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# Regularity results for the solutions of a non-local model of traffic flow

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## Abstract

We consider a non-local traffic model involving a convolution product. Unlike other studies, the considered kernel is discontinuous on  $\mathbb{R}$ . We prove Sobolev estimates and prove the convergence of approximate solutions solving a viscous and regularized non-local equation. It leads to weak,  $\mathbf{C}([0, T], \mathbf{L}^2(\mathbb{R}))$ , and smooth,  $\mathbf{W}^{2,2N}([0, T] \times \mathbb{R})$ , solutions for the non-local traffic model.

**Key words:** Scalar conservation laws; Anisotropic non-local flux; Traffic flow models; Viscous approximation; Sobolev estimates.

## 1 Introduction

We consider the non-local traffic model introduced in [4, 8] to account for the reaction of drivers to downstream traffic conditions. It consists in the following scalar conservation law, where the traffic velocity depends on a weighted mean of the density:

$$\partial_t \rho + \partial_x(\rho v(\rho * \omega)) = 0, \quad (1.1)$$

where

$$(\rho * \omega)(t, x) = \int_0^\eta \rho(t, x + y)\omega(y) dy = \int_x^{x+\eta} \rho(t, y)\omega(y - x) dy. \quad (1.2)$$

We make the following assumptions for  $k = 1, 2, 3$ :

( $\mathbf{A}_\omega^k$ )  $\omega \in \mathbf{C}^k([0, \eta])$  is non-negative with support in  $[0, \eta]$  and is non-increasing on  $[0, \eta]$ .

( $\mathbf{A}_v^k$ )  $v \in \mathbf{C}^k(\mathbb{R}^+)$  with  $v', \dots, v^{(k)}$  bounded.

For traffic flow applications, it is reasonable to assume that  $v$  is non-increasing, even if monotonicity is not required in this paper. We also recall that a similar model, considering a weighted mean of downstream speeds, has been recently introduced in [7]. More generally, model (1.1) belongs to the class of conservation laws with non-local flux functions, which appear in several applications, see for example [3, 6, 9, 10, 15]. We remark that most of the available well-posedness results concern equations involving smooth convolution kernels [1, 2], and are based on the construction of finite-volume approximations and the use of Kružkov's doubling of variable technique [12]. In particular, these results rely on the concept of entropy solutions. Only recently, alternative proofs based on fixed point theorems have been proposed for specific cases [11, 14], allowing to get rid of the entropy requirement.

In general, solutions to non-local equations may be discontinuous [13], despite the expected regularizing effect of the convolution product. Therefore, given any initial datum  $\rho^0 \in \mathbf{L}^\infty(\mathbb{R}) \cap \mathbf{L}^1(\mathbb{R})$ , the solutions to the Cauchy problem for (1.1) are usually intended in the following weak form

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**Definition 1.** A function  $\rho \in (\mathbf{L}^\infty \cap \mathbf{L}^1)(\mathbb{R}^+ \times \mathbb{R})$  is a solution of (1.1) with initial datum  $\rho^0$  if

$$\int_0^{+\infty} \int_{-\infty}^{+\infty} (\rho \partial_t \varphi + \rho v(\rho * \omega) \partial_x \varphi)(t, x) dx dt + \int_{-\infty}^{+\infty} \rho^0(x) \varphi(0, x) dx = 0, \quad (1.3)$$

1 for all  $\varphi \in \mathbf{C}_c^\infty(\mathbb{R}^2)$ .

In this paper, we are interested in deriving regularity properties of solutions to (1.1). To this end, we will consider approximate solutions satisfying the viscous and regularized non-local equation

$$\partial_t \rho_\varepsilon + \partial_x(\rho_\varepsilon v(\rho_\varepsilon * \omega_\varepsilon)) = \varepsilon \partial_{xx}^2 \rho_\varepsilon, \quad (1.4)$$

2 where, for any  $\varepsilon \in ]0, 1]$ , the smooth function  $\omega_\varepsilon$  is an extension of  $\omega$  with the following regularities:

( $\mathbf{A}_{\omega_\varepsilon}^k$ )  $\omega_\varepsilon \in \mathbf{C}^k(\mathbb{R})$  is non-negative with a support in  $[-\varepsilon, \eta + \varepsilon]$ , is non-decreasing on  $[-\varepsilon, x_\varepsilon]$ , for some  $x_\varepsilon \in ]-\varepsilon, 0]$ , is non-increasing on  $[x_\varepsilon, \eta + \varepsilon]$  and  $\omega_\varepsilon = \omega$  on  $[0, \eta]$ .

We set  $W_\varepsilon := \omega_\varepsilon(x_\varepsilon)$  and we assume that  $\lim_{\varepsilon \rightarrow 0} W_\varepsilon = \omega(0)$ . Without loss of generality we can assume

$$W_\varepsilon \leq 2\omega(0). \quad (1.5)$$

3 ( $\mathbf{B}_{\omega_\varepsilon}^k$ )  $\omega_\varepsilon^{(j)}(-\varepsilon) = \omega_\varepsilon^{(j)}(\eta + \varepsilon) = 0$  for  $j = 1, \dots, k$  and  $|\omega_\varepsilon'(u)| \leq 2W_\varepsilon/\varepsilon$  on  $[-\varepsilon, 0]$  and  $|\omega_\varepsilon'(u)| \leq$   
4  $2\omega(\eta)/\varepsilon$  on  $[\eta, \eta + \varepsilon]$ .

5 **Remark 1.** Given  $\omega$  satisfying ( $\mathbf{A}_\omega^k$ ), we can construct a function  $\omega_\varepsilon$  satisfying ( $\mathbf{A}_{\omega_\varepsilon}^k$ ) and ( $\mathbf{B}_{\omega_\varepsilon}^k$ ).  
6 To construct such extensions, for example in the simplest case where the derivatives of  $\omega$  vanish at  
7 0 and  $\eta$ , we use the function  $\varphi$  which is zero for  $x \leq -1$ , 1 for  $x \geq 0$ , non-decreasing and of class  
8  $C^\infty$  and we define  $\omega_\varepsilon(x) = \omega(0)\varphi(x/\varepsilon)$  for  $x < 0$  and  $\omega(0)\varphi((\eta - x)/\varepsilon)$  for  $x > \eta$ .

9 Notice that a similar approximation was used in [5] to establish a convergence property for the  
10 singular limit where the (smooth) convolution kernel is replaced by a Dirac delta, in the viscous  
11 case. Here, we will study the properties of smooth solutions  $\rho_\varepsilon$  of this equation corresponding to a  
12 fixed initial datum  $\rho^0$ , and then we will recover properties for  $\rho$  passing to the limit as  $\varepsilon \rightarrow 0$ .

13 We have the following result.

14 **Theorem 1.** We assume ( $\mathbf{A}_\omega^2$ )-( $\mathbf{A}_v^3$ ). Let  $\rho_\varepsilon$  be a solution of (1.4) with initial datum  $\rho^0$ . We  
15 assume  $\rho^0 \in \mathbf{W}^{1,4}(\mathbb{R}) \cap \mathbf{H}^2(\mathbb{R})$ . Then, for  $T > 0$  sufficiently small,  $\rho_\varepsilon$  converges in  $\mathbf{L}_{\text{loc}}^2([0, T] \times \mathbb{R})$   
16 to a solution  $\rho \in \mathbf{C}([0, T], \mathbf{L}^2(\mathbb{R}))$  to equation (1.1) with initial datum  $\rho^0$ . Furthermore, if  $\rho^0 \in$   
17  $\mathbf{W}^{1,2N}(\mathbb{R})$ ,  $N \in \mathbb{N}^*$ , then  $\rho \in \mathbf{W}^{1,2N}([0, T] \times \mathbb{R})$ , and if  $\rho^0 \in \mathbf{W}^{1,4N}(\mathbb{R}) \cap \mathbf{H}^1(\mathbb{R}) \cap \mathbf{W}^{2,2N}(\mathbb{R})$ , then  
18  $\rho \in \mathbf{W}^{2,2N}([0, T] \times \mathbb{R})$ .

19 In particular, this provides an alternative proof of existence of weak solutions, locally in time.  
20 To prove this result, in Section 2 we first establish estimates on the non-local term and we derive  
21  $\mathbf{L}^p(\mathbb{R})$ ,  $p > 1$ , estimates for  $\rho_\varepsilon$ , then we get estimates in  $\mathbf{W}^{1,2N}(\mathbb{R})$  for  $\rho_\varepsilon$  with respect to  $x$ . This  
22 allows to prove that there exists  $T > 0$  such that the sequence  $\rho_\varepsilon$  is uniformly bounded with respect  
23 to  $\varepsilon$  in  $\mathbf{L}^\infty(\mathbb{R})$  on  $[0, T]$ . Then we prove uniform space estimates in  $\mathbf{W}^{2,2N}(\mathbb{R})$  for  $\rho_\varepsilon$ , which allows  
24 to derive estimates on  $\partial_t \rho_\varepsilon$ . The proof of Theorem 1 is deferred to Section 3.

## 2 Estimates

Here and in the following sections, we will denote by

$$\|\rho\|_\infty := \|\rho\|_{\mathbf{L}^\infty([0,T] \times \mathbb{R})}$$

and by

$$\|\rho(t, \cdot)\|_\infty := \|\rho(t, \cdot)\|_{\mathbf{L}^\infty(\mathbb{R})}.$$

Moreover, notice that we have

$$\begin{aligned} (\rho * \omega_\varepsilon)(t, x) &= \int_{\mathbb{R}} \rho(t, x+y) \omega_\varepsilon(y) dy = \int_{\mathbb{R}} \rho(t, y) \omega_\varepsilon(y-x) dy \\ &= \int_{-\varepsilon}^{\eta+\varepsilon} \rho(t, x+y) \omega_\varepsilon(y) dy = \int_{x-\varepsilon}^{x+\eta+\varepsilon} \rho(t, y) \omega_\varepsilon(y-x) dy. \end{aligned}$$

### 2.1 Estimates of the non-local term

We start by proving the following estimates on the non-local term.

