Hölder stability for an inverse medium problem with internal data

Mourad Choulli, Faouzi Triki

HÖLDER STABILITY FOR AN INVERSE MEDIUM PROBLEM WITH INTERNAL DATA

MOURAD CHOULLI AND FAOUZI TRIKI

ABSTRACT. We are interested in an inverse medium problem with internal data. This problem is originated from multi-waves imaging. We aim in the present work to study the well-posedness of the inversion in terms of the boundary conditions. We precisely show that we have actually a stability estimate of Hölder type. For sake of simplicity, we limited our study to the class of Helmholtz equations $\Delta + V$ with bounded potential V.

Mathematics subject classification : 35R30.

Key words : Helmholtz equation, inverse medium problem, internal data, Hölder stability, unique continuation.

1. INTRODUCTION

Let Ω be a C^2-smooth bounded domain of $\mathbb{R}^n$, $n = 2, 3$, with boundary Γ.

Set

$$\mathscr{D} = \{V \in L^\infty(\Omega);\ 0 \text{ is not an eigenvalue of } A_V\},$$

where $A_V : L^2(\Omega) \to L^2(\Omega)$ is the unbounded operator defined by

$$A_V = -\Delta - V \text{ with } D(A_V) = H^2(\Omega) \cap H_0^1(\Omega).$$

Note that

$$\mathscr{D} \supset \{V;\ V(x) = \lambda,\ x \in \Omega,\ \text{for some } \lambda \in \mathbb{R}\backslash\sigma(A_0)\}.$$

Here A_0 is A_V when $V = 0$ and $\sigma(A_0)$ is the spectrum of A_0.

Let $0 < k < 1$ be given and let $\overline{V} \in \mathscr{D}$ so that $2v_0 \leqslant \overline{V}$, where $v_0 > 0$ is fixed. Consider then $\mathscr{D}_0(k, v_0, \overline{V})$, the subset of $\mathscr{D}$ of those functions $V \in L^\infty(\Omega)$ satisfying

$$\|V - \overline{V}\|_{L^\infty(\Omega)} \leqslant \min\left(k/\|A_{\overline{V}}^{-1}\|_{\mathscr{B}(L^2(\Omega))}, v_0\right).$$

Pick $p > n$ and fix $h \in W^{2-1/p,p}(\Gamma)$ non identically equal to zero and denote by u_V, $V \in \mathscr{D}_0(k, v_0, \overline{V})$, the solution of the following BVP for the Helmholtz equation:

$$\Delta u + Vu = 0 \text{ in } \Omega \quad \text{and} \quad u = h \text{ on } \Gamma.$$

According to [17, Theorem 3.1, page 1782], $u_V \in W^{2,p}(\Omega)$ and the following estimate holds

$$\|u_V\|_{W^{2,p}(\Omega)} \leqslant M = M\left(\Omega, p, v_0, k, \overline{V}, h\right), \quad V \in \mathscr{D}_0(k, v_0, \overline{V}). \tag{1.1}$$

We are mainly interested in determining the absorption coefficient V from the internal data

$$I_V = Vu_V^2.$$

The authors are supported by the grant ANR-17-CE40-0029 of the French National Research Agency ANR (project MultiOnde).

This inverse problem is originated from multi-waves imaging. The term multi-waves refers to the fact that two types of physical waves are used to probe the medium under study. Usually, the first wave is sensitive to the contrast of the desired parameter, the other types can carry the information revealed by the first type of waves to the boundary of the medium where measurements can be taken. In the present work, we assume that the first inversion has been performed, that is the internal data I_V is retrieved, and we focus on the second step. We refer to [4, 5, 7, 8, 9, 10, 17] and reference therein for further details.

Choose $K > 0$ sufficiently large in such a way that

$$\mathscr{D}_1(k, v_0, \overline{V}, K) = \mathscr{D}_0(k, v_0, \overline{V}) \cap \{V \in C^{0,1}(\overline{\Omega});\ \|V\|_{C^{0,1}(\overline{\Omega})} \leqslant K\} \neq \varnothing.$$

Note that, according to Rademacher's theorem, $C^{0,1}(\overline{\Omega})$ is continuously embedded in $W^{1,\infty}(\Omega)$.

For the inverse problem under consideration we are going to prove various Hölder stability estimates.

Theorem 1.1. (*interior stability*) *Let $\omega \Subset \Omega$. Then, there exist two constants $C = C\left(\Omega, p, v_0, k, \overline{V}, h, K, \omega\right) > 0$ and $\mu = \mu\left(\Omega, p, v_0, k, \overline{V}, h, K, \omega\right)$ so that, for any $V, \tilde{V} \in \mathscr{D}_1(k, v_0, \overline{V}, K)$ satisfying $V = \tilde{V}$ on Γ, we have*

$$\|V - \tilde{V}\|_{L^\infty(\omega)} \leqslant C \left\| I_V^{1/2} - I_{\tilde{V}}^{1/2} \right\|_{H^1(\Omega)}^{\mu}.$$

A similar result was already proved by G. Alessandrini in [1] under different assumptions.

When h does not vanish on a part of Γ, we obtain improved results which we state in the following theorems.

Theorem 1.2. *Under the assumption $|h| > \kappa > 0$ on Γ, there exist two constants $C = C\left(\Omega, p, v_0, k, \overline{V}, h, K\right) > 0$ and $\mu = \mu\left(\Omega, p, v_0, k, \overline{V}, h, K\right)$ so that, for any $V, \tilde{V} \in \mathscr{D}_1(k, v_0, \overline{V}, K)$ satisfying $V = \tilde{V}$ on Γ, we have*

$$\|V - \tilde{V}\|_{L^\infty(\Omega)} \leqslant C \left\| I_V^{1/2} - I_{\tilde{V}}^{1/2} \right\|_{H^1(\Omega)}^{\mu}.$$

Theorem 1.3. *Let $\tilde{\Omega} \subset \Omega$ be a C^2-smooth domain with boundary $\tilde{\Gamma}$ such that $\tilde{\gamma} := \tilde{\Gamma} \cap \Gamma$ satisfies $\mathring{\tilde{\gamma}} \neq \varnothing$, and let $\omega \Subset \tilde{\Omega}$. Under the assumption $|h| > \kappa > 0$ on Γ, there exist two constants $C = C\left(\Omega, p, v_0, k, \overline{V}, h, K, \omega, \tilde{\Omega}\right) > 0$ and $\mu = \mu\left(\Omega, p, v_0, k, \overline{V}, h, K, \omega, \tilde{\Omega}\right)$ so that, for any $V, \tilde{V} \in \mathscr{D}_1(k, v_0, \overline{V}, K)$ satisfying $V = \tilde{V}$ on Γ and $\nabla V = \nabla \tilde{V}$ on $\tilde{\gamma}$, we have*

$$\|V - \tilde{V}\|_{L^\infty(\omega)} \leqslant C \left\| I_V^{1/2} - I_{\tilde{V}}^{1/2} \right\|_{H^1(\tilde{\Omega})}^{\mu}.$$

In the following result we allow the Dirichlet boundary condition h to vanish on Γ. Denote

$$\Gamma_+ = \{x \in \Gamma;\ |h(x)| > 0\} \quad \text{and} \quad \Gamma_0 = \{x \in \Gamma;\ h(x) = 0\}.$$

Theorem 1.4. *Let γ be a compact subset of $\Gamma_+ \cup \mathring{\Gamma}_0$ and $\omega \Subset \Omega \cup \gamma$. Then, there exist two constants $C = C\left(\Omega, p, v_0, k, \overline{V}, h, K, \omega\right) > 0$ and $\mu = \mu\left(\Omega, p, v_0, k, \overline{V}, h, K, \omega\right)$ so that, for any $V, \tilde{V} \in \mathscr{D}_1(k, v_0, \overline{V}, K)$ satisfying $V = \tilde{V}$ on Γ, we have*

$$\|V - \tilde{V}\|_{L^\infty(\omega)} \leqslant C \left\| I_V^{1/2} - I_{\tilde{V}}^{1/2} \right\|_{H^1(\Omega)}^{\mu}.$$

We deduce from Theorem 1.3 that it is possible to recover the potential V on a small set ω of Ω if the medium is probed starting from $\tilde{\gamma}$, the part of the boundary where h is intense, and by covering a neighboring region $\tilde{\Omega}$ that contains ω. Unfortunately h may in general settings be zero on some parts of the boundary Γ, and Theorem 1.4 is an attempt to improve the results of Theorems 1.2 and 1.3 in this direction.

