CONVEX DUALITY AND ORLICZ SPACES IN EXPECTED UTILITY MAXIMIZATION

Sara Biagini and Aleš Černý

In this paper we report further progress towards a complete theory of state-independent expected utility maximization with semimartingale price processes for arbitrary utility function. Without any technical assumptions we establish a surprising Fenchel duality result on conjugate Orlicz spaces, offering a new economic insight into the nature of primal optima and providing fresh perspective on the classical papers of Kramkov and Schachermayer (1999, 2003). The analysis points to an intriguing interplay between no-arbitrage conditions and standard convex optimization and motivates study of the Fundamental Theorem of Asset Pricing (FTAP) for Orlicz tame strategies.

Keywords: utility maximization, Orlicz space, Fenchel duality, supermartingale deflator, effective market completion.

1. introduction

Utility maximization is a fundamental tenet of normative economic theory and, as its most classical embodiment, "expected utility remains the primary model in numerous areas of economics dealing with risky decisions" (Moscati, 2016). Although a rigorous axiomatic foundation of expected utility appeared early (Von Neumann and Morgenstern, 1944) there remains a long-standing open problem in the theoretical description of expected utility maximization in a purely financial dynamic stochastic setting. Our aim is to offer new insights in this direction.

The paper studies the mechanics of wealth transfer from initial date 0 to some terminal date T. A single agent whose preferences over terminal wealth are represented by expected utility under given subjective probability P decides, continuously in time, how to allocate her wealth among one risk-free and finitely many risky assets modelled by a semimartingale price process S. There is no intermediate consumption and no production or labour income. The main concern of the paper is finding a suitable class of trading strategies that makes the problem well-defined. This is a non-trivial task because, as observed by Harrison and Kreps (1979), unrestricted trading in continuous time permits so-called doubling strategies that create something out of nothing with certainty even when trading on a martingale.

In this paper we make three distinct contributions to the literature. Firstly, the Orlicz space framework unifies different strands of currently fragmented literature on utility maximization and absence of arbitrage. Coupled with convex duality it also conveys strong economic intuition. The unifying framework, its economic interpretation, and links to the relevant literature are presented in Sections 1.1-1.9.

LUISS Guido Carli, Rome, sbiagini@luiss.it

Cass Business School, City, University of London, ales.cerny.1@city.ac.uk

Our second contribution is a new Fenchel duality result (Theorem 2.1) which allows us to remove singular parts in the dual problem and offer new interpretation of the resulting duality as an 'effective completion' of the market. Effective completion means that the complete market represented by the dual optimizer does not contain the entirety of the original opportunity set but only those elements that have finite expected utility (Definition 2.2).

Immediate consequences of this new result are discussed in Sections 2.1–2.3. What emerges very strongly is that effective completions are linked to 'corner solutions' in the primal problem whereby, based on marginal utility considerations, the economic agent would like to increase her exposure to risky assets in a particular direction but this is not possible because any further exposure takes the agent out of the effective domain of expected utility.

Our third contribution is a new construction of the optimal trading strategy (Propositions 3.5, 4.1, 5.3, Definition 5.1, and Theorem 5.4) where we avoid reliance on the dual optimizer altogether. This permits, for the first time in a semimartingale setting, construction of optimal portfolios for monotone mean variance preferences. The new construction also covers the previously unresolved case where the utility function is finite on the whole real line but the dual optimizer is only an effective completion. We establish existence of an optimal trading strategy under mild assumptions that reduce to the minimal assumptions of Kramkov and Schachermayer (2003) in the L^∞ case. The challenges of this construction are summarized in Section 1.10.

The paper is organised as follows: in the remainder of Section 1 we introduce the necessary concepts and notation, and discuss their economic and mathematical significance. Without going into too much technical detail, we also set out the different elements of our research strategy and explain how they fit together. Sections 2-5 implement our research programme and Section 6 concludes. For reader's convenience Appendix A collects useful known results in convex analysis. Appendix B constructs an explicit example of a corner solution in a continuous model with Lévy dynamics and proves the dual optimizer cannot be linked to a supermartingale deflator in this case. Appendix C provides an explicit example where duality over full market completions fails and links it to the structure of the underlying Orlicz space.

1.1. *Utility function U and Orlicz space $L^{\hat{U}}$*

A utility function U in this paper is a proper, concave, non-decreasing, upper semi-continuous function. Its effective domain is the non-empty set

$$\operatorname{dom} U = \{x \mid U(x) > -\infty\}. \tag{1.1}$$

The lower bound of the effective domain of U is denoted by

$$\underline{x} = \inf(\operatorname{dom} U). \tag{1.2}$$

Upper semicontinuity of U means that at $\underline{x}$, which is the only possible point of discontinuity for U, the utility function must be right-continuous.

The bliss point of utility is defined by

$$\overline{x} = \inf\{x \mid U(x) = U(+\infty)\}, \tag{1.3}$$

where $U(+\infty) = \lim_{x\to+\infty} U(x)$. For strictly increasing utility functions $\overline{x} = +\infty$, while for truncated utility functions, which feature for example in monotone mean-variance portfolio allocation (Černý et al., 2012), $\overline{x} < +\infty$ represents a point where further increase in wealth does not produce additional enjoyment in terms of utility. In economics this is interpreted as the point of maximum satisfaction, or bliss.

By construction $\underline{x} \leq \overline{x}$ and the equality arises only when U is constant on its entire effective domain in which case the utility maximization problem is trivial since 'doing nothing' is always optimal. Therefore, up to a translation, the following convention entails no loss of generality and simply means that initial endowment has been normalized to 0.

CONVENTION 1.1 *$\underline{x} < 0 < \overline{x}$ and $U(0) = 0$.*

Fixing a filtered probability space $(\Omega, \mathcal{F}_T, P)$, the *left* tail of utility function U gives rise to the Orlicz space of random variables

$$L^{\hat{U}}(\Omega, \mathcal{F}_T, P) = \{X \in L^0(\Omega, \mathcal{F}_T, P) \mid E[\hat{U}(\lambda|X|)] < \infty \text{ for some } \lambda \geq 0\}.$$

In the theory of Orlicz spaces[1] the convex function $\hat{U}(x) = -U(-|x|)$ is known as the Young function. We write $L^{\hat{U}}(P)$ or $L^{\hat{U}}$ for short when no confusion can arise.

With X interpreted as net trading gain one has $X \in L^{\hat{U}}$ if and only if any *sufficiently small* position in X, both long and short, has finite expected utility. Orlicz space $L^{\hat{U}}$ contains a smaller subspace $M^{\hat{U}}$ (known as the Orlicz heart)[2] of financial positions whose expected utility remains finite with arbitrary scaling,

$$M^{\hat{U}} = \{X \in L^{\hat{U}} \mid E[\hat{U}(\lambda|X|)] < \infty \text{ for all } \lambda \geq 0\}.$$

It is convenient to equip $L^{\hat{U}}$ with a Minkowski gauge norm,

$$\|X\|_{\hat{U}} = \inf\{\lambda > 0 \mid E[\hat{U}(|X|/\lambda)] \leq 1\},$$

[1] For a minimal overview of Orlicz spaces in the context of utility maximization see, for example, Biagini and Černý (2011, Section 2.2). A compact exposition (35 pages) appears in Edgar and Sucheston (1992, Sections 2.1 and 2.2). Monographic references include Krasnosel'skiĭ and Rutickiĭ (1961) and Rao and Ren (1991).

[2] The terminology 'Orlicz heart' appears to originate with Edgar and Sucheston (1989). It emphasizes $M^{\hat{U}}$ as a subspace of $L^{\hat{U}}$, which is a point of view important in our context. $M^{\hat{U}}$ can also be understood as a self-standing Banach space, going back to Morse and Transue (1950, Section 8). Some authors use 'Morse-Transue (sub)space' or merely 'Morse subspace' when referring to $M^{\hat{U}}$.

which coincides with the classical L^p norm when $\hat{U}(|x|) = |x|^p$. In this construction the space $L^{\hat{U}}$ always satisfies the embeddings

$$L^\infty \hookrightarrow L^{\hat{U}} \hookrightarrow L^1, \tag{1.4}$$

and for quadratic utility, in particular, one obtains the natural setting where $L^{\hat{U}}$ is isomorphic to L^2 ($L^{\hat{U}} \sim L^2$).

While the construction involving space $L^{\hat{U}}$ allows one to formulate a unified treatment for all utility functions, for topological reasons it is at times necessary to distinguish among three cases based on the behaviour of U at $-\infty$. We flag up the three cases here for reader's convenience.

Case L-F (linear, therefore finite) Utility decays asymptotically linearly,

$$0 < \lim_{x\to-\infty} U(x)/x = \lim_{x\to-\infty} U'_+(x) < \infty.$$

A typical example is the Domar-Musgrave piecewise linear utility (Richter, 1960, Fig. 3). The relevan space is $L^1(P)$.

Case SL-F (super-linear and finite) Examples include truncated quadratic utility and exponential utility. The relevant space $L^{\hat{U}}(P)$ depends on the specific U but it is always strictly larger than $L^\infty(P)$ and strictly smaller than $L^1(P)$,

$$L^\infty(P) \hookrightarrow\!\!\!\!\rightarrow L^{\hat{U}}(P) \hookrightarrow\!\!\!\!\rightarrow L^1(P).$$

Case SL-INF (left tail of U equals $-\infty$) Utility functions in this category include logarithmic utility as well as power utility functions with negative exponent. The relevant space is $L^\infty(P)$.

Coarser classifications, such as **F** vs. **INF** or **L** vs. **SL**, will be used in appropriate places. The intermediate case **SL-F** will lead to further sub-classification which will emerge partly in the introduction and fully in the main body of the paper. Speaking very roughly, the case $M^{\hat{U}} = L^{\hat{U}}$ will require less work than the case $M^{\hat{U}} \subsetneq L^{\hat{U}}$; see also Table I.

1.2. *Primal problem and tame strategies*

The pioneering work of Merton (1969, 1971, 1973) emphasized tractability of optimal portfolio allocation for diffusive models of asset prices. However, Harrison and Kreps (1979, Section 6) pointed out that unrestricted stochastic integration, implicit in Merton's work, allows for so-called doubling strategies that lead to arbitrage opportunities in essentially any continuous-time model of asset prices. To prevent such economically anomalous but mathematically plausible behaviour a consensus emerged to define 'tame' strategies $\mathscr{T}$ as those whose wealth is bounded below by an arbitrary constant, see Harrison and Kreps (1979, p. 400), Harrison and Pliska (1981, Section 3.3), Dybvig and Huang (1988, Theorem 1) and Karatzas and Shreve (1998, Definition 2.4). We will subsequently refer to these strategies as L^∞-tame,

$$\mathscr{T}_\infty = \{H \in L(S) \mid \inf_{t\in[0,T]} H \cdot S_t \in L^\infty\},$$

where $L(S)$ is the set of all predictable S-integrable processes and symbol $H \cdot S_t$ stands for a stochastic integral $\int_{(0,t]} H dS$.

For a general utility funciton U the primal portfolio allocation problem is to compute the supremum, denoted by u, of expected utility over the set of, as yet unspecified, tame trading strategies $\mathscr{T}$,

$$u(B) = \sup_{H\in\mathscr{T}} E[U(B + H \cdot S_T)]. \tag{1.5}$$

Here $B \in L^{\hat{U}}$ is a random variable representing a random endowment available at time T.

In the remainder of the paper B is fixed and in this introduction we take $B = 0$ for simplicity, resuming the general case from Section 2 onwards. We tacitly assume, along with all related literature in this area, that there is a risk-free asset with constant value 1 at all times. We treat the problem (1.5) as given, with semimartingale S, utility U, and filtered probability space $(\Omega, \mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}, P)$ supplied exogenously. The class of tame strategies $\mathscr{T}$ will be determined in response to these three inputs, independently of the choice of B.

1.3. *Tame and admissible strategies*

It is known from the deep results of Kramkov and Schachermayer (1999, 2003), that for utility functions with $\underline{x}$ finite (case **INF**, including log utility and HARA utilities with negative exponent treated by Merton in a lognormal setting) one can build a satisfactory framework for any arbitrage-free semimartingale price process S by restricting the agent to L^∞-tame strategies. However, for utility functions with $\underline{x} = -\infty$ (case **F**), such as quadratic or negative exponential utility, two difficulties arise that render the above approach unsatisfactory.

The first problem is that a uniform bound on wealth rules out, for example, normally distributed returns in any one-period model. Biagini and Frittelli (2005, 2007, 2008) remedy the situation by allowing tameness to depend on the utility function U, so that maximal loss of *all* tame strategies is controlled by *one* exogenously chosen element of $L^{\hat{U}}$. In this paper we allow losses to be controlled by *any* element of $L^{\hat{U}}$. This leads to a wider class of $L^{\hat{U}}$-tame strategies whose maximal loss belongs to the Orlicz space $L^{\hat{U}}$,

$$\mathscr{T} = \{H \in L(S) \mid \inf_{t\in[0,T]} H \cdot S_t \in L^{\hat{U}}\}.$$

The second difficulty in the case **F** is that L^∞-tame strategies, and even $L^{\hat{U}}$-tame strategies as defined here, may not contain the optimizer. Our task is to design a larger class of *admissible* strategies $\mathscr{A}$ (which again depends on S, U, and P, and this time also on B) that attain the supremum $u(B)$ in (1.5) without exceeding it. This is done by requiring that each strategy in $\mathscr{A}$ is approximated in a natural sense by a sequence of strategies in $\mathscr{T}$, see Definition 5.1 below and also Biagini and Černý (2011, Definition 1.1) to whom we refer the reader for further background and references.

We have shown previously (Biagini and Černý, 2011, Theorem 4.10) that such definition of admissibility is satisfactory (at least in the case $B = 0$) when the optimal solution in the dual problem, which we proceed to describe below, is a σ-martingale measure. In this paper we take the extra step to cover also the difficult case where the dual optimizer is not a σ-martingale measure. We remark that in the setting of Kramkov and Schachermayer (1999, 2003) the present framework yields $L^{\hat{U}} = L^{\infty}$, tame strategies $\mathscr{T}$ and admissible strategies $\mathscr{A}$ coincide and they are precisely those strategies whose wealth is bounded below by some constant.

1.4. *Economic duality* $(L^{\hat{U}}, L^{\hat{V}})$

Duality has a venerable history in economic literature. Clasically, it describes relationship between an indirect utility function and an expenditure function or between a production function and a cost function, see Hotelling (1932); Shephard (1953); Blackorby and Diewert (1979); Diewert (1981). Although not presented in this way historically, Blume (2008a,b) points out that microeconomic duality can be elegantly summarized using the language of convex duality. We, too, use convex duality as a unifying theme throughout the paper.

To avoid heavy notation, some symbols are overloaded as suggested in the approach of Rockafellar (1974). For concave f the *conjugate function* f^* is defined as the concave function $f^*(y) = \inf_x\{\langle x, y\rangle - f(x)\}$, while the same symbol for convex f means the convex function $f^*(y) = \sup_x\{\langle x, y\rangle - f(x)\}$. Here $\langle x, y\rangle$ is a bilinear form defined over appropriate spaces, for example $\langle x, y\rangle = xy$ when $x, y \in \mathbb{R}$. Similarly, the *effective domain* is defined as $\operatorname{dom} f = \{x : f(x) > -\infty\}$ for concave f while for convex f one has $\operatorname{dom} f = \{x : f(x) < +\infty\}$; cf. equation (1.1). A self-contained technical exposition of convex duality and its key results appears in Appendix A. We suggest Blume (2008a,b) as an economic primer.

The duality constructs in this paper and related literature are somewhat different to the classical microeconomic results surveyed above. The basic idea here is to embed the incomplete financial market generated by tame trading in S into a statically complete financial market in which every terminal wealth distribution in $L^{\hat{U}}$ is available at a known cost at time 0. This is very similar in spirit to one of the steps in the construction of general equilibrium in Arrow and Debreu (1954). We now proceed with the detailed description of the dual pricing rules.

The concave utility U has a concave conjugate U^* and in line with notation in Kramkov and Schachermayer (1999, 2003) we let $V = -U^*$. To represent pricing rules as random variables one must be able to express prices, and with them also the bilinear form appearing in convex duality, by an expectation operator,

$$(1.6) \qquad \langle X, Y \rangle = E[XY],$$

which implies (Zaanen, 1983, Theorems 132.2 and 132.4) that dual variables will be taken from the Orlicz space $L^{\hat{V}}$ determined by the *right* tail of function V. Here $\hat{V} \equiv \hat{U}^*$ is known as the conjugate Young function.

Fix $Y \in L^{\hat{V}}$ and on $L^{\hat{U}}$ define a pricing rule $p_Y(X) = E[XY]$. Assuming $p_Y(1) = 1$ one can interpret this pricing rule as a risk-neutral expectation, $p_Y(X) = E^{Q_Y}[X]$ with $dQ_Y/dP = Y$. Here Q_Y describes time-0 prices of Arrow-Debreu securities in the statically complete market with payoffs in $L^{\hat{U}}(\Omega, \mathcal{F}_T, P)$ in the sense that for any contingency $A \in \mathcal{F}_T$ and the corresponding elementary security with payoff 1_A the price of state A is given by $Q_Y(A) \equiv E^{Q_Y}[1_A]$.

Such a market is arbitrage-free if and only if $Y > 0$ P-a.s. which is the same as saying that measure Q^Y is equivalent to measure P. We denote the set of all possible absolutely continuous state price measures by

$$(1.7) \qquad P_{\hat{V}} = \{Q \mid dQ/dP \in L^{\hat{V}}_+\}.$$

We will observe later that each probability measure $Q \in P_{\hat{V}}$ describes prices of Arrow-Debreu securities in a statically complete market that is bliss-free for all utility functions with the same left tail and arbitrary bounded right tail. The set of all equivalent (arbitrage-free) pricing measures is denoted by

$$P^e_{\hat{V}} = \{Q \in P_{\hat{V}} \mid dQ/dP > 0 \ P\text{-a.s.}\}.$$

1.5. *Topology and duality: a caution*

In economics, duality is taken to mean a juxtaposition of two related objects such as indirect utility function and expenditure function. In mathematics, duality frequently refers to the choice of pairing between dual spaces. We will now address aspects of duality in the latter sense. This will lead to introduction of three dual spaces with three corresponding conjugation symbols,

$$\star, \quad \circledast, \quad * \ .$$

One typically thinks of the Orlicz space $L^{\hat{U}}$ as a Banach space endowed with an appropriate norm. Let us denote the norm dual[3] of $L^{\hat{U}}$ by $(L^{\hat{U}})^\star$. The chief difficulty facing us is that the finest topology on $L^{\hat{U}}$ compatible[4] with the economic duality $(L^{\hat{U}}, L^{\hat{V}})$ of Section 1.4 may be strictly coarser than the norm topology on $L^{\hat{U}}$. For example, in the case **INF** studied by Kramkov and Schachermayer

[3]The set of all linear functionals on $L^{\hat{U}}$ that are continuous for the norm topology on $L^{\hat{U}}$.

