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Abstract

In (Kalitine, 1982), the use of semi definite Lyapunov functions for exploring the local stability of autonomous dynamical systems has been introduced. In this paper we give an extension of the results of (Kalitine, 1982) that allows to study the local stability of nonautonomous differential systems. We give an application to the Algebraic Riccati Equation.

Key words: Lyapunov functions, nonlinear systems, stability, semidefinite functions, Riccati equations.

1 Introduction

The most efficient tool for the study of the stability of a given nonlinear system is provided by Lyapunov theory. This theory is based on the use of positive definite functions that are nonincreasing along the solutions of the considered system. It can be summarized as follows: let be given a differential system

$$\begin{cases} \dot{x}(t) = X(x(t)) \ , \\ X(0) = 0 \end{cases} \tag{1}$$

where X is assumed to be locally Lipschitz continuous on some neighborhood of the origin of $\mathbb{R}^n$. If there exists a positive definite function V whose derivative, $\dot{V}$, along the trajectories of system (1) is negative semidefinite then the system is stable and it is asymptotically stable if $\dot{V}$ is negative definite.But finding an appropriate positive definite Lyapunov function is in general a difficult task. Thanks to LaSalle's invariance principle (LaSalle & Lefschetz, 1961; Krasovski, 1959), the assumption on $\dot{V}$ in the asymptotic stability theorem has been considerably relaxed : the definiteness of $\dot{V}$ is no more required. It is sufficient to have $\dot{V} \leq 0$ and the largest positively invariant set contained in the locus $\dot{V} = 0$ must be reduced to the equilibrium point. This is very useful in practice, for it is easier to find Lyapunov functions satisfying these assumptions than it is to find Lyapunov functions which satisfy the assumptions of the original Lyapunov Theorem. In (Kalitine, 1982), a new generalization of Lyapunov's theorems for dynamical systems defined on a locally compact metric space has been derived : the author has relaxed the definiteness requirement not only on $\dot{V}$ but also on the Lyapunov function used in the stability theorem as well as in the asymptotic stability theorem. Roughly speaking, it has been proved in (Kalitine, 1982) that if there exists a function $V \geq 0$ such that $\dot{V} \leq 0$ and the set $\{x : V(x) = 0\}$ does not contain any complete negative orbit except the trivial one $x \equiv 0$ then the system is Lyapunov stable and it is asymptotically stable if no solution of the considered system can stay for all negative time in the set where $\dot{V}$ vanishes, other than the trivial one $x \equiv 0$. This result has been proved, in the case where the motions are assumed to define a group, by using the properties of backward solutions of (1). So the proof can not be used for discrete-time systems.

According to (Kalitine, 1982) this result has been first established for ODE (asymptotic stability theorem) and published in Russian (Boulgakov & Kalitine, 1979). It has been improved and published in English in (Iggidr, Kalitine & Outbib, 1996) where some applications to the stabilization problem has been given as well as the connection with existing results in the western literature. Unfortunately, the reference (Kalitine, 1982) has not been mentioned in (Iggidr, Kalitine & Outbib, 1996).

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So we would like to precise here that, according to (Kalitine, 1982), the initial work that has introduced the use of semi definite Lyapunov functions is (Boulgakov & Kalitine, 1979).

An extension to a particular class of nonautonomous differential equations has been given in (Kalitine, 1995), it concerns systems that can be written:

$$\begin{cases} \dot{x} = f(x,y,t) & x \in \mathbb{R}^p \\ \dot{y} = g(x,y,t) & y \in \mathbb{R}^q, \ t \in \mathbb{R}^+, \end{cases} \tag{2}$$

where f and g are locally Lipschitz with respect to (x,y) uniformly in t and for which there exists a Lyapunov function V satisfying $a_1(\| y - \varphi(x,t) \|) \leq V(x,y,t) \leq a_2(\| y - \varphi(x,t) \|)$ where a_i is a positive monotonic function which is zero at zero and φ is continuous function that is locally Lipschitz with respect to x uniformly in t.

In this paper, we give an other extension to nonautonomous systems. We do not assume any particular structure but we suppose the existence of an autonomous nonnegative Lyapunov function. Roughly stated, we prove the following: If for a given system $\dot{x} = X(x,t)$, there exists a function $V(x) \geq 0$ such that $\dot{V}(x,t) \leq 0$ then the system is uniformly stable if it is uniformly asymptotically stable with respect to perturbations belonging to the set where V vanishes and it is uniformly asymptotically stable if moreover the set where $\dot{V}$ vanishes is equal to the set where V vanishes. Our results and their proofs do not assume any group structure for the motions so when we write them for autonomous systems we obtain new formulation of the results of (Kalitine, 1982) as well as new proofs. We give an application to the algebraic Riccati Equation.