In [2], the authors were able to prove a lower bound for the gradient of solutions near the boundary when the boundary data is "qualitatively unimodal" (see [2] for the definition). Roughly speaking, the key in their proof is that even if the tangential gradient of solutions vanishes, there is, according to Hopf's maximum principle, a non zero contribution of the derivative of solutions in the normal direction. Unfortunately, there is no similar arguments that can be used in order to get lower bound for solutions on the boundary.

The rest of this text consists in two sections and an appendix. We establish in Section 2 weighted interpolation inequalities that are our main ingredient for proving stability estimates. That is what we do in Section 3. We end by an appendix that is devoted to prove a technical result we used in Section 2. This result which is essential in our analysis gives a lower bound of the L^2-norm of solutions on a small ball, away from the boundary, in term of the radius of the ball.

2. Weighted interpolation inequalities

We start with some preliminaries involving the so-called frequency function. If $\mathbf{d}$ is the diameter of Ω with respect to the euclidean metric and $0 < \delta < \mathbf{d}$, let

$$\Omega^\delta = \{x \in \Omega;\ \mathrm{dist}(x,\Gamma) \geqslant \delta\} \quad \text{and} \quad \Omega_\delta = \{x \in \Omega;\ \mathrm{dist}(x,\Gamma) \leqslant \delta\}.$$

For $0 < v_0 \leqslant V_0$ and $M > 0$, set $\mathscr{V}(v_0,V_0) = \{V \in L^\infty(\Omega);\ v_0 \leqslant V \leqslant V_0\}$. Define then

$$\mathscr{S}(v_0,V_0) = \{u \in H^2(\Omega);\ \ \Delta u + Vu = 0,\ \text{for some } V \in \mathscr{V}(v_0,V_0)\}.$$

Let $u \in \mathscr{S}(v_0,V_0)$ and $x_0 \in \Omega^\delta$. We define, for $0 < r < \delta$,

$$\begin{aligned} H_u(x_0,r) &= \int_{S(x_0,r)} u^2(x)dS(x),\\ D_u(x_0,r) &= \int_{B(x_0,r)} \left\{|\nabla u(x)|^2 + V(x)u^2(x)\right\} dx,\\ K_u(x_0,r) &= \int_{B(x_0,r)} u^2(x)dS(x). \end{aligned}$$

Here $S(x_0,r)$ is the sphere of centrer x_0 and radius r.

Henceforth, the first Dirichlet eingenvalue of the Laplace operator in the domain D is denoted by $\lambda_1(D)$.

Prior to define the frequency function, we need to prove the following lemma, where

$$\rho_0 = \sqrt{\lambda_1(B(0,1))/V_0}. \tag{2.1}$$

Lemma 2.1. *Let ρ_0 be as in (2.1). Then, for any $u \in \mathscr{S}(v_0,V_0)$, $u \neq 0$, $x_0 \in \Omega^\delta$ and $0 < r < \min(\delta,\rho_0)$, we have $H_u(x_0,r) > 0$.*

Proof. We proceed by contradiction. Pick then $u \in \mathscr{S}(v_0, V_0)$, $u \neq 0$, $x_0 \in \Omega^\delta$ and assume then that $H_u(x_0, r) = 0$ for some $0 < r < \min(\delta, \rho_0)$. That is, $u = 0$ on $S(x_0, r)$. Therefore Green's formula gives

$$\int_{B(x_0,r)} |\nabla u|^2 dx = \int_{B(x_0,r)} V u^2 dx \leqslant V_0 \int_{B(x_0,r)} u^2. \tag{2.2}$$

On the other hand, bearing in mind that $\lambda_1(B(x_0, r)) = \lambda_1(B(0, 1))/r^2$, Poincaré's inequality yields

$$\begin{aligned} \int_{B(x_0,r)} u^2 dx &\leqslant \frac{1}{\lambda_1(B(x_0, r))} \int_{B(x_0,r)} |\nabla u|^2 dx \\ &\leqslant \frac{r^2}{\lambda_1(B(0, 1))} \int_{B(x_0,r)} |\nabla u|^2 dx. \end{aligned} \tag{2.3}$$

Note that we used here that $u \in H_0^1(B(x_0, r))$.

Inequality (2.3) in (2.2) produce

$$\left(1 - \frac{r^2 V_0}{\lambda_1(B(0, 1))}\right) \int_{B(x_0,r)} |\nabla u|^2 dx \leqslant 0.$$

As $1 - r^2 V_0/\lambda_1(B(0, 1)) > 0$, we conclude that $u = 0$ in $B(x_0, r)$ and hence u is identically equal to zero by the unique continuation property. This leads to the expected contradiction. □

According to Lemma 2.1, if $u \in \mathscr{S}(v_0, V_0)$, $u \neq 0$, we can define the frequency function N_u, corresponding to u, by

$$N_u(x_0, r) = \frac{r D_u(x_0, r)}{H_u(x_0, r)}.$$

The following two lemmas can be deduced from the calculations developed in [19, 20] (see also [14]).

Lemma 2.2. *For $u \in \mathscr{S}(v_0, V_0)$, $u \neq 0$ and $x_0 \in \Omega^\delta$, we have*

$$K_u(x_0, r) \leqslant r H_u(x_0, r), \quad 0 < r < \delta_0 = \min(\rho_0, \rho_1, \delta),$$

where ρ_0 is as in (2.1) *and $\rho_1 = \sqrt{(n-1)/V_0}$.*

Lemma 2.3. *Let $u \in \mathscr{S}(v_0, V_0)$, $u \neq 0$ and $x_0 \in \Omega^\delta$. Then*

$$N_u(x_0, r) \leqslant C \max(N_u(x_0, \delta_0), 1), \quad 0 < r < \delta_0,$$

with a constant $C = C(\Omega, V_0) > 0$ and δ_0 is as in Lemma 2.2.

Fix $0 < \alpha \leqslant 1$. We say that $W \subset L^1_+(\Omega) = \{w \in L^1(\Omega);\ w \geqslant 0\}$ is a uniform set of weights for the weighted interpolation inequality

$$\|f\|_{L^\infty(\Omega)} \leqslant C \|f\|^{1-\mu}_{C^{0,\alpha}(\overline{\Omega})} \|fw\|^\mu_{L^1(\Omega)}, \tag{2.4}$$

if the constants $C > 0$ and $0 < \mu < 1$ in (2.4) can be chosen independently in $w \in W$ and $f \in C^{0,\alpha}(\overline{\Omega})$.

Similarly, we will say that $W \subset L^1_+(\Omega)$ is a uniform set of interior weights for the interior weighted interpolation inequality, where $\omega \Subset \Omega$ is arbitrary,

$$\|f\|_{L^\infty(\omega)} \leqslant C \|f\|^{1-\mu}_{C^{0,\alpha}(\overline{\Omega})} \|fw\|^\mu_{L^1(\Omega)} \tag{2.5}$$

if the constants $C > 0$ and $0 < \mu < 1$ in (2.5), depending on ω, can be chosen independently in $w \in W$ and $f \in C^{0,\alpha}(\overline{\Omega})$.

Remark 2.1. Let $W \subset L^1_+(\Omega)$ be a uniform set of weights for the weighted interpolation inequality (2.4). Pick $w \in W$ and consider $Z = \{x \in \overline{\Omega};\ w(x) = 0\}$. We claim that Z has empty interior. Otherwise, if $\mathring{Z} \neq \varnothing$ then we would find $f_0 \in C_0^\infty(\mathring{Z})$ non identically equal to zero. But (2.4) with $f = f_0$ would entail that $f_0 = 0$. That is we have a contradiction and our claim is proved.