**Proposition 1.** 1. We assume  $(\mathbf{A}_{\omega_\varepsilon}^1)$  and that  $\rho$  is a continuous function. For any  $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$ , we have

$$|\partial_x(\rho * \omega_\varepsilon)(t, x)| \leq 2\|\rho(t, \cdot)\|_\infty W_\varepsilon. \quad (2.1)$$

2. We assume  $(\mathbf{A}_{\omega_\varepsilon}^2) - (\mathbf{B}_{\omega_\varepsilon}^1)$  and that  $\rho$  is a  $C^1$  function. For any  $t \geq 0$ ,  $p > 1$ , we have

$$\begin{aligned} \left( \int_{\mathbb{R}} |\partial_{xx}^2(\rho * \omega_\varepsilon)|^p(t, x) dx \right)^{1/p} &\leq \eta^{1/p} \left( \int_0^\eta |\omega''(u)|^{p/(p-1)} du \right)^{1-1/p} \left( \int_{\mathbb{R}} \rho^p(t, y) dy \right)^{1/p} \\ &\quad + (|\omega'(\eta-)| + |\omega'(0+)|) \left( \int_{\mathbb{R}} \rho^p(t, x) dx \right)^{1/p} \\ &\quad + 2(\omega(\eta) + W_\varepsilon) \left( \int_{\mathbb{R}} |\partial_x \rho(t, x)|^p dx \right)^{1/p}. \end{aligned} \quad (2.2)$$

3. We assume  $(\mathbf{A}_\omega^2) - (\mathbf{B}_{\omega_\varepsilon}^2)$  that  $\rho$  is a  $C^2$  function. For any  $t \geq 0$ ,  $p > 1$ , we have

$$\begin{aligned} \left( \int_{\mathbb{R}} |\partial_{xxx}^3(\rho * \omega_\varepsilon)|^p(t, x) dx \right)^{1/p} &\leq \eta^{1/p} \left( \int_0^\eta |\omega''(u)|^{p/(p-1)} du \right)^{1-1/p} \left( \int_{\mathbb{R}} |\partial_x \rho(t, y)|^p dy \right)^{1/p} \\ &\quad + (|\omega'(\eta-)| + |\omega'(0+)|) \left( \int_{\mathbb{R}} |\partial_x \rho(t, x)|^p dx \right)^{1/p} \\ &\quad + 2(\omega(\eta) + W_\varepsilon) \left( \int_{\mathbb{R}} |\partial_{xx}^2 \rho(t, x)|^p dx \right)^{1/p}. \end{aligned} \quad (2.3)$$

*Proof.* 1. From

$$\begin{aligned} \partial_x(\rho * \omega_\varepsilon)(t, x) &= - \int_{x-\varepsilon}^{x+\eta+\varepsilon} \rho(t, y) \omega'_\varepsilon(y-x) dy + \rho(t, x+\eta+\varepsilon) \omega_\varepsilon(\eta+\varepsilon) - \rho(t, x-\varepsilon) \omega_\varepsilon(-\varepsilon) \\ &= - \int_{x-\varepsilon}^{x+\eta+\varepsilon} \rho(t, y) \omega'_\varepsilon(y-x) dy = - \int_{-\varepsilon}^{\eta+\varepsilon} \rho(t, u+x) \omega'_\varepsilon(u) du, \end{aligned}$$

we obtain

$$\begin{aligned}
|\partial_x(\rho * \omega_\varepsilon)(t, x)| &\leq \|\rho(t, \cdot)\|_\infty \int_{-\varepsilon}^{\eta+\varepsilon} |\omega'_\varepsilon(u)| du \\
&\leq \|\rho(t, \cdot)\|_\infty \left( \int_{-\varepsilon}^{x_\varepsilon} \omega'_\varepsilon(u) du - \int_{x_\varepsilon}^{\eta+\varepsilon} \omega'_\varepsilon(u) du \right) \\
&\leq 2\|\rho(t, \cdot)\|_\infty W_\varepsilon.
\end{aligned}$$

2. From

$$\begin{aligned}
\partial_{xx}^2(\rho * \omega_\varepsilon)(t, x) &= \int_{x-\varepsilon}^{x+\eta+\varepsilon} \rho(t, y) \omega''_\varepsilon(y-x) dy - \rho(t, x+\eta+\varepsilon) \omega'_\varepsilon(\eta+\varepsilon) + \rho(t, x-\varepsilon) \omega'_\varepsilon(-\varepsilon) \\
&= \int_{x-\varepsilon}^{x+\eta+\varepsilon} \rho(t, y) \omega''_\varepsilon(y-x) dy = \int_{-\varepsilon}^{\eta+\varepsilon} \rho(t, x+u) \omega''_\varepsilon(u) du \\
&= \int_{-\varepsilon}^0 \rho(t, x+u) \omega''_\varepsilon(u) du + \int_0^\eta \rho(t, x+u) \omega''_\varepsilon(u) du + \int_\eta^{\eta+\varepsilon} \rho(t, x+u) \omega''_\varepsilon(u) du \\
&= \rho(t, x) \omega'_\varepsilon(0) - \rho(t, x-\varepsilon) \omega'_\varepsilon(-\varepsilon) - \int_{-\varepsilon}^0 \partial_x \rho(t, x+u) \omega'_\varepsilon(u) du \\
&\quad + \int_0^\eta \rho(t, x+u) \omega''_\varepsilon(u) du \\
&\quad + \rho(t, x+\eta+\varepsilon) \omega'_\varepsilon(\eta+\varepsilon) - \rho(t, x+\eta) \omega'_\varepsilon(\eta) - \int_\eta^{\eta+\varepsilon} \partial_x \rho(t, x+u) \omega'_\varepsilon(u) du \\
&= \rho(t, x) \omega'(0+) - \rho(t, x+\eta) \omega'(\eta-) + \int_0^\eta \rho(t, x+u) \omega''_\varepsilon(u) du \\
&\quad - \int_{-\varepsilon}^0 \partial_x \rho(t, x+u) \omega'_\varepsilon(u) du - \int_\eta^{\eta+\varepsilon} \partial_x \rho(t, x+u) \omega'_\varepsilon(u) du,
\end{aligned}$$

we have

$$\begin{aligned}
&\left( \int_{\mathbb{R}} |\partial_{xx}^2(\rho * \omega_\varepsilon)|^p(t, x) dx \right)^{1/p} \\
&\leq \left( \int_{\mathbb{R}} \left| \int_{-\varepsilon}^0 \partial_x \rho(t, x+u) \omega'_\varepsilon(u) du \right|^p dx \right)^{1/p} \\
&\quad + \left( \int_{\mathbb{R}} \left| \int_\eta^{\eta+\varepsilon} \partial_x \rho(t, x+u) \omega'_\varepsilon(u) du \right|^p dx \right)^{1/p} \\
&\quad + \left( \int_{\mathbb{R}} \left| \int_0^\eta \rho(t, x+u) \omega''_\varepsilon(u) du \right|^p dx \right)^{1/p} \\
&\quad + \left( \int_{\mathbb{R}} \rho(t, x)^p |\omega'(0+)|^p dx \right)^{1/p} + \left( \int_{\mathbb{R}} \rho(t, x+\eta)^p |\omega'(\eta-)|^p dx \right)^{1/p}.
\end{aligned}$$

Notice that

$$\left( \int_{\mathbb{R}} \left| \int_{-\varepsilon}^0 \partial_x \rho(t, x+u) \omega'_\varepsilon(u) du \right|^p dx \right)^{1/p}$$

$$\begin{aligned}
&\leq \left( \int_{\mathbb{R}} \left( \int_{-\varepsilon}^0 |\partial_x \rho(t, x+u)|^p du \right) \left( \int_{-\varepsilon}^0 |\omega'_\varepsilon(u)|^q dy \right)^{p/q} dx \right)^{1/p} \\
&\leq \left( \int_{-\varepsilon}^0 |\omega'_\varepsilon(u)|^{p/(p-1)} du \right)^{1-1/p} \left( \int_{\mathbb{R}} \left( \int_{-\varepsilon}^{\eta+\varepsilon} |\partial_x \rho(t, x+u)|^p du \right) dx \right)^{1/p} \\
&\leq \left( \int_{-\varepsilon}^0 \left( 2 \frac{W_\varepsilon}{\varepsilon} \right)^{p/(p-1)} du \right)^{1-1/p} \left( \int_{\mathbb{R}} \int_{x-\varepsilon}^x |\partial_x \rho(t, y)|^p dy dx \right)^{1/p} \\
&\leq 2 \frac{W_\varepsilon}{\varepsilon} \varepsilon^{1-1/p} \left( \int_{\mathbb{R}} \int_y^{y+\varepsilon} dx |\partial_x \rho(t, y)|^p dy \right)^{1/p} \\
&\leq 2 \frac{W_\varepsilon}{\varepsilon} \varepsilon^{1-1/p} \varepsilon^{1/p} \left( \int_{\mathbb{R}} |\partial_x \rho(t, y)|^p dy \right)^{1/p} = 2W_\varepsilon \left( \int_{\mathbb{R}} |\partial_x \rho(t, y)|^p dy \right)^{1/p}
\end{aligned}$$

using Hölder's inequality with  $q = p/(p-1)$  the conjugated exponent of  $p$ . Similarly

$$\left( \int_{\mathbb{R}} \left| \int_{\eta}^{\eta+\varepsilon} \partial_x \rho(t, x+u) \omega'_\varepsilon(u) du \right|^p dx \right)^{1/p} \leq 2\omega(\eta) \left( \int_{\mathbb{R}} |\partial_x \rho(t, y)|^p dy \right)^{1/p}.$$

