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Abstract

A graph G of order n is *implicit claw–heavy* if in every induced copy of $K_{1,3}$ in G there are two non–adjacent vertices with sum of their implicit degrees at least n. We study various implicit degree conditions (including, but not limiting to, Ore– and Fan–type conditions) imposing of which on specific induced subgraphs of a 2–connected implicit claw–heavy graph ensures its hamiltonicity. In particular, we improve a recent result of [X. Huang, *Implicit degree condition for hamiltonicity of 2–heavy graphs*, Discrete Appl. Math. **219** (2017) 126–131] and complete the characterizations of pairs of o–heavy and f–heavy subgraphs for hamiltonicity of 2–connected graphs.

Keywords: Implicit degree; Implicit o–heavy; Implicit f–heavy; Implicit c–heavy; Hamilton cycle.

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1 Introduction

We use [3] for terminology and notation not defined here. In the paper only finite, simple and undirected graphs are considered.

Let G be a graph and H be a subgraph of G. For a vertex $u \in V(G)$, the *neighbourhood* of u in H is denoted by $N_H(u) = \{v \in V(H) \colon uv \in E(G)\}$ and the *degree* of u in H is denoted by $d_H(u) = |N_H(u)|$. For two vertices $u, v \in V(H)$, the *distance* between u and v in H, denoted by $d_H(u, v)$, is

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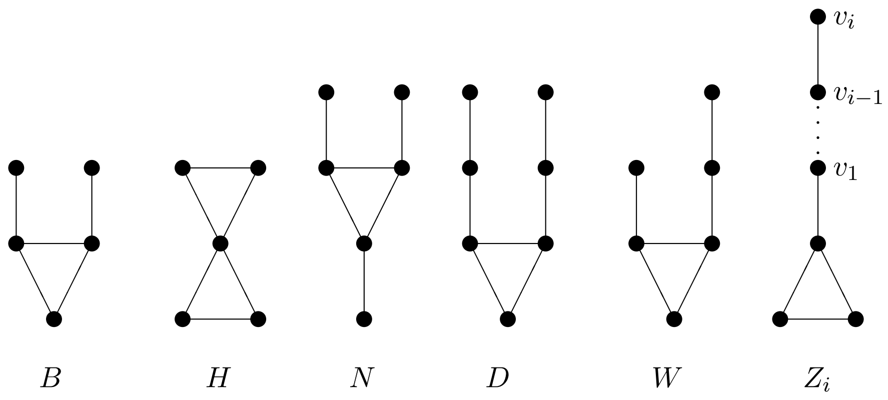

Fig. 1: Graphs B (bull), H (hourglass), N (net), D (deer), W (wounded) and Z_i.

the length of a shortest (u, v)–path in H (if there are no (u, v)–paths in H, then $d_H(u, v) := +\infty$). When there is no danger of ambiguity, we can use $N(u), d(u)$ and $d(u, v)$ in place of $N_G(u), d_G(u)$ and $d_G(u, v)$, respectively. We use $N_2(u)$ to denote the set of vertices which are at distance two from u, i.e., $N_2(u) = \{v \in V(G) : d(u, v) = 2\}$.

Let S be a graph. If there are no induced copies of S in G, then G is said to be *S–free*. Similarly, for a family $\mathcal{S}$ of graphs, G is *$\mathcal{S}$–free* if it is S–free for every $S \in \mathcal{S}$. If one demands G being $\mathcal{S}$–free, then the family $\mathcal{S}$ is *forbidden* in G. A cycle in a graph G is called its *Hamilton cycle* (or *hamiltonian cycle*), if it contains all vertices of G, and G is called *hamiltonian* if it contains a Hamilton cycle. Forbidden subgraph conditions and degree conditions are two important types of sufficient conditions for the existence of Hamilton cycles in graphs.

The only connected graph of order at least three forbidding of which in a 2–connected graph G implies hamiltonicity of G, is the path P_3 (we use P_i for a path with i vertices). When disconnected subgraphs are also considered, forbidding of $3K_1$ also ensures hamiltonicity. The former fact can be deduced from [17] and the latter from Chvátal–Erdős theorem [13]. Actually, the graphs P_3 and $3K_1$ are the only graphs of order at least three having this property. In [26], Li and Vrána proved the necessity part of the following theorem.

Theorem 1 (Li and Vrána [26]). *Let G be a 2–connected graph and S be a graph of order at least three. Then G being S–free implies that G is hamiltonian if and only if S is P_3 or $3K_1$.*

The case with pairs of forbidden subgraphs other than P_3 and $3K_1$ is much more interesting. The complete characterization of forbidden pairs of connected subgraphs for hamiltonicity, based partially on results from [5], [14], [18] and [19], was obtained by Bedrossian in [1]. The 'only if' part of the following theorem is due to Faudree and Gould.