[4]Topology on $L^{\hat{U}}$ such that the set of linear functionals on $L^{\hat{U}}$ that are continuous in this topology can be identified with random variables in $L^{\hat{V}}$.

(1999, 2003) one has $L^{\hat{U}} = L^{\infty}, L^{\hat{V}} = L^1$, while $(L^{\infty})^{\star}$ is the space of finitely additive measures that strictly contains all linear functionals generated by random variables in L^1.

Topologies compatible with the economic duality $(L^{\hat{U}}, L^{\hat{V}})$ can be characterised in more detail (see Appendices A and C.1), but for now it suffices to bear in mind that norm topology of $L^{\hat{U}}$ may not be one of them. It turns out that the norm topology is (trivially) compatible with economic duality on $L^{\hat{U}}$ in finite-dimensional models while in all other cases this happens if and only if $L^{\hat{U}}$ coincides with its Orlicz heart $M^{\hat{U}}$ (Theorem C.1), requiring that all financial positions in $L^{\hat{U}}$ can be scaled up arbitrarily, long and short, while retaining finite expected utility, which in particular covers the case **L**. It is also known that in the case **F** one has $(M^{\hat{U}})^{\star} = L^{\hat{V}}$ (Edgar and Sucheston, 1992, Theorem 2.2.11), therefore the norm topology of $L^{\hat{U}}$ is compatible with economic duality on the smaller space $M^{\hat{U}}$, whether or not the equality $M^{\hat{U}} = L^{\hat{U}}$ holds.

The flipside is that on $L^{\hat{U}} \supsetneq M^{\hat{U}}$ one generally loses access to many helpful properties associated with the Orlicz space $L^{\hat{U}}$ as a Banach space when working in the economic duality $(L^{\hat{U}}, L^{\hat{V}})$. It also means (bi)conjugates computed in the norm topology are in general different from those computed in the economic duality $(L^{\hat{U}}, L^{\hat{V}})$. To avoid possible ambiguity we reserve the symbol $\star$ for the former and $\circledast$ for the latter. One should bear in mind that the bilinear form in the $\star$ duality generally cannot be expressed by means of an expectation under measure P as in equation (1.6), but merely as an abstract action of a linear functional from $(L^{\hat{U}})^{\star}$ on an element in $L^{\hat{U}}$.

There is another level of subtlety in the general case that is not visible in the Kramkov and Schachermayer (1999, 2003) setting where $L^{\hat{U}}$ is the norm-dual of $L^{\hat{V}}$. In general this is not true, but in the case **SL** one can recover this relationship when $L^{\hat{V}}$ is replaced with the Orlicz heart $M^{\hat{V}}$. Conjugation in the duality $(L^{\hat{U}}, M^{\hat{V}})$ will be denoted by asterisk $*$. For readers familiar with functional analysis the duality $(L^{\hat{U}}, M^{\hat{V}})$ is compatible with the weak-star topology on $L^{\hat{U}}$. To summarize, in the order in which the relevant dual spaces range from the largest to the smallest $(L^{\hat{U}})^{\star} \hookleftarrow L^{\hat{V}} \hookleftarrow M^{\hat{V}}$ the conjugation symbols read $\star$, $\circledast$ and $*$. The last two dualities use the same bilinear form (1.6), given by an expectation operator, and for any function f on $L^{\hat{U}}$ the conjugates $f^{\circledast}$ and f^* coincide on $M^{\hat{V}}$.

It is known that the closure of a convex set does not depend on the choice of a specific compatible topology, but only on the dual pair (Aliprantis and Border, 2006, Theorem 5.98). To emphasize this fact we use notation $\mathrm{cl}^{\star}A$, $\mathrm{cl}^{\circledast}A$, $\mathrm{cl}^{*}A$, to denote closure of a convex set A in the three dualities. In particular, for a convex cone $\mathcal{C}$ one has $\mathrm{cl}^{\star}\mathcal{C} = \mathcal{C}^{\star\star}$, $\mathrm{cl}^{\circledast}\mathcal{C} = \mathcal{C}^{\circledast\circledast}$ and $\mathrm{cl}^{*}\mathcal{C} = \mathcal{C}^{**}$, where $\mathcal{C}^{\star}, \mathcal{C}^{\circledast}, \mathcal{C}^{*}$ are polar cones in the appropriate duality (see equation 1.10).

Existing literature has very little to say on the case $L^{\hat{V}} \supsetneq M^{\hat{V}}$, even though the case is logically no less important and no less prevalent in the universe of possible utility functions than $L^{\hat{V}} = M^{\hat{V}}$. The reader is likely to be familiar with L^q spaces, $1 \le q < \infty$, for which one always has $L^q = M^q$. One might therefore think that $L^{\hat{V}} \supsetneq M^{\hat{V}}$ only occurs when $\hat{V}(y)$ grows faster than any power y^q. Indeed, the right-to-left implication always holds (Krasnosel'skiĭ and Rutickiĭ, 1961, eq. (I.4.7)). But the left-to-right implication is not true at all; for any $q \ge 1$ one may construct $L^{\hat{V}}$ that is not equal to its Orlicz heart and such that $L^{q+\varepsilon} \hookrightarrow L^{\hat{V}} \hookrightarrow L^q$ (Salekhov, 1968, Teorema 4). This means that cases $L^{\hat{V}} \supsetneq M^{\hat{V}}$ are interspersed in between L^q spaces where $L^{\hat{V}} = M^{\hat{V}}$ and a theory able to cover both is essential.

The contribution of our paper is significant already in the case $\circledast = *$. Equating $\circledast$ with $*$ throughout the paper amounts to an additional assumption $L^{\hat{V}} = M^{\hat{V}}$, which is certainly justified for all utility functions in the HARA class.

1.6. *Market completion - first attempt*

We are now in a position to describe what, in the field of financial economics, is classically meant by a market completion. Denote by $\mathcal{K}$ the cone of tame terminal wealths with zero initial capital,

$$\mathcal{K} = \{H \cdot S_T : H \in \mathscr{T}\}, \tag{1.8}$$

and let $\mathcal{C}$ be the convex cone of terminal wealths that are super-replicable with zero initial capital,

$$\mathcal{C} = \left(\mathcal{K} - L^0_+\right) \cap L^{\hat{U}}.$$

Recall the notion of Arrow-Debreu state price measure $Q \in P_{\hat{V}}$ introduced in Section 1.4. With each $Q \in P_{\hat{V}}$, too, we associate a cone of claims that are super-replicable with zero initial capital in the statically complete market Q,

$$\mathcal{C}_Q = \{X \in L^{\hat{U}} \mid E^Q[X] \equiv \langle X, dQ/dP\rangle \le 0\}. \tag{1.9}$$

Definition 1.2 *We say that probability measure Q is a (static)* market completion / separating measure[5] *if $\mathcal{C} \subseteq \mathcal{C}_Q$, which is equivalent to $dQ/dP \in \mathcal{C}^{\circledast}$, where $\mathcal{C}^{\circledast}$ is the polar set to $\mathcal{C}$ in the economic duality $(L^{\hat{U}}, L^{\hat{V}})$,*

$$\mathcal{C}^{\circledast} = \{Y \in L^{\hat{V}} \mid E[XY] \equiv \langle X, Y\rangle \le 0 \text{ for all } X \in \mathcal{C} \subset L^{\hat{U}}\}. \tag{1.10}$$

At this point we digress a little to clarify the terminology. There is a subtle distinction between market *completion* and market *extension* which disappears

[5] In the context of arbitrage theory 'separation' refers to the separation of the set of attainable claims $\mathcal{K}$ from the set of arbitrage opportunities $L^{\hat{U}}_+$. In the context of utility theory one is separating $\mathcal{K}$ from sufficiently high upper level sets of expected utility.

when $\mathcal{K}$ is a linear subspace of $L^{\hat{U}}$. This observation applies to any set $\mathcal{K}$ of 'attainable claims', not just the specific set in (1.8). It is commonly said (Ross, 1978; Harrison and Kreps, 1979) that measure Q is a market extension if it correctly prices all attainable claims, that is if $E^Q[X] = 0$ for all $X \in \mathcal{K}$. One easily verifies that when $\mathcal{K}$ is linear, as in the two references above, every market completion is also a market extension and vice versa.

Earlier literature worked exclusively with linear $\mathcal{K}$. In the context of quadratic preferences, $L^{\hat{U}} \sim L^2$, Chamberlain and Rothschild (1983) and Magill and Quinzii (2000) identify the importance of continuous extension of the pricing functional to $\mathcal{K}^{\circledast\circledast}$. In no-arbitrage pricing, $L^{\hat{U}} \sim L^p, p \in [1, \infty]$, this theme is followed up by extensions to the whole of $L^{\hat{U}}$ in Kreps (1981), Clark (1993) and Schachermayer (1992, 1994).

Starting with Delbaen and Schachermayer (1994) and Kabanov (1997) the no-arbitrage literature considers $\mathcal{K}$ that is a cone, but no longer necessarily a linear subspace, or indeed a subset of $L^{\hat{U}}$. This leads to situations where one may have $X \in \mathcal{K}$ such that $-X \notin \mathcal{K}$, as in the case of shortselling constraints[6]. To such an X a separating measure may assign a strictly negative price, $E^Q[X] < 0$ and therefore one cannot say that Q is a 'pricing measure' or a market extension. However, we may say that Q is a market completion since claims in $\mathcal{K}$ are attainable in the completed market at a cost not exceeding 0.

The mathematical necessity of using the set of super-replicable wealths $\mathcal{C}$ instead of tame wealths $\mathcal{K}$ stems from the fact that one may perversely have no arbitrage over $\mathcal{K}^{\circledast\circledast}$ while there is arbitrage over $\mathcal{C}^{\circledast\circledast}$ (Schachermayer, 1994, Example 3.1). It is the statement of the Kreps-Yan theorem[7] that an arbitrage-free market completion exists ($\mathcal{C}^{\circledast}$ contains a strictly positive element) if and only if there is no arbitrage opportunity in $\mathcal{C}^{\circledast\circledast}$ ($\mathcal{C}^{\circledast\circledast} \cap L^{\hat{U}}_+ = \{0\}$).

It is the statement of the even deeper Fundamental Theorem of Asset Pricing (Delbaen and Schachermayer, 1998, Theorems 1.1 and 4.1) that in the case $L^{\hat{U}} = L^\infty$ there is no arbitrage in $\mathcal{C}^{\circledast\circledast}$ if and only if there is no arbitrage in the smaller norm-closure $\mathcal{C}^{\star\star}$ and in such case $\mathcal{C}^{\circledast\circledast} = \mathcal{C}^{\star\star} = \mathcal{C}$ (!) and an equivalent σ-martingale measure for S exists.

[6]With continuous trading $\mathcal{K}$ in (1.8) may not be a linear subspace even though no explicit short-selling constraints have been imposed. This is the situation encountered in Delbaen and Schachermayer (1994). Kabanov (1997) observes that one may add explicit constraints and relax assumption on S without affecting the conclusion that $\mathcal{C}^{\star\star} \cap L^{\hat{U}}_+ = \{0\}$ implies $\mathcal{C} = \mathcal{C}^{\star\star} = \mathcal{C}^{**}$ when $L^{\hat{U}} \sim L^\infty$.

[7]See Gao and Xanthos (2017, Proposition 3.5) for the Orlicz space version of the theorem and Schachermayer (2002) for historical notes.

1.7. *Complete market duality*

There is a dual formula (Biagini and Černý, 2011, Lemma 4.3) that describes maximal utility in the complete market Q in terms of its state price density,

$$u_Q(x) \equiv \sup_{X \in \mathcal{C}_Q} I_U(x+X) = \min_{y \geq 0} \left\{ I_V\left(y\frac{dQ}{dP}\right) + xy \right\}, \text{ when } u_Q(x) \in \mathbb{R}. \tag{1.11}$$

Here I_f denotes an integral functional $I_f(X) = E[f(X)]$. Formula (1.11) arises naturally if one considers maximization of $I_U(X)$ subject to a budget constraint $E^Q[X] = 0$ with λ being the Lagrange multiplier, see Pliska (1986).

We say that the statically complete market $Q \in P_{\hat{V}}$ is *bliss-free* if $u_Q(0) < U(\infty)$. In this case the dual formula reads

$$u_Q(0) = \min_{\lambda > 0} I_V(\lambda dQ/dP). \tag{1.12}$$

We denote the set of all bliss-free state price measures for utility U by

$$P_V = \{Q \ll P \mid u_Q(0) < U(\infty)\} = \{Q \ll P \mid \exists \lambda > 0; I_V(\lambda \frac{dQ}{dP}) < \infty\}, \tag{1.13}$$

where the set equality is hinted at in the dual formula (1.12) and follows rigorously from Biagini and Černý (2011, Proposition 4.6). In parallel, recall the set of all absolutely continuous state price measures $P_{\hat{V}}$ in equation (1.7) and note that the definition of the Orlicz space $L^{\hat{V}}$ allows it to be restated as

$$P_{\hat{V}} = \{Q \ll P \mid \exists \lambda > 0; I_{\hat{V}}(\lambda dQ/dP) < \infty\}. \tag{1.14}$$

On comparing (1.13) and (1.14) one observes that not all complete markets $Q \in P_{\hat{V}}$ are bliss-free since V may be unbounded near zero. It can be shown, however, that any $Q \in P_{\hat{V}}$ is bliss-free as long as $U(\infty) \equiv V(0)$ is finite, *ibid* proof i) $\Rightarrow$ ii). This underscores the economic significance of the space $L^{\hat{V}}$ as the space of complete market pricing functionals that are bliss-free for all utility functions sharing the same left tail and having an arbitrary but bounded right tail. This is true for *any* initial wealth level, as long as the initial wealth level is in the interior of $\operatorname{dom} U$ and below its bliss point $\overline{x}$, *ibid.*

1.8. *Martingale measures and supermartingale deflators*

So far we have supressed the dynamic nature of portfolio selection. To capture the temporal dimension of the problem the no-arbitrage literature operates with *σ-martingale measures*[8] for S, whose totality is denoted by

$$\mathcal{M} = \{Q \ll P \mid S \text{ is a } Q\text{-}\sigma\text{-martingale}\}. \tag{1.15}$$

[8]See Emery (1980) and Delbaen and Schachermayer (1998, Propositions 2.5 and 2.6)

Note that S itself may not be a tame wealth process, that is $H = 1$ may not be a tame strategy in general. For this reason we also introduce the set of *supermartingale measures*[9] for tame wealth processes,

$$\mathcal{S} = \{Q \ll P \mid H \cdot S \text{ is a } Q\text{-supermartingale for all } H \in \mathscr{T}\}. \tag{1.16}$$

On a filtered probability space every probability measure generates so-called *density process* ξ^Q whose values satisfy $\xi_t^Q = E[dQ/dP \mid \mathcal{F}_t]$ and therefore ξ^Q is a uniformly integrable P-martingale. In probabilistic terms ξ_t^Q is the Radon-Nikodym derivative of Q restricted to $\mathcal{F}_t$ with respect to P restricted to $\mathcal{F}_t$. For an equivalent measure $Q \sim P$ one can use ξ^Q to evaluate a conditional price[10] p_t^Q of an Arrow-Debreu security 1_A via the Bayes formula,

$$p_t^Q(1_A) = Q(A|\mathcal{F}_t) \equiv E^Q[1_A|\mathcal{F}_t] = E[\xi_T^Q 1_A|\mathcal{F}_t]/\xi_t^Q. \tag{1.17}$$

It follows from (1.17) that $\xi^Q p^Q(1_A)$ is a uniformly integrable P-martingale. In these circumstances we say that ξ^Q is a *martingale deflator* for the price process $p^Q(1_A)$. Similar notion can be applied to the wealth of tame trading strategies.

Definition 1.3 *Semimartingale ξ is a (strong super)martingale deflator if*

$$\xi(x + H \cdot S) \text{ is a } P\text{–(super)martingale for all } H \in \mathscr{T} \text{ and all } x \in \mathbb{R}.$$

We say that ξ is a weak supermartingale deflator if instead for all $x \in \mathbb{R}$ the supermartingale condition holds only for some $x > 0$.

Remark 1.4 *The set $\mathscr{Y}$ in Kramkov and Schachermayer (1999, 2003) corresponds to the set of all supermartingale deflators for L^∞-tame strategies. We will see in Section 2.3 that weak supermartingale deflators are not a robust concept and only strong supermartingale deflators survive the generalization from L^∞ to $L^{\hat{U}}$.*

It turns out that each $Q \in P_{\hat{V}} \cap \mathcal{M}$ is a supermartingale measure, $P_{\hat{V}} \cap \mathcal{M} \subseteq \mathcal{S}$ (Proposition 5.2). This in turn implies that every $Q \in P_{\hat{V}} \cap \mathcal{M}$ is a market completion / separating measure as per Definition 1.2, and the cone generated by σ-martingale densities in $P_{\hat{V}}$ (denoted with a slight abuse by $\mathcal{C}_\sigma^\circledast$),

$$\mathcal{C}_\sigma^\circledast = \{\lambda dQ/dP \mid \lambda \geq 0, Q \in \mathcal{M} \cap P_{\hat{V}}\}, \tag{1.18}$$

is a subset of the cone $\mathcal{C}^\circledast$ generated by separating densities. In the case **INF** it is additionally known that every equivalent separating measure is a supermartingale measure (Delbaen and Schachermayer, 1998, Proposition 4.7).

[9] Despite their superficial similarity the two notions 'σ-martingale measures' and 'supermartingale measures' refer to two very different sets of test processes. The former relates to S only; the latter refers to all tame wealth processes $\{H \cdot S \mid H \in \mathscr{T}\}$.

[10] Existence of conditional pricing rules is discussed, for example, in Hansen and Richard (1987).

The converse is not true – not every element of $\mathcal{C}^{\circledast}$ gives rise to a σ-martingale measure for S unless S is sufficiently well behaved. For our purposes it is enough to know that σ-martingale densities are $L^{\hat{V}}$ norm-dense in the set of separating densities. This is true in the case **INF** ($L^{\hat{V}} \sim L^1$) by Kabanov (1997, Theorem 2); at present the status of this conjecture in the case **F** is unknown. Therefore we make the following

Assumption 1.5 *$\mathcal{C}^{\circledast}_{\sigma}$ is $\|\cdot\|_{\hat{V}}$-dense in $\mathcal{C}^{\circledast}$, that is for $Q \ll P$ with density $dQ/dP \in \mathcal{C}^{\circledast}$ and for every $\varepsilon > 0$ there is a σ-martingale measure $\tilde{Q} \sim Q$ such that $\|d\tilde{Q}/dP - dQ/dP\|_{\hat{V}} \leq \varepsilon$.*

There is a mild sufficient condition to guarantee that *every* separating measure is a σ-martingale measure which in turn implies validity of Assumption 1.5. For this to hold the asset price process S must be sufficiently integrable with respect to the utility function, namely

$$S \in \mathscr{S}^{\hat{U}}_{\sigma}, \tag{1.19}$$

that is S belongs σ-locally[11] to the class of processes whose maximal process at the terminal date is in $L^{\hat{U}}$, see Sections 2.3, 2.4, Assumption 3.1, and Lemma 6.4 in Biagini and Černý (2011)[12]. In particular, any continuous S is locally bounded ($S \in \mathscr{S}^{\infty}_{\text{loc}}$) and therefore satisfies Assumption 1.5 for *any* utility U due to the embedding (1.4).