Let $\varkappa$ be the norm of the imbedding $H^2(\Omega) \hookrightarrow C(\overline{\Omega})$. We introduce two sets, where $0 < \kappa \leqslant \varkappa M$ are given constants,

$$\mathscr{S}_w(v_0, V_0, \kappa, M) = \{u \in H^2(\Omega);\ \Delta u + Vu = 0,\ \text{for some } V \in \mathscr{V}(v_0, V_0) \\ \text{and } \|u\|_{H^2(\Omega)} \leqslant M,\ \|u\|_{L^\infty(\Gamma)} \geqslant \kappa\},$$

and

$$\mathscr{S}_s(v_0, V_0, \kappa, M) = \{u \in H^2(\Omega);\ \Delta u + Vu = 0,\ \text{for some } V \in \mathscr{V}(v_0, V_0) \\ \text{and } \|u\|_{H^2(\Omega)} \leqslant M,\ |u| \geqslant \kappa \text{ on } \Gamma\}.$$

The main ingredient in establishing weighted interpolation inequalities is the following theorem. Its proof, which is quite technical, is given in Appendix A.

Theorem 2.1. *Let $0 < \delta < \mathbf{d}$. Then there exists a constants $c = c(\Omega, v_0, V_0, \kappa, M, \delta) > 0$ so that, for any $x_0 \in \Omega^\delta$ and $u \in \mathscr{S}_w(v_0, V_0, \kappa, M)$, we have*

$$e^{-e^{c/\delta}} \leqslant \|u\|_{L^2(B(x_0,\delta))}. \tag{2.6}$$

Theorem 2.2. *(1) The set $\mathscr{W}_w(v_0, V_0, \kappa, M) = \{w = u^2;\ u \in \mathscr{S}_w(v_0, V_0, \kappa, M)\}$ is a uniform set of interior weights for the interior weighted interpolation inequality* (2.5).
(2) The set $\mathscr{W}_s(v_0, V_0, \kappa, M) = \{w = u^2;\ u \in \mathscr{S}_s(v_0, V_0, \kappa, M)\}$ is a uniform set of weights for the weighted interpolation inequality (2.4).

Before proving this theorem, we establish some preliminaries.

Lemma 2.4. *Let $0 < \delta < \mathbf{d}$. There exists a constant $C = C(\Omega, v_0, V_0, \kappa, M, \delta) > 0$ so that, for any $u \in \mathscr{S}_w(v_0, V_0, \kappa, M)$, we have*

$$\|N_u\|_{L^\infty(\Omega^\delta \times (0,\delta_0))} \leqslant C.$$

Here δ_0 is as in Lemma 2.2.

Proof. Let $x_0 \in \Omega^\delta$. From Theorem 2.1

$$K_u(x_0, \delta_0) = \|u\|^2_{L^2(B(x_0,\delta_0))} \geqslant C. \tag{2.7}$$

Combined with Lemma 2.2, this estimate yields

$$H_u(x_0, \delta_0) \geqslant C. \tag{2.8}$$

In light of Lemma 2.3, we end up getting

$$N_u(x_0, r) \leqslant C, \quad 0 < r < \delta_0,$$

which leads immediately to the expected inequality. □

Proposition 2.1. *Let $0 < \delta < \mathbf{d}$. There exist two constants $C = C(\Omega, v_0, V_0, \kappa, M, \delta) > 0$ and $c = c(\Omega, v_0, V_0, \kappa, M, \delta) > 0$ so that, for any $u \in \mathscr{S}_w(v_0, V_0, \kappa, M)$, we have*

$$Cr^c \leqslant \|u\|_{L^2(B(x_0,r))}, \quad x_0 \in \Omega^\delta,\ 0 < r < \delta_0,$$

where δ_0 is as in Lemma 2.2.

Proof. Pick $u \in \mathscr{S}_w(v_0, V_0, \kappa, M)$ and $x_0 \in \Omega^\delta$. For simplicity's sake, set $H = H_u$ and $N = N_u$. From the calculations carried out in [19, 20] (see also [14]), we have

$$\partial_r H(x_0, r) = \frac{n-1}{r} H(x_0, r) + 2D(x_0, r).$$

Whence

$$\partial_r \left(\ln \frac{H(x_0, r)}{r^{n-1}} \right) = \frac{\partial_r H(x_0, r)}{H(x_0, r)} - \frac{n-1}{r} = \frac{2N(x_0, r)}{r}.$$

This and Lemma 2.4 entail

$$\partial_r \left(\ln \frac{H(x_0, r)}{r^{n-1}} \right) \leqslant \frac{C}{r}, \quad 0 < r < \delta_0.$$

Thus

$$\int_{sr}^{s\delta_0} \partial_t \left(\ln \frac{H(x_0, t)}{t^{n-1}} \right) dt = \ln \frac{H(x_0, s\delta_0) r^{n-1}}{H(x_0, sr)\delta_0^{n-1}} \leqslant \ln \frac{\delta_0^C}{r^C}, \quad 0 < s < 1,\ 0 < r < \delta_0.$$

Hence

$$H(x_0, s\delta_0) \leqslant \frac{C}{r^c} H(x_0, sr), \quad 0 < r < \delta_0,$$

and then

$$\begin{aligned} \|u\|^2_{L^2(B(x_0,\delta_0))} &= \delta_0^{n-1} \int_0^1 H(x_0, s\delta_0) s^{n-1} ds \\ &\leqslant \frac{C}{r^c} r^{n-1} \int_0^1 H(x_0, sr) s^{n-1} ds = \frac{C}{r^c} \|u\|^2_{L^2(B(x_0,r))}, \quad 0 < r < \delta_0. \end{aligned}$$

Combined with (2.7), this estimate yields

$$Cr^c \leqslant \|u\|_{L^2(B(x_0,r))}, \quad 0 < r < \delta_0,$$

as expected. □

Proof of Theorem 2.2. (1) Let $w \in \mathscr{W}_w(v_0, V_0, \kappa, M)$ and $u \in \mathscr{S}_w(v_0, V_0, \kappa, M)$ so that $w = u^2$. Fix $\omega \Subset \Omega$. We need to prove that (2.5) holds with constants C and μ that are independent of w and $f \in C^{0,\alpha}(\overline{\Omega})$. By homogeneity it is enough to establish (2.5) when $\|f\|_{C^{0,\alpha}(\overline{\Omega})} = 1$. To this end, take $f \in C^{0,\alpha}(\overline{\Omega})$ satisfying $\|f\|_{C^{0,\alpha}(\overline{\Omega})} = 1$.

Fix $0 < \delta < \mathbf{d}$ so that $\overline{\omega} \subset \Omega^\delta$ and pick $x_0 \in \overline{\omega}$ so that $|f(x_0)| = \|f\|_{L^\infty(\omega)}$. According to Proposition 2.1, there exist two constants $C = C(\Omega, v_0, V_0, \kappa, M, \delta) > 0$ and $c = c(\Omega, v_0, V_0, \kappa, M, \delta) > 0$ so that, for any $u \in \mathscr{S}_w(v_0, V_0, \kappa, M)$, we have

$$Cr^c \leqslant \|u\|_{L^2(B(x_0,r))}, \quad 0 < r < \delta_0, \tag{2.9}$$

where δ_0 is as in Lemma 2.2.