Then we get

$$\begin{aligned}
&\left( \int_{\mathbb{R}} |\partial_{xx}^2(\rho * \omega_\varepsilon)|^p(t, x) dx \right)^{1/p} \\
&\leq 2(\omega(\eta) + W_\varepsilon) \left( \int_{\mathbb{R}} |\partial_x \rho(t, y)|^p dy \right)^{1/p} \\
&\quad + \left( \int_{\mathbb{R}} \left( \int_x^{x+\eta} \rho^p(t, y) dy \right) \left( \int_0^\eta |\omega''_\varepsilon(u)|^q du \right)^{p/q} dx \right)^{1/p} \\
&\quad + (|\omega'(\eta+)| + |\omega'(0-)|) \left( \int_{\mathbb{R}} \rho^p(t, x) dx \right)^{1/p}.
\end{aligned}$$

Furthermore

$$\begin{aligned}
&\int_{\mathbb{R}} \left( \int_x^{x+\eta} \rho^p(t, y) dy \right) \left( \int_0^\eta |\omega''(u)|^q dy \right)^{p-1} dx \\
&\leq \left( \int_0^\eta |\omega''(u)|^{p/(p-1)} du \right)^{p-1} \int_{\mathbb{R}} \int_{y-\eta}^y \rho^p(t, y) dx dy \\
&\leq \eta \left( \int_0^\varepsilon |\omega''(u)|^{p/(p-1)} du \right)^{p-1} \int_{\mathbb{R}} \rho^p(t, y) dy,
\end{aligned}$$

1 then we get the announced formula.

3. Remark that, since  $\omega''_\varepsilon(-\varepsilon) = \omega''_\varepsilon(\eta + \varepsilon) = 0$ , we have

$$\partial_{xxx}^3(\rho * \omega_\varepsilon)(t, x) = - \int_{-\varepsilon}^{\eta+\varepsilon} \rho(t, x+u) \omega_\varepsilon^{(3)}(u) du = \int_{-\varepsilon}^{\eta+\varepsilon} \partial_x \rho(t, x+u) \omega''_\varepsilon(u) du$$

$$= \partial_{xx}^2 (\partial_x \rho * \omega_\varepsilon)(t, x),$$

1 then applying 2., we get

$$\begin{aligned} \left( \int_{\mathbb{R}} |\partial_{xxx}^3 (\rho * \omega)|^p(t, x) dx \right)^{1/p} &\leq \eta^{1/p} \left( \int_0^\eta |\omega''(u)|^{p/(p-1)} du \right)^{1-1/p} \left( \int_{\mathbb{R}} |\partial_x \rho(t, y)|^p dy \right)^{1/p} \\ &\quad + (|\omega'(\eta-)| + |\omega'(0+)|) \left( \int_{\mathbb{R}} |\partial_x \rho(t, x)|^p dx \right)^{1/p} \\ &\quad + 2(\omega(\eta) + W_\varepsilon) \left( \int_{\mathbb{R}} |\partial_{xx}^2 \rho(t, x)|^p dx \right)^{1/p}. \end{aligned}$$

2

□

### 3 2.2 $\mathbf{L}^p$ estimates for the viscous case

4 We turn now to estimates on solutions solving the viscous and regularized non-local equation. First,  
5 we deal with  $\mathbf{L}^p$  estimates.

**Proposition 2.** *We assume  $(\mathbf{A}_\omega^1)$ - $(\mathbf{A}_v^1)$ . Let  $\rho_\varepsilon$  be the solution of (1.4) with initial datum  $\rho^0 \in \mathbf{L}^p(\mathbb{R})$ . If  $\rho_\varepsilon \in \mathbf{L}^\infty([0, T] \times \mathbb{R})$  for some  $T > 0$ , then*

$$\rho_\varepsilon \in \mathbf{L}^\infty([0, T], \mathbf{L}^p(\mathbb{R})) \cap \mathbf{L}^p([0, T] \times \mathbb{R}).$$

*Proof.* The equation (1.4) can be rewritten as

$$\partial_t \rho_\varepsilon + v(\rho_\varepsilon * \omega_\varepsilon) \partial_x \rho_\varepsilon + \rho_\varepsilon v'(\rho_\varepsilon * \omega_\varepsilon) \partial_x (\rho_\varepsilon * \omega_\varepsilon) = \varepsilon \partial_{xx}^2 \rho_\varepsilon. \quad (2.4)$$

6 Multiplying (2.4) by  $\rho_\varepsilon^{p-1}$ , then integrating with respect to  $x$ , we obtain

$$\begin{aligned} \frac{1}{p} \frac{d}{dt} \int_{\mathbb{R}} \rho_\varepsilon^p(t, x) dx &= - \int_{\mathbb{R}} \rho_\varepsilon^{p-1}(t, x) v((\rho_\varepsilon * \omega_\varepsilon)(t, x)) \partial_x \rho_\varepsilon(t, x) dx \\ &\quad - \int_{\mathbb{R}} \rho_\varepsilon^p(t, x) v'((\rho_\varepsilon * \omega_\varepsilon)(t, x)) \partial_x (\rho_\varepsilon * \omega_\varepsilon)(t, x) dx \\ &\quad + \varepsilon \int_{\mathbb{R}} \rho_\varepsilon^{p-1}(t, x) \partial_{xx}^2 \rho_\varepsilon(t, x) dx. \end{aligned}$$

7 We observe that

$$\begin{aligned} \int_{\mathbb{R}} \rho_\varepsilon^{p-1} v(\rho_\varepsilon * \omega_\varepsilon) \partial_x \rho_\varepsilon dx &= \int_{\mathbb{R}} \partial_x \left( \frac{\rho_\varepsilon^p}{p} \right) v(\rho_\varepsilon * \omega_\varepsilon) dx \\ &= - \int_{\mathbb{R}} \frac{\rho_\varepsilon^p}{p} \partial_x (v(\rho_\varepsilon * \omega_\varepsilon)) dx \\ &= - \int_{\mathbb{R}} \frac{\rho_\varepsilon^p}{p} v'(\rho_\varepsilon * \omega_\varepsilon) \partial_x (\rho_\varepsilon * \omega_\varepsilon) dx \end{aligned}$$

8 and

$$\int_{\mathbb{R}} \rho_\varepsilon^{p-1} \partial_{xx}^2 \rho_\varepsilon dx = -(p-1) \int_{\mathbb{R}} \rho_\varepsilon^{p-2} (\partial_x \rho_\varepsilon)^2 dx \leq 0,$$

therefore

$$\frac{d}{dt} \int_{\mathbb{R}} \rho_{\varepsilon}^p(t, x) dx \leq (1-p) \int_{\mathbb{R}} \rho_{\varepsilon}^p(t, x) v'((\rho_{\varepsilon} * \omega_{\varepsilon})(t, x)) \partial_x(\rho_{\varepsilon} * \omega_{\varepsilon})(t, x) dx. \quad (2.5)$$

We use (2.1) to control the right hand side of (2.5) and we get

$$\frac{d}{dt} \int_{\mathbb{R}} \rho_{\varepsilon}^p(t, x) dx \leq C_1^{\varepsilon, p} \int_{\mathbb{R}} \rho_{\varepsilon}^p(t, x) dx, \quad (2.6)$$

which implies

$$\int_{\mathbb{R}} \rho_{\varepsilon}^p(t, x) dx \leq e^{C_1^{\varepsilon, p} t} \int_{\mathbb{R}} \rho_{\varepsilon}^p(0, x) dx, \quad (2.7)$$

with

$$C_1^{\varepsilon, p} = 2(p-1) \|\rho_{\varepsilon}\|_{\infty} W_{\varepsilon} \|v'\|_{\infty}.$$

It gives

$$\sup_{t \in [0, T]} \int_{\mathbb{R}} \rho_{\varepsilon}^p(t, x) dx \leq e^{C_1^{\varepsilon, p} T} \int_{\mathbb{R}} \rho_{\varepsilon}^p(0, x) dx. \quad (2.8)$$

By integration of (2.7) with respect to  $t \in [0, T]$ , we get

$$\int_0^T \int_{\mathbb{R}} \rho_{\varepsilon}^p(t, x) dx dt \leq \frac{1}{C_1^{\varepsilon, p}} \left( e^{C_1^{\varepsilon, p} T} - 1 \right) \int_{\mathbb{R}} \rho_{\varepsilon}^p(0, x) dx. \quad (2.9)$$

1

□

### 2.3 $\mathbf{W}^{1,p}$ estimates for $p = 2N$ in the viscous case

3 We turn now to Sobolev estimates. Let  $N \in \mathbb{N}^*$  and set  $p = 2N$ .

**Proposition 3.** *We assume  $(\mathbf{A}_{\omega}^2)$ - $(\mathbf{A}_{\mathbf{v}}^2)$ . Let  $\rho_{\varepsilon}$  be the solution of (1.4) with initial datum  $\rho^0 \in \mathbf{W}^{1,2N}(\mathbb{R})$ . If  $\rho_{\varepsilon} \in \mathbf{L}^{\infty}([0, T] \times \mathbb{R})$  for some  $T > 0$ , then*

$$\rho_{\varepsilon} \in \mathbf{L}^{\infty}([0, T], \mathbf{W}^{1,2N}(\mathbb{R})) \quad \text{and} \quad \rho_{\varepsilon}, \partial_x \rho_{\varepsilon} \in \mathbf{L}^{\infty}([0, T], \mathbf{L}^{2N}(\mathbb{R})) \cap \mathbf{L}^{2N}([0, T] \times \mathbb{R}).$$