1.9. *Duality over state price densities*

It is clear from the construction of market completion that $u_Q(0)$ will overestimate utility of tame trading in the original market, $u_Q(0) \geq u(0)$, for any state price density $dQ/dP \in \mathcal{C}^{\circledast}$. We say that there is 'no duality gap' if one can complete the market in such a way that the increase in utility is arbitrarily small,

$$u(0) = \sup_{X \in \mathcal{C}} I_U(X) = \inf_{Y \in \mathcal{C}^{\circledast}} I_V(Y). \tag{1.20}$$

It will transpire later that (1.20) is crucial for the task we have set out to accomplish – which is to prove that admissible strategies contain an optimizer. To emphasize convex duality we can write the desirable property (1.20) as

$$\sup_{X \in \mathcal{C}} I_U(X) = \inf_{Y \in \mathcal{C}^{\circledast}} -I_{U^*}(Y), \tag{1.21}$$

[11] See Kallsen (2003) for definition and properties of σ-localization. Further relevant properties can be found in Biagini and Černý (2011, Section 2.4).

[12] The requirement $Q \in P_V$ therein can be relaxed to $Q \in P_{\hat{V}}$.

or even more symmetrically as

$$\sup_{X\in L^{\hat U}}\{I_U(X)-\delta_{\mathcal C}(X)\}=\inf_{Y\in L^{\hat V}}-\left\{I_U^{\circledast}(Y)-\delta_{\mathcal C}^{\circledast}(Y)\right\}, \tag{1.22}$$

where δ is convex set indicator function (zero on the set, $+\infty$ outside) and $I_U^{\circledast}$, $\delta_{\mathcal C}^{\circledast}$ denote conjugate functions in the duality $(L^{\hat U}, L^{\hat V})$. It is the consequence of the careful choice of dual spaces that $I_U^{\circledast}=I_{U^*}=-I_V$. Since $\mathcal C$ is a cone one easily obtains $\delta_{\mathcal C}^{\circledast}=\delta_{\mathcal C^{\circledast}}$ and this relationship shows equivalence between (1.21) and (1.22).

Results of the type (1.22) are known as the *Fenchel duality*. For example, I_U is norm-continuous at $0\in\mathcal C$ (Biagini and Frittelli, 2008, Proposition 16) which allows application of the Fenchel duality in the norm topology (Brezis, 2011, Theorem 1.12),

$$\sup_{X\in L^{\hat U}}\{I_U(X)-\delta_{\mathcal C}(X)\}=\min_{Y\in(L^{\hat U})^\star}-\{I_U^\star(Y)-\delta_{\mathcal C}^\star(Y)\}. \tag{1.23}$$

This is formally the same formula as (1.22) but with a larger dual space.

In our previous work we had to assume that the dual minimizer on the right-hand side of (1.23) was an element of $\mathcal C^{\circledast}$, i.e. a separating measure. Here we remove that assumption. Our first step is to rewrite the known result (1.23) for the norm duality $(L^{\hat U},(L^{\hat U})^\star)$, in terms of the economic duality $(L^{\hat U}, L^{\hat V})$,

$$\sup_{X\in L^{\hat U}}\{I_U(X)-\delta_{\mathcal C\cap\mathcal D}(X)\}=\min_{Y\in L^{\hat V}_+}-\left\{I_U^{\circledast}(Y)-\delta_{\mathcal C\cap\mathcal D}^{\circledast}(Y)\right\}, \tag{1.24}$$

where $\mathcal D=\operatorname{dom} I_U$ is the effective domain of expected utility (see Theorem 2.1).

Crucially for our story $\mathcal C\cap\mathcal D$ may be a strict subset of $\mathcal C$ and therefore the dual optimizer in (1.24) need not be an element of $\mathcal C^{\circledast}$ and thus not a separating measure, even though it necessarily must be a state price density in $L^{\hat V}_+$.

1.10. *Optimal trading strategy*

Armed with the previous observation, we adopt a radically new approach that bypasses the dual optimizer entirely. Instead, we construct the candidate optimal trading strategy from a supermartingale compactness result of Delbaen and Schachermayer (1998, Theorem D), using one arbitrary σ-martingale measure whose existence we assume. Having proved in Section 3 that the utility of wealth of the maximizing sequence can be chosen to have an integrable lower bound (Proposition 3.5) the difficulty is then showing that the expected utility of the candidate terminal wealth does not *exceed* $u(0)$, the expected utility attainable by tame trading.

By carefully rethinking the arguments of Biagini and Černý (2011, Proposition 3.8) we observe that the candidate optimal wealth process is a supermartingale under every σ–martingale measure. Consequently, the utility of the candidate wealth is majorized by the utility of every σ–martingale measure and the new construction goes through as long as *there is no duality gap over separating measures*, that is (1.20) holds, and σ–martingale measures are suitably dense among separating measures. The desired duality is proved in Section 4.

The advantage of the proposed construction is twofold — it allows us to deal with the case when the dual optimizer is not a separating measure and it also covers the case where the optimal wealth is in the algebraic interior of the effective domain but the dual maximizer, which now must be a separating measure, is not equivalent to P. In the latter case the utility function is not strictly monotone. To give an example, truncated quadratic utility plays an important role in the computation of monotone mean-variance optimal portfolios, see Maccheroni et al. (2009) and Černý et al. (2012).

The remaining sections implement the research programme outlined above.

2. FENCHEL DUALITY OVER STATE PRICE DENSITIES

To allow for random endowments, the set $\mathcal{C}$ is replaced by the set $B+\mathcal{C}$. As 0 may not be an element of $B+\mathcal{C}$ the arguments leading to (1.23) may fail. However, I_U is norm-continuous not only at zero but everywhere on the algebraic interior of $\mathcal{D}=\operatorname{dom} I_U$ (Rockafellar, 1974, Corollary 8B). In the present setting the algebraic interior is given explicitly as

$$\operatorname{core}\mathcal{D}=\{X\in L^{\hat{U}}:\exists\lambda>1; I_U(\lambda X)>-\infty\}. \tag{2.1}$$

Financially these are the positions that allow for proportional increase while maintaining finite utility level.

THEOREM 2.1 *Assume $B\in L^{\hat{U}}$, $(B+\mathcal{C})\cap\operatorname{core}\mathcal{D}\neq\emptyset$, and let $\mathcal{A}=B+\mathcal{C}$. Then one has*

$$\begin{aligned}\sup_{X\in\mathcal{A}} I_U(X) &= \sup_{X\in\operatorname{cl}^\star\mathcal{A}} I_U(X) = \sup_{X\in\mathcal{A}\cap\mathcal{D}} I_U(X) = \sup_{X\in\operatorname{cl}^\circledast(\mathcal{A}\cap\mathcal{D})} I_U(X)\\ &= \min_{Y\in L_+^{\hat{V}}}\left\{I_V(Y)+\delta^\circledast_{\mathcal{A}\cap\mathcal{D}}(Y)\right\},\end{aligned}\tag{2.2}$$

where the so-called support function $\delta^\circledast_{\mathcal{G}}(Y)$ has the explicit form

$$\delta^\circledast_{\mathcal{G}}(Y)=\sup_{X\in\mathcal{G}} E[XY].$$

PROOF: *Step 1)* The Fenchel inequality $I_U(X)\leq I_V(Y)+E[XY]$ for $X\in L^{\hat{U}}, Y\in L_+^{\hat{V}}$ yields

$$u(B)=\sup_{X\in\mathcal{A}\cap\mathcal{D}} I_U(X)\leq I_V(Y)+\sup_{X\in\mathcal{A}\cap\mathcal{D}} E[XY] \text{ for all } Y\in L_+^{\hat{V}}. \tag{2.3}$$

When $u(B) = \infty$ we necessarily have $U(\infty) \equiv V(0) = \infty$ and the duality (2.2) therefore holds trivially with $Y = 0$.

Step 2) Consider the remaining case $u(B) < \infty$. The Fenchel duality in the norm topology (Brezis, 2011, Theorem 1.12) gives

$$
\begin{aligned}
u(B) &= \sup_{X\in\mathcal{A}} I_U(X) = \min_{\mu\in(L^{\hat{U}})^\star} -\{I_U^\star(\mu) - \delta_{\mathcal{A}}^\star(\mu)\} \\
&= \min_{\mu\in(L^{\hat{U}})^\star} \{-I_U^\star(\mu) + \mu(B) + \sup_{X\in\mathcal{C}} \mu(X)\},
\end{aligned}
\tag{2.4}
$$

where $\mu(X) = \int X(\omega)\mu(d\omega)$. We now invoke finiteness of $u(B)$ and observe, because $\mathcal{C}$ is a cone, that the right-hand side is finite only if $\mu \in \mathcal{C}^\star$ which yields

$$
u(B) = \min_{\mu\in\mathcal{C}^\star} \{-I_U^\star(\mu) + \mu(B)\}. \tag{2.5}
$$

One can repeat the same argument starting with $\mathrm{cl}^\star\mathcal{A} = B + \mathrm{cl}^\star\mathcal{C}$ in place of $\mathcal{A}$ to find the right-hand side in (2.5) remains unchaged. This proves

$$
\sup_{X\in\mathcal{A}} I_U(X) = \sup_{X\in\mathrm{cl}^\star\mathcal{A}} I_U(X).
$$

Step 3) By Kozek (1979, Theorem 2.6) the conjugate $I_U^\star$ on the norm-dual of $L^{\hat{U}}$ is given explicitly by

$$
-I_U^\star(\mu) = I_{-U^*}(d\mu_r/dP) + \delta_{\mathcal{D}}^\star(-\mu_s),
$$

where $\delta_{\mathcal{D}}^\star(\mu) = \sup_{X\in\mathcal{D}} \mu(X)$ is the convex conjugate of the convex indicator function $\delta_{\mathcal{D}}$ (Rockafellar, 1974, equation (3.13)); $\mu = \mu_r + \mu_s$ is a unique decomposition of μ into a regular and singular part (Zaanen, 1983, Theorem 133.6); and $Y = d\mu_r/dP \in L^{\hat{V}}$. Therefore (2.5) can be written as

$$
u(B) = \sup_{X\in\mathcal{A}} I_U(X) = \min_{\mu\in\mathcal{C}^\star} \{I_V(Y) + \mu(B) + \delta_{\mathcal{D}}^\star(-\mu_s)\}, \tag{2.6}
$$

see also Biagini et al. (2011, Theorem 3.8).

Step 3) Denote the minimizer on the right-hand side of (2.6) by $\hat{\mu}$, with $\hat{Y} = \frac{d\hat{\mu}_r}{dP}$,

$$
\begin{aligned}
u(B) &= I_V(\hat{Y}) + \hat{\mu}(B) + \sup_{X\in\mathcal{D}} -\hat{\mu}_s(X) \\
&= I_V(\hat{Y}) + \hat{\mu}(B) + \sup_{X\in\mathcal{D}-B} -\hat{\mu}_s(X+B) \\
&= I_V(\hat{Y}) + E[\hat{Y}B] + \sup_{X\in\mathcal{D}-B} -\hat{\mu}_s(X).
\end{aligned}
\tag{2.7}
$$

Rephrase (2.3) as

$$
u(B) = \sup_{X\in\mathcal{A}\cap\mathcal{D}-B} I_U(X+B) \leq I_V(\hat{Y}) + \sup_{X\in\mathcal{A}\cap\mathcal{D}-B} E[(X+B)\hat{Y}], \tag{2.8}
$$

and combine this with (2.7) to obtain

$$\sup_{X\in\mathcal{D}-B} -\hat{\mu}_s(X) \le \sup_{X\in\mathcal{A}\cap\mathcal{D}-B} E[X\hat{Y}]. \tag{2.9}$$

Step 4) Recall the notation $\mathcal{A} = B + \mathcal{C}$. Recall $\hat{\mu} \in \mathcal{C}^\star$ and, because of the polar relationship $\hat{\mu}(X) \le 0$ for all $X \in \mathcal{C}$, we have $-\hat{\mu}_s(X) \ge E[X\hat{Y}]$ for all $X \in \mathcal{C} \supset \mathcal{A}\cap\mathcal{D} - B$ which yields

$$\sup_{X\in\mathcal{A}\cap\mathcal{D}-B} -\hat{\mu}_s(X) \ge \sup_{X\in\mathcal{A}\cap\mathcal{D}-B} E[X\hat{Y}]. \tag{2.10}$$

From (2.9-2.10) we obtain the following chain of inequalities

$$\begin{aligned}\sup_{X\in\mathcal{D}-B} -\hat{\mu}_s(X) &\ge \sup_{X\in\mathcal{A}\cap\mathcal{D}-B} -\hat{\mu}_s(X)\\ &\ge \sup_{X\in\mathcal{A}\cap\mathcal{D}-B} E[X\hat{Y}] \ge \sup_{X\in\mathcal{D}-B} -\hat{\mu}_s(X),\end{aligned} \tag{2.11}$$

which are therefore equalities. On combining (2.7) and (2.11), together with an explicit expression for the support function (Rockafellar, 1974, equation (3.13)) one obtains equality in (2.8),

$$u(B) = I_V(\hat{Y}) + E[\hat{Y}B] + \sup_{X\in\mathcal{A}\cap\mathcal{D}-B} E[X\hat{Y}] = I_V(\hat{Y}) + \delta^{\circledast}_{\mathcal{A}\cap\mathcal{D}}(\hat{Y}). \tag{2.12}$$

Step 5) By continuity of the bilinear form $\langle X, \hat{Y}\rangle \equiv E[X\hat{Y}]$ in the $\circledast$ duality one has

$$\sup_{X\in\mathcal{A}\cap\mathcal{D}} E[X\hat{Y}] = \sup_{X\in\mathrm{cl}^{\circledast}(\mathcal{A}\cap\mathcal{D})} E[X\hat{Y}],$$

which when combined with Fenchel inequality and (2.12) yields

$$u(B) \le \sup_{X\in\mathrm{cl}^{\circledast}(\mathcal{A}\cap\mathcal{D})} I_U(X) \le I_V(\hat{Y}) + \sup_{X\in\mathrm{cl}^{\circledast}(\mathcal{A}\cap\mathcal{D})} E[X\hat{Y}] = u(B).$$

This completes the proof in the remaining case $u(B) < \infty$. *Q.E.D.*

Observe that Theorem 2.1 does not claim $\sup_{X\in\mathcal{A}} I_U(X) = \sup_{X\in\mathrm{cl}^{\circledast}\mathcal{A}} I_U(X)$. Appendix C.2 gives an example with $B = 0$ where one obtains strict inequality $\sup_{X\in\mathcal{C}} I_U(X) < \sup_{X\in\mathcal{C}^{\circledast\circledast}} I_U(X)$. Nonetheless, Theorem 2.1 continues to hold for $\mathcal{A} = \mathcal{C}$ as well as for $\mathcal{A} = \mathcal{C}^{\circledast\circledast}$ except each case must by necessity have a different dual optimizer.

We remark that (2.2) can be equivalently rephrased as

$$\sup_{X\in L^{\hat{U}}} \{I_U(X) - \delta_{\mathcal{A}\cap\mathcal{D}}(X)\} = \min_{Y\in L^{\hat{V}}} -\left\{I_U^{\circledast}(Y) - \delta^{\circledast}_{\mathcal{A}\cap\mathcal{D}}(Y)\right\}, \tag{2.13}$$

which signifies that the left-hand side and the right-hand side form a relationship known as strong Fenchel duality. The new result (2.13) is mathematically significant because standard regularity conditions for Fenchel duality require $L^{\hat{U}}$ to be normed or at least metric while in the strongest available topology for the pair $(L^{\hat{U}}, L^{\hat{V}})$ the space $L^{\hat{U}}$ generally fails to be barrelled (*tonnelé* in Rockafellar, 1966) and therefore cannot be compatible with metric or norm topology.

Boţ (2010) Theorems 2.2, 15.2, and Remark 7.8 summarize regularity conditions under which (2.13) is known to hold, but in the present case none of these conditions applies. The conditions in Theorem 2.2 are not applicable since $L^{\hat{U}}$ may not be a Riesz (metric) space in any topology compatible with duality $(L^{\hat{U}}, L^{\hat{V}})$; those in Remark 7.8 and Theorem 15.2 fail because $\mathcal{C}$ is not necessarily $\circledast$-closed.

In the case **F** with $B = 0$ the dual formula (1.24) was obtained independently by Gushchin et al. (2014). In comparison, our approach is more direct, covering both **F** and **INF** case in one go and producing a proof that is, even just in the **F** case, significantly shorter, while allowing for random endowment.

In the literature on utility maximization with random endowment the case **INF** is covered by Cvitanić et al. (2001) who assume $B \in L^\infty$ therefore $x + B \in \operatorname{core} \mathcal{D}$ for $\underline{x} < x$. Their dual formula, containing singular parts, corresponds to our equation (2.6) with B replaced by $x + B$ once we realize that in their setting $\mathcal{D}$ is the set of strictly positive random variables in L^∞ and therefore $\delta^\star_{\mathcal{D}}(-\mu_s) = 0$. In the same setting, Hugonnier and Kramkov (2004) remove the singular parts from the dual, using methods similar to those of Kramkov and Schachermayer (2003).

In the case **F**, Biagini et al. (2011, Definition 3.1) have a condition equivalent to $B \in \operatorname{core} \mathcal{D}$ which is stronger than our assumption $(B + \mathcal{C}) \cap \operatorname{core} \mathcal{D} \neq \emptyset$, and just like Cvitanić et al. (2001) their dual problem contains singular parts.

For an immediate consequence of duality (2.13) recall that the largest linear subspace of $L^{\hat{U}}$ contained in $\operatorname{dom} I_{\hat{U}}$ is known as the Orlicz heart $M^{\hat{U}}$. It is now evident from (2.13) and from the inclusion $\operatorname{dom} I_{\hat{U}} \subseteq \operatorname{dom} I_U \equiv \mathcal{D}$ that the dual optimizer will correspond to a separating measure if $L^{\hat{U}} = M^{\hat{U}}$ or if at least $B + \mathcal{C} \subseteq M^{\hat{U}}$ because then $(B + \mathcal{C}) \cap \mathcal{D} = B + \mathcal{C}$. In particular, when working with locally bounded processes one may opt for L^∞-tame strategies controlled from both sides (Biagini and Černý, 2011) whereby $\mathcal{C} \subseteq L^\infty$. Since in the case **F** one has $L^\infty \hookrightarrow M^{\hat{U}}$, Fenchel duality (2.13) then yields for any $B \in M^{\hat{U}}$ a utility-based Fundamental Theorem of Asset Pricing, previously obtained under an additional assumption stronger than $M^{\hat{V}} = L^{\hat{V}}$ in Owen and Žitković (2009, Theorem 1.2).

Further important consequences of the new formula (2.13) are described in the next two sections.