But

$$|f(x_0)| = \|f\|_{L^\infty(\omega)} \leqslant |f(x)| + r^\alpha, \text{ for any } x \in B(x_0, r).$$

Therefore

$$\begin{aligned} \|f\|_{L^\infty(\omega)} \int_{B(x_0,r)} u(x)^2 dx &\leqslant 2 \int_{B(x_0,r)} |f(x)| u(x)^2 dx + 2r^\alpha \int_{B(x_0,r)} u(x)^2 dx \\ &\leqslant 2 \int_\Omega |f(x)| u(x)^2 dx + 2r^\alpha \int_{B(x_0,r)} u(x)^2 dx. \end{aligned}$$

Note that, according to the unique continuation property,

$$\int_{B(x_0,r)} u(x)^2 dx \neq 0, \quad 0 < r < \delta_0.$$

Hence

$$\|f\|_{L^\infty(\Omega)} \leqslant 2\frac{\|fu^2\|_{L^1(\Omega)}}{\|u\|^2_{L^2(B(x_0,r))}} + 2r^\alpha.$$

Combined with (2.9), this estimate yields

$$\|f\|_{L^\infty(\omega)} \leqslant C(\|fu^2\|_{L^1(\Omega)} r^{-c} + r^\alpha), \;\; 0 < r < \delta_0. \tag{2.10}$$

When $\|fu^2\|_{L^1(\Omega)} < \delta_0^{c+\alpha}$, we can take $r = \|fu^2\|_{L^1(\Omega)}^{1/(c+\alpha)}$ in (2.10) in order to get

$$\|f\|_{L^\infty(\omega)} \leqslant C\|fu^2\|^\mu_{L^1(\Omega)}, \tag{2.11}$$

with $\mu = \frac{\alpha}{c+\alpha}$.

If $\|fu^2\|_{L^1(\Omega)} \geqslant \delta_0^{c+\alpha}$, we have

$$\|f\|_{L^\infty(\omega)} \leqslant \delta_0^{-(c+\alpha)}\|fu^2\|_{L^1(\Omega)} \leqslant \delta_0^{-(c+\alpha)} M^{2-2\mu}\|fu^2\|^\mu_{L^1(\Omega)}. \tag{2.12}$$

The expected inequality follows then from (2.11) and (2.12).

(2) Let $w \in \mathscr{W}_s(v_0, V_0, \kappa, M)$ and $u \in \mathscr{S}_s(v_0, V_0, \kappa, M)$ so that $w = u^2$. As in (1), we have to prove that (2.4) holds with constants C and μ independent on w and $f \in C^{0,\alpha}(\overline{\Omega})$. As we have seen before, by homogeneity, it is enough to establish (2.4) when $\|f\|_{C^{0,\alpha}(\overline{\Omega})} = 1$. Let then $f \in C^{0,\alpha}(\overline{\Omega})$ satisfying $\|f\|_{C^{0,\alpha}(\overline{\Omega})} = 1$.

Since $H^2(\Omega)$ is continuously embedded in $C^{0,1/2}(\overline{\Omega})$, there exists a constant $a = a(\Omega) > 0$ so that

$$[u]_{1/2} = \sup\left\{\frac{|u(x) - u(y)|}{|x-y|^{1/2}};\; x, y \in \overline{\Omega}\; x \neq y\right\} \leqslant a\|u\|_{H^2(\Omega)} \leqslant aM.$$

Fix $\delta_1 \leqslant (\kappa/(2aM))^2$. Then a straightforward computation gives

$$|u| \geqslant \kappa/2 \quad \text{in } \Omega_{\delta_1}.$$

From the Hölder's continuity of f, we get, where $\eta = \delta_1/4$,

$$\|f\|_{L^\infty(\Omega^\eta)} = |f(x_0)| \leqslant |f(x)| + r^\alpha, \quad x \in B(x_0, r),\; 0 < r < \delta_0 = \delta_0(\eta).$$

Whence, proceeding as in (1), we get

$$\|f\|_{L^\infty(\Omega^\eta)} \leqslant \frac{\|fu^2\|_{L^1(\Omega)}}{\|u\|^2_{L^2(B(x_0,r))}} + r^\alpha.$$

This and Proposition 2.1 yield

$$\|f\|_{L^\infty(\Omega^\eta)} \leqslant C(\|fu^2\|_{L^1(\Omega)} r^{-c} + r^\alpha), \quad 0 < r < \delta_0. \tag{2.13}$$

On the other hand, noting that $B(y, r) \cap \Omega \subset \Omega_{\delta_1}$ when $y \in \Omega_{2\eta}$ and $0 < r < \eta$, we find

$$\begin{aligned}\|f\|_{L^\infty(\Omega_{2\eta})} \int_{B(y_0,r)\cap\Omega} u(x)^2 dx &= |f(y_0)| \int_{B(y_0,r)\cap\Omega} u(x)^2 dx \\ &\leqslant \int_{B(y_0,r)\cap\Omega} |f(x)| u(x)^2 dx + r^\alpha \int_{B(y_0,r)\cap\Omega} u(x)^2 dx.\end{aligned}$$

We have

$$\int_{B(y_0,r)\cap\Omega} u(x)^2dx \geqslant \kappa^2|B(y_0,r)\cap\Omega|$$

Proceeding similarly to the proof of [15, Appendix A], we find $\aleph=\aleph(\Omega)>0$ and $0<r_0=r_0(\Omega,\kappa,M)<\eta$ so that

$$|B(y_0,r)\cap\Omega|\geqslant \aleph r^n,\quad 0<r<r_0.$$

Whence

$$\|f\|_{L^\infty(\Omega_{2\eta})}\leqslant (1/\aleph)r^{-n}\|fu^2\|_{L^1(\Omega)}+r^\alpha,\quad 0<r<r_0.$$

This means that an estimate of the form (2.13) holds with $\|f\|_{L^\infty(\Omega^\eta)}$ substituted by $\|f\|_{L^\infty(\Omega_{2\eta})}$. In consequence,

$$\|f\|_{L^\infty(\Omega)}\leqslant C(\|fu^2\|_{L^1(\Omega)}r^{-c}+r^\alpha),\quad 0<r<\min(r_0,\delta_0).$$

We can then mimic the end of the proof of (1) in order to obtain the expected inequality. □

3. Proof of the main results

Proof of Theorem 1.1. Pick $\omega\Subset\Omega$. We firstly observe that, according to (1.1),

$$\{u_V;\ V\in\mathscr{D}_0(k,v_0,\overline{V})\}\subset\mathscr{S}_w(v_0,V_0,\kappa,M),$$

with $\kappa=\|h\|_{L^\infty(\Gamma)}$, $V_0=v_0+\|\overline{V}\|_{L^\infty(\Omega)}$ and $M=M\left(\Omega,p,v_0,k,\overline{V},h\right)$ is as in (1.1). We then apply (1) of Theorem 2.2 in order to obtain

$$\|V-\tilde V\|_{L^\infty(\omega)}\leqslant C\|(V-\tilde V)u_V^2\|_{L^1(\Omega)}^{2\mu}, \tag{3.1}$$

with $C=C\left(\Omega,p,v_0,k,\overline{V},h,\omega\right)$ and $\mu=\mu\left(\Omega,p,v_0,k,\overline{V},h,\omega\right)$.

On the other hand, we have from [17, Theorem 2.2, page 1781]

$$\|I_V^{1/2}(V-\tilde V)\|_{H^1(\Omega)}\leqslant C\|I_V^{1/2}-I_{\tilde V}^{1/2}\|_{H^1(\Omega)}^{1/2}. \tag{3.2}$$

Set, for simplicity's sake, $u=u_V$ (resp. $I=I_V$) and $\tilde u=u_{\tilde V}$ (resp. $\tilde I=I_{\tilde V}$). Then

$$\begin{aligned}(V-\tilde V)u^2&=Vu^2-\tilde V\tilde u^2+\tilde V(u^2-\tilde u^2)\\ &=I-\tilde I+\tilde V(|u|+|\tilde u|)(|u|-|\tilde u|).\end{aligned}$$

Hence

$$\|(V-\tilde V)u^2\|_{L^1(\Omega)}\leqslant C\left(\|I^{1/2}-\tilde I^{1/2}\|_{L^1(\Omega)}+\||u|-|\tilde u|\|_{L^1(\Omega)}\right). \tag{3.3}$$

But

$$|\tilde u|-|u|=\frac{1}{\tilde V}\left(I-\tilde I\right)+\frac{I^{1/2}}{V\tilde V}\left[\left(V-\tilde V\right)I^{1/2}\right]$$

implying

$$\|(V-\tilde V)u^2\|_{L^1(\Omega)}\leqslant C\left(\|I^{1/2}-\tilde I^{1/2}\|_{L^1(\Omega)}+\|(V-\tilde V)I^{1/2}\|_{L^1(\Omega)}\right). \tag{3.4}$$

Now a combination of (3.2), (3.3) and (3.4) yields

$$\|V-\tilde V\|_{L^\infty(\omega)}\leqslant C\|I^{1/2}-\tilde I^{1/2}\|_{H^1(\Omega)}^{\mu},$$

which is the expected inequality. □

Proof of Theorem 1.2. Quite similar to that of Theorem 1.1. We have only to apply (2) of Theorem 2.2 instead of (1) of Theorem 2.2. □

Proof of Theorem 1.3. The proof relies on an improvement of a weighted stability estimate obtained in [17, Theorem 2.2, page 1781].