4 *Proof.* We differentiate (2.4) with respect to  $x$ , it gives

$$\begin{aligned} & \partial_t \partial_x \rho_{\varepsilon} + 2v'(\rho_{\varepsilon} * \omega_{\varepsilon}) \partial_x(\rho_{\varepsilon} * \omega_{\varepsilon}) \partial_x \rho_{\varepsilon} + v(\rho_{\varepsilon} * \omega_{\varepsilon}) \partial_{xx}^2 \rho_{\varepsilon} \\ & + \rho_{\varepsilon} v''(\rho_{\varepsilon} * \omega_{\varepsilon}) (\partial_x(\rho_{\varepsilon} * \omega_{\varepsilon}))^2 + \rho_{\varepsilon} v'(\rho_{\varepsilon} * \omega_{\varepsilon}) \partial_{xx}^2(\rho_{\varepsilon} * \omega_{\varepsilon}) = \varepsilon \partial_{xxx}^3 \rho_{\varepsilon}. \end{aligned} \quad (2.10)$$

5 Multiplying this relation by  $(\partial_x \rho_{\varepsilon})^{p-1}$ , then integrating with respect to  $x$ , we have

$$\begin{aligned} & \frac{1}{p} \frac{d}{dt} \int_{\mathbb{R}} (\partial_x \rho_{\varepsilon})^p dx + 2 \int_{\mathbb{R}} v'(\rho_{\varepsilon} * \omega_{\varepsilon}) \partial_x(\rho_{\varepsilon} * \omega_{\varepsilon}) (\partial_x \rho_{\varepsilon})^p dx + \int_{\mathbb{R}} v(\rho_{\varepsilon} * \omega_{\varepsilon}) \partial_{xx}^2 \rho_{\varepsilon} (\partial_x \rho_{\varepsilon})^{p-1} dx \\ & + \int_{\mathbb{R}} \rho_{\varepsilon} v''(\rho_{\varepsilon} * \omega_{\varepsilon}) (\partial_x(\rho_{\varepsilon} * \omega_{\varepsilon}))^2 (\partial_x \rho_{\varepsilon})^{p-1} dx + \int_{\mathbb{R}} \rho_{\varepsilon} v'(\rho_{\varepsilon} * \omega_{\varepsilon}) \partial_{xx}^2(\rho_{\varepsilon} * \omega_{\varepsilon}) (\partial_x \rho_{\varepsilon})^{p-1} dx \\ & = \varepsilon \int_{\mathbb{R}} (\partial_x \rho_{\varepsilon})^{p-1} \partial_{xxx}^3 \rho_{\varepsilon} dx. \end{aligned}$$

6 Notice that

$$\int_{\mathbb{R}} v(\rho_{\varepsilon} * \omega_{\varepsilon}) \partial_{xx}^2 \rho_{\varepsilon} (\partial_x \rho_{\varepsilon})^{p-1} dx = \frac{1}{p} \int_{\mathbb{R}} v(\rho_{\varepsilon} * \omega_{\varepsilon}) \partial_x ((\partial_x \rho_{\varepsilon})^p) dx$$



$$= -\frac{1}{p} \int_{\mathbb{R}} v'(\rho_\varepsilon * \omega_\varepsilon) \partial_x(\rho_\varepsilon * \omega_\varepsilon) (\partial_x \rho_\varepsilon)^p dx$$

1 and, since  $p$  is even,

$$\int_{\mathbb{R}} (\partial_x \rho_\varepsilon)^{p-1} \partial_{xxx}^3 \rho_\varepsilon dx = -(p-1) \int_{\mathbb{R}} (\partial_{xx}^2 \rho_\varepsilon)^2 (\partial_x \rho_\varepsilon)^{p-2} dx \leq 0,$$

2 thus

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} (\partial_x \rho_\varepsilon)^p dx &\leq (1-2p) \int_{\mathbb{R}} v'(\rho_\varepsilon * \omega_\varepsilon) \partial_x(\rho_\varepsilon * \omega_\varepsilon) (\partial_x \rho_\varepsilon)^p dx \\ &\quad - p \int_{\mathbb{R}} \rho_\varepsilon v''(\rho_\varepsilon * \omega_\varepsilon) (\partial_x(\rho_\varepsilon * \omega_\varepsilon))^2 (\partial_x \rho_\varepsilon)^{p-1} dx \\ &\quad - p \int_{\mathbb{R}} \rho_\varepsilon v'(\rho_\varepsilon * \omega_\varepsilon) \partial_{xx}^2(\rho_\varepsilon * \omega_\varepsilon) (\partial_x \rho_\varepsilon)^{p-1} dx \\ &=: I_1^\varepsilon + I_2^\varepsilon + I_3^\varepsilon. \end{aligned}$$

3 We estimate now each of these terms.

- By (2.1) we get

$$|I_1^\varepsilon| = \left| (1-2p) \int_{\mathbb{R}} v'(\rho_\varepsilon * \omega_\varepsilon) \partial_x(\rho_\varepsilon * \omega_\varepsilon) (\partial_x \rho_\varepsilon)^p dx \right| \leq 2(2p-1) \|v'\|_\infty \|\rho_\varepsilon\|_\infty W_\varepsilon \int_{\mathbb{R}} |\partial_x \rho_\varepsilon|^p dx.$$

- 4 • Again by (2.1) we get

$$\begin{aligned} |I_2^\varepsilon| &= \left| p \int_{\mathbb{R}} \rho_\varepsilon v''(\rho_\varepsilon * \omega_\varepsilon) (\partial_x(\rho_\varepsilon * \omega_\varepsilon))^2 (\partial_x \rho_\varepsilon)^{p-1} dx \right| \\ &\leq p \|v''\|_\infty (2\|\rho_\varepsilon\|_\infty W_\varepsilon)^2 \int_{\mathbb{R}} \rho_\varepsilon |\partial_x \rho_\varepsilon|^{p-1} dx \\ &\leq 4 \|v''\|_\infty \|\rho_\varepsilon\|_\infty^2 W_\varepsilon^2 \left( \int_{\mathbb{R}} \rho_\varepsilon^p dx + (p-1) \int_{\mathbb{R}} |\partial_x \rho_\varepsilon|^p dx \right), \end{aligned}$$

5 where we have used Young's inequality  $uv \leq \frac{1}{p}u^p + \frac{1}{q}v^q$  with  $q = p/(p-1)$ .

- 7 • Similarly,

$$\begin{aligned} |I_3^\varepsilon| &= \left| p \int_{\mathbb{R}} \rho_\varepsilon v'(\rho_\varepsilon * \omega_\varepsilon) \partial_{xx}^2(\rho_\varepsilon * \omega_\varepsilon) (\partial_x \rho_\varepsilon)^{p-1} dx \right| \\ &\leq p \|v'\|_\infty \|\rho_\varepsilon\|_\infty \int_{\mathbb{R}} |\partial_{xx}^2(\rho_\varepsilon * \omega_\varepsilon)| |\partial_x \rho_\varepsilon|^{p-1} dx \\ &\leq \|v'\|_\infty \|\rho_\varepsilon\|_\infty \left( \int_{\mathbb{R}} |\partial_{xx}^2(\rho_\varepsilon * \omega_\varepsilon)|^p dx + (p-1) \int_{\mathbb{R}} |\partial_x \rho_\varepsilon|^p dx \right). \end{aligned}$$

We now observe that

$$(u+v+w)^p \leq 3^p(u^p+v^p+w^p) \quad \text{for any } u, v, w > 0 \text{ and } p > 0. \quad (2.11)$$

8 Indeed, from the binomial expansion we get

$$(u+v+w)^p = (u+(v+w))^p$$

$$\begin{aligned}
&= \sum_{k=0}^p \binom{p}{k} u^k (v+w)^{p-k} \\
&= \sum_{k=0}^p \binom{p}{k} u^k \sum_{l=0}^{p-k} \binom{p-k}{l} v^l w^{p-k-l}.
\end{aligned}$$

1 Observing that  $u^k v^l w^{p-k-l} \leq u^p + v^p + w^p$  and that  $\sum_{l=0}^{p-k} \binom{p-k}{l} = (1+1)^{p-k}$  and

2  $\sum_{k=0}^p \binom{p}{k} 1^k 2^{p-k} = 3^p$ , we get the result.