2 Notations and Preliminaries

We consider the nonautonomous differential equation

$$\dot{x} = X(t,x) \ , \tag{3}$$

with

$$\begin{aligned} X : I \times \Omega &\longrightarrow \quad \mathbb{R}^n \\ (t,x) &\longrightarrow X(t,x) \end{aligned}$$

where $I =]\tau, +\infty[$ for some $\tau \in \mathbb{R}^+$ and Ω is an open connected set of $\mathbb{R}^n$, containing the origin. $X(t,0) = 0$ for all $t \in I$. We assume that X satisfies the following:

Assumption 1 *X is locally Lipschitz in x uniformly in t. That is, for all $x \in \Omega$, there exists U_x a neighborhood of x, and $K_x \geq 0$ such that*

$$\begin{gathered} \forall x', x'' \in U_x, \ \forall t \in I : \\ \| X(t,x') - X(t,x'') \| \leq K_x \| x' - x'' \| \end{gathered}$$

For $(t_0, x_0) \in I \times \Omega$, we denote by $X_t(t_0, x_0)$ the unique solution of (3) with initial conditions (t_0, x_0).

Throughout this paper, we shall use the following notations :

$B_\epsilon(y) = \{x \in \mathbb{R}^n : \| x - y \| < \epsilon\}$, $\overline{B_\epsilon}(y) = \{x \in \mathbb{R}^n : \| x - y \| \leq \epsilon\}$, $B_\epsilon = B_\epsilon(0) = \{x \in \mathbb{R}^n : \| x \| < \epsilon\}$.

$\mathbb{R}^+ = \{t \in \mathbb{R} : t \geq 0\}$ is the set of nonnegative real numbers and $\mathbb{R}^- = \{t \in \mathbb{R} : t \leq 0\}$ the set of nonpositive real numbers.

K is the set of real-valued functions that are continuous, zero at zero and strictly increasing.

We first recall the various stability concepts that we need (see Rouche, Habets & Laloy (1977)).

Definition 2 (uniform stability) *The origin of $\mathbb{R}^n$ is an uniformly stable equilibrium point for (3) if for all t_0, $X_t(t_0, x_0)$ is defined for all $t \geq t_0$ and if*

$$\begin{gathered} \forall \epsilon > 0, \quad \exists \eta(\epsilon), \quad \forall t_0 \in I, \quad \forall t \geq t_0 \quad \forall x_0 \in \Omega : \\ \| x_0 \| < \eta \implies \| X_t(t_0, x_0) \| < \epsilon. \end{gathered}$$

Definition 3 (uniform attractivity) *The origin of $\mathbb{R}^n$ is an uniformly attractive equilibrium point if for all t_0, $X_t(t_0, x_0)$ is defined for all $t \geq t_0$ and if*

$$\begin{gathered} \exists \eta > 0, \quad \forall t_0 \in I, \quad \forall x_0 \in B_\eta, \\ \lim_{t \to +\infty} X_t(t_0, x_0) = 0 \\ (\textit{uniformly with respect to } t_0 \textit{ and } x_0). \end{gathered}$$

i.e.,

$$\begin{gathered} \exists \eta > 0, \quad \forall \epsilon > 0, \quad \exists T(\epsilon) > 0, \quad \forall t_0 \in I, \quad \forall x_0 \in B_\eta, \\ \forall t \geq t_0 + T(\epsilon), \quad \| X_t(t_0, x_0) \| < \epsilon. \end{gathered}$$

Definition 4 (uniform asymptotic stability) *The origin is uniformly asymptotically stable if it is uniformly stable and uniformly attractive.*

3 Main results

3.1 Stability theorems

Theorem 5 *If on a neighborhood Ω of the origin there exists a function $V \in C^1(\Omega, \mathbb{R})$ such that*

- *$V(x) \geq 0$ for all $x \in \Omega$ and $V(0) = 0$.*
- *$\dot{V}(t,x) = \langle \nabla V(x), X(t,x) \rangle \leq 0$ for all $x \in \Omega$ and all $t \geq 0$.*
- *On the positively invariant set $M = \{x \in \Omega : V(x) = 0\}$ The restriction of X is uniformly asymptotically stable.*

Then the origin is an uniformly stable equilibrium point for system (3).