Lemma 3.1. *Assume that the assumptions of Theorem 1.3 hold. Then there exist* $C = C(v_0, \overline{V}, h, K, \omega, \tilde{\Omega}) > 0$ *and* $1 > \mu' = \mu'(v_0, \overline{V}, h, K, \omega, \tilde{\Omega}) > 0$ *such that*

$$\|I_V^{1/2}(V - \tilde{V})\|_{L^2(\omega)} \leqslant C\|I_V^{1/2} - I_{\tilde{V}}^{1/2}\|_{H^1(\tilde{\Omega})}^{\mu'}. \tag{3.5}$$

Proof. We set $\theta = V^{-1/2}$ and $J = I_V^{1/2}$. We deduce from the proof of [17, Theorem 2.2, page 1781] that θ satisfies

$$J\Delta\left(J\theta\right) = -\frac{J^2}{\theta} \quad \text{in } \Omega. \tag{3.6}$$

Referring to [22], we see, as u_V is non identically equal to zero, that the set where u_V vanishes is of zero measure. Therefore, $J = Vu_V^2$ has the same property and hence θ verifies

$$\Delta\left(J\theta\right) = -\frac{J}{\theta} \quad \text{in } \Omega. \tag{3.7}$$

Let $\tilde{\theta} = \tilde{V}^{-1/2}$ and $\tilde{J} = I_{\tilde{V}}^{1/2}$. Identity (3.7), with V substituted by $\tilde{V}$, yields

$$\Delta\left(\tilde{J}\tilde{\theta}\right) = -\frac{\tilde{J}}{\tilde{\theta}} \quad \text{in } \Omega. \tag{3.8}$$

Taking the difference side by side of equations (3.7) and (3.8), we obtain

$$\Delta\left(J(\theta - \tilde{\theta})\right) - \frac{1}{\theta\tilde{\theta}}J(\theta - \tilde{\theta}) = \frac{\tilde{J} - J}{\tilde{\theta}} + \Delta\left(\tilde{\theta}(\tilde{J} - J)\right). \tag{3.9}$$

As $J(\theta - \tilde{\theta})$ has zero Cauchy data on $\tilde{\gamma}$, we deduce from [3, Theorem 1.7] that there exists $C(v_0, \overline{V}, h, K, \omega, \tilde{\Omega}) > 0$ and $1 > \mu'(v_0, \overline{V}, h, K, \omega, \tilde{\Omega}) > 0$ so that

$$\|J(\theta - \tilde{\theta})\|_{L^2(\omega)} \leqslant C\left(\left\|\frac{\tilde{J} - J}{\tilde{\theta}}\right\|_{L^2(\tilde{\Omega})} + \left\|\tilde{\theta}(\tilde{J} - J)\right\|_{H^1(\tilde{\Omega})}\right)^{\mu'}. \tag{3.10}$$

Whence the expected inequality follows. □

The rest of the proof is quite similar to that of Theorem 1.1. We apply again (1) of Theorem 2.2 to inequality (3.5). □

Proof of Theorem 1.4. We split γ into two components $\gamma_+ = \gamma \cap \Gamma_+$ and $\gamma_0 = \gamma \cap \Gamma_0$.

Let V and $\tilde{V}$ be as in the statement of Theorem 1.4. As $p > n$, $W^{2,p}(\Omega)$ is continuously embedded in $C^1(\overline{\Omega})$. Whence $u_V^2 \in W^{2,p}(\Omega)$. Inspecting the proof of [6, Proposition 3.1], we get that there exists a constant $\delta = \delta\left(\Omega, p, v_0, k, \overline{V}, h, , \gamma_0\right)$ and a neighborhood $\mathcal{U}_0$ of γ_0 in $\omega \cup \Gamma_0$ so that $|u_V|^{-\delta} \in L^1(\mathcal{U}_0)$. We get from the proof of [6, Lemma 1.3] that there exists $C_0 = C_0\left(\Omega, p, v_0, k, \overline{V}, h, \gamma_0\right)$ such that

$$\|V - \tilde{V}\|_{L^2(\mathcal{U}_0)} \leqslant C_0\|(V - \tilde{V})u_V^2\|_{L^1(\mathcal{U}_0)}^{\delta/(2+\delta)}.$$

In light of [16, Lemma B.1], this inequality entails

$$\|V - \tilde{V}\|_{L^\infty(\mathcal{U}_0)} \leqslant C_0\|(V - \tilde{V})u_V^2\|_{L^1(\mathcal{U}_0)}^{2\mu_0},$$

for some $\mu_0 = \mu_0\left(\Omega, p, v_0, k, \overline{V}, h, , \gamma_0\right)$.

As in Theorem 1.1, this inequality leads to the following one

$$\|V - \tilde{V}\|_{L^\infty(\mathcal{U}_0)} \leqslant C_0\|I_V^{1/2} - I_{\tilde{V}}^{1/2}\|_{H^1(\Omega)}^{\mu_0}. \tag{3.11}$$

On the other hand, we easily check that

$$|u_V| \geqslant \frac{1}{2}\min_{\gamma_+}|u_V| = \frac{1}{2}\min_{\gamma_+}|h| \ (>0)$$

in a neighborhood $\mathcal{U}_+$ of γ_+ in $\omega \cup \Gamma_+$, depending only Ω, p, v_0, k,$\overline{V}$, h and γ_+. We apply once again [16, Lemma B.1] in order to get

$$\|V-\tilde{V}\|_{L^\infty(\mathcal{U}_+)} \leqslant C_+\|(V-\tilde{V})u_V^2\|_{L^1(\mathcal{U}_+)}^{\mu_+},$$

for some $\mu_+ = \mu_+\left(\Omega, p, v_0, k, \overline{V}, h, , \gamma_+\right)$ and $C_+ = C_+\left(\Omega, p, v_0, k, \overline{V}, h, , \gamma_+\right)$.

From this inequality we deduce, again similarly as in the proof of Theorem 1.1,

$$\|V-\tilde{V}\|_{L^\infty(\mathcal{U}_+)} \leqslant C_+\|I_V^{1/2}-I_{\tilde{V}}^{1/2}\|_{H^1(\Omega)}^{\mu_+}. \tag{3.12}$$

Let $\tilde{\omega} \Subset \Omega$ so that $\omega \subset \tilde{\omega}\cup\mathcal{U}_0\cup\mathcal{U}_+$. By the interior stability estimate in Theorem 1.1, there exists $\tilde{\mu} = \tilde{\mu}\left(\Omega, p, v_0, k, \overline{V}, h, , \omega\right)$ and $C = C\left(\Omega, p, v_0, k, \overline{V}, h, \omega\right)$ so that

$$\|V-\tilde{V}\|_{L^\infty(\tilde{\omega})} \leqslant C\|I_V^{1/2}-I_{\tilde{V}}^{1/2}\|_{H^1(\Omega)}^{\tilde{\mu}}. \tag{3.13}$$

We end up getting the expected inequality by combining (3.11), (3.12) and (3.11), with $\mu = \min(\mu_0, \mu_+, \tilde{\mu})$. □

Appendix A

In this appendix, Ω is a bounded domain of $\mathbb{R}^n$, $n \geqslant 2$, with Lipschitz boundary Γ. Let

$$L = \mathrm{div}(A\nabla\,\cdot) + V,$$

where $V \in L^\infty(\Omega)$, $A = (a^{ij})$ is a symmetric matrix with coefficients in $W^{1,\infty}(\Omega)$ and there exist $\kappa > 0$ and $\Lambda > 0$ so that

$$A(x)\xi\cdot\xi \geqslant \kappa|\xi|^2,\quad x \in \Omega,\ \xi \in \mathbb{R}^n, \tag{A.1}$$

and

$$\|V\|_{L^\infty(\Omega)} + \|a^{ij}\|_{W^{1,\infty}(\Omega)} \leqslant \Lambda,\quad 1 \leqslant i,j \leqslant n. \tag{A.2}$$

Recall the following three-ball interpolation inequality, proved in [11] when $V = 0$ but still holds for any bounded V (see also [14]).