3 Estimate (2.2) of Proposition 1 and inequality (2.11) give

$$\begin{aligned}
\int_{\mathbb{R}} |\partial_{xx}^2(\rho_\varepsilon * \omega)|^p dx &\leq 3^p \eta \left( \int_0^\eta |\omega''(u)|^{p/(p-1)} du \right)^{p-1} \int_{\mathbb{R}} \rho_\varepsilon^p(t, x) dx \\
&\quad + 3^p (|\omega'(\eta-)| + |\omega'(0+)|)^p \int_{\mathbb{R}} \rho_\varepsilon^p(t, x) dx \\
&\quad + 6^p (\omega(\eta) + W_\varepsilon)^p \int_{\mathbb{R}} |\partial_x \rho_\varepsilon(t, x)|^p dx,
\end{aligned} \tag{2.12}$$

thus

$$\left| p \int_{\mathbb{R}} \rho_\varepsilon v'(\rho * \omega_\varepsilon) \partial_{xx}^2(\rho_\varepsilon * \omega_\varepsilon) (\partial_x \rho_\varepsilon)^{p-1} dx \right| \leq C_2^{\varepsilon, p} \int_{\mathbb{R}} \rho_\varepsilon^p dx + C_3^{\varepsilon, p} \int_{\mathbb{R}} |\partial_x \rho_\varepsilon|^p dx,$$

with

$$C_2^{\varepsilon, p} = \|v'\|_\infty \|\rho_\varepsilon\|_\infty 3^p \left[ \eta \left( \int_0^\eta |\omega''(u)|^{p/(p-1)} du \right)^{p-1} + (|\omega'(\eta-)| + |\omega'(0+)|)^p \right]$$

and

$$C_3^{\varepsilon, p} = \|v'\|_\infty \|\rho_\varepsilon\|_\infty \left[ p-1 + 6^p (\omega(\eta) + W_\varepsilon)^p \right].$$

4 These bounds give finally the estimate

$$\frac{d}{dt} \int_{\mathbb{R}} (\partial_x \rho_\varepsilon(t, x))^p dx \leq C_4^{\varepsilon, p} \int_{\mathbb{R}} |\partial_x \rho_\varepsilon(t, x)|^p dx + C_5^{\varepsilon, p} \int_{\mathbb{R}} \rho_\varepsilon^p(t, x) dx.$$

with

$$C_4^{\varepsilon, p} = 2(2p-1) \|v'\|_\infty \|\rho_\varepsilon\|_\infty W_\varepsilon + 4(p-1) \|v''\|_\infty \|\rho_\varepsilon\|_\infty^2 W_\varepsilon^2 + C_3^\varepsilon$$

and

$$C_5^{\varepsilon, p} = 4 \|v''\|_\infty \|\rho_\varepsilon\|_\infty^2 W_\varepsilon^2 + C_2^\varepsilon.$$

5 With (2.6), we get

$$\begin{aligned}
&\frac{d}{dt} \left( \int_{\mathbb{R}} |\partial_x \rho_\varepsilon(t, x)|^{2N} dx + \int_{\mathbb{R}} \rho_\varepsilon^{2N}(t, x) dx \right) \\
&\leq C_6^{\varepsilon, p} \left( \int_{\mathbb{R}} |\partial_x \rho_\varepsilon(t, x)|^{2N} dx + \int_{\mathbb{R}} \rho_\varepsilon^{2N}(t, x) dx \right),
\end{aligned} \tag{2.13}$$

6 with  $C_6^{\varepsilon, p} = \max(C_4^{\varepsilon, p}, C_5^{\varepsilon, p} + C_1^{\varepsilon, p})$ , which implies

$$\int_{\mathbb{R}} |\partial_x \rho_\varepsilon(t, x)|^{2N} dx + \int_{\mathbb{R}} \rho_\varepsilon^{2N}(t, x) dx$$

$$\leq e^{C_6^{\varepsilon,p}t} \left( \int_{\mathbb{R}} |\partial_x \rho_\varepsilon(0, x)|^{2N} dx + \int_{\mathbb{R}} \rho_\varepsilon^{2N}(0, x) dx \right). \quad (2.14)$$

1 Then

$$\begin{aligned} & \sup_{t \in [0, T]} \left( \int_{\mathbb{R}} |\partial_x \rho_\varepsilon(t, x)|^{2N} dx + \int_{\mathbb{R}} \rho_\varepsilon^{2N}(t, x) dx \right) \\ & \leq e^{C_6^{\varepsilon,p}T} \left( \int_{\mathbb{R}} |\partial_x \rho_\varepsilon(0, x)|^{2N} dx + \int_{\mathbb{R}} \rho_\varepsilon^{2N}(0, x) dx \right). \end{aligned} \quad (2.15)$$

2 Integrating (2.14) with respect to  $t$  on  $[0, T]$ , we get

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}} |\partial_x \rho_\varepsilon(t, x)|^{2N} dx dt + \int_0^T \int_{\mathbb{R}} \rho_\varepsilon^{2N}(t, x) dx dt \\ & \leq \frac{1}{C_6^{\varepsilon,p}} \left( e^{C_6^{\varepsilon,p}T} - 1 \right) \left( \int_{\mathbb{R}} |\partial_x \rho_\varepsilon(0, x)|^{2N} dx + \int_{\mathbb{R}} \rho_\varepsilon^{2N}(0, x) dx \right). \end{aligned} \quad (2.16)$$

3

□

#### 4 **2.4 $\mathbf{L}^\infty$ bound on an interval $[0, T]$**

5 With the previous estimates, we are now able to prove an  $\mathbf{L}^\infty$  bounds for the sequence  $\{\rho_\varepsilon\}_\varepsilon$  on an  
6 interval  $[0, T]$ .

**Proposition 4.** *We assume  $(\mathbf{A}_w^2)$ - $(\mathbf{A}_v^2)$ . Let  $\rho_\varepsilon$  be the solution of (1.4) with initial datum  $\rho^0 \in \mathbf{H}^1$ . Then there exists a constant  $\bar{T} > 0$  such that  $\rho_\varepsilon \in \mathbf{L}^\infty([0, T] \times \mathbb{R})$  for any  $\varepsilon > 0$ ,  $T < \bar{T}$ . Furthermore*

$$\rho_\varepsilon \in \mathbf{L}^\infty([0, T], \mathbf{W}^{1,2N}(\mathbb{R})) \quad \text{and} \quad \rho_\varepsilon, \partial_x \rho_\varepsilon \in \mathbf{L}^\infty([0, T], \mathbf{L}^{2N}(\mathbb{R})) \cap \mathbf{L}^{2N}([0, T] \times \mathbb{R}) \quad (2.17)$$

7 and this sequence is uniformly bounded in these spaces with respect to  $\varepsilon$ .

8 *Proof.* Let  $\rho_\varepsilon$  be a smooth solution of (1.4) with the same initial datum  $\rho^0 \in \mathbf{H}^1$ . The relation  
9 (2.13) for  $N = 1$  gives

$$\begin{aligned} & \frac{d}{dt} \left( \int_{\mathbb{R}} |\partial_x \rho_\varepsilon(t, x)|^2 dx + \int_{\mathbb{R}} \rho_\varepsilon^2(t, x) dx \right) \\ & \leq C \max \left\{ 1, \|\rho_\varepsilon(t, \cdot)\|_\infty^2 \right\} \left( \int_{\mathbb{R}} |\partial_x \rho_\varepsilon(t, x)|^2 dx + \int_{\mathbb{R}} \rho_\varepsilon^2(t, x) dx \right), \end{aligned}$$

for some constant  $C$  that does not depend on  $\varepsilon$  (since  $W_\varepsilon$  is uniformly bounded). If no uniform  $\mathbf{L}^\infty$ -bound on  $\rho_\varepsilon$  is available, we can use the Sobolev injection of  $\mathbf{H}^1(\mathbb{R})$  in  $\mathbf{L}^\infty(\mathbb{R})$  and get

$$\frac{d}{dt} \|\rho_\varepsilon(t, \cdot)\|_{\mathbf{H}^1}^2 \leq C \|\rho_\varepsilon(t, \cdot)\|_{\mathbf{H}^1}^2 + C \|\rho_\varepsilon(t, \cdot)\|_{\mathbf{H}^1}^4,$$

eventually updating the constant  $C$ . We set  $u_\varepsilon(t) = \|\rho_\varepsilon(t, \cdot)\|_{\mathbf{H}^1}^2$ , then  $u'_\varepsilon \leq C(u_\varepsilon + u_\varepsilon^2)$ , which leads to

$$\frac{u'_\varepsilon}{u_\varepsilon} - \frac{u'_\varepsilon}{1 + u_\varepsilon} \leq C.$$

We obtain

$$u_\varepsilon(t) \leq \frac{C_0 e^{Ct}}{1 - C_0 e^{Ct}}, \quad \text{for any } 0 \leq t < -\frac{\ln C_0}{C},$$

with  $C_0 = \frac{u_0}{1+u_0} < 1$ ,  $u_0 = \|\rho^0\|_{\mathbf{H}^1}^2$ . Notice that the initial datum is the same for all the sequence and then  $u_0$  and  $C_0$  do not depend on  $\varepsilon$ . Setting  $T < \bar{T} := -\frac{\ln C_0}{C}$ , we have

$$\|\rho_\varepsilon(t, \cdot)\|_{\mathbf{H}^1}^2 \leq C \|\rho^0\|_{\mathbf{H}^1}^2, \quad \text{for any } 0 \leq t \leq T, \quad \varepsilon > 0.$$

1 Therefore, by Sobolev injection,  $\rho_\varepsilon \in \mathbf{L}^\infty([0, T] \times \mathbb{R})$ . Using the estimates of Propositions 2 and 3,  
 2 we get (2.17) with bounds independents of  $\varepsilon$ .  $\square$

### 3 **2.5 $\mathbf{W}^{2,p}$ estimate for $p = 2N$**

4 To pass to the limit, we need also estimates in  $\mathbf{W}^{2,p}$ , which will provide, in the next section, with  
 5 the help of the equation, the necessary regularity in time. As in Section 2.3, let  $N \in \mathbb{N}^*$  and set  
 6  $p = 2N$ .

**Proposition 5.** *We assume  $(\mathbf{A}_\omega^2)$ - $(\mathbf{A}_v^3)$ . Let  $\rho_\varepsilon$  be the solution of (1.4) with initial datum  $\rho^0 \in \mathbf{W}^{1,4N}(\mathbb{R}) \cap \mathbf{H}^1(\mathbb{R}) \cap \mathbf{W}^{2,2N}(\mathbb{R})$ . Let  $T > 0$  as in Proposition 4. Then*

$$\rho_\varepsilon \in \mathbf{L}^\infty\left([0, T], \mathbf{W}^{2,2N}(\mathbb{R})\right) \quad \text{and} \quad \rho_\varepsilon, \partial_x \rho_\varepsilon, \partial_{xx}^2 \rho_\varepsilon \in \mathbf{L}^\infty\left([0, T], \mathbf{L}^{2N}(\mathbb{R})\right) \cap \mathbf{L}^{2N}([0, T] \times \mathbb{R})$$

7 and this sequence is bounded in these spaces with respect to  $\varepsilon$ .