Proof. Suppose that the origin is not uniformly stable. Then there exist an $\epsilon > 0$ for which we can construct a sequence of initial conditions $(x_n^0)_{n\in\mathbb{N}} \subset B_\epsilon$, $\lim\limits_{n\to\infty} x_n^0 = 0$ such that for each n there exists an initial time $t_n^0 \geq 0$ in such a way that the solution of (3) issued from x_n^0 at t_n^0 does not stay within B_ϵ for all time $t \geq t_n^0$. In other words, there exists a time $t_n > 0$ for which one has

$$\begin{cases} 0 \leq t < t_n \implies \| X_{t_n^0+t}(t_n^0, x_n^0) \| < \epsilon \,, \\ \| X_{t_n^0+t_n}(t_n^0, x_n^0) \| = \epsilon \;\; \forall n \in \mathbb{N}. \end{cases} \tag{4}$$

$t_n^0 + t_n$ is nothing but the first exit time from B_ϵ for the solution of (3) with initial condition x_n^0 at t_n^0.

The origin being uniformly asymptotically stable on M, there exist $\delta > 0$ and $T > 0$ such that one has

$$\forall t_0 \in I, \quad \forall\, x \in M \cap B_\delta, \quad \forall t \geq t_0 + T, \\ \| X_t(t_0, x) \| < \tfrac{\epsilon}{2}.$$

We take $\epsilon < \delta$ so that we can write

$$\forall t_0 \in I,\ \forall t \geq t_0 + T \ \text{ and } \ \forall y \in \overline{B_\epsilon} \cap M, \\ \| X_t(t_0, y) \| < \tfrac{\epsilon}{2}. \tag{5}$$

It must be emphasized that the time T depends only on ϵ and not on $y \in \overline{B_\epsilon} \cap M$.

Thanks to Assumption 1 and the compactness of $\overline{B_\epsilon}$ (see for instance Aeyels & Peutman (1998), Lemma 1) , there exists $\eta > 0$ such that

$$\forall (x,y) \in \overline{B_\epsilon} \times \overline{B_\epsilon} \;, \quad \forall t_0 \in I, \quad \forall t \in [t_0,\ t_0 + T] : \\ \| x - y \| < \eta \implies \| X_t(t_0, x) - X_t(t_0, y) \| < \tfrac{\epsilon}{2}. \tag{6}$$

The sequence $(x_n^0)_{n\in\mathbb{N}}$ tends to the origin as n tends to $+\infty$, so there exists $n_0 \in \mathbb{N}$ such that $\|x_n^0\| < \eta$ for all $n \geq n_0$. Thus, by (6), one has

$$\forall t_0 \in I, \quad \forall t \in [t_0,\ t_0 + T], \forall n \geq n_0 : \\ \| X_t(t_0, x_n^0) \| < \tfrac{\epsilon}{2}.$$

This is true in particular for $t_0 = t_n^0$. So, by (4), this implies that $T < t_n$ for all $n \geq n_0$. Therefore, we have $0 < t_n - T < t_n$ for all $n \geq n_0$. Taking into account (4), we get

$$\forall n \geq n_0, \quad \| X_{t_n^0+t_n-T}(t_n^0, x_n^0) \| < \epsilon.$$

So, by extracting subsequence we can assume that the sequence $(u_n)_{n\geq n_0}$ defined by $u_n = X_{t_n^0+t_n-T}(t_n^0, x_n^0)$ converges to $z \in \overline{B_\epsilon}$ as n tends to $+\infty$. Since V is assumed to be continuous, we have

$$0 \leq V(z) = \lim_{n\to+\infty} V(u_n) \\ = \lim_{n\to+\infty} V(X_{t_n^0+t_n-T}(t_n^0, x_n^0)) \leq \lim_{n\to+\infty} V(x_n^0) = 0.$$

Hence, z belongs to $\overline{B_\epsilon} \cap M$ and then (5) yields

$$\forall t_0 \in I, \quad \| X_{t_0+T}(t_0, z) \| < \frac{\epsilon}{2}. \tag{7}$$