Theorem A.1. *Let $0 < k < \ell < m$. There exist $C > 0$ and $0 < s < 1$, only depending on Ω, k, ℓ, m, κ and Λ, such that*

$$\|v\|_{L^2(B(y,\ell r))} \leqslant C\|v\|_{L^2(B(y,kr))}^s\|v\|_{L^2(B(y,mr))}^{1-s}, \tag{A.3}$$

for all $v \in H^1(\Omega)$ satisfying $Lv = 0$ in Ω, $y \in \Omega$ and $0 < r < dist(y,\Gamma)/m$.

We know from [21, Theorem 2.4.7, page 53] that any Lipschitz domain has the uniform interior cone condition, abbreviated to **UICP** in the sequel. In particular, there exist $R > 0$ and $\theta \in \left]0, \frac{\pi}{2}\right[$ so that, to any $\tilde{x} \in \Gamma$ corresponds $\xi = \xi(\tilde{x}) \in \mathbb{S}^{n-1}$ for which

$$\mathcal{C}(\tilde{x}) = \{x \in \mathbb{R}^n;\ 0 < |x-\tilde{x}| < R,\ (x-\tilde{x})\cdot\xi > |x-\tilde{x}|\cos\theta\} \subset \Omega.$$

Define the geometric distance d_g^D, on a bounded domain D of $\mathbb{R}^n$, by

$$d_g^D(x,y) = \inf\left\{\ell(\psi);\ \psi : [0,1] \to D \text{ Lipschitz path joining } x \text{ to } y\right\},$$

where

$$\ell(\psi) = \int_0^1 |\dot{\psi}(t)|dt$$

is the length of ψ.

Note that, according to Rademacher's theorem, any Lipschitz continuous function $\psi : [0,1] \to D$ is almost everywhere differentiable with $|\dot{\psi}(t)| \leqslant k$ a.e. $t \in [0,1]$, where k is the Lipschitz constant of ψ.

Lemma A.1. *Let D be a bounded Lipschitz domain of $\mathbb{R}^n$. Then $d_g^D \in L^\infty(D \times D)$.*

A proof of this lemma can be found in [18].

In the rest of this text

$$\mathbf{d}_g = \|d_g^\Omega\|_{L^\infty(\Omega\times\Omega)}.$$

Proof of Theorem 2.1. In this proof C denote a generic constant that can depend only on Ω, v_0, V_0, κ and M.

Step 1. Let $y, y_0 \in \Omega^{3\delta}$ and $\psi : [0,1] \to \Omega$ be a Lipschitz path joining y_0 to y so that $\ell(\psi) \leqslant d_g^\Omega(y_0, y) + 1$. Let $t_0 = 0$ and $t_{k+1} = \inf\{t \in [t_k, 1];\ \psi(t) \notin B(\psi(t_k), \delta)\}$, $k \geqslant 0$. We claim that there exists an integer $N \geqslant 1$ so that $\psi(1) \in B(\psi(t_N), \delta)$. If not, we would have $\psi(1) \notin B(\psi(t_k), \delta)$ for any $k \geqslant 0$. As the sequence (t_k) is non decreasing and bounded from above by 1, it converges to $\hat{t} \leqslant 1$. In particular, there exists an integer $k_0 \geqslant 1$ so that $\psi(t_k) \in B\left(\psi(\hat{t}), \delta/2\right)$, $k \geqslant k_0$. But this contradicts the fact that $|\psi(t_{k+1}) - \psi(t_k)| = \delta$, $k \geqslant 0$.

Let us check that $N \leqslant N_0$, where N_0 only depends on $\mathbf{d}_g$ and δ. Pick $1 \leqslant j \leqslant n$ so that

$$\max_{1\leqslant i\leqslant n} |\psi_i(t_{k+1}) - \psi_i(t_k)| = |\psi_j(t_{k+1}) - \psi_j(t_k)|.$$

Then

$$\delta \leqslant n\,|\psi_j(t_{k+1}) - \psi_j(t_k)| = n\left|\int_{t_k}^{t_{k+1}} \dot{\psi}_j(t)dt\right| \leqslant n\int_{t_k}^{t_{k+1}} |\dot{\psi}(t)|dt.$$

Consequently, where $t_{N+1} = 1$,

$$(N+1)\delta \leqslant n\sum_{k=0}^{N}\int_{t_k}^{t_{k+1}} |\dot{\psi}(t)|dt = n\ell(\psi) \leqslant n(\mathbf{d}_g + 1).$$

Therefore

$$N \leqslant N_0 = \left[\frac{n(\mathbf{d}_g+1)}{\delta}\right].$$

Here $[n(\mathbf{d}_g+1)/\delta]$ is the integer part of $n(\mathbf{d}_g+1)/\delta$.

Let $y_k = \psi(t_k)$, $0 \leqslant k \leqslant N$. If $|z - y_{k+1}| < \delta$ then

$$|z - y_k| \leqslant |z - y_{k+1}| + |y_{k+1} - y_k| < 2\delta.$$

In other words, $B(y_{k+1}, \delta) \subset B(y_k, 2\delta)$.

We get from Theorem A.1

$$\|u\|_{L^2(B(y_j,2\delta))} \leqslant C\|u\|^{1-s}_{L^2(B(y_j,3\delta))}\|u\|^s_{L^2(B(y_j,\delta))}, \quad 0 \leqslant j \leqslant N. \tag{A.4}$$

Set $I_j = \|u\|_{L^2(B(y_j,\delta))}$, $0 \leqslant j \leqslant N$ and $I_{N+1} = \|u\|_{L^2(B(y,\delta))}$. Since $B(y_{j+1}, \delta) \subset B(y_j, 2\delta)$, $1 \leqslant j \leqslant N-1$, estimate (A.4) implies

$$I_{j+1} \leqslant CM_0^{1-s}I_j^s, \quad 0 \leqslant j \leqslant N, \tag{A.5}$$

where we set $M_0 = \|u\|_{L^2(\Omega)}$.

Let $C_1 = C^{1+s+\ldots+s^{N+1}}$ and $\beta = s^{N+1}$. Then, by a simple induction argument, estimate (A.5) yields

$$I_{N+1} \leqslant C_1 M_0^{1-\beta} I_0^{\beta}. \tag{A.6}$$

Without loss of generality, we assume in the sequel that $C \geqslant 1$ in (A.5). Using that $N \leqslant N_0$, we have

$$\begin{aligned}
&\beta \geqslant \beta_0 = s^{N_0+1} \geqslant s e^{-\frac{\kappa}{\delta}} = \psi(\delta), \quad \text{where } \kappa = n(\mathbf{d}_g + 1)|\ln s|, \\
&C_1 \leqslant C^{\frac{1}{1-s}}, \\
&\left(\frac{I_0}{M_0}\right)^{\beta} \leqslant \left(\frac{I_0}{M_0}\right)^{\beta_0}.
\end{aligned}$$

These estimates in (A.6) gives

$$\frac{I_{N+1}}{M_0} \leqslant C \left(\frac{I_0}{M_0}\right)^{\psi(\delta)}.$$

In other words,

$$\frac{\|u\|_{L^2(B(y,\delta))}}{\|u\|_{L^2(\Omega)}} \leqslant C \left(\frac{\|u\|_{L^2(B(y_0,\delta))}}{\|u\|_{L^2(\Omega)}}\right)^{\psi(\delta)}.$$

Applying Young's inequality, we get from this inequality

$$\|u\|_{L^2(B(y,\delta))} \leqslant C \left(\epsilon^{\frac{1}{1-\psi(\delta)}} \|u\|_{L^2(\Omega)} + \epsilon^{-\frac{1}{\psi(\delta)}} \|u\|_{L^2(B(y_0,\delta))}\right), \tag{A.7}$$

$\epsilon > 0$, $y, y_0 \in \Omega^{3\delta}$.