8 *Proof.* We differentiate (2.10) with respect to  $x$ , which gives

$$\begin{aligned} & \partial_t \partial_{xx}^2 \rho_\varepsilon + 3v''(\rho_\varepsilon * \omega_\varepsilon) (\partial_x(\rho_\varepsilon * \omega_\varepsilon))^2 \partial_x \rho_\varepsilon + 3v'(\rho_\varepsilon * \omega_\varepsilon) \partial_{xx}^2(\rho_\varepsilon * \omega_\varepsilon) \partial_x \rho_\varepsilon \\ & + 3v'(\rho_\varepsilon * \omega_\varepsilon) \partial_x(\rho_\varepsilon * \omega_\varepsilon) \partial_{xx}^2 \rho_\varepsilon + v(\rho_\varepsilon * \omega_\varepsilon) \partial_{xxx}^3 \rho_\varepsilon + \rho_\varepsilon v^{(3)}(\rho_\varepsilon * \omega_\varepsilon) (\partial_x(\rho_\varepsilon * \omega_\varepsilon))^3 \\ & + 3\rho_\varepsilon v''(\rho_\varepsilon * \omega_\varepsilon) \partial_x(\rho_\varepsilon * \omega_\varepsilon) \partial_{xx}^2(\rho_\varepsilon * \omega_\varepsilon) + \rho_\varepsilon v'(\rho_\varepsilon * \omega_\varepsilon) \partial_{xxx}^3(\rho_\varepsilon * \omega_\varepsilon) = \varepsilon \partial_{xxxx}^4 \rho_\varepsilon. \end{aligned} \quad (2.18)$$

9 Multiplying this relation by  $(\partial_{xx}^2 \rho_\varepsilon)^{p-1}$ , then integrating with respect to  $x$ , we obtain

$$\begin{aligned} & \frac{1}{p} \frac{d}{dt} \int_{\mathbb{R}} (\partial_{xx}^2 \rho_\varepsilon)^p dx + 3 \int_{\mathbb{R}} v''(\rho_\varepsilon * \omega_\varepsilon) (\partial_x(\rho_\varepsilon * \omega_\varepsilon))^2 \partial_x \rho_\varepsilon (\partial_{xx}^2 \rho_\varepsilon)^{p-1} dx \\ & + 3 \int_{\mathbb{R}} v'(\rho_\varepsilon * \omega_\varepsilon) \partial_{xx}^2(\rho_\varepsilon * \omega_\varepsilon) \partial_x \rho_\varepsilon (\partial_{xx}^2 \rho_\varepsilon)^{p-1} dx + 3 \int_{\mathbb{R}} v'(\rho_\varepsilon * \omega_\varepsilon) \partial_x(\rho_\varepsilon * \omega_\varepsilon) (\partial_{xx}^2 \rho_\varepsilon)^p dx \\ & + \int_{\mathbb{R}} v(\rho_\varepsilon * \omega_\varepsilon) \partial_{xxx}^3 \rho_\varepsilon (\partial_{xx}^2 \rho_\varepsilon)^{p-1} dx + \int_{\mathbb{R}} \rho_\varepsilon v^{(3)}(\rho_\varepsilon * \omega_\varepsilon) (\partial_x(\rho_\varepsilon * \omega_\varepsilon))^3 (\partial_{xx}^2 \rho_\varepsilon)^{p-1} dx \\ & + 3 \int_{\mathbb{R}} \rho_\varepsilon v''(\rho_\varepsilon * \omega_\varepsilon) \partial_x(\rho_\varepsilon * \omega_\varepsilon) \partial_{xx}^2(\rho_\varepsilon * \omega_\varepsilon) (\partial_{xx}^2 \rho_\varepsilon)^{p-1} dx \\ & + \int_{\mathbb{R}} \rho_\varepsilon v'(\rho_\varepsilon * \omega_\varepsilon) \partial_{xxx}^3(\rho_\varepsilon * \omega_\varepsilon) (\partial_{xx}^2 \rho_\varepsilon)^{p-1} dx \\ & = \varepsilon \int_{\mathbb{R}} \partial_{xxxx}^4 \rho_\varepsilon (\partial_{xx}^2 \rho_\varepsilon)^{p-1} dx. \end{aligned}$$

10 Now

$$\begin{aligned} \int_{\mathbb{R}} v(\rho_\varepsilon * \omega_\varepsilon) \partial_{xxx}^3 \rho_\varepsilon (\partial_{xx}^2 \rho_\varepsilon)^{p-1} dx &= \frac{1}{p} \int_{\mathbb{R}} v(\rho_\varepsilon * \omega_\varepsilon) \partial_x \left( (\partial_{xx}^2 \rho_\varepsilon)^p \right) dx \\ &= -\frac{1}{p} \int_{\mathbb{R}} v'(\rho_\varepsilon * \omega_\varepsilon) \partial_x(\rho_\varepsilon * \omega_\varepsilon) (\partial_{xx}^2 \rho_\varepsilon)^p dx, \end{aligned}$$

1 therefore

$$\begin{aligned}
\frac{d}{dt} \int_{\mathbb{R}} (\partial_{xx}^2 \rho_\varepsilon)^p dx &= -3p \int_{\mathbb{R}} v''(\rho_\varepsilon * \omega_\varepsilon) (\partial_x(\rho_\varepsilon * \omega_\varepsilon))^2 \partial_x \rho_\varepsilon (\partial_{xx}^2 \rho_\varepsilon)^{p-1} dx \\
&\quad -3p \int_{\mathbb{R}} v'(\rho_\varepsilon * \omega_\varepsilon) \partial_{xx}^2(\rho_\varepsilon * \omega_\varepsilon) \partial_x \rho_\varepsilon (\partial_{xx}^2 \rho_\varepsilon)^{p-1} dx \\
&\quad + (1-3p) \int_{\mathbb{R}} v'(\rho_\varepsilon * \omega_\varepsilon) \partial_x(\rho_\varepsilon * \omega_\varepsilon) (\partial_{xx}^2 \rho_\varepsilon)^p dx \\
&\quad -p \int_{\mathbb{R}} \rho_\varepsilon v^{(3)}(\rho_\varepsilon * \omega_\varepsilon) (\partial_x(\rho_\varepsilon * \omega_\varepsilon))^3 (\partial_{xx}^2 \rho_\varepsilon)^{p-1} dx \\
&\quad -3p \int_{\mathbb{R}} \rho_\varepsilon v''(\rho_\varepsilon * \omega_\varepsilon) \partial_x(\rho_\varepsilon * \omega_\varepsilon) \partial_{xx}^2(\rho_\varepsilon * \omega_\varepsilon) (\partial_{xx}^2 \rho_\varepsilon)^{p-1} dx \\
&\quad -p \int_{\mathbb{R}} \rho_\varepsilon v'(\rho_\varepsilon * \omega_\varepsilon) \partial_{xxx}^3(\rho_\varepsilon * \omega_\varepsilon) (\partial_{xx}^2 \rho_\varepsilon)^{p-1} dx \\
&\quad + \varepsilon p \int_{\mathbb{R}} \partial_{xxxx}^4 \rho_\varepsilon (\partial_{xx}^2 \rho_\varepsilon)^{p-1} dx \\
&=: J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + J_7.
\end{aligned}$$

2 We estimate now each of these terms.

3 • Using (2.1) and Young's inequality, we get

$$\begin{aligned}
|J_1| &= 3p \left| \int_{\mathbb{R}} v''(\rho_\varepsilon * \omega_\varepsilon) (\partial_x(\rho_\varepsilon * \omega_\varepsilon))^2 \partial_x \rho_\varepsilon (\partial_{xx}^2 \rho_\varepsilon)^{p-1} dx \right| \\
&\leq 3p \|v''\|_\infty (2\|\rho_\varepsilon\|_\infty W_\varepsilon)^2 \int_{\mathbb{R}} |\partial_x \rho_\varepsilon| |\partial_{xx}^2 \rho_\varepsilon|^{p-1} dx \\
&\leq 12 \|v''\|_\infty \|\rho_\varepsilon\|_\infty^2 W_\varepsilon^2 \left( \int_{\mathbb{R}} |\partial_x \rho_\varepsilon|^p dx + (p-1) \int_{\mathbb{R}} |\partial_{xx}^2 \rho_\varepsilon|^p dx \right)
\end{aligned}$$

•

$$\begin{aligned}
|J_2| &= 3p \left| \int_{\mathbb{R}} v'(\rho_\varepsilon * \omega_\varepsilon) \partial_{xx}^2(\rho_\varepsilon * \omega_\varepsilon) \partial_x \rho_\varepsilon (\partial_{xx}^2 \rho_\varepsilon)^{p-1} dx \right| \\
&\leq 3p \|v'\|_\infty \int_{\mathbb{R}} |\partial_{xx}^2(\rho_\varepsilon * \omega_\varepsilon)| |\partial_x \rho_\varepsilon| |\partial_{xx}^2 \rho_\varepsilon|^{p-1} dx \\
&\leq 3 \|v'\|_\infty \left( \frac{1}{2} \int_{\mathbb{R}} |\partial_{xx}^2(\rho_\varepsilon * \omega_\varepsilon)|^{2p} dx + \frac{1}{2} \int_{\mathbb{R}} |\partial_x \rho_\varepsilon|^{2p} dx + (p-1) \int_{\mathbb{R}} |\partial_{xx}^2 \rho_\varepsilon|^p dx \right)
\end{aligned}$$

using the inequality

$$uvw \leq \frac{1}{p_1} u^{p_1} + \frac{1}{p_2} v^{p_2} + \frac{1}{p_3} w^{p_3}, \quad \text{with } \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1, \quad (2.19)$$