Since $z = \lim\limits_{n\to+\infty} X_{t_n^0+t_n-T}(t_n^0, x_n^0)$, there exists $p \geq n_0$ such that

$$\| z - X_{t_p^0+t_p-T}(t_p^0, x_p^0) \| < \eta.$$

So, by (6), we get

$$\| X_{t_p^0+t_p}(t_p^0 + t_p - T, z) \\ - X_{t_p^0+t_p}\big(t_p^0 + t_p - T, X_{t_p^0+t_p-T}(t_p^0, x_p^0)\big) \| < \tfrac{\epsilon}{2}.$$

Since

$$X_{t_p^0+t_p}\big(t_p^0 + t_p - T, X_{t_p^0+t_p-T}(t_p^0, x_p^0)\big) = X_{t_p^0+t_p}\big(t_p^0, x_p^0\big),$$

we deduce that

$$\| X_{t_p^0+t_p}(t_p^0 + t_p - T, z) - X_{t_p^0+t_p}\big(t_p^0, x_p^0\big) \| < \frac{\epsilon}{2}.$$

Combining the latest inequality with (7) and invoking the triangular inequality leads to

$$\| X_{t_p^0+t_p}\big(t_p^0, x_p^0\big) \| < \epsilon.$$

But this is a contradiction to (4). So the origin is uniformly stable. ∎

Theorem 6 *If on a neighborhood Ω of the origin there exist a function $V \in C^1(\Omega, \mathbb{R})$ and a function $a \in K$ such that*

- $V(x) \geq 0$ *for all* $x \in \Omega$ *and* $V(0) = 0$.
- $\dot{V}(t,x) = \langle \nabla V(x), X(t,x) \rangle \leq -a(V(x))$ *for all* $x \in \Omega$ *and all* $t \geq 0$.
- *On the positively invariant set* $M = \{x \in \Omega : V(x) = 0\}$ *The restriction of* X *is uniformly asymptotically stable.*

Then the null solution of (3) is uniformly asymptotically stable.

Proof. System (3) being uniformly asymptotically stable on M, there exists $\delta > 0$ such that the following holds:

$$\begin{aligned} &\forall \epsilon > 0, \quad \exists T_\epsilon > 0, \quad \forall t_0 \in I, \quad \forall x_0 \in B_\delta \cap M, \\ &\quad \forall t \geq t_0 + T_\epsilon, \quad \| X_t(t_0, x_0) \| < \epsilon. \end{aligned} \tag{8}$$

Thanks to Theorem 5, the origin is uniformly stable so one can find $\gamma > 0$ in such a way that $X_t(t_0, B_\gamma) \subseteq B_\delta$ for all t_0 and all $t \geq t_0$. To prove the uniform attractivity of the origin, we shall show that the following holds:

$$\begin{aligned} &\forall \epsilon > 0, \quad \exists T_\epsilon > 0, \quad \forall t_0 \in I, \quad \forall x_0 \in B_\gamma, \\ &\quad \forall t \geq t_0 + T_\epsilon, \quad \| X_t(t_0, x_0) \| < \epsilon. \end{aligned}$$

To this end, let ϵ be any positive real number. By uniform stability, it is possible to find $\eta > 0$ satifying

$$\forall t_0 \in I, \quad \forall t \geq t_0, \quad X_t(t_0, B_\eta) \subseteq B_\epsilon. \tag{9}$$

Using (8), we can write

$$\begin{aligned} &\exists T_\eta > 0, \quad \forall t_0 \in I, \quad \forall z \in B_\delta \cap M \quad \forall t \geq t_0 + T_\eta, \\ &\quad \| X_t(t_0, z) \| < \tfrac{\eta}{2}. \end{aligned} \tag{10}$$

On the one hand, Assumption 1 implies the existence of $\alpha > 0$ satisfying :

$$\begin{aligned} &\forall (x,y) \in \overline{B_\delta} \times \overline{B_\delta} \ , \quad \forall t_0 \in I \ : \| x - z \| < \alpha \Longrightarrow \\ &\quad \| X_{t_0+T_\eta}(t_0, x) - X_{t_0+T_\eta}(t_0, z) \| < \tfrac{\eta}{2}. \end{aligned} \tag{11}$$

On the other hand, we know (LaSalle (1976), Theorem2) that all the solutions of (3) starting in B_γ converge uniformly (as t goes to infinity) to $M \cap \overline{B_\delta}$. Hence, the following holds:

$$\begin{aligned} &\exists T_\alpha > 0, \quad \forall t_0 \in I, \quad \forall x_0 \in B_\gamma \quad \forall t \geq t_0 + T_\alpha, \\ &\quad d(X_t(t_0, x_0), B_\delta \cap M) < \alpha. \end{aligned}$$