Step 2. Fix $\tilde{x} \in \Gamma$ so that $|u(\tilde{x})| = \|u\|_{L^\infty(\Gamma)}$. Let $\xi = \xi(\tilde{x})$ be as in the definition of the **UICP**. Let $x_0 = \tilde{x} + \delta\xi$, $\delta \leqslant R/2$, $d_0 = |x_0 - \tilde{x}| = \delta$ and $\rho_0 = d_0 \sin\theta/3$. Note that $B(x_0, 3\rho_0) \subset \mathcal{C}(\tilde{x})$.

By induction in k, we construct a sequence of balls $(B(x_k, 3\rho_k))$, contained in $\mathcal{C}(\tilde{x})$, as follows

$$\begin{cases}
x_{k+1} = x_k - \alpha_k \xi, \\
\rho_{k+1} = \mu\rho_k, \\
d_{k+1} = \mu d_k,
\end{cases}$$

where

$$d_k = |x_k - \tilde{x}|, \quad \rho_k = \vartheta d_k, \quad \alpha_k = (1-\mu) d_k,$$

with

$$\vartheta = \frac{\sin\theta}{3}, \quad \mu = \frac{3 - 2\sin\theta}{3 - \sin\theta}.$$

Note that this construction guarantees that, for each k, $B(x_k, 3\rho_k) \subset \mathcal{C}(\tilde{x})$ and

$$B(x_{k+1}, \rho_{k+1}) \subset B(x_k, 2\rho_k). \tag{A.8}$$

We get, by applying Theorem A.1, that there exist $C > 0$ and $0 < s < 1$, only depending on Ω, v_0 and V_0, so that

$$\begin{aligned}
\|u\|_{L^2(B(x_k,2\rho_k))} &\leqslant C \|u\|_{L^2(B(x_k,3\rho_k))}^{1-s} \|u\|_{L^2(B(x_k,\rho_k))}^{s} \\
&\leqslant C M^{1-s} \|u\|_{L^2(B(x_k,\rho_k))}^{s}.
\end{aligned} \tag{A.9}$$

In light of (A.8), (A.9) gives

$$\|u\|_{L^2(B(x_{k+1},\rho_{k+1}))} \leqslant C M^{1-s} \|u\|_{L^2(B(x_k,\rho_k))}^{s}. \tag{A.10}$$

Let $J_k = \|u\|_{L^2(B(x_k,\rho_k))}$, $k \geqslant 0$. Then (A.10) is rewritten as follows

$$J_{k+1} \leqslant CM^{1-s}J_k^s.$$

An induction in k yields

$$J_k \leqslant C^{1+s+\ldots+s^{k-1}}M^{(1-s)(1+s+\ldots+s^{k-1})}J_0^{s^k}.$$

That is

$$J_k \leqslant \left[C^{\frac{1}{1-s}}M\right]^{1-s^k}J_0^{s^k}. \tag{A.11}$$

Applying Young's inequality we obtain, for any $\epsilon > 0$,

$$\begin{aligned} J_k &\leqslant (1-s^k)\epsilon^{\frac{1}{1-s^k}}C^{\frac{1}{1-s}}M + s^k\epsilon^{-\frac{1}{s^k}}J_0 \\ &\leqslant \epsilon^{\frac{1}{1-s^k}}C^{\frac{1}{1-s}}M + \epsilon^{-\frac{1}{s^k}}J_0 \\ &\leqslant C\epsilon^{\frac{1}{1-s^k}}M + \epsilon^{-\frac{1}{s^k}}J_0. \end{aligned} \tag{A.12}$$

Now, since $u \in C^{0,1/2}(\overline{\Omega})$,

$$|u(\tilde{x})| \leqslant [u]_{1/2}|\tilde{x}-x|^{1/2} + |u(x)|, \quad x \in B(x_k,\rho_k).$$

Hence

$$|\mathbb{S}^{n-1}|\rho_k^n|u(\tilde{x})|^2 \leqslant 2[u]_{1/2}^2\int_{B(x_k,\rho_k)}|\tilde{x}-x|dx + 2\int_{B(x_k,\rho_k)}|u(x)|^2dx.$$

Or equivalently

$$|u(\tilde{x})|^2 \leqslant 2|\mathbb{S}^{n-1}|^{-1}\rho_k^{-n}\left([u]_{1/2}^2\int_{B(x_k,\rho_k)}|\tilde{x}-x|dx + \int_{B(x_k,\rho_k)}|u(x)|^2dx\right).$$

A simple computation shows that $d_k = \mu^k d_0$. Then

$$|\tilde{x}-x| \leqslant |\tilde{x}-x_k| + |x_k - x| \leqslant d_k + \rho_k = (1+\vartheta)d_k = (1+\vartheta)\mu^k d_0.$$

Therefore,

$$|u(\tilde{x})|^2 \leqslant 2\left(M^2(1+\vartheta)^{1/2}d_0^{1/2}\mu^k + |\mathbb{S}^{n-1}|^{-1}(\vartheta d_0)^{-n}\mu^{-nk}\|u\|_{L^2(B(x_k,\rho_k))}^2\right)$$

implying, when $d_0(=\delta) \leqslant 1$,

$$|u(\tilde{x})| \leqslant C\left(M\mu^{k/2} + \mu^{-nk/2}\delta^{-n/2}J_k\right). \tag{A.13}$$

Inequalities (A.12) and (A.13) gives

$$|u(\tilde{x})| \leqslant C\left(\mu^{k/2}M + \mu^{-nk/}\epsilon^{1/(1-s^k)}\delta^{-n/2}M + \mu^{-nk/2}\epsilon^{-1/s^k}\delta^{-n/2}J_0\right). \tag{A.14}$$

We get, by choosing $\epsilon = \mu^{(1-s^k)(n+1)k/2}$ in (A.14),

$$|u(\tilde{x})| \leqslant C\left(\mu^{k/2}M + \mu^{k/2}\delta^{-n/2}M + \mu^{-(n+1)k/(2s^k)+k/2}\delta^{-n/2}J_0\right).$$

Hence

$$|u(\tilde{x})| \leqslant C\delta^{-n/2}\left(\mu^{k/2}M + \mu^{-(n+1)k/(2s^k)}J_0\right), \tag{A.15}$$

by using $\delta \leqslant \mathrm{diam}(\Omega)$.

Let $t > 0$ and k be the integer so that $k \leqslant t < k+1$. It follows from (A.15)

$$|u(\tilde{x})| \leqslant C\delta^{-n/2}\left(\mu^{t/2}M + \mu^{-t(n+1)/(2s^t)}J_0\right). \tag{A.16}$$

Let $p = (n+1)/2 + |\ln s|$. Then (A.16) yields

$$|u(\tilde{x})| \leqslant C\delta^{-n/2}\left(\mu^{t/2}M + \mu^{-e^{pt}}J_0\right). \tag{A.17}$$

Putting $e^{pt} = 1/\epsilon$, $0 < \epsilon < 1$, we get from (A.17)

$$|u(\tilde{x})| \leqslant C\delta^{-n/2}\left(\epsilon^{\beta}M + e^{|\ln\mu|/\epsilon}J_0\right), \tag{A.18}$$

where $\beta = |\ln\mu|/(2p)$.