2.1. *Market completion - a new definition*

Denoting an optimizer on the right-hand side of the Fenchel duality (1.24) by $\hat{Y}$ and setting $d\hat{Q}/dP = \hat{Y}/E[\hat{Y}]$, in view of equation (1.11) we may interpret the right-hand side expression in (1.24) as maximal utility in a bliss-free complete market $\hat{Q}$ with initial endowment increased by the amount $\delta^{\circledast}_{\mathcal{C}\cap\mathcal{D}}(d\hat{Q}/dP)$. This market completion is somewhat unusual since we are not completing the entire market $\mathcal{C}$, merely the part where the expected utility is finite, $\mathcal{C}\cap\mathcal{D}$, and extra initial endowment is required.

Definition 2.2 *We say $Q \in P_{\hat{V}}$ is a completion of market $\mathcal{C}$ if $\mathcal{C}\cap\mathcal{D} \subseteq x+\mathcal{C}_Q$ for some $x \in [0,\infty)$. When x can be chosen equal to zero we say Q is a* full completion*, otherwise we say Q is an* effective completion*.*

It follows that Q is a completion if and only if $dQ/dP \in \operatorname{dom}\delta^{\circledast}_{\mathcal{C}\cap\mathcal{D}} \supseteq \operatorname{dom}\delta^{\circledast}_{\mathcal{C}} \equiv \mathcal{C}^{\circledast}$. The terminology *full completion* is justified by the equivalence

$$\mathcal{C}\cap\mathcal{D} \subseteq \mathcal{C}_Q \Leftrightarrow \mathcal{C} \subseteq \mathcal{C}_Q, \tag{2.14}$$

which follows from the observation that 0 is in the norm interior of $\mathcal{D}$ implying $\mathcal{C} = \operatorname{cone}(\mathcal{C}\cap\mathcal{D})$. Full completion Q is therefore precisely the classical completion discussed in Section 1.6, that is a separating measure.

We can now interpret the Fenchel duality formula (1.24) as a market completion theorem: market $\mathcal{C}$ is bliss-free if and only if $\mathcal{C}\cap\mathcal{D}$ can be embedded in a bliss-free complete market with the same expected utility.

2.2. *Boundary solutions, corner solutions, and separating measures*

For the purpose of this section we assume V is strictly convex on $\operatorname{dom} V$. Since V is closed (Proposition A.9) it follows by Rockafellar (1970, Theorem 26.3) this is equivalent to U being essentially smooth, that is differentiable on $(\underline{x}, +\infty)$ and satisfying $\lim_{x\searrow\underline{x}} U'(x) = \infty$, which explains the origin of technical conditions customarily imposed on U in the literature. Strict convexity of V means the dual optimizer in (1.24) is necessarily unique. We denote it by $\hat{Y}$ and let

$$d\hat{Q}/dP = \hat{Y}/E[\hat{Y}].$$

While the emergence of the set $\mathcal{C}\cap\mathcal{D}$ in formula (1.24) is unexpected, *post hoc* it has a natural economic interpretation. The fact that $\mathcal{C}\cap\mathcal{D}$ may be a strict subset of $\mathcal{C}$ implies that the primal optimum $\hat{X}$ (supposing it exists in $L^1(\hat{Q})$-closure of $\mathcal{C}\cap\mathcal{D}$ as discussed in Section 3) may be a 'boundary solution' in the sense that $\theta\hat{X} \notin \mathcal{D}$ for $\theta > 1$.

Consider the constrained optimization $\max_{\theta\leq 1} I_U(\theta\hat{X})$. When $E[\hat{X}U'(\hat{X})] > 0$ the constraint $\theta \leq 1$ is binding and the Lagrange multiplier associated with this constraint is exactly equal to $E[\hat{X}U'(\hat{X})]$. We will refer to this situation as a 'corner solution'. It is interesting to note that the constraint in question is not exogenous, rather the corner arises implicitly due to the boundedness of the effective domain $\mathcal{D}$ in some directions. Under the current hypotheses the following statements are equivalent:

1. $\hat{Q}$ is not a separating measure (i.e. $\hat{Q}$ is only an effective completion);
2. $\delta^{\circledast}_{\mathcal{C}\cap\mathcal{D}}(\hat{Y}) = E[\hat{X}\hat{Y}] = E[U'(\hat{X})\hat{X}] > 0$.

As can be expected, boundary solution is *not synonymous* with $\delta^{\circledast}_{\mathcal{C}\cap\mathcal{D}}(\hat{Y}) > 0$. In particular, when optimizing over a complete market one always has $\delta^{\circledast}_{\mathcal{C}\cap\mathcal{D}}(\hat{Y}) = 0$ (see equation 1.12), while the primal solution may lie on the edge of the effective domain $\mathcal{D}$. The converse statement that non-boundary primal optimizer corresponds to a full completion in the dual problem appears, with extra technical condition, in Biagini and Frittelli (2008, Proposition 31).

2.3. *Implications for supermartingale deflators*

In Kramkov and Schachermayer (1999, 2003) the dual optimizer $\hat{Y}$ is interpreted as a terminal value of a supermartingale deflator. Now suppose that the optimal terminal wealth $\hat{X}$ introduced in Section 2.2 has an optimal strategy $\hat{H}$ associated with it, $\hat{X} = \hat{H}\cdot S_T$. We already know that $\hat{Y}$ is an effective completion (that is, not a separating measure) if and only if $\hat{X}$ is a corner solution, in which case

$$\hat{H}\cdot S_0 = 0 < E[\hat{X}\hat{Y}] = E[\hat{Y}(\hat{H}\cdot S_T)].$$

This inequality means that there can be *no strong* supermartingale deflator for $\hat{H}\cdot S$ with terminal value $\hat{Y}$, if $\hat{Y}$ is an effective completion.

In Kramkov and Schachermayer (1999, 2003) (case **INF**) an effective completion $\hat{Y}$ can be turned into a *weak* supermartingale deflator by setting

$$\xi_t = \frac{E[\hat{Y}(x+\hat{H}\cdot S_T)|\mathcal{F}_t]}{x+\hat{H}\cdot S_t},$$

with $\underline{x} = 0$ and $x > 0$ in the notation of Section 1.1. However, when U is finite everywhere (case **F**) for $\hat{Y} \notin \mathcal{C}^{\circledast}$ there may be *no* supermartingale deflator with terminal value $\hat{Y}$ at all. An example of such situation is given in Appendix B.4. This shows that associaton of a supermartingale deflator with a dual optimizer not in $\mathcal{C}^{\circledast}$ is an ad-hoc construction.

One can robustly characterize effective completion $\hat{Q}$ as a *"submartingale"* measure in the sense that for the optimal trading strategy $\hat{H}$ one will have

$$E^{\hat{Q}}[\hat{H}\cdot S_T] = \delta^{\circledast}_{\mathcal{C}\cap\mathcal{D}}(d\hat{Q}/dP) = \sup_{X\in\mathcal{C}\cap\mathcal{D}} E^{\hat{Q}}[X] > 0 = \hat{H}\cdot S_0.$$

We conjecture that for an effective completion $\hat{Q}$ the submartingale property $E^{\hat{Q}}[\hat{H}\cdot S_u|\mathcal{F}_t] \geq \hat{H}\cdot S_{t\wedge u}$ holds for $u = T$ and arbitrary t although not necessarily for all u and t. The relationship $E^{\hat{Q}}[\hat{H}\cdot S_T] > \hat{H}\cdot S_0$ is universal across utility functions and robust to arbitrary translation of initial wealth.

3. OPTIMAL TERMINAL WEALTH

Our next step is to show that there is an optimizing sequence $\{X_n\}$ of terminal wealth distributions in $\mathcal{C}$ which converges pointwise P-a.s. to a limit $\hat{X}$ and such that $U(B+X_n)$ approximate $U(B+\hat{X})$ in $L^1(P)$. This means that $\hat{X}$ necesarily attains maximal utility $u(B)$. The desired convergence requires uniform integrability of the sequence $\{U(B+X_n)\}$ which in general fails to materialize, even in the 'nice' case $M^{\hat{U}} = L^{\hat{U}}$.

A complete study of minimal conditions for uniform integrability of the utility of maximizing sequence is beyond the scope of this paper. We remark that the key tools in that direction are results of Andô (1962) on compactness in the economic duality $(L^{\hat{U}}, L^{\hat{V}})$. In this section we proceed by introducing comparatively simple sufficient conditions encompassing all results available to date.

Denote $a = U'_+(0)$. Recall that $V = -U^*$ and due to $U(0) = 0$ function V is decreasing on $[0, a]$ and increasing on $[a, \infty)$. Recall $\hat{U}(x) = -U(-x)$ for $x \geq 0$ and $+\infty$ otherwise and $X \in L^{\hat{U}}$ if there is $\lambda > 0$ such that $I_{\hat{U}}(\lambda |X|) < \infty$. We have $\hat{V}(y) = \hat{U}^*(y) = V(y \vee a)$.

The first important ingredient is the requirement that the elements of $B + \mathcal{C}$ with high expected utility must have negative parts of bounded $L^{\hat{U}}$ norm. This requirement is satisfied trivially in the case **INF** ($L^{\hat{U}} \sim L^\infty$) and not just over $B + \mathcal{C}$ but over the entire space L^∞ since in that case $I_U(X) > -\infty \Rightarrow X \geq \underline{x} \iff \|X^-\|_\infty \leq -\underline{x}$. The following concept appears to be new.

Definition 3.1 *We say that expected utility I_U is* norm-coercive in losses on a set $\mathcal{G} \subseteq L^{\hat{U}}$ *if*

$$\lim_{\|X^-\|_{\hat{U}} \to \infty, X \in \mathcal{G}} I_U(X) = -\infty. \tag{3.1}$$

Equivalently, expected utility is norm-coercive in losses on $\mathcal{G}$ if and only if for every $k \in \mathbb{R}$ there is $l > 0$ such that $I_U(X) > k$ implies $\|X^-\|_{\hat{U}} \leq l$ for all $X \in \mathcal{G}$.

We continue with a lemma that establishes boundedness properties for cost-constrained subsets of upper level sets of expected utility and leads to sufficient conditions that imply norm coercivity in losses. For U bounded above, expected utility is trivially norm-coercive in losses over the entire space $L^{\hat{U}}$. This can be seen also in the lemma below by setting $\tilde{Y} = 0$, which is possible in the bounded case thanks to $U(\infty) = V(0) < \infty$.

Lemma 3.2 *Consider a set $\mathcal{G} \subseteq L^{\hat{U}}$ and suppose there is $\tilde{Y} \in L^{\hat{V}}$ such that $\{E[X\tilde{Y}]\}_{X \in \mathcal{G}}$ is bounded from above, and $\lambda\tilde{Y} \in \operatorname{dom} I_V$ for two distinct values of $\lambda > 0$. Consider further an arbitrary $\widetilde{\mathcal{G}} \subseteq \mathcal{G}$ such that $\{I_U(X)\}_{X \in \widetilde{\mathcal{G}}}$ is bounded from below. The following statements hold:*

i) $\{|X|\,\tilde{Y}\}_{X\in\widetilde{\mathcal{G}}}$ *is* $L^1(P)$*-bounded;*
ii) $\{|U(X)|\}_{X\in\widetilde{\mathcal{G}}}$ *is* $L^1(P)$*-bounded;*
iii) I_U *is norm-coercive in losses on* $\mathcal{G}$*.*

PROOF: *i)* Consider $\lambda_2 > \lambda_1 > 0$ such that $\lambda_i\tilde{Y} \in \operatorname{dom} I_V$, $i = 1, 2$. By Fenchel inequality

$$U\left(X^+\right) \le V(\lambda_1 Y) + \lambda_1 X^+ Y, \tag{3.2}$$

$$U\left(-X^-\right) \le V\left((\lambda_2 Y) \vee a\right) - \lambda_2 X^- Y. \tag{3.3}$$

On taking expectations

$$I_U(X) \le I_V(\lambda_1\tilde{Y}) + I_V(\lambda_2\tilde{Y}) + \lambda_1 E[X\tilde{Y}] - (\lambda_2 - \lambda_1)\, E[X^-\tilde{Y}]. \tag{3.4}$$

Note that $\tilde{Y} \in L^{\hat{V}}$ and $|X\tilde{Y}| \in L^1(P)$ for any $X \in L^{\hat{U}}$ by Orlicz space Hölder inequality (Rao and Ren, 1991, eq. 3.3.4). Since $\{I_U(X)\}_{X\in\widetilde{\mathcal{G}}}$ is bounded below, the assumed upper bound on $\{E[X\tilde{Y}]\}_{X\in\mathcal{G}}$ and (3.4) imply $\{X^-\tilde{Y}\}_{X\in\widetilde{\mathcal{G}}}$ is $L^1(P)$-bounded, therefore $\{X^+\tilde{Y}\}_{X\in\widetilde{\mathcal{G}}}$ is $L^1(P)$-bounded and claim i) follows.

ii) Having proved i) $\{U\left(X^+\right)\}_{X\in\widetilde{\mathcal{G}}}$ is $L^1(P)$-bounded by (3.2), and hence by the assumed lower bound on $\{I_U(X)\}_{X\in\widetilde{\mathcal{G}}}$ the set $\{U(-X^-)\}_{X\in\widetilde{\mathcal{G}}}$ is also $L^1(P)$-bounded .

iii) Item ii) implies norm-boundedness of $\{X^-\}_{X\in\widetilde{\mathcal{G}}}$ by equivalence of gauge norms (Caruso, 2001, Proposition 2). Since $\widetilde{\mathcal{G}}$ was arbitrary, item iii) follows by contradiction. *Q.E.D.*

COROLLARY 3.3 *Suppose* $\{U(X)^-\}_{X\in\widetilde{\mathcal{G}}}$ *is* $L^1(P)$*-bounded. For any* $Y \in L^{\hat{V}}$ *such that* $\{E[XY]\}_{X\in\widetilde{\mathcal{G}}}$ *is bounded above* $\{|X|\,Y\}_{X\in\widetilde{\mathcal{G}}}$ *is also* $L^1(P)$*-bounded.*

PROOF: $L^1(P)$-boundedness of $\{X^-Y\}_{X\in\widetilde{\mathcal{G}}}$ follows from the Fenchel inequality (3.3) where we take λ_2 such that $\lambda_2 Y \in \operatorname{dom} I_{\hat{V}}$. $L^1(P)$-boundedness of $\{X^+Y\}_{X\in\widetilde{\mathcal{G}}}$ now follows from the assumed upper bound on $\{E[XY]\}_{X\in\widetilde{\mathcal{G}}}$. *Q.E.D.*

As the final ingredient we must ensure uniform integrability of $\{U(A_n^+)\}$ for a maximizing sequence $\{A_n\} = B + \{X_n\}$. Define indirect utility $\overline{u} : \mathbb{R}_+ \to \mathbb{R}$ by maximizing $u(B)$ over all random endowments B whose $L^{\hat{U}}$ norm is bounded above by x,

$$\overline{u}(x) = \sup_{\|B\|_{\hat{U}} \le x} u(B) = \sup\{I_U(X + Z) \mid X \in \mathcal{C}, \|Z\|_{\hat{U}} \le x\}. \tag{3.5}$$

To obtain uniform integrability of positive parts of utility we will require

$$\lim_{x\to\infty} \overline{u}(x)/x = 0. \tag{3.6}$$

The construction involving $\overline{u}$ also appears to be new.

Note that for $L^{\hat{U}} \sim L^\infty$ one has $\overline{u}(x) = u(x)$ for $x \geq 0$ and therefore condition (3.6) exactly coincides with the minimal condition in Kramkov and Schachermayer (2003, Note 1). The significance of the condition (3.6) is captured by the following statement.

LEMMA 3.4 *Condition* $\lim_{x\to\infty} \overline{u}(x)/x = 0$ *implies that for*

$$\mathcal{Z}(k_1, k_2) = \{X + Z \mid X \in \mathcal{C}, Z \in L^{\hat{U}}, \|X^-\|_{\hat{U}} \leq k_1, \|Z\|_{\hat{U}} \leq k_2\},$$

the set $\{U(\mathcal{Z}(k_1, k_2)^+)\}$ *is uniformly integrable for every* $k_1, k_2 > 0$.

PROOF: It follows from the Eberlein-Šmulian and Dunford-Pettis theorems (Bogachev, 2007, Theorems 4.7.10 and 4.7.18) that uniform integrability (UI) of a set is equivalent to UI of sequences in the set (see also Diestel, 1991, pages 45 and 50). Now we can proceed as in Kramkov and Schachermayer (2003) but with the new notion $\overline{u}$ in place of u which allows us to handle the general case where the unit ball of $L^{\hat{U}}$ does not have an upper bound. We also replace polarity arguments of the original proof (unavailable here) with simpler set inclusions.

Arguing by contradiction assume that for $X_i \in \mathcal{C}, \|X_i^-\|_{\hat{U}} \leq k_1$ and $\|Z_i\|_{\hat{U}} \leq k_2$ the sequence $\{U((X_i + Z_i)^+)\}$ is not uniformly integrable. Then there are disjoint sets $D_i \in \mathcal{F}_T$ and a constant $\alpha > 0$ such that

$$E[U((X_i^+ + Z_i^+)1_{D_i})] \geq E[U((X_i + Z_i)^+ 1_{D_i})] \geq \alpha.$$

Note that for $X \in \mathcal{C}, \|X^-\|_{\hat{U}} \leq k_1$ one has

$$X_i^+ = X_i + X_i^- \in \mathcal{Z}(k_1, k_1),$$

which implies

$$\sum_{i=1}^n (X_i^+ + Z_i^+)1_{D_i} \leq \sum_{i=1}^n (X_i^+ + Z_i^+) \in \mathcal{Z}(nk_1, n(k_1 + k_2))$$

and consequently

$$n\alpha \leq \sum_{i=1}^n I_U(X_i^+ + Z_i^+)1_{D_i} = I_U(\sum_{i=1}^n (X_i^+ + Z_i^+)1_{D_i}) \leq \overline{u}(n(k_1 + k_2)).$$

From here $\alpha/(k_1 + k_2) \leq \overline{u}(n(k_1 + k_2))/\left(n(k_1 + k_2)\right)$, and for $n \to \infty$ the right-hand side converges to 0 by hypothesis which gives the desired contradiction. *Q.E.D.*

We are now in a position to prove existence of optimal terminal wealth with the desired approximation property.

Proposition 3.5 *Assume $B \in L^{\hat U}$ and $(B+\mathcal{C}) \cap \operatorname{core}\mathcal{D} \neq \emptyset$. Assume further there is no arbitrage over $\mathcal{C}^{\circledast\circledast}$; $\lim_{x\to\infty} \overline{u}(x)/x = 0$; and the dual minimizer $\hat Y$ in (2.2) satisfies $\lambda \hat Y \in \operatorname{dom} I_V$ for some $\lambda > 1$ (this is automatic when $M^{\hat V} = L^{\hat V}$). Then there is a sequence $\{X_n\} \in \mathcal{C}$ with $I_U(B+X_n) \nearrow u(B) < \infty$ and a random variable $\hat X$ such that $X_n \overset{P\text{-a.s.}}{\to} \hat X$,*

$$U(B+\hat X) - (B+\hat X)\hat Y = V(\hat Y), \tag{3.7}$$

and

$$U(B+X_n) \overset{L^1(P)}{\to} U(B+\hat X).$$

Moreover, the sequence $\{X_n\}$ can be chosen such that $U(B+X_n) \geq R$ with $0 \geq R \in L^1(P)$.

