Feedback stabilization of stochastic nonlinear composite systems

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Abstract: In this paper, we study the global stabilization, by means of smooth state feedback, of partially linear composite stochastic systems.

Keywords: nonlinear stochastic systems, feedback, global stabilization, Lyapunov's function.

1 Introduction

Many recent papers (see [1, 2, 3] and references therein) addressed the problem of The global stabilization, by means of state feedback, of deterministic nonlinear control systems of the form :

$$\begin{cases} \dot{x} = f(x,y) & x \in \mathbb{R}^n \\ \dot{y} = Ay + Bu & y \in \mathbb{R}^p \end{cases} \tag{1}$$

where $u \in \mathbb{R}^k$ is the control, $A \in \mathcal{M}_{p,p}(\mathbb{R}), B \in \mathcal{M}_{p,k}(\mathbb{R})$ and f is a smooth vector field such that :

(h1) *The pair (A, B) is stabilizable.*

(h2) *The equilibrium $x = 0$ of $\dot{x} = f(x, 0)$ is globally asymptotically stable (G.A.S).*

In [3], the authors assumed that the dependence of $f(x, y)$ on y is of the form :

(h3) $f(x, y) = f(x, 0) + G(x, y).Cy.$

with $C \in \mathcal{M}_{k,p}(\mathbb{R})$. They gave conditions on the linear subsytem

$$\begin{cases} \dot{y} = Ay + Bu \\ \tilde{y} = Cy\,, & \tilde{y} \in \mathbb{R}^k \end{cases}$$

under which there exist a matrix $K \in \mathcal{M}_{k,p}(\mathbb{R})$ and a symmetric positive definite matrix $P \in \mathcal{M}_{p,p}(\mathbb{R})$ satisfying the following three conditions :

(H1) $P(A + BK) + (A + BK)^T P = -Q$ *, with Q symmetric positive (^{T} = transpose).*

(H2) $(Q^{1/2}, A + BK)$ *detectable.*

(H3) $B^T P = C.$

Using the above assumptions, they proved that the system (1) is globally asymptotically stabilizable and they gave the stabilizing feedback

$$u(x, y) = Ky - \frac{1}{2}\left(G(x, y)\right)^T \nabla V(x)$$

where V is a smooth Lyapunov function satisfying

$$\langle\, \nabla V, f(x, 0)\, \rangle < 0, \quad \forall x \in \mathbb{R}^n\,,\ x \neq 0 \tag{2}$$

The goal of our work is to show that the result of [3] can be extended when the nonlinear part of the system (1) is corrupted by a noise which satisfies the same hypothesis **(h3)** as f. We prove that the stochastic system

$$\begin{cases} dx_t = f(x_t, y_t)dt + g(x_t, y_t)dw_t \\ dy_t = (Ay_t + Bu)dt \end{cases}$$

where both f and g are of the form **(h3)**, is globally asymptotically stabilizable in probability, if **(H1)**, **(H2)**, **(H3)** and the condition

(h'2) *the solution $x_t \equiv 0$ of $dx_t = f(x_t, 0)dt + g(x_t, 0)dw_t$ is globally asymptotically stable in probability,*

hold.

Notice that the systems of the form

$$\begin{cases} dx_t = f(x_t, y_t)dt + g(x_t)dw_t \\ dy_t = (Ay_t + Bu)dt \end{cases} \tag{3}$$

have been studied in [4]. Under conditions on the dependence on y of the vector field f, the authors proved that (3) is exponentially stabilizable in mean square if

(h"2) *the solution $x_t \equiv 0$ of $dx_t = f(x_t, 0)dt + g(x_t)dw_t$ is exponentially stable in mean square.*

Remark that **(h"2)** is stronger than **(h'2)**.

2 Stochastic stability

The aim of this section is to recall the main definitions and results proved by Has'minskii (see [5], chapter V) for the zero state of a stochastic differential equation to be stable in probability.

Let $(\Omega, \mathcal{F}, P)$ be an usual probability space and denote by w a standard $\mathbb{R}^m$–valued Wiener process defined on this space. Denote by $(\mathcal{F}_t)_{t\geq 0}$ the complete right–continuous filtration generated by the standard Wiener process w. Let $x_t \in \mathbb{R}^n$ be the stochastic process solution of the stochastic differential equation written in the sense of Itô,

$$x_t = x_0 + \int_o^t b(x_s)\, ds + \sum_{k=1}^m \int_0^t \sigma_k(x_s)\, o\, dw_s^k \tag{4}$$

where b and σ_k, $1 \leq k \leq m$, are Lipschitz functions mapping $\mathbb{R}^n$ into $\mathbb{R}^n$ such that

1. $b(0) = 0$, $\sigma_k(0) = 0$, $1 \leq k \leq m$.
2. There exists a non–negative constant K such that

$$|b(x)| + \sum_{k=1}^m |\sigma_k(x)| \leq K(1 + |x|)$$

   for every x in $\mathbb{R}^n$.

