h_{top}}(\upsilon) < \mathrm{h_{top}}(q)$, it is easy to check that in this case $\Lambda(f) = q$. If $\mathrm{h_{top}}(\upsilon) > \mathrm{h_{top}}(q)$, then $\mathrm{h_{top}}(f) > \mathrm{h_{top}}(q)$ and $\Lambda(f)$ cannot be equal to q. The case $\mathrm{h_{top}}(\upsilon) = \mathrm{h_{top}}(q)$ is more subtle and it is not clear whether $\Lambda(f)$ is q or not. However, the next lemma shows that $\Lambda(f)$ and q are similar.

Lemma 6.2. *For any $q \in \mathcal{L}_{\mathcal{X}}$*

$$\tilde{\Gamma}_q^{-1}(\tilde{\mathcal{P}}_{\mathcal{X}_q}(\leq \mathrm{h_{top}}(q))) \subset \bigcup_{q' \approx q} \Lambda^{-1}(q').$$

Proof. Take $f \in \tilde{\Gamma}_q^{-1}(\tilde{\mathcal{P}}_{\mathcal{X}_q}(\leq \mathrm{h_{top}}(q)))$. By the definition of Γ_q we know that f and q are semi-conjugate. It is also clear that $\mathrm{h_{top}}(f) = \mathrm{h_{top}}(q)$. Let $q' = \Lambda(f)$. Again, by the definition of Λ, f and q' are semi-conjugate and have the same topological entropy. Thus, $q \approx q'$ and we are done. □

This lemma implies that for any $\tilde{q} \in \tilde{\mathcal{L}}_{\mathcal{X}}$

$$\tilde{\Lambda}^{-1}(\tilde{q}) \quad = \quad \bigcup_{\Psi(q)=\tilde{q}} \tilde{\Gamma}_q^{-1}(\tilde{\mathcal{P}}_{\mathcal{X}_q}(\leq \mathrm{h_{top}}(\tilde{q}))). \tag{6}$$

It is straightforward that the left hand side of equality (6) is a subset of the right hand side. The opposite inclusion follows directly from Lemma 6.2.

7 Proof of the main result (Theorem A)

In this section we prove that for every $h \geq 0$ the set $\mathcal{P}_{\mathcal{X}}(= h)$ is connected. We will do it by induction with respect to the total number of critical points $|l|$ and N.

If $|l| = 0$, then the topological entropy of every map in $\mathcal{P}_{\mathcal{X}}$ is zero and we have nothing to do.

Assume that for any $h \geq 0$ the set $\mathcal{P}_{\mathcal{X}'}(= h)$ is connected where $\mathcal{X}' = \{N', \sigma', l', s', \mathsf{b}'\}$ with $|l'| \leq L - 1$.

Fix some $h_0 > 0$. Take $\mathcal{X} = \{N, \sigma, l, s, \mathsf{b}\}$ where $|l| = L$, σ is cyclic, and $N > \log(2)\frac{L}{h_0}$. In this case every $f \in \mathcal{P}_\mathcal{X}$ has topological entropy less than h_0 and, therefore, $\mathcal{P}_\mathcal{X}(= h) = \emptyset$, $\mathcal{P}_\mathcal{X}(\leq h) = \mathcal{P}_\mathcal{X}$ are connected sets for $h \geq h_0$.

Now assume that for any $h \geq h_0$ the set $\mathcal{P}_{\mathcal{X}'}(= h)$ is connected where $\mathcal{X}' = \{N', \sigma', l', s', \mathsf{b}'\}$ with either $|l'| \leq L - 1$ or $|l'| = L$, σ' is cyclic and $N' \geq N + 1$. This is our induction assumption. At this stage the induction will be done with respect to N backwards.

The space $\mathcal{P}_{\mathcal{X}'}$ is connected and the topological entropy continuously on maps in $\mathcal{P}_{\mathcal{X}'}$. This implies that since $\mathcal{P}_{\mathcal{X}'}(= h)$ is connected, then $\mathcal{P}_{\mathcal{X}'}(\leq h)$ is connected as well.

If the primary decomposition of the space $\mathcal{P}_\mathcal{X}$ is

$$\mathcal{P}_\mathcal{X} = \mathcal{P}_{\mathcal{X}_1} \times \cdots \times \mathcal{P}_{\mathcal{X}_m}$$

and we know that for any $h \geq h_0$ and $i = 1, \dots, m$ the sets $\mathcal{P}_{\mathcal{X}_i}(= h)$ and $\mathcal{P}_{\mathcal{X}_i}(\leq h)$ are connected, then the sets $\mathcal{P}_\mathcal{X}(= h)$ and $\mathcal{P}_\mathcal{X}(\leq h)$ are connected as well. This implies that we can assume that the sets $\mathcal{P}_{\mathcal{X}'}(= h)$ and $\mathcal{P}_{\mathcal{X}'}(\leq h)$ are connected if $\mathcal{X}'$ is subordinate to $\mathcal{X} = \{N, \sigma, l, s, \mathsf{b}\}$, where $|l| = L$ and σ is cyclic.

Fix cyclic $\mathcal{X} = \{N, \sigma, l, s, \mathsf{b}\}$ and take $\tilde{q} \in \tilde{\mathcal{L}}_\mathcal{X}$ with $\mathrm{h_{top}}(\tilde{q}) = h \geq h_0$. Take some $q \in \Psi^{-1}(\tilde{q})$. Due to Lemma 5.1(5) we know that $\mathcal{X}_q$ is subordinate to $\mathcal{X}$, and then from the induction assumption it follows that $\mathcal{P}_{\mathcal{X}_q}(\leq h)$ is connected. The map $\Upsilon : \mathcal{P}_{\mathcal{X}_q} \to \tilde{\mathcal{P}}_{\mathcal{X}_q}$ is continuous, surjective and preserves topological entropy, therefore $\tilde{\mathcal{P}}_{\mathcal{X}_q}(\leq h) = \Upsilon(\mathcal{P}_{\mathcal{X}_q}(\leq h))$ is connected.

The map $\tilde{\Gamma}_q : \mathcal{SH}^\mathcal{P}_\mathcal{X}(q) \to \tilde{\mathcal{P}}_{\mathcal{X}_q}$ is continuous, surjective and monotone (because of Lemma 5.1(4)), therefore the set $\tilde{\Gamma}_q^{-1}(\tilde{\mathcal{P}}_{\mathcal{X}_q}(\leq h))$ is connected due to Lemma 12.1.

Take two similar maps $q_1, q_2 \in \Psi^{-1}(\tilde{q})$, so $q_1 \approx q_2$. By definition there exists $p \in \mathcal{P}_\mathcal{X}$ which has the same entropy as q_1 and q_2 and which is semi-conjugate to q_1 and q_2. This implies that $p \in \tilde{\Gamma}_{q_i}^{-1}(\tilde{\mathcal{P}}_{\mathcal{X}_{q_i}}(\leq h))$, where $i = 1, 2$, and therefore the set

$$\tilde{\Gamma}_{q_1}^{-1}(\tilde{\mathcal{P}}_{\mathcal{X}_{q_1}}(\leq h)) \bigcup \tilde{\Gamma}_{q_2}^{-1}(\tilde{\mathcal{P}}_{\mathcal{X}_{q_2}}(\leq h))$$

is connected. Using equality (6) we get that the set

$$\tilde{\Lambda}^{-1}(\tilde{q}) = \bigcup_{\Psi(q) = \tilde{q}} \tilde{\Gamma}_q^{-1}(\tilde{\mathcal{P}}_{\mathcal{X}_q}(\leq h))$$

is connected as well.

The set $\mathcal{L}_\mathcal{X}(= h)$ is connected, so is the set $\tilde{\mathcal{L}}_\mathcal{X}(= h)$. The map $\tilde{\Lambda} : \mathcal{P}_\mathcal{X} \to \tilde{\mathcal{L}}_\mathcal{X}$ is continuous, surjective and monotone as we just have proved. Thus, due to Lemma 12.1 $\mathcal{P}_\mathcal{X}(= h) = \tilde{\Lambda}^{-1}(\tilde{\mathcal{L}}_\mathcal{X}(= h))$ is connected and we are done.