Proof: *Step 1)* We will first exhibit a random variable $\tilde Y > 0$ P-a.s. such that $\lambda \tilde Y \in \operatorname{dom} I_V$ for two distinct values of λ and $\sup_{X\in\mathcal{A}\cap\mathcal{D}}\{E[X\tilde Y]\} < \infty$. We distinguish two mutually exclusive cases.

a) When U is bounded from above then $V(y)$ is bounded from above for y near zero. By the Kreps-Yan theorem (Gao and Xanthos, 2017, Proposition 3.5) no arbitrage over $\mathcal{C}^{\circledast\circledast}$ implies existence of $\tilde Y \in \mathcal{C}^{\circledast}, \tilde Y > 0$ P-a.s. Since $\mathcal{C}^{\circledast}$ is a cone, without loss of generality we may assume $\tilde Y \in \operatorname{dom} I_{\hat V}$. Recalling that $\hat V(y) = V(y \vee a)$ while $V(y \wedge a)$ is bounded we conclude $\lambda \tilde Y \in \operatorname{dom} I_V$ for all $0 < \lambda \leq 1$. By construction $\sup_{X\in\mathcal{C}}\{E[X\tilde Y]\} \leq 0$ which implies $\sup_{X\in\mathcal{A}}\{E[X\tilde Y]\} < \infty$.

b) By condition (3.6) $u(B) < \infty$. When U is unbounded from above then $V(0) = \infty$ and therefore necessarily the dual optimizer in (2.2) satisfies $\hat Y > 0$ P-a.s. as well as $\sup_{X\in\mathcal{A}\cap\mathcal{D}}\{E[X\hat Y]\} < \infty$ and $\hat Y \in \operatorname{dom} I_V$. In this case we let $\tilde Y = \hat Y$.

Step 2) By definition of supremum there is a sequence $\{A_n\}$ in $\mathcal{A}$ with $\{I_U(A_n)\}$ bounded below and $I_U(A_n) \nearrow u(B)$. The random variable $\tilde Y$ from step 1) and the sets $\mathcal{G} = \mathcal{A} \cap \mathcal{D}$ and $\widetilde{\mathcal{G}} = \operatorname{conv}\{A_n\}$ therefore satisfy the hypotheses of Lemma 3.2. We thus conclude that $\{U(A_n^+)\}$, $\{U(-A_n^-)\}$, $\{A_n^+\tilde Y\}$, $\{A_n^-\tilde Y\}$ are $L^1(P)$-bounded.

Step 3) Construct $\hat A \tilde Y$ as the pointwise limit of tail convex combinations of $A_n \tilde Y$. By abuse of notation denote these convex combinations again $A_n \tilde Y$. By construction $A_n \to \hat A$ P-a.s. Note that the new sequence $\{A_n\}$ satisfies the same hypotheses as the old one: all elements are in $\mathcal{A}$ and $I_U(A_n)$ is bounded from below and converges to $u(B)$. Let $X_n = A_n - B$ and $\hat X = \hat A - B$.

Step 4) From here onwards we pass to a subsequence such that $I_U(A_n^+)$, $I_U(-A_n^-)$, $E[A_n^+\hat Y]$, and $E[A_n^-\hat Y]$ all have a finite limit.

Step 5) By assumption there is $\lambda_0 > 1$ such that for all $\lambda \in [1, \lambda_0]$ we have $\lambda \hat Y \in \operatorname{dom} I_V$. Fatou lemma yields

$$\begin{aligned}\lim_{n\to\infty} I_U(A_n) - \lambda \lim_{n\to\infty} E[A_n\hat Y] &= \lim_{n\to\infty}\{I_U(A_n) - \lambda E[A_n\hat Y]\}\\ &\leq I_U(\hat A) - \lambda E[\hat A\hat Y] \leq I_V(\lambda\hat Y),\end{aligned}$$

which means

$$u(B) - \lambda \lim_{n\to\infty} E[A_n\hat{Y}] \leq I_U(\hat{A}) - \lambda E[\hat{A}\hat{Y}] \leq I_V(\lambda\hat{Y}). \tag{3.8}$$

Step 6) By Theorem 2.1

$$u(B) = I_V(\hat{Y}) + \sup_{A\in\mathcal{A}\cap\mathcal{D}} E[A\hat{Y}]. \tag{3.9}$$

Substitute this into (3.8) with $\lambda = 1$ to obtain

$$I_V(\hat{Y}) + \sup_{A\in\mathcal{A}\cap\mathcal{D}} E[A\hat{Y}] - \lim_{n\to\infty} E[A_n\hat{Y}] \leq I_V(\hat{Y}).$$

This implies $\sup_{A\in\mathcal{A}\cap\mathcal{D}} E[A\hat{Y}] - \lim_n E[A_n\hat{Y}] \leq 0$ but since $A_n \in \mathcal{A}\cap\mathcal{D}$ this is only possible if

$$\lim_{n\to\infty} E[A_n\hat{Y}] = \sup_{A\in\mathcal{A}\cap\mathcal{D}} E[A\hat{Y}]. \tag{3.10}$$

Therefore for $\lambda = 1$ the inequalities in (3.8) are actually equalities

$$u(B) - \lim E[A_n\hat{Y}] = I_U(\hat{A}) - E[\hat{A}\hat{Y}] = I_V(\hat{Y}). \tag{3.11}$$

Equality (3.11) implies that the Fenchel inequality $U(\hat{A}) - \hat{A}\hat{Y} \leq V(\hat{Y})$ is in fact a P-a.s. equality which proves (3.7).

Step 7) Subtract (3.11) from (3.8) to obtain

$$(1-\lambda) \lim_{n\to\infty} E[A_n\hat{Y}] \leq (1-\lambda)E[\hat{A}\hat{Y}] \leq I_V(\lambda\hat{Y}) - I_V(\hat{Y}).$$

Taking $\lambda > 1$ we have

$$\lim_{n\to\infty} E[A_n\hat{Y}] \geq E[\hat{A}\hat{Y}]. \tag{3.12}$$

On combining (3.12) with (3.9) and (3.10) we finally conclude

$$u(B) \geq I_U(\hat{A}) = I_V(\hat{Y}) + E[\hat{A}\hat{Y}]. \tag{3.13}$$

Step 8) Observe that a sequence $Z_n \overset{P\text{-a.s.}}{\to} Z$ is uniformly integrable iff

$$E[|Z_n|] \to E[|Z|] \text{ iff } E[|Z_n - Z|] \to 0,$$

see Scheffé lemma (Bogachev, 2007, Theorem 2.8.9) and Lebesgue-Vitali convergence theorem (Bogachev, 2007, Theorem 4.5.4). Fatou lemma yields

$$\lim_n I_U(-A_n^-) \leq I_U(-\hat{A}^-). \tag{3.14}$$

In order to obtain $L^1(P)$-convergence of $\{I_U(A_n)\}$ in view of (3.13) and (3.14) it suffices to prove $I_U(A_n^+) \to I_U(\hat{A}^+)$ or equivalently that the sequence $\{U(A_n^+)\}$ is uniformly integrable.

Step 9) By step 2) $\sup_n \|(B + X_n)^-\|_{\hat{U}} < \infty$. Let $k_1 = \|B\|_{\hat{U}}$. We have $X_i^- \le (B + X_i)^- + B^+$ and therefore $\|X_i^-\|_{\hat{U}} \le \sup_n \|(B + X_n)^-\|_{\hat{U}} + \|B\|_{\hat{U}} =: k_2 < \infty$. We conclude that $A_n = B + X_n \in \mathcal{Z}(k_1, k_2)$ and the sequence $\{U(A_n^+)\}$ is uniformly integrable by Lemma 3.4. By step 8) $U(A_n) \to U(\hat{A})$ in $L^1(P)$.

Step 10) This means the non-positive sequence $\{U(-A_n^-)\}$ is Cauchy in $L^1(P)$ and we can find a subsequence, here denoted by $\tilde{A}_n$, and a random variable $R \in L^1(P)$ such that $0 \ge U(-\tilde{A}_n^-) \ge R$. *Q.E.D.*

REMARK 3.6 *Corollary 3.10 in Delbaen and Owari (2016) shows, under the assumption that $\hat{V}$ satisfies the Δ_2–condition, that every $L^{\hat{U}}$ norm-bounded sequence admits a pointwise-convergent sequence of forward convex combinations whose $\hat{U}$ is dominated by an integrable random variable. Here the dominated convergence of forward convex combinations is shown to exist for the negative parts of the sequence of terminal wealths, $\{A_n^-\}$, without necessarily assuming the Δ_2–condition on $\hat{V}$. Our starting sequence, however, is maximizing and therefore not arbitrary.*

4. DUALITY OVER SEPARATING MEASURES

We have argued in the introductory Section 1.10 that for the construction of the optimal trading strategy it is important to know there exists a full market completion whose utility is arbitrarily close to $u(B)$,

$$u(B) = \sup_{X \in \mathcal{C}} I_U(B + X) = \inf_{Y \in \mathcal{C}^{\circledast}} \{I_V(Y) + E[YB]\}. \tag{4.1}$$

The case $M^{\hat{U}} = L^{\hat{U}}$ is immediately very nice in this respect: one automatically has $\star = \circledast$ so the norm duality (2.4) yields the desired result (4.1). This covers the case **L** where the utility function is asymptotically linear near $-\infty$ and $L^{\hat{U}} \sim L^1$.

The remaining case is **SL** with $M^{\hat{U}} \subsetneq L^{\hat{U}}$. The norm duality (2.4) implies that utility cannot increase by going from $B + \mathcal{C}$ to its norm-closure $B + \mathcal{C}^{\star\star}$ while the economic duality (2.13) implies that utility does not increase by going from $(B + \mathcal{C}) \cap \mathcal{D}$ to $\mathrm{cl}^{\circledast}((B + \mathcal{C}) \cap \mathcal{D})$. However, these facts do not in themselves prevent a utility gap between $B + \mathcal{C}$ and $B + \mathcal{C}^{\circledast\circledast}$. Appendix C.2 provides a counterexample illustrating that with $M^{\hat{U}} \subsetneq L^{\hat{U}}$ one can generically expect to find situations where

$$\sup_{X \in \mathcal{C}} I_U(B + X) < \sup_{X \in \mathcal{C}^{\circledast\circledast}} I_U(B + X). \tag{4.2}$$

Obviously, if the gap (4.2) emerges then by Fenchel inequality (4.1) cannot hold.

This observation highlights the importance of the classical 'small market' fundamental theorem of asset pricing (FTAP) which asserts that in the absence of arbitrage over $\mathcal{C}^{\star\star}$ in the case **INF** one necessarily obtains $\mathcal{C} = \mathcal{C}^{\circledast\circledast}$. Our counterexample also shows that in a 'large financial market' the link between absence of arbitrage over $\mathcal{C}^{\circledast\circledast}$ and the equality $\mathcal{C} = \mathcal{C}^{\circledast\circledast}$ is broken, and the case **INF** is no exception.

Having made the necessary preparations, it turns out that the following weaker alternative of (4.1) is already sufficient for our purposes.

PROPOSITION 4.1 *Assume $B \in L^{\hat{U}}$, $(B + \mathcal{C}) \cap \mathcal{D} \neq \emptyset$, and*

$$u(B) = \sup_{X \in \mathcal{C}} I_U(B + X) = \sup_{X \in \mathcal{C}^{**}} I_U(B + X). \tag{4.3}$$

Then in the case ***SL*** *($L^\infty(P) \hookrightarrow L^{\hat{U}}(P) \hookrightarrow\!\!\!\!\rightarrow L^1(P)$), there is a sequence $\{Z_n\} \in M^{\hat{V}}$ with $\|Z_n\|_{\hat{V}} \to 0$, and a sequence of $\{Y_n\} \in \mathcal{C}^*$ such that*

$$\lim_{n\to\infty} I_V\left(Y_n + Z_n\right) + E[(Y_n + Z_n)B] = u(B).$$

PROOF: By Proposition A.15 I_U is $*$-u.s.c. Because I_U is finite-valued at 0 it is proper by Proposition A.7, and therefore $*$-closed by Definition A.8. Likewise $\delta_{\mathcal{C}^{**}}$ is a $*$-closed function since $\mathcal{C}^{**}$ is a closed (convex) set in the duality $(L^{\hat{U}}, M^{\hat{V}})$. Taking $f(X) = I_U(B+X)$ and $g = -\delta_{\mathcal{C}^{**}}$ we have $f+g$ is $*$-u.s.c. by Proposition A.5. Since $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$ the sum is also proper and therefore closed. We have $f^*(Y) = -I_V(Y) - E[YB]$ and $g^* = -\delta_{-\mathcal{C}^*}$. Since $f^*(1)$ and $g^*(0)$ are finite f^*, g^* are proper and by Lemma A.14

$$u(B) = \sup_{X \in L^{\hat{U}}} \{I_U(B + X) - \delta_{\mathcal{C}^{**}}(X)\} = \operatorname{lsc}\left(-f^* \Box \delta_{-\mathcal{C}^*}\right)(0). \tag{4.4}$$

Due to $L^{\hat{U}} = (M^{\hat{V}})^\star$ we may evaluate the lower semicontinuous hull in the norm topology on $M^{\hat{V}}$, see Theorem A.6. Therefore there exists a sequence Z_n in $M^{\hat{V}}$ norm-convergent to 0 such that $\lim_{n\to\infty}(-f^* \Box \delta_{-\mathcal{C}^*})(Z_n) = u(B)$ which completes the proof on recalling the formula for infimal convolution, see Definition A.13. *Q.E.D.*

REMARK 4.2 *In the case $M^{\hat{V}} = L^{\hat{V}}$ one has $\circledast = *$ hence the assumption (4.3) is absolutely necessary to prevent the utility gap in (4.2). In contrast, with $M^{\hat{V}} \subsetneq L^{\hat{V}}$ condition (4.3) is no longer economically innocuous because there are complete market examples where $\mathcal{C}^{**} = L^{\hat{U}}$ while $\mathcal{C}^{\circledast\circledast}$ is arbitrage-free. Nonetheless, assumption (4.3) gives, by some margin, the best result available to date.*

At present the only works in the literature that allow $M^{\hat{V}} \subsetneq L^{\hat{V}}$ are Biagini and Frittelli (2005) and Biagini and Černý (2011) who require $\mathcal{C}^{\circledast} = \mathcal{C}^*$ which forces $\mathcal{C}^{\circledast\circledast} = \mathcal{C}^{**}$ and so implies (4.3). Biagini and Frittelli (2008) assume $M^{\hat{V}} = L^{\hat{V}}$ and Biagini and Frittelli (2007), Schachermayer (2001, 2003), and Owen and Žitković (2009) require reasonable asymptotic elasticity at $-\infty$ (Schachermayer, 2001, Definition 1.4) which is stronger than $M^{\hat{V}} = L^{\hat{V}}$ (Schachermayer, 2001, Proposition 4.1(iii)).

For completeness we now prove the full duality over separating measures (4.1) which requires stronger assumptions.

THEOREM 4.3 *Assume either i)* $M^{\hat{U}} = L^{\hat{U}}$*; or ii)* $\mathcal{C} = \mathcal{C}^{**}$*;* $\lim_{x\to\infty} \overline{u}(x)/x = 0$*; there is* $0 < \bar{Y} \in \mathcal{C}^{\circledast}$ *(no arbitrage over* $\mathcal{C}^{\circledast\circledast}$*); and, only in the case* ***F-SL****, there is* $\tilde{Y} \in \mathcal{C}^{\circledast}$ *such that* $\lambda\tilde{Y} \in \operatorname{dom} I_V$ *for two distinct values of* $\lambda \geq 0$*. Then duality over separating measures (4.1) holds for all* $B \in L^{\hat{U}}$*.*

PROOF: *Step 1)* For $M^{\hat{U}} = L^{\hat{U}}$ the claim follows from Theorem 2.1. This covers case **L.** It remains to prove the case **SL** under assumption ii). Recall $u : L^{\hat{U}} \to \mathbb{R} \cup \{-\infty\}$ is the maximal expected utility as a function of random endowment $Z \in L^{\hat{U}}$,

$$u(Z) = \sup_{X\in\mathcal{C}} \{I_U(X + Z)\} = (I_U \Box - \delta_{-\mathcal{C}})(Z),$$

where $\Box$ denotes the supremal convolution (Definition A.13).

Since both I_U and $-\delta_{-\mathcal{C}}$ are proper (Definition A.1), by Lemma A.14

$$u^*(Y) = I_U^*(Y) - \delta_{\mathcal{C}}^*(Y) = -I_V(Y) - \delta_{\mathcal{C}^*}(Y)$$

and by definition of conjugate function

$$u^{**}(Z) = \inf_{Y\in M^{\hat{V}}} \{E[YZ] + I_V(Y) + \delta_{\mathcal{C}^*}(Y)\} = \inf_{Y\in\mathcal{C}^*} \{E[YZ] + I_V(Y)\}. \tag{4.5}$$

Step 2) By virtue of (4.5) the proof will be complete if we can show $u(B) = u^{**}(B)$. By Proposition A.7 and Theorem A.10 this is equivalent to demonstrating that u is $*$-u.s.c. at B. This line of reasoning is the essence of the conjugate duality construction proposed in Rockafellar (1974). We will show a stronger property, namely that u is $*$-u.s.c. globally. By Proposition A.12 u is $*$-u.s.c. if and only if for arbitrary norm-bounded sequence $\{Z_n\} \in L^{\hat{U}}$ such that $Z_n \overset{P\text{-a.s.}}{\to} Z \in L^{\hat{U}}$ one has $\limsup_{n\to\infty} u(Z_n) \leq u(Z)$.

Step 3) If $\limsup_{n\to\infty} u(Z_n) = -\infty$ there is nothing to prove. In the remaining case $\limsup_{n\to\infty} u(Z_n) =: \tilde{u} > -\infty$. By definition of supremum there is a subsequence (still denoted $Z_n \in L^{\hat{U}}$) and a corresponding sequence of $X_n \in \mathcal{C}$ such that $I_U(X_n + Z_n)$ is bounded below and

$$I_U(X_n + Z_n) \nearrow \tilde{u}.$$

Denote by $\widetilde{\mathcal{G}}$ the convex hull of $\{X_n + Z_n\}$. By convexity of upper level sets I_U is bounded below on $\widetilde{\mathcal{G}}$.

Step 4) We claim that $k\mathcal{B} + \mathcal{C}$ is norm-coercive in losses (see Definition 3.1) for arbitrary $k > 0$, where $\mathcal{B}$ is the unit ball in $L^{\hat{U}}$. For $L^{\hat{U}} \sim L^{\infty}$ this is true trivially. In the remaining case **F-SL** the set $\mathcal{G} = k\mathcal{B} + \mathcal{C}$ and separating density $\tilde{Y}$ satisfy assumptions of Lemma 3.2 and the claim follows. As a result $\widetilde{\mathcal{G}}$ is norm-bounded in losses and in view of norm-boundedness of $\{Z_n\}$ the set $\{U(\widetilde{\mathcal{G}}^+)\}$ is uniformly integrable by Lemma 3.4. By Lemma 3.2 $\{|X_n + Z_n|\tilde{Y}\}$ is $L^1(P)$-bounded, while $\{|X_n|\tilde{Y}\}$, too, is $L^1(P)$-bounded by Hölder inequality. The same holds for $\{|X_n + Z_n|\bar{Y}\}$, $\{|X_n|\bar{Y}\}$ by Corollary 3.3.