Furthermore, for any $t \geq 0$ and $x_0 \in \mathbb{R}^n$, denote by $x_t(x_0)$, $t \leq t$, the solution at time t of the equation (4) starting from the state x_0 .

Then, the main notions of stochastic stability we are dealing with in this paper may be defined by

Definition 1 *The solution $x_t \equiv 0$ of the stochastic differential equation (4) is said to be stable in probability if for any $\epsilon > 0$ there exists $\delta > 0$ such that*

$$|x_0| < \delta \Rightarrow \; P\left(\sup_{t>0} |x_t(x_0)| > \epsilon\right) = 0.$$

If, in addition, there exists a neighbourhood D of the origin such that

$$P\left(\lim_{t\to+\infty} |x_t(x_0)| = 0\right) = 1, \; \forall x_0 \in D$$

the solution $x_t \equiv 0$ of the stochastic differential equation (4) is said to be asymptotically stable in probability. It is globally asymptotically stable in probability (G.A.S.P) if

$$P\left(\lim_{t\to+\infty} |x_t(x_0)| = 0\right) = 1, \; \forall x_0 \in \mathbb{R}^n$$

Therefore, denoting by L the infinitesimal generator associated with the stochastic differential equation (4) defined for any function Ψ in $C^2(\mathbb{R}^n)$ by

$$L\Psi(x) = \sum_{i=1}^{n} b^i(x)\frac{\partial \Psi}{\partial x_i}(x) + \frac{1}{2}\sum_{i,j=1}^{n} a^{i,j}(x)\frac{\partial^2 \Psi}{\partial x_i \partial x_j}(x) \tag{5}$$

where $a^{i,j}(x) = \sum_{k=1}^{m} \sigma_k^i(x)\sigma_k^j(x)$, $1 \leq i,j \leq n$, one can prove the following stochastic Lyapunov Theorem (see [5], [6]).

Theorem 1 *Let D be a neighbourhood of the point $x = 0$ which is contained in $\mathbb{R}^n$ together with its boundary, and assume that there exists a Lyapunov function V defined in D (i.e. a proper function V positive definite mapping D into $\mathbb{R}$) such that*

$$LV(x) \leq 0 \;\; (\textit{respectively} \;\; LV(x) < 0), \quad \forall x \in D, \;\; x \neq 0$$

Then, the solution $x_t \equiv 0$ of the stochastic differential equation (4) is stable (respectively asymptotically stable) in probability. It is G.A.S.P if

$$LV(x) < 0, \; \forall x \in \mathbb{R}^n, \; x \neq 0$$

In this paper, we shall make use of the latter Theorem and a stochastical version of Lassalle's invariance principle (see [7]), in order to prove that the class of nonlinear stochastic control systems introduced in the following section is globally asymptotically stabilizable in probability.

3 Main result

The systems considered here are of the form

$$\begin{cases} x_t = x_0 + \displaystyle\int_0^t f(x_s, y_s)\, ds + \int_0^t g(x_s, y_s)\, dw_s \\ y_t = y_0 + \displaystyle\int_0^t (Ay_s + Bu)\, ds \end{cases} \tag{6}$$

where the dependance of g on y is analogous to the one of f given by **(h3)**, that is

$$g(x,y) = g(x,0) + H(x,y)\, Cy \tag{7}$$

We assume that the solution $x_t \equiv 0$ is G.A.S.P for

$$x_t = x_0 + \int_0^t f(x_s, 0)\, ds + \int_0^t g(x_s, 0)\, dw_s$$

and a positive definite and proper function satisfying

$$L.V(x) = \langle\, f(x,0), \nabla V(x)\,\rangle + \frac{1}{2}\mathrm{Tr}\left(g(x,0)\big(g(x,0)\big)^T \frac{\partial^2 V}{\partial x^2}(x)\right) < 0$$

is known. Then we can state :

Theorem 2 *If there exist a matrix $K \in \mathcal{M}_{k,p}(\mathbb{R})$ and a symmetric positive definite matrix $P \in \mathcal{M}_{p,p}(\mathbb{R})$ such that* **(H1)**, **(H2)** *and* **(H3)** *hold then the system (6) is globally asymptotically stabilzable in probability thanks to the following feedback*

$$u = Ky - (G(x,y))^T\, \nabla V(x) - \frac{1}{2}\big(H(x,y)\big)^T \left(\frac{\partial^2 V}{\partial x^2}(x)\right)^T H(x,y)Cy \tag{8}$$