Finally, notice that the same argument proves that the set $\mathcal{P}_\mathcal{X}(\leq h)$ is connected for any $h > 0$. Then the set

$$\mathcal{P}_\mathcal{X}(= 0) = \bigcap_{h > 0} \mathcal{P}_\mathcal{X}(\leq h)$$

is connected as an intersection of compact connected nested sets.

8 Case of non-degenerate maps (proof of Theorem B)

In this section we will modify the proof of Theorem A given in the previous section and prove Theorem B.

First, for given $\epsilon > 0$ let us define the space $\mathcal{L}^\epsilon_\mathcal{X} \subset \mathcal{L}_\mathcal{X}$ as the set of all maps $q \in \mathcal{L}_\mathcal{X}$ such that the distance between any turning points and the distance from the turning points to the boundaries of I_k are greater or equal than ϵ. This space is closed and the set $\mathcal{L}^\epsilon_\mathcal{X}(=h)$ is connected. Indeed, to describe the set of parameters of $\mathcal{L}^\epsilon_\mathcal{X}(=h)$ we have to solve inequalities similar to (1) and (2). More precisely, the inequality (2) stays the same and (1) should be replaced by

$$a_0 + \epsilon \le c_1^1 \le c_1^1 + \epsilon \le c_1^2 \le \cdots \le c_1^{l(1)-1} + \epsilon \le c_1^{l(1)} \le a_1 - \epsilon \le a_1 + \epsilon \le c_2^1 \le \cdots \le a_N - \epsilon \quad (7)$$

Again $\mathcal{L}^\epsilon_\mathcal{X}(=h)$ is described by linear inequalities in $\mathbb{R}^D$, and it is connected as an intersection of finitely many connected convex subsets of $\mathbb{R}^D$.

Next we define the space $\tilde{\mathcal{L}}^\epsilon_\mathcal{X}$ in the exactly same way as we did in Section 6. More precisely, $\tilde{\mathcal{L}}^\epsilon_\mathcal{X}$ is a subset of $\tilde{\mathcal{L}}_\mathcal{X}$ such that each equivalence class in $\tilde{\mathcal{L}}^\epsilon_\mathcal{X}$ contains an element of $\mathcal{L}^\epsilon_\mathcal{X}$. Since $\mathcal{L}^\epsilon_\mathcal{X}$ is compact, the space $\tilde{\mathcal{L}}^\epsilon_\mathcal{X}$ is compact too. We can also define the space $\mathcal{P}^\epsilon_\mathcal{X} \subset \mathcal{P}^0_\mathcal{X}$ by setting it to be equal to $\tilde{\Lambda}^{-1}(\tilde{\mathcal{L}}^\epsilon_\mathcal{X})$. Since $\tilde{\Lambda}$ is continuous, $\mathcal{P}^\epsilon_\mathcal{X}$ is compact. Using the same prove as in the previous section without any alterations one can show that the set $\mathcal{P}^\epsilon_\mathcal{X}(=h)$ is connected.

For any $\epsilon_2 > \epsilon_1 > 0$ it is clear that $\mathcal{P}^{\epsilon_2}_\mathcal{X} \subset \mathcal{P}^{\epsilon_1}_\mathcal{X}$ and $\mathcal{P}^{\epsilon_2}_\mathcal{X}(=h) \subset \mathcal{P}^{\epsilon_1}(=h)$. Since $\mathcal{P}^\epsilon_\mathcal{X}(=h)$ are connected we get that the union $\cup_{\epsilon>0}\mathcal{P}^\epsilon_\mathcal{X}(=h)$ is connected as well.

Let us denote $\cup_{\epsilon>0}\mathcal{P}^\epsilon_\mathcal{X}$ by $\mathcal{P}^+_\mathcal{X}$. This set is a subset of $\mathcal{P}^0_\mathcal{X}$ but does not coincide with it. Let us see the structure of $\mathcal{P}^0_\mathcal{X} \setminus \mathcal{P}^+_\mathcal{X}$.

There are polynomials with all critical points non-degenerate, but which are semi-conjugate to a piece-wise linear maps with collided turning points. Take $p_0 \in \mathcal{P}^0_\mathcal{X}(=h) \setminus \mathcal{P}^+_\mathcal{X}$ and let $q_0 = \Lambda(p_0)$. The map q_0 cannot have all its turning points distinct because otherwise q_0 would belong to $\mathcal{L}^\epsilon_\mathcal{X}$ for some $\epsilon > 0$ and p_0 would belong to $\mathcal{P}^+_\mathcal{X}$. Hence, q_0 must have some collided turning points and it belongs to the boundary of $\mathcal{L}_\mathcal{X}$. Moreover, these collided turning points of q must be periodic, otherwise p_0 would have a degenerate critical point. Recall that $\mathcal{SH}^\mathcal{P}_\mathcal{X}(q_0)$ denotes all polynomials in $\mathcal{P}_\mathcal{X}$ which are semi-conjugate to q_0. Thus $p_0 \in \mathcal{SH}^\mathcal{P}_\mathcal{X}(q_0) \cap \mathcal{P}^0_\mathcal{X}(=h)$ and we have the following decomposition formula for $\mathcal{P}^0_\mathcal{X}(h)$:

$$\mathcal{P}^0_\mathcal{X}(=h) \;=\; \mathcal{P}^+_\mathcal{X}(=h) \bigcup \left(\cup_{q\in\mathcal{L}^\partial_\mathcal{X}(=h)} \left(\mathcal{SH}^\mathcal{P}_\mathcal{X}(q) \cap \mathcal{P}^0_\mathcal{X}(=h)\right)\right), \quad (8)$$

where $\mathcal{L}^\partial_\mathcal{X}$ denotes all maps in $\mathcal{L}_\mathcal{X}$ which have collided periodic turning points.

We now going to finish the proof that $\mathcal{P}^0_\mathcal{X}(=h)$ is connected. This will be done by induction similar to one in the previous section. We again fix $h_0 > 0$ and $\mathcal{X}$, and assume that $\mathcal{P}^0_{\mathcal{X}'}(=h)$ and $\mathcal{P}^0_{\mathcal{X}'}(\le h)$ are connected for all $h \ge h_0$ and all $\mathcal{X}'$ subordinate to $\mathcal{X}$.

Fix a map q_0 as in a paragraph above, i.e. $q_0 = \Lambda(p_0)$ where $p_0 \in \mathcal{P}^0_\mathcal{X}(=h)$. The map q_0 has some periodic collided turning points. Denote one of these turning points by t, its period denote by n, and suppose that the number of turning points collided at t is k_t. To simplify the exposition we will assume that q_0 is increasing at t, all other turning points of q_0 are simple (i.e. all colliding turning points are concentrated at t) and non-periodic (in particular, the orbit of t does not contain other turning points). The arguments below are quite general and these restrictions can be easily dropped.

By the definition we know that a partial conjugacy maps critical points onto critical points and it preserves the order of critical points. This implies that if a polynomial in a given partial conjugacy class has all critical points non-degenerate, then all other polynomials from this partial conjugacy class have all critical points non-degenerate too. Thus, the set $\tilde{\mathcal{P}}^0_{\mathcal{X}'} = \Upsilon(\mathcal{P}^0_{\mathcal{X}'})$