This implies

$$\begin{aligned} &\exists T_\alpha > 0, \quad \forall t_0 \in I, \quad \forall x_0 \in B_\gamma, \quad \exists z \in B_\delta \cap M \ : \\ &\quad \| X_{t_0+T_\alpha}(t_0, x_0) - z \| < \alpha. \end{aligned} \tag{12}$$

Combining (11) and (12), we get

$$\begin{aligned} \Big\| X_{t_0+T_\alpha+T_\eta}\Big(t_0 + T_\alpha, X_{t_0+T_\alpha}(t_0, x_0)\Big) \\ - X_{t_0+T_\alpha+T_\eta}\Big(t_0 + T_\alpha, z\Big) \Big\| < \tfrac{\eta}{2} \end{aligned}$$

This can also be written

$$\begin{aligned} &\| X_{t_0+T_\alpha+T_\eta}(t_0, x_0)) \\ &\quad - X_{t_0+T_\alpha+T_\eta}(t_0 + T_\alpha, z) \| < \tfrac{\eta}{2}. \end{aligned} \tag{13}$$

By (10), we have $\| X_{t_0+T_\alpha+T_\eta}(t_0 + T_\alpha, z) \| < \frac{\eta}{2}$. This with (13) leads to

$$\| X_{t_0+T_\alpha+T_\eta}(t_0, x_0) \| < \eta.$$

Thanks to uniform stability (9), we deduce that

$$\begin{aligned} &\forall t \geq t_0 + T_\alpha + T_\eta, \\ &\Big\| X_t\Big(t_0 + T_\alpha + T_\eta, X_{t_0+T_\alpha+T_\eta}(t_0, x_0)\Big) \Big\| < \epsilon. \end{aligned}$$

Since $X_t\Big(t_0 + T_\alpha + T_\eta, X_{t_0+T_\alpha+T_\eta}(t_0, x_0)\Big) = X_t(t_0, x_0)$, we have then proved the following

$$\begin{aligned} &\exists \gamma > 0, \forall \epsilon > 0, \ \exists T_\epsilon = T_\alpha + T_\eta > 0, \\ &\forall t_0 \in I, \ \forall x_0 \in B_\gamma, \ \forall t \geq t_0 + T_\epsilon, \ \|X_t(t_0, x_0)\| < \epsilon. \end{aligned}$$

This shows that system (3) is uniformly asymptotically stable. The proof is thereby completed. ■

Remark 7 In the proof of Theorem 5, we do not use the fact that V is C^1, all what we need is that V is continuous and that it is not increasing along the solutions of the system. We stated the result with the assumption that V is C^1 because in this case it is easy to see if V is not increasing.

Remark 8 Our result does not require any regularity assumption on the Jacobian matrix of V evaluated in 0. If the jacobian matrix of V at the origin is of full rank then

$$V(x) = 0 \Longleftrightarrow x = (y, z) \text{ and } z = \phi(y).$$

Hence, in this case, system (3) can be written

$$\begin{cases} \dot{y} = f(x, y, t) \\ \dot{z} = g(x, y, t). \end{cases}$$

And so, in this case, our result can be derived from (Kalitine, 1995).

Remark 9 Theorem 5 extend the results of (Aeyels & Sepulchre, 1992) to nonautonomous systems that admit autonomous first integrals.

3.2 Autonomous systems

If we write Theorem 5 and its proof for an autonomous system

$$\begin{cases} \dot{x}(t) = X(x(t)) \;, \\ X(0) = 0 \end{cases} \tag{14}$$

where X is assumed to be locally Lipschitz continuous. Then we obtain the following formulation of the results of (Kalitine, 1982) as well as a new proof.

Theorem 10 *If on a neighborhood Ω of the origin there exists a function $V \in C^1(\Omega, \mathbb{R})$ such that*

- *$V(x) \geq 0$ for all $x \in \Omega$ and $V(0) = 0$.*
- *$\dot{V}(x) = X.V(x) = \langle \nabla V(x), X(x) \rangle \leq 0$ for all $x \in \Omega$.*
- *The restriction of X is asymptotically stable on the positively invariant set $M = \{x \in \Omega : V(x) = 0\}$.*

Then the origin is a Lyapunov stable equilibrium point for system (14).