4 setting  $p_1 = 2p = p_2$  and  $p_3 = p/(p-1)$ . Estimate (2.19) can be derived applying twice the  
5 classical Young's inequality to  $uvw = u(vw)$ . Using now the relation (2.12) with  $2p$  at the  
6 place of  $p$ , we get

$$|J_2| \leq \frac{3^{2p+1}}{2} \|v'\|_\infty \eta \left( \int_0^\eta |\omega''(u)|^{2p/(2p-1)} du \right)^{2p-1} \int_{\mathbb{R}} \rho_\varepsilon^{2p} dx$$

$$\begin{aligned}
& + \frac{3^{2p+1}}{2} \|v'\|_\infty (|\omega'(\eta-)| + |\omega'(0+)|)^{2p} \int_{\mathbb{R}} \rho_\varepsilon^{2p} dx \\
& + 3^{2p+1} 2^{2p-1} \|v'\|_\infty (\omega(\eta) + 2\omega(0))^{2p} \int_{\mathbb{R}} |\partial_x \rho_\varepsilon|^{2p} dx \\
& + \frac{3}{2} \|v'\|_\infty \int_{\mathbb{R}} |\partial_x \rho_\varepsilon|^{2p} dx + 3(p-1) \|v'\|_\infty \int_{\mathbb{R}} |\partial_{xx}^2 \rho_\varepsilon|^p dx.
\end{aligned}$$

•

$$\begin{aligned}
|J_3| & = (3p-1) \left| \int_{\mathbb{R}} v'(\rho_\varepsilon * \omega_\varepsilon) \partial_x(\rho_\varepsilon * \omega_\varepsilon) (\partial_{xx}^2 \rho_\varepsilon)^p dx \right| \\
& \leq 2(3p-1) \|v'\|_\infty \|\rho_\varepsilon\|_\infty W_\varepsilon \int_{\mathbb{R}} |\partial_{xx}^2 \rho_\varepsilon|^p dx.
\end{aligned}$$

•

$$\begin{aligned}
|J_4| & = p \left| \int_{\mathbb{R}} \rho_\varepsilon v^{(3)}(\rho_\varepsilon * \omega_\varepsilon) (\partial_x(\rho_\varepsilon * \omega_\varepsilon))^3 (\partial_{xx}^2 \rho_\varepsilon)^{p-1} dx \right| \\
& \leq p \|v^{(3)}\|_\infty (2\|\rho_\varepsilon\|_\infty W_\varepsilon)^3 \int_{\mathbb{R}} \rho_\varepsilon |\partial_{xx}^2 \rho_\varepsilon|^{p-1} dx \\
& \leq 8 \|v^{(3)}\|_\infty \|\rho_\varepsilon\|_\infty^3 W_\varepsilon^3 \left( \int_{\mathbb{R}} \rho_\varepsilon^p dx + (p-1) \int_{\mathbb{R}} |\partial_{xx}^2 \rho_\varepsilon|^p dx \right)
\end{aligned}$$

1 using Young's inequality.

•

$$\begin{aligned}
|J_5| & = 3p \left| \int_{\mathbb{R}} \rho_\varepsilon v''(\rho_\varepsilon * \omega_\varepsilon) \partial_x(\rho_\varepsilon * \omega_\varepsilon) \partial_{xx}^2(\rho_\varepsilon * \omega_\varepsilon) (\partial_{xx}^2 \rho_\varepsilon)^{p-1} dx \right| \\
& \leq 6p \|v''\|_\infty \|\rho_\varepsilon\|_\infty^2 W_\varepsilon \int_{\mathbb{R}} |\partial_{xx}^2(\rho_\varepsilon * \omega_\varepsilon)| |\partial_{xx}^2 \rho_\varepsilon|^{p-1} dx \\
& \leq 6 \|v''\|_\infty \|\rho_\varepsilon\|_\infty^2 W_\varepsilon \left( \int_{\mathbb{R}} |\partial_{xx}^2(\rho_\varepsilon * \omega_\varepsilon)|^p dx + (p-1) \int_{\mathbb{R}} |\partial_{xx}^2 \rho_\varepsilon|^p dx \right) \\
& \leq 6 \|v''\|_\infty \|\rho_\varepsilon\|_\infty^2 W_\varepsilon \left( 3^p \eta \left( \int_0^\eta |\omega''(u)|^{p/(p-1)} du \right)^{p-1} \int_{\mathbb{R}} \rho_\varepsilon^p dx \right. \\
& \quad \left. + 3^p (|\omega'(\eta-)| + |\omega'(0+)|)^p \int_{\mathbb{R}} \rho_\varepsilon^p dx \right. \\
& \quad \left. + 6^p (\omega(\eta) + 2\omega(0))^p \int_{\mathbb{R}} |\partial_x \rho_\varepsilon|^p dx + (p-1) \int_{\mathbb{R}} |\partial_{xx}^2 \rho_\varepsilon|^p dx \right)
\end{aligned}$$

2 using relation (2.12).

•

$$\begin{aligned}
|J_6| & = p \left| \int_{\mathbb{R}} \rho_\varepsilon v'(\rho_\varepsilon * \omega_\varepsilon) \partial_{xxx}^3(\rho_\varepsilon * \omega_\varepsilon) (\partial_{xx}^2 \rho_\varepsilon)^{p-1} dx \right| \\
& \leq p \|\rho_\varepsilon\|_\infty \|v'\|_\infty \int_{\mathbb{R}} |\partial_{xxx}^3(\rho_\varepsilon * \omega_\varepsilon)| |\partial_{xx}^2 \rho_\varepsilon|^{p-1} dx
\end{aligned}$$

$$\leq \|\rho_\varepsilon\|_\infty \|v'\|_\infty \left( \int_{\mathbb{R}} |\partial_{xxx}^3(\rho_\varepsilon * \omega_\varepsilon)|^p dx + (p-1) \int_{\mathbb{R}} |\partial_{xx}^2 \rho_\varepsilon|^p dx \right).$$

1 Estimate (2.3) of Proposition 1 and the inequality (2.11) give

$$\begin{aligned} \int_{\mathbb{R}} |\partial_{xxx}^3(\rho_\varepsilon * \omega_\varepsilon)|^p(t, x) dx &\leq 3^p \eta \left( \int_0^\eta |\omega''(u)|^{p/(p-1)} du \right)^{p-1} \int_{\mathbb{R}} |\partial_x \rho_\varepsilon(t, x)|^p dx \\ &\quad + 3^p (|\omega'(\eta-)| + |\omega'(0+)|)^p \int_{\mathbb{R}} |\partial_x \rho_\varepsilon(t, x)|^p dx \\ &\quad + 6^p (\omega(\eta) + 2\omega(0))^p \int_{\mathbb{R}} |\partial_{xx}^2 \rho_\varepsilon(t, x)|^p dx. \end{aligned}$$

2 Then

$$\begin{aligned} |J_6| &\leq \|\rho_\varepsilon\|_\infty \|v'\|_\infty \left( 3^p \eta \left( \int_0^\eta |\omega''(u)|^{p/(p-1)} du \right)^{p-1} \int_{\mathbb{R}} |\partial_x \rho_\varepsilon(t, x)|^p dx \right. \\ &\quad + 3^p (|\omega'(\eta-)| + |\omega'(0+)|)^p \int_{\mathbb{R}} |\partial_x \rho_\varepsilon(t, x)|^p dx \\ &\quad + 6^p (\omega(\eta) + 2\omega(0))^p \int_{\mathbb{R}} |\partial_{xx}^2 \rho_\varepsilon(t, x)|^p dx \\ &\quad \left. + (p-1) \int_{\mathbb{R}} |\partial_{xx}^2 \rho_\varepsilon(t, x)|^p dx \right). \end{aligned}$$

•

$$J_7 = \varepsilon p \int_{\mathbb{R}} \partial_{xxxx}^4 \rho_\varepsilon (\partial_{xx}^2 \rho_\varepsilon)^{p-1} dx = -\varepsilon p(p-1) \int_{\mathbb{R}} (\partial_{xxx}^3 \rho_\varepsilon)^2 (\partial_{xx}^2 \rho_\varepsilon)^{2(N-1)} dx \leq 0.$$

3 The above estimates give an estimate of the form

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} (\partial_{xx}^2 \rho_\varepsilon(t, x))^p dx &\leq C_7^{\varepsilon, p} \left( \int_{\mathbb{R}} |\partial_{xx}^2 \rho_\varepsilon(t, x)|^p dx + \int_{\mathbb{R}} |\partial_x \rho_\varepsilon(t, x)|^p dx + \int_{\mathbb{R}} \rho_\varepsilon^p(t, x) dx \right. \\ &\quad \left. + \int_{\mathbb{R}} |\partial_x \rho_\varepsilon(t, x)|^{2p} dx + \int_{\mathbb{R}} \rho_\varepsilon^{2p}(t, x) dx \right), \end{aligned}$$

4 where  $C_7^p = C_7^p(p, \|v'\|_\infty, \|v''\|_\infty, \|v^{(3)}\|_\infty, \sup_\varepsilon \{\|\rho_\varepsilon\|_\infty W_\varepsilon\}, C_\omega^p, C_\omega^{2p})$  and

$$C_\omega^p = \max \left\{ 3^p \eta \left( \int_0^\eta |\omega''(u)|^{p/(p-1)} du \right)^{p-1}, 3^p (|\omega'(\eta-)| + |\omega'(0+)|)^p, 6^p (\omega(\eta) + 2\omega(0))^p \right\}.$$