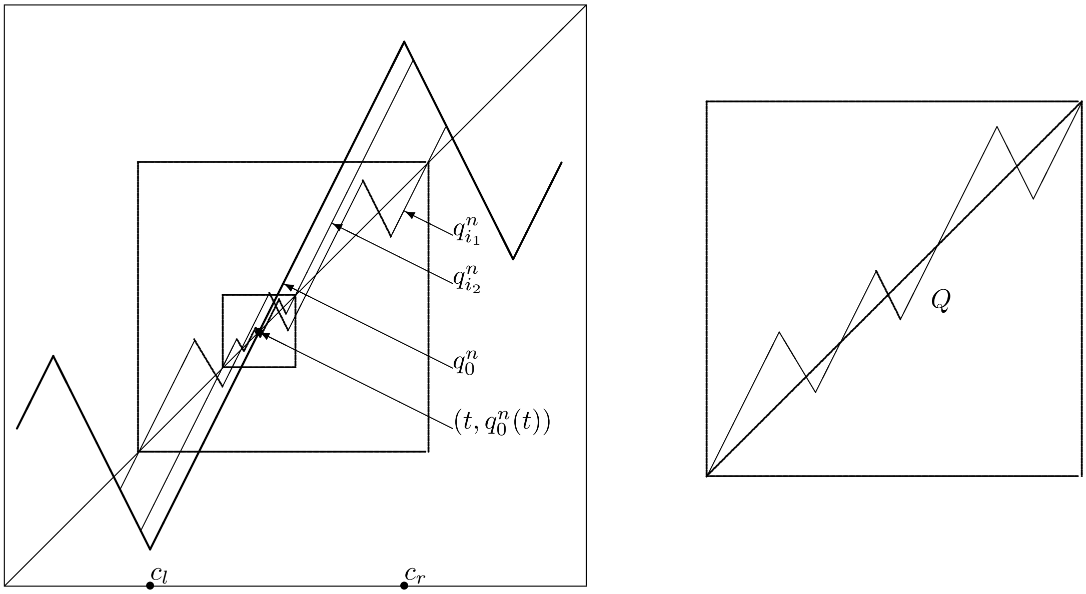

Figure 1: Perturbations of q_0.

is well defined and by the induction assumption the sets $\tilde{\mathcal{P}}^0_{\mathcal{X}'}(= h)$ and $\tilde{\mathcal{P}}^0_{\mathcal{X}'}(\le h)$ are connected for all $\mathcal{X}'$ subordinate to $\mathcal{X}$.

We already know that the map $\tilde{\Gamma}_{q_0} : \mathcal{SH}^{\mathcal{P}}_{\mathcal{X}}(q_0) \to \tilde{\mathcal{P}}_{\mathcal{X}_{q_0}}$ is surjective, monotone and continuous. As in the previous section we can argue that $\mathcal{X}_{q_0}$ is subordinate to $\mathcal{X}$, and using Lemma 12.1 for this map we obtain that the set $\tilde{\Gamma}^{-1}_{q_0}(\tilde{\mathcal{P}}^0_{\mathcal{X}_{q_0}}(\le h)) = \mathcal{SH}^{\mathcal{P}}_{\mathcal{X}}(q_0) \cap \mathcal{P}^0_{\mathcal{X}}(= h)$ is connected. Notice that the topological entropy of maps in $\mathcal{SH}^{\mathcal{P}}_{\mathcal{X}}(q_0)$ is at least $h = \mathrm{h_{top}}(q_0)$, so the sets $\mathcal{SH}^{\mathcal{P}}_{\mathcal{X}}(q_0) \cap \mathcal{P}^0_{\mathcal{X}}(= h)$ and $\mathcal{SH}^{\mathcal{P}}_{\mathcal{X}}(q_0) \cap \mathcal{P}^0_{\mathcal{X}}(\le h)$ coincide.

Now we are going to construct a sequence of maps $q_i \in \mathcal{L}_{\mathcal{X}}$ converging to q_0 which satisfies the following properties:

1. All turning points of q_i are distinct and non-periodic. Because of the polynomial rigidity this implies that for any q_i there exists unique $f_i \in \mathcal{P}^0_{\mathcal{X}}$ which is semi-conjugate to q_i.

2. The dynamics of turning points of q_i will prevent the critical points of f_i to collapse in the limit. It will be clearer later what this means precisely.

First, make a piece-wise linear map $Q : [0, 1] \to \mathbb{R}$ with constant slopes equal to $\pm \exp(h)$ which fixes the boundary points 0, 1, and has exactly k_t turning points $t_1, \dots, t_{k_t}$. Moreover, one can construct Q in such a way that none of the turning points is fixed by Q and the positions of the turning values on the graph of Q are oscillating around the diagonal, i.e. if $Q(t_i) > t_i$, then $Q(t_{i+1}) < t_{i+1}$, and if $Q(t_i) < t_i$, then $Q(t_{i+1}) > t_{i+1}$. The last property is equivalent to the following: each interval $[t_i, t_{i+1}]$ contains a fixed point of Q. An example of such a map is shown on Figure 1 on the right. Notice that the turning values are not required to belong to the interval $[0, 1]$.

In a short while we are going to use the following property of the map Q: for any two consecutive turning points t_i and t_{i+1} the interval $[Q(t_i), Q(t_{i+1})]$ contains one of these turning points. Indeed, the interval $[t_i, t_{i+1}]$ contains a fixed point of Q, so does $[Q(t_i), Q(t_{i+1})]$. If the last interval does not contain t_i and t_{i+1}, then $|Q(t_i) - Q(t_{i+1})| < |t_i - t_{i+1}|$ and the slope of the corresponding branch is less than one which is a contradiction.

Now we are ready to construct the sequence q_i. Let c_l and c_r be the turning points of q_0 to the left and right of t (or one of c_l, c_r can be a boundary point of $\mathrm{Dom}(q_o)$ if there is no turning point there). We are going to glue a scaled copy of Q at the point t and shift branches of q_0 defined on the intervals $[c_r, t]$ and $[t, c_r]$ up and down, see Figure 1. More precisely, we take the graph of the map $x \mapsto \exp(-h(n-1))\, K^{-1}Q(Kx)$ defined on $[0, K^{-1}]$ for large values of K, place it on the graph of q_0 around the point $(t, q_0(t))$, and then adjust branches of q_0 in such a way that we obtain a graph of a map in $\mathcal{L}_{\mathcal{X}}(=h)$. By taking a small perturbation of the obtained map if necessary we can assume that all its turning points are non-periodic. (Indeed, all maps in $\mathcal{L}_{\mathcal{X}}(=h)$ having a periodic turning point lie on a countable number of codimension one planes, so maps without periodic turning points are dense in $\mathcal{L}_{\mathcal{X}}(=h)$.) A sequence of maps in $\mathcal{L}_{\mathcal{X}}(=h)$ obtained in this way for larger and larger values of K and tending to q_0 we will denote by q_i.

The polynomials in $\mathcal{P}_{\mathcal{X}}$ semi-conjugate to q_i will be denoted by f_i. Once again, because of the rigidity these polynomials are unique. Since all turning points of maps q_i are distinct, the critical points of f_i are distinct as well, so they are quadratic. Moreover, in this case the semi-conjugacies between f_i and q_i are, in fact, just conjugacies. This implies that the topological entropies of f_i and q_i coincide and all these polynomials belong to $\mathcal{P}^+_{\mathcal{X}}(=h)$. By taking a subsequence we can assume that the sequence f_i converges to a polynomial $f_0 \in \mathcal{P}_{\mathcal{X}}$.

By the continuity of the topological entropy we know that $\mathrm{h_{top}}(f_0) = h$. From Lemma 6.1 it follows that f_0 is semi-conjugate to q_0 and, therefore, $f_0 \in \mathcal{SH}^{\mathcal{P}}_{\mathcal{X}}(q_0)$. Now we will show that all critical points of f_0 are distinct. It is obvious that if c is a critical point of f_0 which is mapped onto a simple turning point of q_0, it has to be quadratic. Next, suppose that f_0 has a degenerate critical point c_* which is mapped onto t by the semi-conjugacy. Consider two cases.

Case 1: the point c_* is periodic of period n, that is $f_0^n(c_*) = c_*$ (recall that t is also periodic of period n). Then c_* is a superattractor of f_0 and all polynomials sufficiently close to f_0 will have their critical points close to c_* converge to a periodic attractor. This means that for each sufficiently large value of i the map f_i has a critical point with periodic itinerary, hence, the map q_i has a periodic turning point, which is a contradiction.

Case 2: $f_0^n(c_*) \neq c_*$. Then there exists a small interval $[a, b]$ containing the point c_* such that $f_0^n([a, b])$ does not intersect $[a, b]$. Take sufficiently large i so that the interval $[a, b]$ contains at least two critical points c_1, c_2 of f_i and so that $f_i^n([c_1, c_2]) \cap [c_1, c_2] = \emptyset$. This is a contradiction because we checked that $Q([t_k, t_{k+1}])$ contains either t_k or t_{k+1}, a similar property holds for q_i because of its construction and maps f_i and q_i are topologically conjugate.