Proof Let $W(x,y) = V(x) + \frac{1}{2}y^T\, P\, y$. Denoting by $\mathcal{L}$ the infinitesimal generator associated with the stochastic differential equation (6-8), and setting $z = \begin{pmatrix} x \\ y \end{pmatrix}$, $Z(z) = \begin{pmatrix} f(x,y) \\ Ay + Bu(x,y) \end{pmatrix}$ and $\tilde{g}(z) = \begin{pmatrix} g(z) \\ 0 \end{pmatrix}$, one has

$$\begin{aligned} \mathcal{L}.W(z) &= Z.W(z) + \frac{1}{2}\mathrm{Tr}\left(\tilde{g}(z)\big(\tilde{g}(z)\big)^T \frac{\partial^2 W}{\partial z^2}(z)\right) \\ &= Z.W(z) + \frac{1}{2}\mathrm{Tr}\left(g(z)\big(g(z)\big)^T \frac{\partial^2 V}{\partial x^2}(x)\right) \end{aligned}$$

According to the decomposition of f given by **(h3)** and the one of g given by (7), one get

$$\begin{aligned} Z.W(x,y) &= \langle\, f(x,0)\,,\, \nabla V(x)\,\rangle - \frac{1}{2} y^T\, Q\, y + \langle\, \nabla V(x)\,,\, G(x,y)Cy\,\rangle \\ &\quad + \left\langle\, y\,,\, -PB\left(\big(G(x,y)\big)^T\, \nabla V(x) + \frac{1}{2}\big(H(x,y)\big)^T \left(\frac{\partial^2 V}{\partial x^2}(x)\right)^T H(x,y)Cy \right) \right\rangle \end{aligned}$$

and

$$\frac{1}{2}\mathrm{Tr}\left(g(z)\big(g(z)\big)^T \frac{\partial^2 V}{\partial x^2}(z)\right) = \frac{1}{2}\mathrm{Tr}\left(g(x,0)\big(g(x,0)\big)^T \frac{\partial^2 V}{\partial x^2}(x) + H(z)Cy\, y^T C^T \big(H(z)\big)^T \frac{\partial^2 V}{\partial x^2}(x)\right)$$

So, from $PB = C^T$ one has

$$\begin{aligned} \mathcal{L}.W(z) &= \langle\, f(x,0)\,,\, \nabla V(x)\,\rangle - \frac{1}{2} y^T\, Q\, y + \frac{1}{2}\mathrm{Tr}\left(g(x,0)\big(g(x,0)\big)^T \frac{\partial^2 V}{\partial x^2}(x)\right) \\ &\quad - \frac{1}{2}\left\langle\, y\,,\, C^T\big(H(x,y)\big)^T \left(\frac{\partial^2 V}{\partial x^2}(x)\right)^T H(x,y)Cy \right\rangle + \frac{1}{2}\mathrm{Tr}\left(H(z)Cy\, y^T C^T\big(H(z)\big)^T \frac{\partial^2 V}{\partial x^2}(x)\right) \end{aligned}$$

and using the fact that

$$\begin{aligned} \mathrm{Tr}\left(H(z)Cy\, y^T C^T\big(H(z)\big)^T \frac{\partial^2 V}{\partial x^2}(x)\right) &= \left\langle\, H(z)Cy\,,\, \left(\frac{\partial^2 V}{\partial x^2}(x)\right)^T H(z)Cy \right\rangle \\ &= \left\langle\, y\,,\, C^T\big(H(z)\big)^T \left(\frac{\partial^2 V}{\partial x^2}(x)\right)^T H(z)Cy \right\rangle \end{aligned}$$

it follows

$$\mathcal{L}.W(z) = L.V(x) - \frac{1}{2} y^T\, Q\, y \le 0$$

According to the stochastical version of Lassale's invariance principle (see [7]), the processus z_t converges in probability to Ω the largeste invariant set whose support is contained in the locus $\mathcal{L}.W(z_t) = 0$. Let (x_t, y_t) be a complete solution of the closed-loop system (6) along which $\mathcal{L}.W(x_t, y_t) = 0$, we must show that $(x_t, y_t) = (0,0)$ for all $t \ge 0$. Since $\mathcal{L}.W(x,y) = 0 \;\Leftrightarrow\; x = 0$ and $y^T Q y = 0$, x_t must be zero for all $t \ge 0$ and y_t will be a solution of $\dot{y}_t = (A + BK)y_t$ and must satisfy $y_t^T Q y_t = 0$ for all $t \ge 0$. By the detectability assumption (H2) this implies $y_t = 0$ for all $t \ge 0$ and, hence, $(x_t, y_t) = (0,0)$. This completes the proof.

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