Thanks to LaSalle Invariance Principle, the hypotheses of Theorem 6 can be weakened for autonomous systems : we do not need to have $\dot{V}(x) \leq -a(V(x))$. Hence this theorem can be formulated as follows:

Theorem 11 *If on a neighborhood Ω of the origin there exists a function $V \in C^1(\Omega, \mathbb{R})$ such that*

- *$V(x) \geq 0$ for all $x \in \Omega$ and $V(0) = 0$.*
- *$\dot{V}(x) = X.V(x) = \langle \nabla V(x), X(x) \rangle \leq 0$ for all $x \in \Omega$.*
- *The restriction of X is asymptotically stable on L the largest positively invariant set contained in $\{x \in \Omega : \dot{V}(x) = 0\}$.*

Then the origin is an asymptotically stable equilibrium point for system (14).

4 Algebraic Riccati Equation

Here we give an application to the Algebraic Riccati Equation.[1] It is well known that if (A, B) is stabilizable and (C, A) is detectable then there exists a unique solution P, positive semi-definite matrix, to the celebrated Algebraic Riccati Equation (ARE) :

$$PA + A^{\mathrm{T}}P - PBR^{-1}B^{\mathrm{T}}P + C^{\mathrm{T}}C = 0 \tag{15}$$

with R a symmetric positive definite matrix.

Furthermore it is shown that the matrix $A - BR^{-1}B^{\mathrm{T}}P$ is Hurwitz (see Wonham, 1985, Theorem 12.2). The remark that this result is true with (C, A) detectable is due to Kŭcera (Kŭcera, 1972). We prove here, in an elementary manner, the following result :

Proposition 12 *If P is a symmetric positive semi-definite solution of (15), with (C, A) detectable, then $A - BR^{-1}B^{\mathrm{T}}P$ is a Hurwitz matrix.*

Proof. We consider the following function $V(x) = x^{\mathrm{T}}Px$. V is a semidefinite positive Lyapunov function for

$$\dot{x} = (A - BR^{-1}B^{\mathrm{T}}P)x. \tag{16}$$

Indeed, $\dot{V}(x) = x^{\mathrm{T}}(PA + A^{\mathrm{T}}P - 2PBR^{-1}B^{\mathrm{T}}P)x$. Using the fact that P is a solution of (15), we get $\dot{V}(x) = -\|Cx\|^2 - \langle R^{-1}B^{\mathrm{T}}Px, B^{\mathrm{T}}Px \rangle$.

Since R is positive definite, we have $\dot{V}(x) \leq 0$. It remains to show that the matrix $A - BR^{-1}B^{\mathrm{T}}P$ is asymptotically stable on L the largest invariant set by $A - BR^{-1}B^{\mathrm{T}}P$ contained in $M = \{x : \dot{V}(x) = 0\}$. But $M = \{x : Cx = 0$ and $B^{\mathrm{T}}P)x = 0\}$. Hence system (16) is governed on M by $\dot{x} = Ax$. Thus it is clear that the largest invariant set contained in M is contained in the unobservable subspace $N = \{x \in \mathbb{R}^n : Cx = CAx = \ldots = CA^{n-1}x = 0\}$.

Detectability implies that A is asymptotically stable on N. Since $L \subset N$, this implies that A is asymptotically stable on L which ends the proof. ■

It is worthwhile to compare this proof with the classical proof in the literature (Sontag, 1990; Wonham, 1985).

Remark 13 It is known (see Brockett (1970)) that there is only one positive semidefinite solution of the ARE (15) that yields a stable closed loop system. Hence our proposition proves immediately that detectability implies uniqueness of the positive semidefinite solution of the ARE (15).

[1] This example has been presented in (Iggidr, Kalitine & Sallet, 1999).

Remark 14 Usually, in the literature, one can find proofs of this result with the pair (C, A) observable and the use of LaSalle's Invariance Principle (see Anderson & Moore (1971)). Our result avoid the use of positive definite solution of the ARE.

Remark 15 Since we do not have any hypothesis on the matrix B, our result (Proposition 12) contains the celebrated Lyapunov criterion :

If $P \geq 0$, (C, A) detectable and $PA + A^{\mathrm{T}}P + C^{\mathrm{T}}C = 0$ then A is stable.

Once more we do not need $P > 0$ and the proof is elementary (compare with Wonham (1985)).

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