So, we have proved that $f_0 \in \mathcal{P}^0_{\mathcal{X}}(=h)$. Combining this and the facts that $f_0 \in \mathcal{SH}^{\mathcal{P}}_{\mathcal{X}}(q_0)$, that the set $\mathcal{P}^+_{\mathcal{X}}(=h)$ is connected and the sequence $f_i \in \mathcal{P}^+_{\mathcal{X}}(=h)$ converges to f_0 and that the set $\mathcal{SH}^{\mathcal{P}}_{\mathcal{X}}(q_0) \cap \mathcal{P}^0_{\mathcal{X}}(=h)$ is connected we get that the sets $\mathcal{P}^+_{\mathcal{X}}(=h)$ and $\mathcal{SH}^{\mathcal{P}}_{\mathcal{X}}(q_0) \cap \mathcal{P}^0_{\mathcal{X}}(=h)$ cannot be separated, so

$$\mathcal{P}^+_{\mathcal{X}}(=h) \cup \left(\mathcal{SH}^{\mathcal{P}}_{\mathcal{X}}(q_0) \cap \mathcal{P}^0_{\mathcal{X}}(=h)\right)$$

is connected. Equality (8) implies that $\mathcal{P}^0_{\mathcal{X}}$ is connected too.

9 On Thurston's question

In the rest of the paper we will argue that most likely the answer to Thurston's question is negative.

In the arguments which follow we will not need multi-interval maps, so we set $N = 1$ from now on. Consider the space $\mathcal{L}_{\mathcal{X}}$ for some combinatorial information $\mathcal{X} = \{1, \sigma, l, s\}$. It is clear that this space is parameterised by $|l|$ parameters. Using notation of Section 2 these parameters are the entropy h and the coefficients b_1^i where $i = 1, \dots, l(1) - 1$. Notice that b_1^0 and $b_1^{l(1)}$ are fixed by the boundary conditions. Since we are going to work with the case $N = 1$ for now we will drop the subscript $\cdot_1$ for the coefficients b_1^i and write b^i instead. The same applies to $l(1)$ and $s(1)$.

The turning points $0 \le c^1 \le \cdots \le c^l \le 1$ partition the interval $I = [0, 1]$ into $l + 1$ open subintervals which we denote by $J^0, \dots, J^l$. Some of these intervals can be degenerate if some turning points collide. Given a map $q \in \mathcal{L}_{\mathcal{X}}$ and a point $x \in I$ we call an infinite sequence of symbols in $\{c^1, \dots, c^l, J^0, \dots, J^l\}$ the *itinerary* of x if the iterate $q^n(x)$ belongs to the corresponding element of the sequence. Notice that if q has collided turning points, the itinerary of a point may be not unique, but this will not cause any problems for us. The *n-itinerary* of x we will call the sequence of the first $n + 1$ elements of the itinerary which control points $q^m(x)$ for $m = 0, \dots, n$.

The itinerary $\tilde{\mathcal{I}} = \{\tilde{I}_m\}$, $m = 0, \dots$, is called *compatible* with the itinerary $\mathcal{I} = \{I_m\}$ if the following holds. For all m

1. if $\tilde{I}_m$ is one of the intervals J^i, then $I_m = \tilde{I}_m$;
2. if $\tilde{I}_m$ is one of the turning points, let it be c^i, then I_m is either c^i or J^{i-1} or J^i.

Take a map $q \in \mathcal{L}_{\mathcal{X}}$ which has a turning point c^{i_0} which is mapped to another turning point c^{i_1} by some iterate q^n and let the orbit $\{q^m(c^{i_0}),\ m = 1, \dots, n-1\}$ not contain other turning points. We do allow the case $i_0 = i_1$ where the turning point becomes periodic. Denote the n-itinerary of c^{i_0} by $\mathcal{I}^{i_0}$. It it easy to see by a direct computation that the equation $q^n(c^{i_0}) = c^{i_1}$ has the form

$$\sum_{i=1}^{l-1} Q_i^{\mathcal{I}^{i_0}}(e^h) b^i = Q_0^{\mathcal{I}^{i_0}}(e^h), \tag{9}$$

where $Q_i^{\mathcal{I}^{i_0}}$ are some polynomials with rational coefficients. These polynomials have some particular structure which we will discuss in Section 11. Here we also used the equalities $c^i = \frac{1}{2}(-1)^i s e^{-h}(b^{i-1} - b^i)$ and $q(c^i) = \frac{1}{2}(b^{i-1} + b^i)$.

This equation we will call the *bifurcation* equation of $\mathcal{I}^{i_0}$ and the polynomials $Q_i^{\mathcal{I}^{i_0}}$ will be called *bifurcation* polynomials. Notice that the bifurcation equation is always well defined for periodic turning points.

Obviously, if another map $\tilde{q} \in \mathcal{L}_{\mathcal{X}}$ has a turning point with the same n-itinerary as the turning point of q under consideration, then the parameters of this map satisfy equation (9). Notice that even if the n-itinerary of this turning point is just compatible with $\mathcal{I}^{i_0}$, then the parameters of $\tilde{q}$ have to satisfy equation (9). This is an important observation which deserves to be formulated as a lemma:

Lemma 9.1. *Let q and $\tilde{q}$ be in $\mathcal{L}_{\mathcal{X}}$, c^{i_0} and $\tilde{c}^{i_0}$ be their turning points with n-itineraries $\mathcal{I}^{i_0}$, $\tilde{\mathcal{I}}^{i_0}$. Moreover, let $\tilde{\mathcal{I}}^{i_0}$ be compatible with $\mathcal{I}^{i_0}$ and $q^n(c^{i_0})$ be a turning point of q (so the bifurcation equation is defined)*[1]*. Then the parameters of the map $\tilde{q}$ satisfy the bifurcation*

[1] Also notice that the compatibility condition implies that in this case $\tilde{q}(\tilde{c}^{i_0})$ is also a turning point

equation of the map q:

$$\sum_{i=1}^{l-1} Q_i^{\mathcal{I}^{i_0}}(e^{\tilde{h}})\tilde{b}^i = Q_0^{\mathcal{I}^{i_0}}(e^{\tilde{h}}).$$

Notice that in the lemma above the maps q and $\tilde{q}$ can have different topological entropies.

Let us make clear that the converse of this lemma does not hold. If for some map its parameters satisfy equation (9), it does not imply that the corresponding turning point has the given n-itinerary: one would have to consider a bunch of inequalities together with equation (9) to guaranty that all the points from the orbit of the turning point fall into appropriate intervals as the itinerary dictates. However, maps close to q and satisfying the bifurcation equation do have a turning point with the same n-itinerary as the turning point c^{i_0} of q as the following lemma claims.

Lemma 9.2. *Let c^{i_0} be a turning point of $q \in \mathcal{L}_{\mathcal{X}}$ such that $q^n(c^{i_0})$ is also a turning point and $n \geq 1$ is minimal with this property. Then there exists a neighbourhood of q in the space $\mathcal{L}_{\mathcal{X}}$ such that every map in this neighbourhood satisfying the corresponding bifurcation equation has a turning point with n itinerary coinciding with n-itinerary of c^{i_0} of the map q.*

Remark. Notice that if $q^n(c^{i_0}) = c^{i_0}$, i.e. c^{i_0} is a periodic turning point, then all the maps in the neighbourhood given by the lemma and satisfying the bifurcation equation will have a periodic turning point of period n with the same itinerary.