Step 5) Letting $d\bar{Q} = \bar{Y}dP$, Komlós theorem yields a sequence of forward convex combinations of $\{X_n\}$ (denoted $\{\hat{X}_n\}$) such that $\hat{X}_n$ converges P-a.s. to some limit $\hat{X}$. We will apply the same convex combinations to $\{Z_n\}$ and denote the resulting sequence by $\{\hat{Z}_n\}$. By construction $\hat{X}_n + \hat{Z}_n \in \widetilde{\mathcal{G}}$, therefore $U((\hat{X}_n + \hat{Z}_n)^+)$ is uniformly integrable by step 4). By concavity of I_U the utility of convex combinations dominates the utility of the original sequence,

$$\tilde{u} \leq \hat{u} = \limsup I_U(\hat{X}_n + \hat{Z}_n),$$

and by passing to a further subsequence we may assume $\hat{u} = \lim I_U(\hat{X}_n + \hat{Z}_n)$.

Step 6) UI of $\{U((\hat{X}_n + \hat{Z}_n)^+)\}$ and Fatou lemma yield

$$\hat{u} = \lim_{n\to\infty} I_U(\hat{X}_n + \hat{Z}_n) = \lim_{k\to\infty} \limsup_{n\to\infty} I_U((\hat{X}_n + \hat{Z}_n) \wedge k)$$
$$\leq \lim_{k\to\infty} I_U((\hat{X} + Z) \wedge k). \tag{4.6}$$

For any $k > 0$ the sequence $\{\hat{X}_n \wedge (k - Z_n)\} \in \mathcal{C}$ is norm-bounded and P-a.s. convergent to $\hat{X} \wedge (k - Z)$. By Gao (2014, Theorem 2.1) we conclude that $\hat{X} \wedge (k - Z) \in \mathcal{C}^{**}$. By assumption $\mathcal{C} = \mathcal{C}^{**}$, therefore

$$I_U((\hat{X} + Z) \wedge k)) = I_U(Z + \hat{X} \wedge (k - Z)) \leq \sup_{X\in\mathcal{C}} I_U(Z + X) = u(Z),$$

and from (4.6) we conclude $\hat{u} \leq u(Z)$.

Step 7) Combining steps 1)-6) we have shown in the case **SL** under assumption ii)

$$\sup_{X\in\mathcal{C}}\{I_U(X + Z)\} = \inf_{Y\in\mathcal{C}^*}\{I_V(Y) + E[YZ]\} \text{ for all } Z \in L^{\hat{U}}.$$

By Fenchel inequality $\sup_{X\in\mathcal{C}}\{I_U(X + Z)\} \leq \inf_{Y\in\mathcal{C}^{\circledast}}\{I_V(Y) + E[YZ]\}$ while

$$\inf_{Y\in\mathcal{C}^{\circledast}}\{I_V(Y) + E[YZ]\} \leq \inf_{Y\in\mathcal{C}^*}\{I_V(Y) + E[YZ]\},$$

due to the inclusion $\mathcal{C}^* = \mathcal{C}^{\circledast} \cap M^{\hat{V}} \subseteq \mathcal{C}^{\circledast}$. This proves (4.1). *Q.E.D.*

5. OPTIMAL ADMISSIBLE STRATEGY

Our soon-to-be-found ability to deal with utility functions that are not strictly monotone prompts a slight modification of the definition of admissibility, compared to Biagini and Černý (2011, Definition 1.1). In this paper we require a tight approximation of the wealth process below the bliss point of the utility function but only a loose one above the bliss point.

We also need to amend the definition of convergence at intermediate times to allow for effective completions as dual optimizers. The limiting process has to be defined not as a pointwise limit of $H^{(n)} \cdot S$ at fixed times but rather as its right-continuous regularization[13] $\mathrm{rqlim}_{n\to\infty} H^{(n)} \cdot S$ defined by

$$(\mathrm{rqlim}_{n\to\infty} H^{(n)} \cdot S)_t = \lim_{q\searrow t, q\in\mathbb{Q}} (\lim_{n\to\infty} H^{(n)} \cdot S_q), \tag{5.1}$$

in line with the supermartingale compactness result in Delbaen and Schachermayer (1998, Theorem D).

Definition 5.1 *A strategy $H \in L(S)$ is admissible ($H \in \mathscr{A}$) if there is a sequence $H^{(n)}$ in $\mathscr{T}$ such that*

1. *$U\left(B + H^{(n)} \cdot S_T\right) \to U\left(B + H \cdot S_T\right)$ in $L^1(P)$;*
2. *on the set $U(B + H \cdot S_T) < U(\infty)$ the approximating tame wealth $H^{(n)} \cdot S$ converges to the admissible wealth $H \cdot S$ in the sense of right-continuous regularization (5.1)*

$$H \cdot S_t = (\mathrm{rqlim}_{n\to\infty} H^{(n)} \cdot S)_t \text{ for all } t \in [0, T].$$

Recall the definition of the set of supermartingale measures $\mathcal{S}$ in equation (1.16). We begin by observing that the wealth process of every tame strategy is a supermartingale under each $Q \in \mathcal{M} \cap P_{\hat{V}}$.

Proposition 5.2 *For $Q \in \mathcal{M} \cap P_{\hat{V}}$ and $H \in \mathscr{T}$ the wealth process $H \cdot S$ is a Q-supermartingale. In other words, $\mathcal{M} \cap P_{\hat{V}} \subseteq \mathcal{S}$.*

Proof: *Step 1)* Since S is a Q-σ-martingale $H \cdot S$ can be written as an integral with respect to a Q-martingale (Emery, 1980, Proposition 2). $H \in \mathscr{T}$ means that for $W_t = \inf_{\tau\in[0,t]}\{H \cdot S_\tau\} \wedge 0$ one has $W_T \in L^{\hat{U}}$. This implies $W_T \in L^1(Q)$ and there is a P-martingale Z^Q such that $Z^Q_T = W_T$. The minimal process W is decreasing and therefore $W \geq Z^Q$.

Step 2) Since $H \cdot S$ is bounded below by the Q-martingale Z^Q it follows from Ansel-Stricker lemma (Ansel and Stricker, 1994, Corollaire 3.5) that $H \cdot S$ is a Q-local martingale. Now $H \cdot S - Z^Q$ is a positive local Q-martingale and by Fatou lemma therefore also Q-supermartingale. Since Z^Q is a true martingale $H \cdot S$ itself must a supermartingale. *Q.E.D.*

In the next step we will construct a candidate optimal trading strategy and prove that its wealth process is a supermartingale under any $Q \in \mathcal{M} \cap P_{\hat{V}}$. Note that the supermartingale property holds over the larger set $P_{\hat{V}}$ rather than just those measures that lead to bliss-free expected utility P_V.

[13] For regularization of submartingales see, for example, Revuz and Yor (1999, Theorem II.2.5 and Proposition II.2.6)

PROPOSITION 5.3 *Assume there is* $\bar{Q} \in \mathcal{M} \cap P^e_{\hat{V}}$. *Under the assumptions of Proposition 3.5 there is a trading strategy* $H \in L(S)$, *a sequence of maximizing tame strategies* $H^{(n)}$ *and a semimartingale* $\tilde{V}$ *such that*

1. $\tilde{V}$ *is* $\bar{Q}$*-supermartingale;*
2. $\tilde{V} = \mathrm{rqlim}_{n\to\infty} H^{(n)} \cdot S$, *see equation (5.1);*
3. $H \cdot S \geq \tilde{V}$ *and* $H \cdot S - \tilde{V}$ *is an increasing process;*
4. *In particular,* $H^{(n)} \cdot S_T \overset{P\text{-a.s.}}{\to} \tilde{V}_T$*;*
5. $U(B + H^{(n)} \cdot S_T) \overset{L^1(P)}{\to} U(B + \tilde{V}_T)$ *and thus* $I_U(B + \tilde{V}_T) = u(B) \in \mathbb{R}$*;*
6. $H \cdot S$ *is a* Q*-supermartingale for any* $Q \in \mathcal{M} \cap P_{\hat{V}}$.

PROOF: *Step 1)* First we prove that there is a maximizing sequence $\tilde{H}^{(n)} \in \mathscr{T}$ such that for any $Q \in \mathcal{S} \cap P_{\hat{V}}$ there is a Q-martingale Z^Q with the property $\tilde{H}^{(n)} \cdot S \geq Z^Q$. This is similar in spirit to Biagini and Černý (2011, Proposition 3.8) but there each Q calls for a different subsequence whereas here the maximizing (sub)sequence will be the same for all Q-s. Proposition 3.5 gives a maximizing sequence $\hat{H}^{(n)} \in \mathscr{T}$, a random variable $\hat{X} \in L^0(P)$, and a uniform lower bound $0 \geq R \in L^1(P)$ such that $R \leq U(B + \hat{H}^{(n)} \cdot S_T) \to U(B + \hat{X})$ in $L^1(P)$ and P-a.s.

Step 2) Let $\tilde{W} = U^{-1}(R) \leq 0$. By Fenchel inequality,

$$-E^Q[\lambda \tilde{W}] \leq E[\hat{V}(\lambda dQ/dP)] - E[U(\tilde{W})] = I_{\hat{V}}(\lambda dQ/dP) - E[R],$$

we conclude that $\tilde{W}$ is in $L^1(Q)$ for any $Q \in P_{\hat{V}}$. The wealth process $\tilde{H}^{(n)} \cdot S$ is a Q-supermartingale for any $Q \in \mathcal{S}$ and hence

$$E^Q_t[B] + \tilde{H}^{(n)} \cdot S_t \geq E^Q_t[B + \tilde{H}^{(n)} \cdot S_T] \geq E^Q_t[-(B + \tilde{H}^{(n)} \cdot S_T)^-] \geq E^Q_t[\tilde{W}], \tag{5.2}$$

therefore $Z^Q_t = E^Q_t[\tilde{W}] - E^Q_t[B]$ yields the lower bound announced in Step 1).

Step 3) Now select a fixed $\bar{Q}$ in $\mathcal{M} \cap P^e_{\hat{V}}$ and apply Theorem D in Delbaen and Schachermayer (1998) to construct processes $\tilde{V}$, H, and a sequence

$$H^{(n)} \in \mathrm{conv}(\tilde{H}^{(n)}, \tilde{H}^{(n+1)}, \ldots), \tag{5.3}$$

with the properties claimed in items (1)–(4). Item (5) follows from $\tilde{V}_T = \hat{X}$ and from the fact that $H^{(n)}$ is still a maximizing sequence.

Step 4) The uniform lower bound Z^Q for $\tilde{H}^{(n)} \cdot S$ obtained in (5.2) also applies to $H^{(n)} \cdot S$ for $H^{(n)}$ from (5.3) and therefore by item (2) also to $\tilde{V}$ since Z^Q can be chosen right-continuous. Consequently $H \cdot S \geq Z^Q$ and by step 2) in the proof of Proposition 5.2 $H \cdot S$ is a Q-supermartingale for every $Q \in \mathcal{M} \cap P_{\hat{V}}$. *Q.E.D.*

In the second step we will prove that the candidate strategy H attains maximal utility $u(B)$. Therefore by item (6) of Proposition 5.3 the optimizer H belongs to the supermartingale class of strategies. It is readily seen that our approach simplifies and generalizes the results of Schachermayer (2001, 2003), in

	$M^{\hat V} = L^{\hat V}$	$M^{\hat V} \subsetneq L^{\hat V}$
$M^{\hat U} = L^{\hat U}$	none required;	$\exists\lambda > 1; \lambda\hat Y \in \operatorname{dom} I_V$
$M^{\hat U} \subsetneq L^{\hat U} \nsim L^\infty$	$u(B) = \sup_{X\in\mathcal{C}^{\circledast\circledast}} I_U(B+X);$	$u(B) = \sup_{X\in\mathcal{C}^{**}} I_U(B+X);$ $\exists\lambda > 1; \lambda\hat Y \in \operatorname{dom} I_V$
$M^{\hat U} \subsetneq L^{\hat U} \sim L^\infty$	none required;	cannot occur

TABLE I
Additional assumptions of Theorem 5.4.

particular we completely sidestep dynamic optimization arguments in the proof of the supermartingale property, see also Owen and Žitković (2009). This is all the more remarkable since the tools we use do not go beyond those pioneered by Schachermayer and his co-authors in the run-up to Schachermayer (2003).

Theorem 5.4 *Assume*

1. $B \in L^{\hat U}$;
2. $(B+\mathcal{C}) \cap \operatorname{core}\mathcal{D} \neq \emptyset$;
3. *no arbitrage over* $\mathcal{C}^{\circledast\circledast}$;
4. $\lim_{x\to\infty} \overline{u}(x)/x = 0$;
5. $\mathcal{C}^{\circledast}_\sigma$ *is norm-dense in* $\mathcal{C}^{\circledast}$ *(Assumption 1.5)*;
6. *and further specific assumptions as detailed in Table I.*

Then Theorem 2.1, Proposition 3.5, Proposition 4.1 and Proposition 5.3 apply and the strategy H from Proposition 5.3 is optimal and admissible.

Proof: *Step 1)* No arbitrage over $\mathcal{C}^{\circledast\circledast}$ implies existence of an equivalent separating measure (Gao and Xanthos, 2017, Proposition 3.5). Obtain $\bar Q \in \mathcal{M} \cap P^e_{\hat V}$ from Assumption 1.5. We know from Proposition 5.3, items (1), (3), and (5), that

$$I_U((B + H\cdot S_T)\wedge 0) \geq I_U((B+\tilde V_T)\wedge 0) > -\infty.$$

Step 2) By item (6) of Proposition 5.3 one has $E[Y(H\cdot S_T)] \leq 0$ for any $Y \in C^{\circledast}_\sigma$. Therefore for any $m \geq 0$ and any $Y \in C^{\circledast}_\sigma$ we obtain

$$\langle (B+H\cdot S_T)\wedge m, Y\rangle \leq \langle B + H\cdot S_T, Y\rangle \leq \langle B, Y\rangle.$$

Step 3) By Proposition 4.1 and Assumption 1.5 there are sequences $W_n \in L^{\hat V}$ and $Y_n \in C^{\circledast}_\sigma$ with the properties $(Y_n + W_n) \in \mathcal{C}^{\circledast}$, $\|W_n\|_{\hat V} \to 0$, and

$$I_V(Y_n + W_n) + \langle B, Y_n + W_n\rangle \to u(B).$$

Step 1) implies $\|(B+H\cdot S_T)\wedge m\|_{\hat U}<\infty$ and Fenchel inequality yields

$$\begin{aligned} I_U((B+H\cdot S_T)\wedge m) &\le I_V(Y_n+W_n)+\langle (B+H\cdot S_T)\wedge m, Y_n+W_n\rangle \\ &\le I_V(Y_n+W_n)+\langle B, Y_n+W_n\rangle \\ &\qquad + \langle (B+H\cdot S_T)\wedge m - B, W_n\rangle, \end{aligned}$$

where we have used $\langle (B+H\cdot S_T)\wedge m, Y_n\rangle \le \langle B, Y_n\rangle$ from step 2).

Step 4) Use Hölder inequality for Orlicz spaces and let $n\to\infty$ in step 3) to obtain $I_U((B+H\cdot S_T)\wedge m)\le u(B)$. By monotone convergence (Bogachev, 2007, Theorem 2.8.2) letting $m\nearrow\infty$ we find $I_U(B+H\cdot S_T)\le u(B)$. Items (3) and (5) of Proposition 5.3 now yield

$$I_U(B+H\cdot S_T)=I_U(B+\tilde V_T)=u(B). \tag{5.4}$$

Step 5) It remains to show that $H\cdot S$ can be approximated by $H^{(n)}\cdot S$ at intermediate times in the sense $\operatorname{rqlim}_{n\to\infty}H^{(n)}\cdot S=\tilde V=H\cdot S$ on the set $B+H\cdot S_T<\overline{x}$, recalling $\overline{x}$ in equation (1.3). When U is strictly monotone, the inequality $H\cdot S_T\ge \tilde V_T=\hat X$ together with equality (5.4) imply

$$H\cdot S_T=\tilde V_T. \tag{5.5}$$

By Proposition 5.3 the process $H\cdot S-\tilde V$ is non-negative and increasing which in view of (5.5) is only possible if $H\cdot S=\tilde V$.

Step 6) When $\overline{x}<\infty$ argue by contradiction. Suppose there is a non-null set A on which $H\cdot S_t>\tilde V_t$ for some t (not necessarily the same on each path) and $B+\tilde V_T<\overline{x}$. Since $H\cdot S-\tilde V$ is increasing it follows $H\cdot S_T>\tilde V_T$ on A and since $B+\tilde V_T<\overline{x}$ on A this contradicts $I_U(B+\tilde V_T)=I_U(B+H\cdot S_T)$. *Q.E.D.*

6. CONCLUSIONS

We have studied expected utility maximization from the point of view of conjugate duality over Orlicz spaces $(L^{\hat U}, L^{\hat V})$ determined by the left tail of the utility function and the right tail of its conjugate, respectively. In this setup objects in $L^{\hat V}$ can be interpreted as complete market state price densities but not necessarily as separating measures. In Theorem 2.1 we have established Fenchel duality over state price densities, applicable also to large financial markets, in circumstances where none of the standard regularity conditions apply. In Theorem 5.4 we have provided construction of the optimal trading strategy that does not rely on the dual maximizer $\hat Q$ being a separating measure or being equivalent to P. In the case $L^{\hat U}\sim L^\infty$ we have achieved this goal under the minimal conditions from the seminal works of Kramkov and Schachermayer (2003) and Hugonnier and Kramkov (2004).

The Fenchel duality formula mentioned above,

$$\sup_{X\in\mathcal{C}} I_U(X)=\min_{Y\in L^{\hat V}}\{I_V(Y)+\delta^*_{\mathcal{C}\cap\mathcal{D}}(Y)\},$$

has an interesting economic interpretation. The quantity $\delta^*_{\mathcal{C}\cap\mathcal{D}}(d\hat{Q}/dP)$ can be interpreted as an increase in the initial endowment required to bring the expected utility in a complete market $\hat{Q}$ to the optimal level $u(0)$. In the same vein the term $\delta^*_{\mathcal{C}\cap\mathcal{D}}(dQ/dP)$ may be interpreted as the shadow price of the implicit trading constraint presented by the finiteness of the effective domain of expected utility $\mathcal{D}$. This provides a fresh perspective on the classical results of Kramkov and Schachermayer (1999, 2003).