Step 3. A combination of (A.7) and (A.18) entails, with $0 < \epsilon < 1$ and $\epsilon_1 > 0$,

$$|u(\tilde{x})| \leqslant C\delta^{-n/2}\left(\epsilon^{\beta}M + e^{|\ln\mu|/\epsilon}\left(\epsilon_1^{1/(1-\psi(\delta))}M + \epsilon_1^{-1/\psi(\delta)}\|u\|_{L^2(B(y_0,\delta))}\right)\right).$$

Hence, where $\ell = n/2$ and $\rho = |\ln\mu|$,

$$\|u\|_{L^\infty(\Gamma)} \leqslant C\delta^{-\ell}\left(\epsilon^{\beta}M + e^{\rho/\epsilon}\left(\epsilon_1^{1/(1-\psi(\delta))}M + \epsilon_1^{-1/\psi(\delta)}\|u\|_{L^2(B(y_0,\delta))^n}\right)\right).$$

In this inequality we take $\epsilon_1 = \epsilon^{\beta(1-\psi(\delta))}e^{-\rho(1-\psi(\delta))/\epsilon}$. Using that

$$\epsilon_1^{-1/\psi(\delta)} \leqslant e^{(\rho+\beta)(1-\psi(\delta))/(\epsilon\psi(\delta))},$$

we obtain in a straightforward manner

$$\|u\|_{L^\infty(\Gamma)} \leqslant C\delta^{-\ell}\left(\epsilon^{\beta}M + e^{(\rho+\beta)(1-\psi(\delta))/(\epsilon\psi(\delta))}\|u\|_{L^2(B(y_0,\delta))^n}\right).$$

If $\phi(\delta) = (\rho+\beta)(1-\psi(\delta))/\psi(\delta)$ then we can rewrite the previous estimate as follows

$$\|u\|_{L^\infty(\Gamma)} \leqslant C\delta^{-\ell}\left(\epsilon^{\beta}M + e^{\phi(\delta)/\epsilon}\|u\|_{L^2(B(y_0,\delta))^n}\right),$$

or equivalently

$$\|u\|_{L^\infty(\Gamma)} \leqslant C\delta^{-\ell}\left(t^{-\beta}M + e^{t\phi(\delta)}\|u\|_{L^2(B(y_0,\delta))}\right), \quad t > 1. \tag{A.19}$$

If $M/\|u\|_{L^2(B(y_0,\delta))} > e^{\phi(\delta)}$, we find $t > 1$ so that $M/\|u\|_{L^2(B(y_0,\delta))} = t^{\beta}e^{t\phi(\delta)}$. The estimate (A.19) with that t yields

$$\|u\|_{L^\infty(\Gamma)} \leqslant C\delta^{-\ell}M\left(\frac{1}{(\beta+\phi(\delta)}\ln\left(\frac{M}{\|u\|_{L^2(B(y_0,\delta))}}\right)\right)^{-\beta}.$$

In light of the inequality $\|u\|_{L^\infty(\Gamma)} \geqslant \eta$, this estimate implies

$$\eta \leqslant C\delta^{-\ell}M\left(\frac{1}{\beta+\phi(\delta)}\ln\left(\frac{M}{\|u\|_{L^2(B(y_0,\delta))}}\right)\right)^{-\beta}.$$

This inequality is equivalent to the following one

$$Me^{-C(\beta+\phi(\delta))\delta^{-\ell/\beta}(M/\eta)^{1/\beta}} \leqslant \|u\|_{L^2(B(y_0,\delta))}. \tag{A.20}$$

Otherwise,

$$Me^{-\phi(\delta)} \leqslant \|u\|_{L^2(B(y_0,\delta))}. \tag{A.21}$$

We derive from (A.20) and (A.21) that, there exist $C > 0$ and δ^* so that

$$e^{-e^{c/\delta}} \leqslant \|u\|_{L^2(B(y_0,\delta))^n}, \ 0 < \delta \leqslant \delta^*.$$

Obviously, a similar estimate holds for $\delta \geqslant \delta^*$. □

References

[1] G. Alessandrini, Global stability for a coupled physics inverse problem, *Inverse Problems* 30 (2014) 075008 (10pp).
[2] G. Alessandrini, M. Di Cristo, E. Francini, S. Vessella, Stability for quantitative photoacoustic tomography with well chosen illuminations, *Ann. Mat. Pura e Appl.* 196 (2) (2017), 395-406.
[3] G. Alessandrini, L. Rondi, E. Rosset, and S. Vessella, The stability for the Cauchy problem for elliptic equations. Inverse problems, 25 (12), 123004 (2009).
[4] H. Ammari, E. Bonnetier, Y. Capdeboscq, M. Tanter and M. Fink, Electrical impedance tomography by elastic deformation, SIAM J. Appl. Math. 68 (6) (2008), 1557-1573.
[5] H. Ammari, Y. Capdeboscq, F. De Gournay, A. Rozanova-Pierrat, F. Triki, Microwave imaging by elastic perturbation, SIAM J. Appl. Math. 71 (6) (2011), 2112-2130.
[6] K. Ammari, M. Choulli and F. Triki, Hölder stability in determining the potential and the damping coefficient in a wave equation, arXiv:1609.06102.
[7] H. Ammari, J. Garnier, L.H. Nguyen and L. Seppecher, Reconstruction of a piecewise smooth absorption coefficient by an acousto-optic process, Commun. Part. Differ. Equat. 38 (10) (2013), 1737-1762.
[8] G. Bal and K. Ren, Non-uniqueness result for a hybrid inverse problem, Contemporary Math. 559 (2011), 29-38.
[9] G. Bal and J. C. Schotland, Inverse scattering and acousto-optics imaging, Phys. Rev. Letters 104 (2010), p. 043902.
[10] G. Bal and G. Uhlmann, Reconstruction of coefficients in scalar second order elliptic equations from knowledge of their solutions, Commun. Pure and Appl. Math. 66 (10) (2013), 1629-1652.
[11] E. Bonnetier, M. Choulli and F. Triki, Stability for quantitative photo-acoustic tomography revisited, preprint.
[12] D. Colton, and R. Kress, Integral equation methods in scattering theory, A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1983.
[13] M. Choulli, Applications of elliptic Carleman inequalities, *BCAM SpringerBriefs*, Springer, Belin, 2016.
[14] M. Choulli, An introduction to the analysis of elliptic partial differential equations, book, to appear.
[15] M. Choulli and L. Kayser, Gaussian lower bound for the Neumann Green function of a general parabolic operator, *Positivity* **19** (3) (2015) 625-646.
[16] M. Choulli and Y. Kian, Logarithmic stability in determining the time-dependent zero order coefficient in a parabolic equation from partial Diriclet-to-Neumann map. Application to the determination of a nonlinear term, to appear in *J. Math. Pure Appl.* 114 (2018), 235-261.
[17] M. Choulli and F. Triki, New stability estimates for the inverse medium problem with internal data, *SIAM J. Math. Anal.* **47** (3) (2015), 1778-1799.
[18] M. Choulli and M. Yamamoto, Logarithmic stability of parabolic Cauchy problems, arXiv:1702.06299
[19] N. Garofalo and F.-H. Lin, Monotonicity properties of variational integrals, A_p weights and unique continuation, Indiana Univ. Math. J. 35 (2) (1986), 245-268.
[20] N. Garofalo and F.-H. Lin, Unique continuation for elliptic operators: a geometric-variational approach, Comm. Pure Appl. Math. 40 (3) (1987), 347-366.
[21] A. Henrot and M. Pierre, Variation et optimisation de formes, vol. 48, SMAI-Springer Verlag, Berlin, 2005.
[22] L. Robbiano, Dimension des zéros d'une solution faible d'un opérateur elliptique, J. Math. Pures Appl. 67 (1988), 339-357.

Mourad Choulli, IECL, UMR CNRS 7502, Université de Lorraine, Boulevard des Aiguillettes BP 70239 54506 Vandoeuvre Les Nancy cedex- Ile du Saulcy - 57 045 Metz Cedex 01 France

E-mail address: `mourad.choulli@univ-lorraine.fr`

Faouzi Triki, Laboratoire Jean Kuntzmann, UMR CNRS 5224, Université Grenoble-Alpes, 700 Avenue Centrale, 38401 Saint-Martin-d'Hères, France

E-mail address: `faouzi.triki@univ-grenoble-alpes.fr`