Proof. Let $\{c^{i_0}, J^{m_1}, \ldots, J^{m_{n-1}}\}$ be the $n-1$ itinerary of c^{i_0}. We know that $q^j(c^{i_0})$ is not a turning point for $j = 1, \ldots, n-1$, so it belongs to the interior of J^{m_j}. Therefore, there exists a neighbourhood of q in the space $\mathcal{L}_{\mathcal{X}}$ such that if a map belongs to this neighbourhood, then the $n-1$ itinerary of the corresponding to c^{i_0} turning point is $\{c^{i_0}, J^{m_1}, \ldots, J^{m_{n-1}}\}$. Then the bifurcation equation ensures that the n itinerary of this point will be $\{c^{i_0}, J^{m_1}, \ldots, J^{m_{n-1}}, c^{i_1}\}$ where $c^{i_1} = q^n(c^{i_0})$. □

The equation (9) is linear in all b^i and as such it is easy to solve. There are several cases to consider:

Case 1. For given h some of the polynomials $Q_i^{\mathcal{I}^{i_0}}$, $i = 1, \ldots, l-1$, are non-zero at the point e^h. Then the parameters of maps in $\mathcal{L}_{\mathcal{X}}(=h)$ satisfying equation (9) form $l-2$ dimensional linear space. This case might be regarded as "generic".

Case 2. For given h we have $Q_i^{\mathcal{I}^{i_0}}(e^h) = 0$ for all $i = 0, 1, \ldots, l-1$. Clearly, all parameters of maps in $\mathcal{L}_{\mathcal{X}}(=h)$ satisfy equation (9). This is a very special case.

Case 3. For given h all the polynomials $Q_i^{\mathcal{I}^{i_0}}$, $i = 1, \ldots, l-1$ vanish at e^h, but $Q_0^{\mathcal{I}^{i_0}}(e^h) \neq 0$. There are no maps in $\mathcal{L}_{\mathcal{X}}(=h)$ which have the turning point c^{i_0} with the given itinerary.

These different cases motivate the following definition:

Definition 2. *Let c^{i_0} be a turning point of $q \in \mathcal{L}_{\mathcal{X}}$ such that $q^n(c^{i_0})$ is also a turning point and $n \geq 1$ is minimal with this property. Then this turning point is called* ordinary *if some of the polynomials $Q_i^{\mathcal{I}^{i_0}}$, $i = 1, \ldots, l-1$ do not vanish at $e^{\mathrm{h_{top}}(q)}$ (so we are in Case 1). If $Q_i^{\mathcal{I}^{i_0}}(e^{\mathrm{h_{top}}(q)}) = 0$ for all $i = 0, \ldots, l-1$, then the turning point c^{i_0} is called* exceptional *(Case 2 above).*

Remark. Since all the polynomials $Q_i^{\mathcal{I}^{i_0}}$ have rational coefficients, it is clear that if an isentrope of entropy level h has an exceptional turning point, then the number e^h is algebraic. In particular, only countably many isentropes can have exceptional turning points.

Soon we will give some examples of ordinary and exceptional turning points, however before that let us demonstrate their relevance to Thurston's question. We need another definition first.

Definition 3. *A turning point of a map $q \in \mathcal{L}_{\mathcal{X}}$ is called* controlled *if it is periodic or is mapped onto a periodic turning point by some iterate of q.*

A map $q \in \mathcal{L}_{\mathcal{X}}$ is called a codimension one hyperbolic *map if it has one turning point whose orbit does not contain any turning points, and all other $l-1$ turning points c^{i_j}, $j = 1, \dots, l-1$, are controlled. Moreover, if the determinant of the matrix $\|Q_i^{i_j}(e^{\mathrm{h_{top}}(q)})\|$, $i, j = 1, \dots, l-1$ formed by the bifurcation polynomials is non-zero, such the map q will be called an* ordinary codimension one hyperbolic *map.*

Similarly, a critical point of a map $p \in \mathcal{P}_{\mathcal{X}}$ is called controlled *if it is contained in the basin of a periodic attracting point.*

A polynomial $p \in \mathcal{P}_{\mathcal{X}}$ is called a codimension one hyperbolic *map if its all periodic points are hyperbolic and it has exactly $l-1$ controlled critical points counted with the multiplicities. Moreover, if p is semi-conjugate to an ordinary codimension one hyperbolic map $q \in \mathcal{L}_{\mathcal{X}}(= \mathrm{h_{top}}(p))$, then p will be called an* ordinary codimension one hyperbolic *map.*

Notice that a codimension one hyperbolic map is not hyperbolic! It has one critical point whose iterates do not converge to a periodic attractor.

Lemma 9.3. *Let $q \in \mathcal{L}_{\mathcal{X}}$ be an ordinary codimension one hyperbolic map and c^{i_j}, $j = 1, \dots, l-1$ be its controlled turning points. Then there exist an interval (h_-, h_+) containing $\mathrm{h_{top}}(q)$ and a function $r : (h_-, h_+) \to \mathcal{L}_{\mathcal{X}}$ such that*

- $\mathrm{h_{top}}(r(h)) = h$ *for all $h \in (h_-, h_+)$;*
- *the parameters b^i of the map $r(h)$ are given by some rational functions $R_i(e^h)$;*
- $r(\mathrm{h_{top}}(q)) = q$*;*
- *for all $h \in (h_-, h_+)$ the itineraries of the controlled turning points c^{i_j} of $r(h)$ coincide with the itineraries of the corresponding turning points of the map q;*
- *the converse also holds: if the itineraries of $l-1$ turning points of a map $q' \in \mathcal{L}_{\mathcal{X}}$ are compatible with the itineraries of the corresponding controlled turning points of q and $\mathrm{h_{top}}(q') \in (h_-, h_+)$, then $q' = r(\mathrm{h_{top}}(q'))$.*

In particular, there are no other than q maps in $\mathcal{L}_{\mathcal{X}}(= \mathrm{h_{top}}(q))$ which have the controlled turning points with the itineraries compatible with the itineraries of the controlled turning points of q.

Proof. From the previous discussion we already know that if the itineraries of the controlled turning points of some map q' are the same (or compatible) as of q, then the parameters of q' must satisfy the $l-1$ bifurcation equations. Notice that we have $l-1$ linear in b equations which also depend on the parameter h. Since the map q is ordinary, the solution of this system of bifurcation equations as a function of h is well defined in some interval around the point $\mathrm{h_{top}}(q)$. Using Lemma 9.2 and by shrinking this interval if necessary we can ensure that maps corresponding to the solutions of this system have the controlled points with the given itineraries. □

Theorem D. *Let $p \in \mathcal{P}_{\mathcal{X}}$ be an ordinary codimension one hyperbolic polynomial map of positive entropy. Then p cannot be approximated by hyperbolic polynomial maps of the same entropy $\mathrm{h_{top}}(p)$.*

Remark 1. There is nothing special about the polynomial space here, this theorem also holds for the space $\mathcal{M}^1_{\mathcal{X}}$.

Remark 2. If one drops the condition that the map is ordinary, the theorem does not hold anymore. Once we construct maps with exceptional critical points in Sections 10.1 and 10.2, one can easily find codimension one hyperbolic maps which can be approximated by hyperbolic maps of the same entropy.

Remark 3. We will see (Theorem F) that if the entropy is larger than $\log 3$, then the corresponding isentrope can contain only ordinary codimension one hyperbolic maps. In the such case if one finds a codimension one hyperbolic map, they do not need to check that it is ordinary, it holds automatically.

Proof. This Theorem is a consequence of Lemma 9.3.