Our analysis motivates the study of the fundamental theorem of asset pricing in Orlicz space setting in a small financial market. The question is whether $\mathcal{C}^{\star\star} \cap L^{\hat{U}}_{+} = \{0\}$ implies $\mathcal{C} = \mathcal{C}^{\circledast\circledast}$ and whether the set of σ-martingale measures with density in $L^{\hat{V}}$ is norm-dense among all separating measures, in the sense of Assumption 1.5. The answer is known to be affirmative when $L^{\hat{U}} = L^\infty$ from the work of Kabanov (1997); Delbaen and Schachermayer (1998).

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APPENDIX A: KEY RESULTS IN CONVEX DUALITY

In this appendix we have collected results from convex analysis required in the main body of the paper, principally in the proofs of Section 4. Unless explicitly specified the functions are defined over a locally convex, Hausdorff topological vector space (E, τ). Let E' denote the topological dual of (E, τ), namely the space of linear, continuous functionals on (E, τ). Any other topology σ on E such that its topological dual coincides with E', meaning $E' = (E, \sigma)'$ is called compatible with the dual pair (Rockafellar, 1974, Section 3). The conjugate functions are defined on E', endowed with a topology τ' compatible with the dual pair, namely such that the dual space of (E', τ') equals E. Taking E and E' as fixed it is known that the coarsest compatible topology on E is the initial topology $\sigma(E, E')$ while the finest compatible topology in E is the Mackey topology $\tau(E, E')$. For $y \in E'$ and $x \in E$ we denote the bilinear form $y(x)$ by $\langle x, y\rangle$.

Definition A.1 *For a concave (resp. convex) function h with values in $[-\infty, \infty]$ its effective domain* $\operatorname{dom} h$ *is defined by* $\operatorname{dom} h = \{x : h(x) > -\infty\}$ *(resp. $\{x : h(x) < \infty\}$). A concave (resp. convex) function h is called proper if $h < \infty$ (resp. $h > -\infty$) and* $\operatorname{dom} h$ *is non-empty.*

Definition A.2 *A function h (not necessarily concave/convex) with values in $[-\infty, \infty]$ is called upper semi-continuous (resp. lower semicontinuous), in short u.s.c. (resp. l.s.c.), if for each $c \in \mathbb{R}$ the set $\{x : h(x) \geq c\}$ (resp. $\{x : h(x) \leq c\}$) is closed.*

DEFINITION A.3 *For a function h we denote by $\operatorname{usc} h$ the upper semicontinuous hull of h, i.e. the smallest upper semicontinuous function that dominates h. Likewise, for a convex function h we denote by $\operatorname{lsc} h$ the lower semicontinuous hull of h, i.e. the greatest lower semicontinuous function dominated by h.*

PROPOSITION A.4 *The upper semicontinuous hull (resp. the lower semicontinuous hull) is given by the formula*

$$\operatorname{usc} h(x) = \sup_{x_\alpha \to x} \limsup_\alpha h(x_\alpha),$$

resp.

$$\operatorname{lsc} h(x) = \inf_{x_\alpha \to x} \liminf_\alpha h(x_\alpha),$$

where nets can be replaced by sequences when (E, τ) is first-countable (in particular normed). A function h is u.s.c. (resp. l.s.c.) if and only if $h \geq \operatorname{usc} h$ (resp. $h \leq \operatorname{lsc} h$).

PROOF: Rockafellar (1974, eq. 3.7) and Aliprantis and Border (2006, Lemma 2.42). *Q.E.D.*

LEMMA A.5 *The sum of two u.s.c. functions with values in $[-\infty, \infty)$ (resp. two l.s.c. functions with values in $(-\infty, \infty]$) is u.s.c. (resp. l.s.c.).*

PROOF: We have

$$\begin{aligned}
\operatorname{lsc}(g+h)(x) &= \inf_{x_\alpha \to x} \liminf_\alpha (g+h)(x_\alpha) \\
&\geq \inf_{x_\alpha \to x} \{\liminf_\alpha g(x_\alpha) + \liminf_\alpha h(x_\alpha)\} \\
&\geq \inf_{x_\alpha \to x} \liminf_\alpha g(x_\alpha) + \inf_{x_\alpha \to x} \liminf_\alpha h(x_\alpha) \\
&= \operatorname{lsc} g(x) + \operatorname{lsc} h(x) = g(x) + h(x)
\end{aligned}$$

and the statement follows by Proposition A.4. *Q.E.D.*

THEOREM A.6 *For a concave (resp. convex) function h one has*

$$\operatorname{usc}_{\sigma(E,E')} h = \operatorname{usc}_{\tau(E,E')} h$$

resp. $\operatorname{lsc}_{\sigma(E,E')} h = \operatorname{lsc}_{\tau(E,E')} h$, meaning that the upper (resp. lower) semicontinuous hull of a concave (resp. convex) function is the same in any compatible topology.

PROOF: See Aliprantis and Border (2006), Theorem 5.98 and Corollary 5.99. *Q.E.D.*

PROPOSITION A.7 *Suppose h is concave (resp. convex). If $\operatorname{usc} h$ (resp. $\operatorname{lsc} h$) is finite-valued at a point then necessarily $\operatorname{usc} h$ (resp. $\operatorname{lsc} h$) is proper.*

PROOF: See Rockafellar (1974, Theorem 4). *Q.E.D.*

DEFINITION A.8 (Rockafellar (1974)) *For a concave function h the upper closure $\operatorname{cl} h$ is defined as*

$$\operatorname{cl} h = \begin{cases} \operatorname{usc} h & \text{if } \operatorname{usc} h < \infty \\ \infty & \text{otherwise} \end{cases}.$$

Likewise, for a convex function h the lower closure $\operatorname{cl} h$ is defined as

$$\operatorname{cl} h = \begin{cases} \operatorname{lsc} h & \text{if } \operatorname{lsc} h > -\infty \\ -\infty & \text{otherwise} \end{cases}.$$

We say that h is closed if $h = \operatorname{cl} h$.

Proposition A.9 *For h concave (resp. convex) $h^* = (\operatorname{usc} h)^*$ (resp. $h^* = (\operatorname{lsc} h)^*$) and $h^* = (\operatorname{cl} h)^*$ is closed.*

Proof: See Rockafellar (1974, Theorem 5) and Zalinescu (2002, Theorem 2.3.1). *Q.E.D.*

Theorem A.10 (Fenchel-Moreau) *For h concave or convex $h^{**} = \operatorname{cl} h$.*

Proof: See Rockafellar (1974, Theorem 5). *Q.E.D.*

Corollary A.11 *A concave (convex) function $\operatorname{cl} h$ is proper if and only if h^* is proper.*

Proposition A.12 *When $L^{\hat{U}} = (M^{\hat{V}})^\star$ a concave (resp. convex) function h on $L^{\hat{U}}$ is u.s.c. (resp. l.s.c.) in the duality $(L^{\hat{U}}, M^{\hat{V}})$ if and only if*

$$h(x) = \operatorname{usc} h(x) = \sup_{\substack{x_n \to x \ P\text{-a.s.} \\ \|x_n\|_{\hat{U}} < K}} \limsup_n h(x_n), \ \text{resp.}$$

$$h(x) = \operatorname{lsc} h(x) = \inf_{\substack{x_n \to x \\ \|x_n\|_{\hat{U}} < K}} \liminf_n h(x_n),$$

for all $x \in L^{\hat{U}}$. That is, in computing a candidate for u.s.c./l.s.c. hull in the $(L^{\hat{U}}, M^{\hat{V}})$ duality nets can be replaced with a.s.-convergent norm-bounded sequences.

Proof: Gao and Xanthos (2018, Theorem 2.4). *Q.E.D.*

Definition A.13 *Suppose f, g are two concave and proper functions. Their supremal convolution $f \Box g : E \to [-\infty, \infty]$ is defined as*

$$f \Box g(x) = \sup_{z \in E} \{f(x-z) + g(z)\}.$$

Likewise, for two proper convex functions f, g their inf(imal) convolution is given by

$$f \Box g(x) = \inf_{z \in E} \{f(x-z) + g(z)\}.$$

Lemma A.14 *For f, g proper concave (convex) one has*

$$(f \Box g)^* = f^* + g^*. \tag{A.1}$$

For concave f and g such that f^* and g^* are proper and $\operatorname{cl} f + \operatorname{cl} g = \operatorname{cl}(f+g)$ one has for all $y \in E'$

$$\operatorname{cl}\left(f^* \Box g^*\right)(y) = (f+g)^*(y) = \inf_{x \in E} \{\langle x, y\rangle - (f(x) + g(x))\}. \tag{A.2}$$

Proof: Formula (A.1) follows from an easy computation (Rockafellar, 1974, eq. 9.30). The same formula applied to f^* and g^* yields

$$(f^* \Box g^*)^* = f^{**} + g^{**} = \operatorname{cl} f + \operatorname{cl} g, \tag{A.3}$$

where the second equality follows by Theorem A.10. By Proposition A.9

$$(f+g)^* = (\operatorname{cl}(f+g))^* = (\operatorname{cl} f + \operatorname{cl} g)^* = (f^* \Box g^*)^{**} = \operatorname{cl}\left(f^* \Box g^*\right),$$

where the last two equalities follow from (A.3) and again Theorem A.10. The last equality in (A.2) is immediate from the definition of conjugate function. *Q.E.D.*

Proposition A.15 *When U decreases superlinearly at $-\infty$ the expected utility functional I_U is $*$-upper semicontinuous. In the linear case I_U is L^1-norm continuous everywhere and therefore $\circledast$-u.s.c.*

Proof: i) In the superlinear case $\lim_{x\to-\infty} U(x)/x = \infty$ one has $(M^{\hat{V}})^\star = L^{\hat{U}}$. By Propositions A.4 and A.12 it suffices to prove that for every pointwise convergent norm-bounded sequence $X_n \to X$ one has

$$\limsup_{n\to\infty} I_U(X_n) \leq I_U(X). \tag{A.4}$$

Since $U(0) = 0$ and U is increasing and concave Fatou lemma gives

$$\limsup_{n\to\infty} I_U(-X_n^-) \leq I_U(-X^-), \tag{A.5}$$

$$\limsup_{n\to\infty} \{I_U(X_n^+) - U'_+(0)E[X_n^+]\} \leq I_U(X^+) - U'_+(0)E[X^+]. \tag{A.6}$$

$M^{\hat{V}}$ with Orlicz norm is an order-continuous Banach lattice. By Gao (2014, Theorem 2.1) X_n is $\sigma(L^{\hat{U}}, M^{\hat{V}})$-convergent to X. By Wickstead (2008, Proposition 3.6) the lattice operations are $\sigma(L^{\hat{U}}, M^{\hat{V}})$-sequentially continuous on norm bounded subsets of $L^{\hat{U}}$ and hence $\lim_{n\to\infty} E[X_n^+] = E[X^+]$. On combining inequalities (A.5, A.6) we thus obtain (A.4), which completes the proof.

ii) In the remaining linear case $\lim_{x\to-\infty} U(x)/x < \infty$ space $L^{\hat{U}}$ is isomorphic to L^1 and I_U is finite everywhere. Since I_U is bounded below on any norm-bounded neighbourhood of 0 it follows (Aliprantis and Border, 2006, Theorem 5.43) that I_U is norm-continuous on L^1. In this case norm topology is compatible with the duality and norm-continuity therefore implies $\circledast$-upper semicontinuity by Theorem A.6. *Q.E.D.*

APPENDIX B: CORNER SOLUTION WITH EXPONENTIAL UTILITY

In this appendix we take $U(x) = -e^{-x}$. A routine calculation yields $V(y) = y \ln y - y$ and

$$\max_{y>0} I_V(yZ) = -E[Z]e^{-E[Z\ln Z]/E[Z]}. \tag{B.1}$$

The model for asset price X and the optimal strategy are described in Sections B.1-B.3. The non-existence of a supermartingale deflator with terminal value $U'(-X_T)$, where $-X_T$ is the optimal terminal wealth and $\hat{Y} = U'(-X_T)$ is the dual optimizer from Theorem 2.1, is shown in Section B.4, where it is also noted that the optimal wealth process $-X$ is a submartingale under $\hat{Q}$, $d\hat{Q}/dP = \hat{Y}/E[\hat{Y}]$.

B.1. *Asset price process*

Let X be a special semimartingale Lévy process with characteristics $(b^X, 0, F^X)$ where $b^X \in \mathbb{R}$ and

$$F^X(dx) = \frac{3}{4\sqrt{\pi}} x^{-5/2} e^{-x} I_{(0,\infty)}(x)dx + \delta_{-1/2}(dx),$$

with δ_x denoting a Dirac measure at point x. Consequently the cumulant generating function of X is given by

$$\begin{aligned}\kappa_X(v) &= b^X v + \int (e^{vx} - 1 - vx)F^X(dx)\\ &= e^{-v/2} + (1-v)^{3/2} - 2 + (2 + b^X)v \text{ for } v \leq 1 \text{ and } +\infty \text{ otherwise.}\end{aligned}$$

X can be interpreted as a sum of a compensated one-sided (positive) tempered stable process with parameters $\beta = 3/2$, $\alpha = 1/\Gamma(-\beta)$, $\lambda = 1$; a compensated Poisson process with intensity 1 and jump size $-1/2$; and a drift component with drift b^X.

In this construction it is important that $\beta > 1$. The choice of tempered stable process for positive jumps is significant only to the extent that its Lévy measure density is exponential divided by a polynomial of sufficiently high degree as $x \to \infty$; any other Lévy measure with this property would do just as well. The convenience of tempered stable formulation is that it yields a simple expression for the cumulant generating function (Küchler and Tappe, 2013) which makes it particularly obvious that we will be dealing with a corner solution.

The choice of Poisson process for the single negative jump is not important, but the jump size being bounded below by $-1/2$ means that $\mathcal{E}(X)$ is strictly positive and so our example could be recast in terms of an exponential Lévy model. We will not pursue this line of exposition here and instead formulate everything as trading on X.

To this end, $\kappa_X(v)$ being finite for $v \leq 1$ and exponential being a submultiplicative function (Sato, 1999, Proposition 25.4) we obtain $\sup_{t\in[0,T]} |X_t| \in L^{\hat{U}}$ (Sato, 1999, Theorem 28.18). By Biagini and Černý (2011, Proposition 6.4) this means every separating measure in $L^{\hat{V}}$ is a local martingale measure for X and by Proposition 5.2 every separating measure is therefore a supermartingale measure for all $L^{\hat{U}}$-tame strategies.

B.2. *Candidate optimal trading strategy*

Consider now optimization over buy-and-hold strategies in X. Assume $X_0 = 0$ so that terminal wealth reads ϑX_T. Expected utility is then $I_U(\vartheta X_T) = -\exp(\kappa_X(-\vartheta))$. Optimization over ϑ yields the following first order condition for interior maximum,

$$0 = \kappa_X'(-v) = -\frac{1}{2}e^{v/2} - \frac{3}{2}(1+v)^{1/2} + 2 + b^X.$$

Provided $b^X < -2 + 1/(2\sqrt{e}) \approx -1.7$, which is what we assume hereafter, there will be no interior optimizer and instead maximum will be achieved at $\vartheta = -1$. For future reference let

$$\kappa_X'(1) = b^X + \int x\,(e^x - 1)\,F^X(dx) = b^X + 2 - 1/(2\sqrt{e}) =: -A < 0.$$

Our task is to prove that $-X_T$ is the optimal wealth and therefore $\vartheta = -1$ is the optimal strategy and $-\exp\left(\kappa^X(1)\right)$ is the maximal utility. The buy-and-hold strategy $\vartheta = -1$ is $L^{\hat{U}}$-tame in view of $\sup_{t\in[0,T]} |X_t| \in L^{\hat{U}}$.

B.3. *Dual optimizing sequence of separating measures*

For $n = 1, 2, \ldots$ define

$$\begin{aligned} W_n(x) &= e^x - 1 + \frac{4\sqrt{\pi}}{3}x^{5/2}e^x K_n 1_{[n,n+1]}(x) \\ &= e^x - 1 + K_n 1_{[n,n+1]}(x)dx/F^X(dx), \\ K_n &= A/(n+1/2). \end{aligned}$$

With this definition one has

$$b^X + \int x W_n(x) F^X(dx) = \underbrace{b^X + \int (e^x - 1)\,x F^X(dx)}_{-A} + (n+1/2)K_n = 0. \tag{B.2}$$

Let J^X be the jump measure associated with process X and define

$$d\mathcal{L}(Z^{(n)}) = dX + (W_n(x) - x)dJ^X. \tag{B.3}$$

Note that $Z^{(n)}$ is a well-defined strictly positive process because jumps on the right-hand side are bounded below by $1/\sqrt{e}-1$. It follows that the P-drift of $\mathcal{L}(Z^{(n)})$ is given by

$$b^{\mathcal{L}(Z^{(n)})} = b^X + \int (W_n(x) - x) F^X(dx), \tag{B.4}$$

and that $Z^{(n)} \exp(-b^{\mathcal{L}(Z^{(n)})} t)$ is a density process of a local martingale measure for X by virtue of Girsanov theorem and (B.2) which may be rewritten as

$$b^X dt + d\langle X, \mathcal{L}(Z^{(n)} \exp(-b^{\mathcal{L}(Z^{(n)})} t))\rangle = 0.$$

Denote this local martingale measure by $Q^{(n)}$, with

$$dQ^{(n)} = Z_T^{(n)} \exp(-b^{\mathcal{L}(Z^{(n)})} T) dP.$$

Here $\langle X, Y\rangle$ now stands for predictable quadratic covariation, i.e. the drift part of process $[X, Y]$ (provided the semimartingale $[X, Y]$ is special). By Itō formula

$$\begin{aligned} b^{\ln Z^{(n)}} &= b^X + \int (\ln(1 + W_n(x)) - x)\, F^X(dx) \\ &= b^X + \underbrace{\int_n^{n+1} \ln(1 + K_n \frac{dx}{F(dx)} e^{-x}) F(dx)}_{B_n \searrow 0}. \end{aligned} \tag{B.5}$$

From the Girsanov theorem the drift of $\ln Z^{(n)}$ under $Q^{(n)}$ is given by

$$\begin{aligned} b_{Q^{(n)}}^{\ln Z^{(n)}} &= b^{\ln Z^{(n)}} + \int W_n(x) \ln(1 + W_n(x)) F^X(dx), \\ &= b^{\ln Z^{(n)}} + \int W_n(x) \ln\left(e^x + K_n \frac{dx}{F^X(dx)} 1_{[n,n+1]}(x)\right) F^X(dx) \\ &= b^X + B_n + \int W_n(x) x F^X(dx) \\ &\quad + \underbrace{\int_n^{n+1} W_n(x) \ln\left(1 + K_n \frac{dx}{F^X(dx)} e^{-x}\right) F^X(dx)}_{C_n \searrow 0} \\ &= B_n + C_n \searrow 0, \end{aligned}$$

where we have substituted for $b^{\ln Z^{(n)}}$ from (B.5) and used (B.2) in the penultimate line.