5 Note that  $C_7^p$  is a constant since  $\|\rho_\varepsilon\|_\infty W_\varepsilon$  is bounded with respect to  $\varepsilon$  thanks to Proposition 4  
6 and 1.5. This estimate, combined with (2.13) and (2.14), give then

$$\frac{d}{dt} \left( \int_{\mathbb{R}} |\partial_{xx}^2 \rho_\varepsilon(t, x)|^{2N} dx + \int_{\mathbb{R}} |\partial_x \rho_\varepsilon(t, x)|^{2N} dx + \int_{\mathbb{R}} \rho_\varepsilon^{2N}(t, x) dx \right)$$

$$\begin{aligned} &\leq C_8^{2N} \left( \int_{\mathbb{R}} |\partial_{xx}^2 \rho_\varepsilon(t, x)|^{2N} dx + \int_{\mathbb{R}} |\partial_x \rho_\varepsilon(t, x)|^{2N} dx + \int_{\mathbb{R}} \rho_\varepsilon^{2N}(t, x) dx \right) \\ &\quad + C_7^{2N} e^{C_9^{2N} t} \left( \int_{\mathbb{R}} |\partial_x \rho(0, x)|^{4N} dx + \int_{\mathbb{R}} |\rho(0, x)|^{4N} dx \right), \end{aligned}$$

where

$$C_8^{2N} = C_7^{2N} + \sup_{\varepsilon} C_6^{\varepsilon, 2N}, \quad C_9^{2N} = \sup_{\varepsilon} C_6^{\varepsilon, 2N}.$$

Note that  $C_6^{\varepsilon, 2N}$  is bounded with respect to  $\varepsilon$  thanks to Proposition 4 and 1.5. Since an inequality of the form

$$u'(t) \leq K_1 u(t) + K_2 e^{K_3 t}$$

implies the estimate

$$u(t) \leq u(0) e^{K_1 t} + K_2 e^{K_1 t} \int_0^t e^{(K_3 - K_1)s} ds \leq u(0) e^{K_1 t} + \frac{K_2}{K_3} \left( e^{(K_1 + K_3)t} - e^{K_1 t} \right),$$

1 we get the estimate

$$\begin{aligned} &\left( \int_{\mathbb{R}} |\partial_{xx}^2 \rho_\varepsilon(t, x)|^{2N} dx + \int_{\mathbb{R}} |\partial_x \rho_\varepsilon(t, x)|^{2N} dx + \int_{\mathbb{R}} \rho_\varepsilon^{2N}(t, x) dx \right) \\ &\leq \left( \int_{\mathbb{R}} |\partial_{xx}^2 \rho(0, x)|^{2N} dx + \int_{\mathbb{R}} |\partial_x \rho(0, x)|^{2N} dx + \int_{\mathbb{R}} \rho^{2N}(0, x) dx \right) e^{C_8^{2N} t} \\ &\quad + \frac{C_7^{2N}}{C_9^{2N}} \left( \int_{\mathbb{R}} |\partial_x \rho(0, x)|^{4N} dx + \int_{\mathbb{R}} |\rho(0, x)|^{4N} dx \right) \left( e^{(C_8^{2N} + C_9^{2N})t} - e^{C_8^{2N} t} \right), \end{aligned}$$

2 which implies

$$\begin{aligned} &\sup_{t \in [0, T]} \left( \int_{\mathbb{R}} |\partial_{xx}^2 \rho_\varepsilon(t, x)|^{2N} dx + \int_{\mathbb{R}} |\partial_x \rho_\varepsilon(t, x)|^{2N} dx + \int_{\mathbb{R}} \rho_\varepsilon^{2N}(t, x) dx \right) \\ &\leq \left( \int_{\mathbb{R}} |\partial_{xx}^2 \rho(0, x)|^{2N} dx + \int_{\mathbb{R}} |\partial_x \rho(0, x)|^{2N} dx + \int_{\mathbb{R}} \rho^{2N}(0, x) dx \right) e^{C_8^{2N} T} \\ &\quad + \frac{C_7^{2N}}{C_9^{2N}} \left( \int_{\mathbb{R}} |\partial_x \rho(0, x)|^{4N} dx + \int_{\mathbb{R}} |\rho(0, x)|^{4N} dx \right) e^{(C_8^{2N} + C_9^{2N})T} \end{aligned}$$

3 and

$$\begin{aligned} &\left( \int_0^T \int_{\mathbb{R}} |\partial_{xx}^2 \rho_\varepsilon(t, x)|^{2N} dx dt + \int_0^T \int_{\mathbb{R}} |\partial_x \rho_\varepsilon(t, x)|^{2N} dx dt + \int_0^T \int_{\mathbb{R}} \rho_\varepsilon^{2N}(t, x) dx dt \right) \\ &\leq \left( \int_{\mathbb{R}} |\partial_{xx}^2 \rho(0, x)|^{2N} dx + \int_{\mathbb{R}} |\partial_x \rho(0, x)|^{2N} dx + \int_{\mathbb{R}} \rho^{2N}(0, x) dx \right) T e^{C_8^{2N} T} \\ &\quad + \frac{C_7^{2N}}{C_9^{2N}} \left( \int_{\mathbb{R}} |\partial_x \rho(0, x)|^{4N} dx + \int_{\mathbb{R}} |\rho(0, x)|^{4N} dx \right) e^{(C_8^{2N} + C_9^{2N})T} T. \end{aligned}$$

4

□



### 3 Proof of Theorem 1

In this section, we pass to the limit as  $\varepsilon \rightarrow 0$  and we show that the limit function  $\rho$  satisfies equation (1.1).

Using Proposition 2, the sequence  $\{\rho_\varepsilon\}_\varepsilon$  is bounded in  $\mathbf{L}^\infty([0, T], \mathbf{L}^2(\mathbb{R}))$ . Using Proposition 3, the sequence  $\{\partial_x \rho_\varepsilon\}_\varepsilon$  is bounded in  $\mathbf{L}^\infty([0, T], \mathbf{L}^2(\mathbb{R}))$ . Using Propositions 1 and 4, the sequences  $\{v(\rho_\varepsilon * \omega_\varepsilon)\}_\varepsilon$ ,  $\{v'(\rho_\varepsilon * \omega_\varepsilon)\}_\varepsilon$  and  $\{\partial_x(\rho_\varepsilon * \omega_\varepsilon)\}_\varepsilon$  are bounded in  $\mathbf{L}^\infty([0, T] \times \mathbb{R})$ . Then

$$\partial_x(\rho_\varepsilon v(\rho_\varepsilon * \omega_\varepsilon)) = \partial_x \rho_\varepsilon \cdot v(\rho_\varepsilon * \omega_\varepsilon) + \rho_\varepsilon v'(\rho_\varepsilon * \omega_\varepsilon) \partial_x(\rho_\varepsilon * \omega_\varepsilon)$$

is bounded in  $\mathbf{L}^\infty([0, T], \mathbf{L}^2(\mathbb{R}))$ . Using Proposition 5, we also have a bound with respect to  $\varepsilon$  for  $\partial_{xx}^2 \rho_\varepsilon$  in the space  $\mathbf{L}^\infty([0, T], \mathbf{L}^2(\mathbb{R}))$ , then

$$\partial_t \rho_\varepsilon = \varepsilon \partial_{xx}^2 \rho_\varepsilon - \partial_x(\rho_\varepsilon v(\rho_\varepsilon * \omega_\varepsilon)) \in \mathbf{L}^\infty([0, T], \mathbf{L}^2(\mathbb{R}))$$

uniformly with respect to  $\varepsilon$ . In particular,  $\rho_\varepsilon \in \mathbf{C}([0, T], \mathbf{L}^2(\mathbb{R}))$  and the sequence is bounded in this space. Since  $\partial_t \rho_\varepsilon, \partial_x \rho_\varepsilon \in \mathbf{L}^\infty([0, T], \mathbf{L}^2(\mathbb{R}))$  with uniform bounds with respect to  $\varepsilon$ , then  $\{\rho_\varepsilon\}_\varepsilon$  is bounded in  $\mathbf{H}_{\text{loc}}^1([0, T] \times \mathbb{R})$ . Up to the extraction of a subsequence, the sequence  $\{\rho_\varepsilon\}_\varepsilon$  converges to some  $\rho$  in  $\mathbf{L}_{\text{loc}}^2([0, T] \times \mathbb{R})$  and a.e. We have now to prove that the limit  $\rho \in \mathbf{C}([0, T], \mathbf{L}^2(\mathbb{R}))$  is a solution of (1.1). Since

$$\begin{aligned} (\rho_\varepsilon * \omega_\varepsilon)(t, x) - (\rho * \omega)(t, x) &= \int_{-\varepsilon}^0 \rho_\varepsilon(t, x + y) \omega_\varepsilon(y) dy + \int_0^\eta (\rho_\varepsilon - \rho)(t, x + y) \omega(y) dy \\ &\quad + \int_\eta^{\eta+\varepsilon} \rho_\varepsilon(t, x + y) \omega_\varepsilon(y) dy \end{aligned}$$

tends to 0 when  $\varepsilon$  goes to zero, we have

$$\rho_\varepsilon v(\rho_\varepsilon * \omega_\varepsilon) \rightarrow \rho v(\rho * \omega) \text{ a.e.}$$

Therefore using dominated convergence Theorem, we get  $\rho_\varepsilon v(\rho_\varepsilon * \omega_\varepsilon) \rightarrow \rho v(\rho * \omega)$  in  $\mathbf{L}_{\text{loc}}^1([0, T] \times \mathbb{R})$ , implying that  $\rho$  is a solution of (1.1).

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