From the definition of ordinary maps we can find an ordinary codimension one hyperbolic map $q \in \mathcal{L}_{\mathcal{X}}(= \mathrm{h_{top}}(p))$ semi-conjugate to p. Since all attracting periodic points of p are hyperbolic, there exists a neighbourhood of p in $\mathcal{P}_{\mathcal{X}}$ where these attracting points persist and the critical points of maps in this neighbourhood corresponding to the controlled critical points of p are also controlled and are in the basins of attraction of the corresponding periodic points. Let p' be in this neighbourhood and $q' \in \mathcal{L}_{\mathcal{X}}$ be semi-conjugate to p' and of the same entropy $\mathrm{h_{top}}(q') = \mathrm{h_{top}}(p')$. Let $c^{i_0}(p)$ be one of the controlled critical points of p, and $c^{i_0}(p')$, $c^{i_0}(q)$ be the corresponding critical (turning) points of p', q'. It is easy to see that the itinerary of $c^{i_0}(q')$ is compatible with the itinerary of $c^{i_0}(q)$. Now assume that $\mathrm{h_{top}}(p') = \mathrm{h_{top}}(p)$. Since the map q is ordinary and due to Lemma 9.3 we know that if $\mathrm{h_{top}}(p') = \mathrm{h_{top}}(p)$ and, therefore, $\mathrm{h_{top}}(q') = \mathrm{h_{top}}(q)$, then q' and q are the same maps. One of the turning points of q is not eventually periodic, hence the map p' has a critical point which is not in the basin of attraction of some periodic attractor. So, the map p' cannot be hyperbolic. □

10 Exceptional isentropes

In this section we study with more details when an isentrope can have an exceptional turning point and partially answer on Question 2. We start with a number of examples. We restrict ourselves to the case of bimodal maps which can be easily generalised. To make computations simpler we will rescale the domain of the definition of maps we consider so our bimodal maps are defined by this formula:

$$q = q_{\lambda,b} : x \mapsto \begin{cases} \lambda x + 1, & \text{if } x \in J^0 = [-a, c^1] \\ -\lambda x + b, & \text{if } x \in J^1 = [c^1, c^2] \\ \lambda x - 1, & \text{if } x \in J^2 = [c^2, a] \end{cases} \tag{10}$$

where $\lambda = e^h$, $a = \frac{1}{\lambda - 1}$, $c^1 = \frac{b-1}{2\lambda}$, $c^2 = \frac{b+1}{2\lambda}$. Also, notice that $\pm a$ are fixed points of q and that $q(c^1) = \frac{1}{2}(b+1)$, $q(c^2) = \frac{1}{2}(b-1)$. We want q to map the interval $[-a, a]$ into itself, this implies that λ and b should satisfy inequalities $\lambda \in [1, 3]$ and $b \in [-\frac{3-\lambda}{\lambda-1}, \frac{3-\lambda}{\lambda-1}]$.

10.1 Exceptional isentropes from unimodal tent maps.

The simplest examples of exceptional isentropes can be constructed using unimodal tent maps with a periodic turning point.

Fix some parameter λ in the interval $(1, 2)$. For such λ there exists a non-degenerate interval of parameters b (which is $[-\frac{3-\lambda}{\lambda-1}, -1]$) such that $q_{\lambda,b}$ maps the interval $[-a, c^2]$ inside itself. The itineraries of the turning point c^1 for all values of b in this interval are the same and coincide with the itinerary of the turning point of the unimodal tent map of entropy $h = \log \lambda$.

Now fix $\lambda = e^h \in (1, 2)$ in such a way that the unimodal tent map of entropy h has a periodic turning point. Then in the bimodal family maps $q_{\lambda,b}$ will have a periodic turning point c_1 of the same itinerary for all $b \in [-\frac{3-\lambda}{\lambda-1}, -1]$. Because of Lemma 9.3 we know that for ordinary turning points we can have at most one parameter b for the given itinerary, therefore c^1 is exceptional. On the other hand, when b varies in this interval, the itinerary of the other turning point c^2 is not constant and there are infinitely many different itineraries of c^2 when this turning point becomes preperiodic. Using arguments similar to ones we use in the first part of the paper (in particular, continuity of the map $\tilde{\Lambda}$) one can show that the isentrope $\mathcal{P}_{\mathcal{X}}(= \log \lambda)$ contains infinitely many combinatorially different hyperbolic maps and that this isentrope contains a codimension one hyperbolic maps which can be approximated by hyperbolic maps.

10.2 Cascades of exceptional itineraries.

There is another mechanism which produces isentropes with exceptional turning points and generalises the previous construction. We start with a concrete example where most of the things can be explicitly computed.

In the bimodal family under consideration let us consider maps which have a periodic turning point c^1 of period 2 and with itinerary $\mathcal{I} = \{c^1, J^2, c^1, \ldots\}$. One can easily compute the bifurcation equation for this itinerary:

$$(\lambda^2 - 1)b = -(\lambda - 1)^2,$$

so $Q_1^{\mathcal{I}}(\lambda) = \lambda^2 - 1$, and $Q_0^{\mathcal{I}}(\lambda) = -(\lambda - 1)^2$. The case of $\lambda = 1$ is always special: it is easy to see that for any itinerary $\mathcal{I}$ one has $Q_1^{\mathcal{I}}(1) = 0$. In our case we have $Q_0^{\mathcal{I}}(1) = 0$ as well, so we can reduce $\lambda - 1$ factor and obtain

$$(\lambda + 1)b = 1 - \lambda.$$

From this equation we can see that there is no exceptional isentropes for the given itinerary $\mathcal{I}$ because the polynomials $\lambda + 1$ and $\lambda - 1$ never vanish at the same time.

Nothing exciting so far. Now let us consider some other itinerary $\mathcal{I}'$ so that $\mathcal{I}$ is compatible to $\mathcal{I}'$. For example, let $\mathcal{I}' = \{c^1, J^2, J^0, J^2, c^1, \ldots\}$. It is clear that if maps with such the itinerary exist, then the turning point c^1 is periodic of period 4. Since $\mathcal{I}$ is compatible with $\mathcal{I}'$, all the solutions of the bifurcation equation for $\mathcal{I}$ are also solutions of the bifurcation equation for $\mathcal{I}'$. This implies that the bifurcation polynomials for $\mathcal{I}'$ can be factorised as

$$\begin{aligned} Q_1^{\mathcal{I}'}(\lambda) &= F(\lambda) Q_1^{\mathcal{I}}(\lambda) \\ Q_0^{\mathcal{I}'}(\lambda) &= F(\lambda) Q_0^{\mathcal{I}}(\lambda) \end{aligned}$$

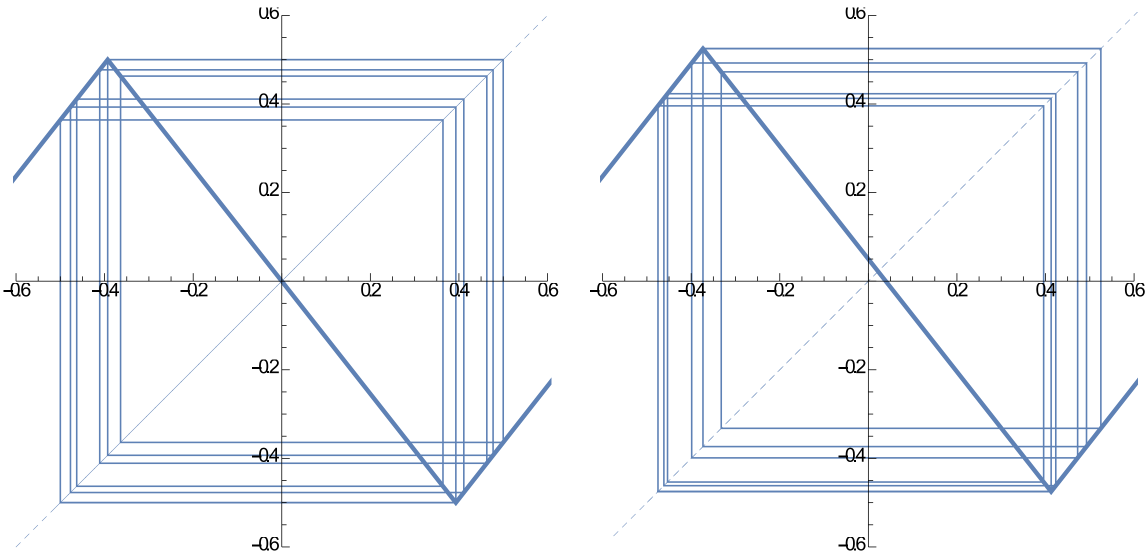

Figure 2: Iterates of the turning points when $\lambda = \lambda_e$, $b = 0$ (left) and $b = 0.05$ (right)

where F is a polynomial. It is easy to see that the degree of the polynomials $Q_{0,1}^{\mathcal{I}'}$ is 4, so the degree of F is two. A simple computation (assisted by Wolfram Mathematica) gives

$$F(\lambda) = \lambda^2 + 1.$$

The roots of F are complex, hence there are no other solutions of the bifurcation equation for $\mathcal{I}'$ in the region of the interest except $b = \frac{1-\lambda}{1+\lambda}$ and there are no bimodal maps in $\mathcal{L}_{\mathcal{X}}$ which realise $\mathcal{I}'$.