Apply the complete market utility formula (1.12), $u_{Q^{(n)}}(0) = \max_{y>0} I_V(y Z_T^{(n)})$, and evaluate it using the expression (B.1), and the help of identities $E[Z_T^{(n)}] = \exp(b^{\mathcal{L}(Z^{(n)})} T)$ and $E[Z_T^{(n)} \ln Z_T^{(n)}]/E[Z_T^{(n)}] = E^{Q^{(n)}}[\ln Z_T^{(n)}] = b_{Q^{(n)}}^{\ln Z^{(n)}} T$,

$$u_{Q^{(n)}}(0) = -E[Z_T^{(n)}] e^{-E[Z_T^{(n)} \ln Z_T^{(n)}]/E[Z_T^{(n)}]} = -\exp(b^{\mathcal{L}(Z^{(n)})} T - b_{Q^{(n)}}^{\ln Z^{(n)}} T).$$

In (B.4) substitute for W_n and rearrange to obtain

$$b^{\mathcal{L}(Z^{(n)})} = b^X + \int (e^x - 1 - x)\, F^X(dx) + \int_n^{n+1} K_n dx \searrow \kappa_X(1).$$

In conclusion,

$$u_{Q^{(n)}}(0) = -\exp((b^{\mathcal{L}(Z^{(n)})} - b_{Q^{(n)}}^{\ln Z^{(n)}}) T) \searrow -\exp(\kappa^X(1)) = E[U(-X_T)],$$

which proves optimality of $\vartheta = -1$ as all tame strategies are supermartingales under the measures $Q^{(n)}$ and hence the utility of any tame strategy may not exceed the expression on the left-hand side.

B.4. *There is no supermartingale deflator ending with $U'(-X_T)$*

The distribution P_{X_t} is absolutely continuous with respect to Lebesgue measure (Sato, 1999, Lemma 27.1, Theorem 27.7), and its support is the entire real line. We wish to investigate whether there is $c \geq 0$ and supermartingale D with

$$D_T = U'(-X_T) = e^{X_T}$$

such that $D(c-X)$ is also a supermartingale. Now X is a process with independent increments so one can evaluate the conditional expectations $E_t[(c-X_T)e^{X_T}]$ explicitly.

Take $W(x) = e^x - 1$ then Z_T defined by (B.3) gives precisely $Z_T = e^{X_T}$ and we can reuse the calculations in the previous section with $K_n = 0$ and $d\hat{Q}/dP = e^{X_T - \kappa_X(1)T} = \hat{Y}/E[\hat{Y}]$ to obtain

$$\begin{aligned} E_t[(c-X_T)e^{X_T}] &= e^{X_t + \kappa^X(1)(T-t)} \left(c - X_t + E_t[(X_t - X_T)e^{X_T - X_t - \kappa^X(1)(T-t)}]\right) \\ &= e^{X_t + \kappa^X(1)(T-t)} \left(c - X_t - b^X_{\hat{Q}}(T-t)\right) \leq D_t(c - X_t). \end{aligned}$$

Now $-b^X_{\hat{Q}} = A > 0$ and so when $c < X_t < c + A$, which happens with non-zero probability thanks to the support of X being the whole real line, the left-hand side is positive while the right-hand side, no matter how one chooses $D_t \geq 0$, is non-positive. Therefore, $U'(-X_T)$ cannot be identified with a terminal value of even a weak supermartingale deflator.

Since X is a Lévy process under $\hat{Q}$ the inequality $-b^X_{\hat{Q}} = A > 0$ implies that $-X$ is a $\hat{Q}$-submartingale.

APPENDIX C: UTILITY MAY INCREASE FROM $\mathcal{C}$ TO $\mathcal{C}^{\circledast\circledast}$

We have observed in Theorem 2.1 that the maximal utility over $\mathcal{C}$ and its norm-closure $\mathcal{C}^{\star\star}$ always coincide. Thus when the norm topology is compatible with the economic duality we are guaranteed that the maximal utility over $\mathcal{C}$ and $\mathcal{C}^{\circledast\circledast}$ is the same. We study equivalent conditions for such compatibility in Theorem C.1 .

In Section C.2 we then provide an explicit *arbitrage-free* example, using exponential utility, of a situation where maximal utility increases by going from $\mathcal{C}$ to $\mathcal{C}^{\circledast\circledast}$. This construction can be applied to arbitrary utility in the setting of statement 8) in Theorem C.1, that is on any probability space that is not purely atomic and such that the Orlicz heart $M^{\hat{U}}$ is strictly contained in the Orlicz space $L^{\hat{U}}$. We thus find that the difficult cases are essentially those where $M^{\hat{U}} \neq L^{\hat{U}}$. The word essentially refers to infinite, purely atomic spaces where it is not known whether in all instances with $M^{\hat{U}} \neq L^{\hat{U}}$ such an example exists, the same way it is not known whether 7) implies 8) in Theorem C.1.

C.1. *When is norm topology compatible with economic duality?*

It turns out that the characterization hinges on the properties of a functional called 'modular'. We therefore begin with a more general concept of an ordered modular space, following the exposition of Nowak (1989a). We then specialize this more general setup to Orlicz spaces used in this paper. Atribute σ should be read as "countably" or "countable".

Let E be a σ-Dedekind complete Riesz space. A functional $\rho : E \to [0, \infty]$ is called a *modular* if the following conditions hold:

(i) $\rho(x) = 0$ iff $x = 0$.
(ii) $|x| < |y|$ implies $\rho(x) < \rho(y)$.
(iii) $\rho(x_1 \vee x_2) < \rho(x_1) + \rho(x_2)$ for $x_1 \geq 0, x_2 \geq 0$.
(iv) $\rho(\lambda x) \to 0$ if $\lambda \to 0$.

One can verify that with this definition ρ is a modular also in the original sense of Musielak and Orlicz (1959). A modular ρ is said to be convex, if $\rho(\lambda_1 x_1 + \lambda_2 x_2) < \lambda_1 \rho(x_1) + \lambda_2 \rho(x_2)$ for

$\lambda_1, \lambda_2 \geq 0$ and $\lambda_1 + \lambda_2 = 1$. A modular ρ is said to be *metrizing* whenever $\rho(x_n) \to 0$ implies $\rho(2x_n) \to 0$ for a sequence $\{x_n\}$ in E. Recall the definition of the corresponding gauge norm, $\|x\|_\rho = \inf\{\lambda > 0 : \rho(x/\lambda) \leq 1\}$. For the definition of modular topology see Nowak (1989a, p. 262).

The following characterization of modular and norm convergence is key. A sequence $\{x_n\}$ in E converges to zero modularly iff there is $\lambda > 0$ such that $\rho(\lambda x_n) \to 0$, while it converges to zero in the norm $\|\cdot\|_\rho$ iff $\rho(\lambda x_n) \to 0$ for all $\lambda > 0$, *ibid.* Therefore ρ fails to be metrizing precisely when there is a sequence that converges to zero modularly but not in the norm.

Let Φ be a Young function, $(\Omega, \mathcal{F}, P)$ a probability measure space. Then $\rho : L^0(\Omega, \mathcal{F}, P) \to [0, +\infty]$

$$\rho(X) = E[\Phi(X)], \tag{C.1}$$

is a convex orthogonally additive modular on the Orlicz space L^Φ, satisfying the σ-Lebesgue property, the σ-Fatou property and the σ-Levi property (Nowak, 1989a, Section 2; Nowak, 1989b, pp. 274-275). The norm $\|\cdot\|_\rho$ is known as the Luxemburg norm in this setting.

Finally, recall the definition of Mackey topology and strong topology for a given dual pair (Aliprantis and Border, 2006, Sections 5.18 and 5.19). The following theorem gives full characterization of circumstances under which norm closure and economic closure coincide.

THEOREM C.1 *Let Φ be a Young function, ρ the corresponding modular from (C.1) and Ψ the conjugate of Φ. Then the strong topology $\beta(L^\Phi, L^\Psi)$ coincides with the norm topology on L^Φ, the Mackey topology $\tau(L^\Phi, L^\Psi)$ coincides with the modular topology and the following are equivalent:*

1. *ρ is metrizing;*
2. *Every sequence $\{X_n\} \in L^\Phi$ modularly convergent to zero is also norm-convergent;*
3. *$\beta(L^\Phi, L^\Psi) = \tau(L^\Phi, L^\Psi)$;*
4. *The gauge norm $\|\cdot\|_\rho$ is order-continuous on L^Φ;*
5. *$(L^\Phi, \tau(L^\Phi, L^\Psi))$ is barrelled;*
6. *$(L^\Phi)^\star = L^\Psi$, that is L^Ψ is the norm-dual of L^Φ;*

   If, furthermore, we exclude the case where the probability space is finite (i.e. we assume P is not *supported on a finite number of atoms) then the following is equivalent to 1)-6)*
7. *$M^\Phi = L^\Phi$.*

   If we also exclude the case where P is purely atomic then the following is equivalent to 1)-7)
8. *Φ satisfies so-called Δ_2–condition at $+\infty$, i.e. there is $K > 0$ and $x_0 > 0$ such that $\Phi(2x) \leq K\Phi(x)$ for all $x > x_0$.*

The implications 8) $\Rightarrow$ 7) $\Rightarrow$ 1)-6) hold without further assumptions.

PROOF: Nowak (1989b, Theorems 3.2 and 4.2) show that $\beta(L^\Phi, L^\Psi)$ is the norm topology on L^Φ and $\tau(L^\Phi, L^\Psi)$ is the modular topology. The equivalence 1) $\iff$ 2) is trivial. Equivalences 1) $\iff$ 3) $\iff$ 4) follow from Nowak (1989a, Theorem 2.3), while 3) $\iff$ 5) follows from a standard result in topology, Husain and Khaleelulla (1978, Corollary II.1.2 and II.2.4). Equivalence of 4) and 6) for Φ finite follows from Zaanen (1983, Ch 15, p. 336 and Ch 19, pp. 572-3), while in the remaining case $L^\Phi \sim L^\infty, L^\Psi \sim L^1$ both 4) and 6) are true if the probability is finite and both are false otherwise.

7) $\Rightarrow$ 6) follows from Edgar and Sucheston (1992, Theorem 2.2.11) for finite Φ while for Φ that jumps to infinity 7) is false and the implication holds trivially.

2) $\Rightarrow$ 7) distinguish two cases: A) When $L^\Phi \sim L^\infty$ use non-finiteness of the probability space to construct a disjoint sequence of events $A_n \in \mathcal{F}, P(A_n) > 0$ and $\sum_{n=1}^\infty P(A_n) = 1$.

Without loss of generality we may assume $\Phi(1) < \infty, \Phi(2) = \infty$. Let $B_n = \bigcup_{k=1}^{n} A_k$ and define $X_n(\omega) = 1_{B_n^c}(\omega)$. Then $I_\Phi(X_n) \to 0$ by dominated convergence while $I_\Phi(2X_n) = \infty$ meaning $\{X_n\}$ converges to zero modularly but not in norm. This shows statement 2) is false and the implication holds trivially. B) When Φ is everywhere finite argue by contradiction. Suppose there is $X \in L^\Phi \setminus M^\Phi$ which in particular means $X \notin L^\infty$ and therefore $(\Omega, \mathcal{F}, P)$ must be a non-finite probability space. Without loss of generality we may suppose $0 \leq X$, $I_\Phi(X) < \infty, I_\Phi(2X) = \infty$. Define $X_n = X1_{X \geq n}$. Once again $I_\Phi(X_n) \to 0$ by dominated convergence while $I_\Phi(2X_n) = \infty$ in contradiction to 2).

The implication 8) $\Rightarrow$ 7) follows from Zaanen (1983, Theorem 131.3). The opposite implication for P that is not purely atomic follows from Rao and Ren (1991, Theorem III.2). *Q.E.D.*

C.2. *Illustrative example*

Let $U(x) = -e^{-x}$ which implies $V(y) = y \ln y - y$. Consider a probability space $(\Omega = \mathbb{Z}, \mathcal{F} = 2^{\mathbb{Z}}, P)$ with $P(\{n\}) = e^{-|n|}$ for $n \in \{1, \pm 2, \pm 3, \ldots\}$, $P(\{-1\}) = e^{-5}$, and

$$P(\{0\}) = 1 - \sum_{|n| \geq 1} P(\{n\}) = 1 - 2(e^2 - e)^{-1} - e^{-1} - e^{-5}.$$

Define a random variable X by setting $X(-1) = -1, X(1) = 1$, and $X = 0$ elsewhere. Let $Y(n) = n$ for $|n| \geq 2$ and $Y = 0$ otherwise. Define a sequence of random variables $\{X_k, Y_k\}_{k \in \mathbb{N}}$ by setting $Y_k = Y1_{|Y| \geq k}$ and $X_k = X + Y_k$. Note that Y, in common with all Y_k, has finite exponential moments in the interval $(-1, 1)$ but not beyond. This means Y_k converge to zero modularly (which by previous theorem means in the economic duality) but not in the norm on $L^{\hat{U}}$. By construction

$$E[Y_k|X] = 0 \text{ for all } k \in \mathbb{N}. \tag{C.2}$$

Think of X_k as an excess return on a traded position. Define the marketed subspace as $\mathcal{K} = \operatorname{span}(\{X_k\}_{k \in \mathbb{N}})$. Probability measure $\hat{Q}$ defined by

$$d\hat{Q}/dP = e^{-2X}/E[e^{-2X}]$$

is a bliss-free completion of the market since, by construction, $E^{\hat{Q}}[X] = E^{\hat{Q}}[Y_k] = 0$ for $k \in \mathbb{N}$ and $\hat{Q}$ has finite entropy,

$$H(\hat{Q}||P) = E\left[\frac{d\hat{Q}}{dP} \ln \frac{d\hat{Q}}{dP}\right] = -\ln E[e^{-2X}].$$

One can readily verify that $2X$ is the optimal wealth in the complete market $\hat{Q}$, since for exponential utility and any bliss-free state price measure Q one has by formula (1.12) and direct calculation

$$u_Q(0) = \sup_{E^Q[X] \leq 0} I_U(X) = \min_{y > 0} I_V(y dQ/dP) = I_V(e^{-H(Q||P)} dQ/dP) = -e^{-H(Q||P)}.$$

However, the maximal utility in the original market $\mathcal{K}$ is strictly lower than u_Q. To see this, consider a finite linear combination $Z \in \mathcal{K}$ with $1 \leq k(1) < k(2) < \ldots < k(N)$ being the indices in ascending order of vectors with non-zero coefficients,

$$\begin{aligned} Z = \sum_{i=1}^{N} \lambda_i X_{k(i)} &= \xi_N X_{k(N)} + \sum_{j=1}^{N-1} \xi_{N-j} \left(X_{k(N-j)} - X_{k(N-j+1)}\right) \\ &= \xi_N (X + Y_{k(N)}) + \sum_{j=1}^{N-1} \xi_{N-j} \left(Y_{k(N-j)} - Y_{k(N-j+1)}\right), \end{aligned}$$

where $\xi_{N-j} = \left(\sum_{i=1}^{N-j} \lambda_i\right)$ for $j = 0, \ldots, N-1$. The random variables X and $\{Y_{k(N-j)} - Y_{k(N-j+1)}\}_{j=1}^{N-1}$ are in L^∞. It follows

$$E[e^{-Z}] < \infty \iff E[\exp\left(-\xi_N Y_{k(N)}\right)] \iff E[\exp\left(-\xi_N Y\right)] \iff |\xi_n| < 1.$$

In view of (C.2) the conditional Jensen's inequality (Mussmann, 1988, Lemma 2.1) yields

$$E[e^{-Z}] \geq E[e^{-\xi_N X}] \geq E[e^{-X}],$$

where the last inequality follows since $E[e^{-\lambda X}] = e^{-5}e^{\lambda} + e^{-1}e^{-\lambda}$ is a strictly convex function of λ attaining global minimum at $\lambda = 2$ and therefore decreasing on $(-\infty, 2]$. It follows that that the maximal utility over $\mathcal{K}$ satisfies

$$u(0) = \sup_{W \in \mathcal{K}} I_U(W) = \sup_{W \in \mathcal{C}} I_U(W) = -E[e^{-X}].$$

Finally, let us examine the economic closure $\mathcal{C}^{\circledast\circledast}$. Since Y_k converge to 0 in the economic duality we have $\lambda X \in \mathcal{C}^{\circledast\circledast}$ for all $\lambda \in \mathbb{R}$. In contrast,

$$\lambda X \in \mathrm{cl}^{\circledast}(\mathcal{C} \cap \mathcal{D}) \text{ if and only if } |\lambda| \leq 1.$$

Consequently,

$$\begin{aligned} -E[e^{-X}] &= \sup_{W \in \mathcal{C}} I_U(W) \\ &< \max_{W \in \mathcal{C}^{\circledast\circledast}} I_U(W) = \min_{Y \in \mathcal{C}^{\circledast}} I_V(Y) = -e^{-H(\hat{Q}||P)} = -E[e^{-2X}]. \end{aligned}$$

Note, however, that Theorem 2.1 continues to hold for $\mathcal{A} = \mathcal{C}$ (as well as for $\mathcal{A} = \mathcal{C}^{\circledast\circledast}$ which is just the last three equalities above):

$$\begin{aligned} -E[e^{-X}] &= \sup_{W \in \mathcal{C}} I_U(W) = \max_{W \in \mathrm{cl}^{\circledast}(\mathcal{C} \cap \mathcal{D})} I_U(W) = \min I_V(Y) + \sup_{W \in \mathcal{C} \cap \mathcal{D}} E[WY] \\ &= I_V(e^{-X}) + \sup_{W \in \mathcal{C} \cap \mathcal{D}} E[We^{-X}] = I_V(e^{-X}) + E[Xe^{-X}], \end{aligned}$$

and the optimal effective completion of $\mathcal{C}$ is given by the state price density

$$d\hat{Q}/dP = e^{-X}/E[e^{-X}].$$

Note that under $\hat{Q}$ the optimal wealth process *increases* in expectation, $E^Q[X] > 0$, hence X is a $\hat{Q}$–submartingale.

In the given example $X \in L^\infty \subseteq M^{\hat{U}}$ possesses all exponential moments and the dual optimizer is therefore a separating measure. One can modify this example by splitting the state $\{0\}$ into countably many states where X is unbounded and such that it only posesses exponential moment of order at most, say 1.5, while maintaining the present inequality $E[Xe^{-1.5X}] > 0$. In this way one may exhibit a situation where

$$\max_{W \in \mathrm{cl}^{\circledast}(\mathcal{C} \cap \mathcal{D})} I_U(W) < \max_{W \in \mathcal{C}^{\circledast\circledast}} I_U(W),$$

the first optimizer is X, the second optimizer is $1.5X$ and each optimizer represents a corner solution. In the first case the corner is caused by the 'nuisance' zero-mean shocks Y_k which do not allow us to increase our position in X beyond 1 while we are trading inside $\mathcal{C}$. These nuisance shocks 'stop contaminating' X as one passes to the economic closure $\mathcal{C}^{\circledast\circledast}$. One is now able to take a position λX with λ above 1. In the second case the corner over $\mathcal{C}^{\circledast\circledast}$ at $\lambda = 1.5$ is inherent in X itself. Seen in this light, duality over separating measures (1.20) signifies that even this corner can be 'removed' by passing to a full completion whose utility is arbitrarily close to that of the original market $\mathcal{C}^{\circledast\circledast}$.