If instead of the itinerary $\{c^1, J^2, J^0, J^2, c^1, \ldots\}$ we considered $\mathcal{I}' = \{c^1, J^2, J^1, J^2, c^1, \ldots\}$, we could argue again that the bifurcation polynomials must have a common factor which can be computed to be $F(\lambda) = \lambda^2 - 1$. The roots of this polynomial are real, but of no interest for us, so again, there are no bimodal maps which realise $\mathcal{I}'$.

Let us move forward and find a nontrivial itinerary compatible to $\mathcal{I}$ which can be realised by some bimodal maps. Consider the itinerary $I' = \{c^1, J^2, J^1, J^2, J^0, J^2, c^1, \ldots\}$. Using the same argument as before we can factorise its bifurcation polynomials and compute the factor to be $F(\lambda) = \lambda^4 - \lambda^2 - 1$. This factor polynomial has two complex roots, one negative root and one positive root $\lambda_e = \sqrt{\frac{1}{2}\left(\sqrt{5}+1\right)} \approx 1.27202$. Furthermore, we can check that for this value of $\lambda = \lambda_e$ if $-0.119726 \leq b \leq 0.346014$, then the turning point c^1 indeed has the itinerary $\mathcal{I}'$. Thus we have found an exceptional itinerary.

Let us make an interesting observation. Because of the symmetry we know that if the turning point c^2 is periodic with the itinerary $\mathcal{I}'' = \{c^2, J^0, J^1, J^2, J^1, J^0, c^2, \ldots\}$, then $\mathcal{I}''$ is exceptional as well for parameters $\lambda = \lambda_e$ and $-0.346014 \leq b \leq 0.119726$. This implies that for all $b \in (-0.119726, 0.119726)$ both turning points are periodic of period 6 with the constant itineraries, and therefore all maps in this parameter interval are combinatorially and topologically conjugate, see Figure 2. We investigate this phenomenon in more details in the next section.

The method of finding exceptional itineraries can obviously applied not only to the itinerary $\mathcal{I} = \{c^1, J^2, c^1, \ldots\}$, but to other periodic itineraries too. We have to do the following. Take some periodic itinerary $\mathcal{I} = \{c^1, J^{i_1}, \ldots, J^{i_m}, c^1, \ldots\}$ and check that this itinerary can be realised by some maps in $\mathcal{L}_\mathcal{X}$. Form another compatible itinerary $\mathcal{I}' = \{c^1, J^{i_1}, \ldots, J^{i_m}, J^{j_1}, J^{i_1}, \ldots, J^{i_m}, J^{j_2}, \ldots, J^{i_1}, \ldots, J^{i_m}, c^1, \ldots\}$, where all $j_1, j_2, \ldots$ are either 0 or 1. As before, the bifurcation polynomials of $\mathcal{I}'$ have a common factor. If this factor has real roots in the interval $(1, 3)$, investigate if for values of these roots the corresponding maps can realise $\mathcal{I}'$. If they can, we have another exceptional itinerary.

In this way we can obtain many exceptional itineraries starting with $\mathcal{I} = \{c^1, J^1, c^1, \ldots\}$. Another example would be $\mathcal{I} = \{c^1, J^1, J^0, J^2, c^1, \ldots\}$. This itinerary is not exceptional and its bifurcation equation is

$$(\lambda - 1)\left(\lambda^3 + \lambda^2 + \lambda - 1\right) b = (\lambda - 1)\left(-\lambda^3 + \lambda^2 - \lambda - 1\right).$$

The compatible itinerary $\mathcal{I}' = \{c^1, J^1, J^0, J^2, J^1, J^1, J^0, J^2, J^0, J^1, J^0, J^2, c^1, \ldots\}$ is exceptional, its bifurcation equation factor is

$$F(\lambda) = \lambda^8 - \lambda^4 - 1$$

which has a root ≈ 1.12784. For this value of λ and for $-0.808065 \leq b \leq -0.720696$ the itinerary of c^1 under the map $q_{\lambda,b}$ is $\mathcal{I}'$.

10.3 Non-rigidity in the bimodal family $\mathcal{L}_\mathcal{X}$.

We know the following fundamental rigidity result for polynomials with all critical points real: if two such polynomials are combinatorially equivalent and do not have periodic attractors, then they are linearly conjugate. In other words, if we consider a normalised parameterisation of the polynomial family, there exists only one parameter with this prescribed combinatorics. For the piece-wise linear maps of constant slope a similar rigidity result holds provided the maps are transitive, see [AM15]. If the transitivity condition does not hold, the rigidity does not necessarily hold either:

Theorem E. *There exists a nonempty open set $\mathcal{E} \subset \mathcal{L}_\mathcal{X}$ such that if the set $\mathcal{E}(= \mathrm{h_{top}})$ is not empty, then all the maps in $\mathcal{E}(= \mathrm{h_{top}})$ are combinatorially equivalent (and therefore topologically conjugate).*

Proof. Let us look at the example described in the previous section, and consider map $q = q_{\lambda_e,0}$. We know that in this case both the turning points are periodic of period 6, see Figure 2. Consider the interval R^1 defined as $[q^4(c^1), q^2(c^1)]$ and notice that $c^1 \in R^1$. It is easy to see that $q^2(R^1) \subset R^1$, so R^1 is a renormalization interval of period two. The interval $R^2 = [q^2(c^2), q^4(c^2)]$ is another renormalization interval around the turning point c^2.

For parameters λ and b close enough to λ_e and 0 the intervals $R^i_{\lambda,b} = [q^4_{\lambda,b}(c^i), q^2_{\lambda,b}(c^i)]$, $i = 1, 2$, will still be renormalization intervals of period two. The maps $q^2_{\lambda,b}|_{R^i_{\lambda,b}}$ are unimodal tent maps and their combinatorics is completely determent by the parameter λ. Thus for fixed λ close to λ_e and all b close to zero all the maps $q_{\lambda,b}$ have the same combinatorics. □

10.4 Non-existence of exceptional isentropes for large entropies.

All examples of isentropes with exceptional turning points we had so far have been given for the parameter λ smaller than two. We will prove that this is always the case in the bimodal case:

Theorem F. *There do not exist isentropes with exceptional turning points in the space of bimodal maps $\mathcal{L}_{1,id,2,s}$ of topological entropies larger than* $\log 2$.

In general, there do not exist exceptional isentropes of topological entropies larger than $\log 3$.

Proof. We start the proof with the general case when l is not necessarily two. We will be using the notation introduced in Section 4, i.e. on the interval J^i the map q is defined as $q(x) = (-1)^i s\lambda x + b^i$. Recall that b^0 and b^l are fixed by the boundary conditions.

Let $\mathcal{I} = \{c^{i_0}, J^{i_1}, \ldots, J^{i_{m-1}}, c^{i_m}, \ldots\}$ be an exceptional itinerary, i.e. there exist parameters λ_0 and b_0 such that the itinerary of $c^{i_0}_{\lambda_0,b_0}$ under the map q_{λ_0,b_0} is $\mathcal{I}$ and all the bifurcation polynomials of $\mathcal{I}$ vanish for $\lambda = \lambda_0$. It is also clear that if the parameter b is close enough to b_0, then the itinerary of $c^{i_0}_{\lambda_0,b}$ will be $\mathcal{I}$ again.

Now consider the iterates of c^{i_0} under the map q. From the definition it is easy to see that

$$q^k_{\lambda_0,b}(c^{i_0}) = \sum_{i=0}^{l} w^i_k b^i$$

where w^i_k are some numbers (which in general depend on λ_0). These numbers are related by recursive formulas of the form

$$w^i_{k+1} = (-1)^{i_k} s\lambda_0 w^i_k + \delta^i_{i_k}$$

where $\delta^i_{i_k}$ is equal to one if $i = i_k$ and zero otherwise. The initial conditions for these recursive formulas are

$$w^i_1 = \frac{1}{2}\left(\delta^{i-1}_{i_0} + \delta^i_{i_0}\right)$$

because $q(c^{i_0}) = \frac{1}{2}(b^{i-1} + b^i)$.

Fix some $\hat{\imath} \neq 0, l$ such that $w^{\hat{\imath}}_1 = \frac{1}{2}$. Notice that if $|x| \geq \frac{1}{2}$, then $|\pm\lambda x| > \frac{3}{2}$ and $|\pm\lambda x + 1| > \frac{1}{2}$ for all $\lambda > 3$. This implies that

$$|w^{\hat{\imath}}_k| > \frac{1}{2} \tag{11}$$

for all $k \geq 2$.

We know that $q^{m+1}(c^{i_0}) = q(c^{i_m})$. Since the level λ_0 is exceptional we also know that the bifurcation polynomial $Q^{\mathcal{I}}_{\hat{\imath}}(\lambda_0) = 0$. This implies that w^i_{m+1} is either $\frac{1}{2}$ or zero. This contradicts inequality (11). Thus λ_0 cannot be larger than three.

The case of bimodal maps where $l = 2$ is dealt with similarly. We will consider the case $s = +1$, the other case $s = -1$ is analogous. The index $\hat{\imath}$ here is just 1, $w^1_1 = \frac{1}{2}$ and the recursive formula is

$$w^1_{k+1} = \begin{cases} \lambda_0 w^1_k & \text{if } i_k \text{ is 0 or 2} \\ -\lambda_0 w^1_k + 1 & \text{if } i_k \text{ is 1} \end{cases}$$

This formula implies that if $\lambda_0 > 2$ and $w^1_k \in (-\infty, 0) \cup [\frac{1}{2}, +\infty)$, then $w^1_{k+1} \in (-\infty, 0) \cup (\frac{1}{2}, +\infty)$. So, arguing as in the general case we get a contradiction. □

11 Codimension one hyperbolic maps in the bimodal family

Let us consider the family of real polynomial maps of degree 3 and their isentropes of entropy larger than $\log 2$. From the previous section we already know that there are no exceptional critical points in this case, so Theorem D implies that if a map has entropy larger than $\log 2$, two critical points, one of which is not controlled and the other is controlled (so it is periodic), then such a map cannot be approximated by hyperbolic maps of the same entropy.

We conjecture that these codimension one hyperbolic maps exist on every isentrope (with some trivial exceptions like $h = \log 3$ for the bimodal maps). Let us see what would happen if this is not the case.

Fix some entropy level $h > \log 2$ and the corresponding isentrope in the space of the bimodal piece-wise linear maps of the constant slopes given by formula (10). Suppose that c^1 is a periodic point of period n. Then equation (9) can be written as

$$Q_1^1(\lambda)b = Q_0^1(\lambda), \tag{12}$$

where $Q_0^1(\lambda) = \sum_{i=0}^{n} \alpha_i^1 \lambda^i$ and $Q_1^1(\lambda) = \sum_{i=0}^{n} \beta_i^1 \lambda^i$ and the coefficients α_i^1 and β_i^1 can be explicitly computed if the itinerary of c^1 is known. Moreover, these coefficients satisfy the following conditions which are easy to obtain by a direct computation: $-\alpha_n^1 = \beta_n^1 = 1$, $\alpha_0^1 = \beta_0^1 = \pm 1$, and for $i = 1, \dots, n-1$ we have $\alpha_i^1, \beta_i^1 \in \{-2, 0, 2\}$ and $|\alpha_i^1| + |\beta_i^1| = 2$. The last condition means that if a coefficient in front of λ^i is non-zero in the polynomial Q_0^1, then the corresponding coefficient in Q_1^1 must be zero and vise verse (however they cannot be both zeros at the same time).

For example, let us consider the case when λ is close to 3. Then the critical value $q(c^1)$ is close to the fixed repelling point a and there exists a bimodal map q such that c^1 is periodic of period n with the itinerary $\{c^1, \underbrace{J^2, J^2, \dots, J^2}_{n-1}, c^1, \dots\}$ (recall that $J^2 = (c^2, a)$). Formula (12) in this case becomes

$$(\lambda^n - 1)b = -\lambda^n + 2\lambda^{n-1} + \dots + 2\lambda - 1.$$

Similarly, if the critical point c^2 is periodic or is mapped onto c^1 by some iterate of the map, the parameters λ and b satisfy

$$Q_1^2(\lambda)b = Q_0^2(\lambda), \tag{13}$$

where $Q_0^2(\lambda) = \sum_{i=0}^{n'} \alpha_i^2 \lambda^i$ and $Q_1^2(\lambda) = \sum_{i=0}^{n'} \beta_i^2 \lambda^i$ and for the coefficients the following holds: $\alpha_{n'}^2 = \beta_{n'}^2 = 1$, $\alpha_0^2 = \pm 1$, $\beta_0^2 = \pm 1$, and for $i = 1, \dots, n'-1$ we have $\alpha_i^2, \beta_i^2 \in \{-2, 0, 2\}$ and $|\alpha_i^2| + |\beta_i^2| = 2$.

Suppose that all maps in some isentrope $\mathcal{P}_{\mathcal{X}}(= h)$, where $h \in (\log 2, \log 3)$, can be approximated by hyperbolic maps. Fix the corresponding value of $\lambda = e^h$ and consider the corresponding isentrope $\mathcal{L}_{\mathcal{X}}(= h)$. Theorem D and Theorem F imply that $\mathcal{L}_{\mathcal{X}}(= h)$ does not contain any codimension one hyperbolic maps. Then for any parameter b such that the turning point c^1 of $q_{\lambda,b} \in \mathcal{L}_{\mathcal{X}}(h)$ is periodic (and, therefore, equality (12) holds) the other critical point c^2 must be controlled as well and equality (13) must holds. This implies that λ satisfies the equality

$$Q_0^1(\lambda)Q_1^2(\lambda) = Q_1^1(\lambda)Q_0^2(\lambda) \tag{14}$$

The parameter λ must satisfy an equality of this type whenever one of the turning points of $q_{\lambda,b}$ is periodic. For this fixed λ there are infinitely many different values of b when this map has a periodic critical point, therefore λ has to satisfy infinitely many different polynomial equalities of type (14). Notice that the involved polynomials Q_j^i are very special (we described properties of their coefficients in the paragraphs above). It seems highly likely that such parameters λ do not exist, but we were unable to prove this.

12 Appendix

Here we will prove the topological fact we have been often using.

Lemma 12.1. *Let X, Y be topological spaces, and X be compact. Let $F : X \to Y$ be continuous. Let B be a subset of Y, $A = F^{-1}(B)$, the set $F(A)$ be connected, and $F|_A$ be monotone. Then A is connected.*

Proof. Suppose that A is not connected, so there exists a separation of A. This means that there are two non-empty subsets A_1 and A_2 such that $A = A_1 \cup A_2$, $\bar{A}_1 \cap A_2 = \emptyset = A_1 \cap \bar{A}_2$. Set $B_k = F(A_k)$ where $k = 1, 2$.

The sets B_1, B_2 are non-empty because A_1 and A_2 are non empty. Since $F(A)$ is connected, B_1 and B_2 cannot form a separation of $F(A)$, thus the closure of one of them should have non-empty intersection with the other. Assume $B_1 \cap \bar{B}_2 \neq \emptyset$ and let $y_0 \in B_1 \cap \bar{B}_2$. Since $y_0 \in B_1$ there exists $x_0 \in A_1$ such that $F(x_0) = y_0$. Also, take a sequence of $y_i \in B_2$ converging to y_0 and let $x_i \in A_2$ be such that $F(x_i) = y_i$. The space X is compact, so we can take a subsequence x_{i_j} converging to some $x_\infty \in \bar{A}_2$. From the continuity of F it follows that $F(x_\infty) = y_0$.

The map $F|_A$ is monotone, therefore $F^{-1}(y_0)$ is connected. We know that $x_0 \in A_1$, hence $F^{-1}(y_0) \subset A_1$. On the other hand x_∞ belongs to both $\bar{A}_2$ and $F^{-1}(y_0)$, so the intersection of A_1 and $\bar{A}_2$ are non-empty. This is a contradiction. □

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