Theorem 2 (Bedrossian [1]; Faudree and Gould [17]). *Let R and S be connected graphs with R, $S \neq P_3$ and let G be a 2-connected graph. Then G being $\{R, S\}$-free implies G is hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4, P_5, P_6, C_3, Z_1, Z_2, B, N$ or W (see Fig.1).*

In [26], Li and Vrána considered pairs of forbidden subgraphs that are not necessarily connected.

Theorem 3 (Li and Vrána [26]). *Let R and S be graphs of order at least three other than P_3 and $3K_1$ and let G be a 2-connected graph. Then G being $\{R, S\}$-free implies G is hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and S is an induced subgraph of P_6, W, N or $K_2 \cup P_4$.*

A widely studied way of relaxing the forbidden subgraph conditions for hamiltonicity is allowing the subgraphs in the graph, but with some requirements regarding degrees of their vertices imposed on them. Some of these extensions exploit the concept of implicit degree, introduced by Zhu et al. in [32].

Definition 1 (Zhu, Li and Deng [32]). Let v be a vertex of a graph G and $d(v) = l + 1$. Set $M_2 = \max\{d(u) \colon u \in N_2(v)\}$. If $N_2(v) \neq \emptyset$ and $d(v) \geq 2$, then let $d_1 \leq d_2 \leq d_3 \leq \ldots \leq d_l \leq d_{l+1} \leq \ldots$ be the degree sequence of vertices of $N(v) \cup N_2(v)$. Define

$$d^*(v) = \begin{cases} d_{l+1}, & \text{if } d_{l+1} > M_2, \\ d_l, & \text{otherwise.} \end{cases}$$

Then the *implicit degree* of v in G is defined as $id(v) = \max\{d(v), d^*(v)\}$. If $N_2(v) = \emptyset$ or $d(v) \leq 1$, then define $id(v) = d(v)$.

Observe that, by the above definition, for every $v \in V(G)$ the inequality $id(v) \geq d(v)$ holds.

Some of the (implicit) degree conditions suitable for relaxing the forbidden subgraph conditions originate from the following classical results.

Theorem 4 (Fan [15]). *Let G be a 2-connected graph of order $n \geq 3$. If*

$$d(u, v) = 2 \Rightarrow \max\{d(u), d(v)\} \geq n/2$$

for every pair of vertices u and v in G, then G is hamiltonian.

Theorem 5 (Ore [31]). *Let G be a graph of order n. If for every pair of its non-adjacent vertices the sum of their degrees is not less than n, then G is hamiltonian.*

The authors of [32] prove a counterpart of Ore's Theorem 5, where the degree sum condition is replaced with the implicit degree sum condition. Similar extension of Thereom 4 can be found in [10]. Theorems 4 and 5, and their extensions, gave rise to notions of f–heavy [30], o–heavy [7], [30], implicit f–heavy [9] and implicit o–heavy graphs. Here, we cite the definitions of o–heavy and f–heavy from [30] which are given as follows. Let G be a graph of order n. A vertex v of G is called *heavy* (or *implicit heavy*) if $d(v) \geq n/2$ (or $id(v) \geq n/2$). If v is not heavy (or not implicit heavy), we call it *light* (*implicit light*, respectively). For a given graph H we say that G is *H–o–heavy* (or *implicit H–o–heavy*) if in every induced subgraph of G isomorphic to H there are two non-adjacent vertices with the sum of their degrees (implicit degrees, respectively) in G at least n. And G is said to be *H–f–heavy* (or *implicit H–f–heavy*), if for every subgraph S of G isomorphic to H, and every two vertices $u, v \in V(S)$ holds

$$d_S(u,\, v) = 2 \Rightarrow \max\{d(u),\, d(v)\} \geq n/2$$

($\max\{id(u),\, id(v)\} \geq n/2$, respectively).

For a family of graphs $\mathcal{H}$, G is said to be *(implicit) $\mathcal{H}$–o–heavy*, if G is (implicit) H–o–heavy for every $H \in \mathcal{H}$. Classes of *$\mathcal{H}$–f–heavy* and *implicit $\mathcal{H}$–f–heavy* graphs are defined similarly. We note that the above definitions about H–f–heavy, $\mathcal{H}$–o–heavy, and $\mathcal{H}$–f–heavy are also all from [30]. When a graph is implicit $K_{1,3}$–o–heavy we will call it *implicit claw–heavy.*

Observe that every H–free graph is trivially H–o–heavy and H–f–heavy. Furthermore, every H–o–heavy (or H–f–heavy) graph is implicit H–o–heavy (implicit H–f–heavy, respectively). Replacing forbidden subgraph conditions with conditions expressed in terms of heavy subgraphs yielded the following extensions of Theorem 2.

Theorem 6 (B. Li, Ryjáček, Wang and S. Zhang [25])**.** *Let R and S be connected graphs with $R \neq P_3$, $S \neq P_3$ and let G be a 2-connected graph. Then G being $\{R,\, S\}$–o–heavy implies G is hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and $S = C_3$, P_4, P_5, Z_1, Z_2, B, N or W.*

Theorem 7. *Let R and S be connected graphs with $R \neq P_3$, $S \neq P_3$ and let G be a 2-connected graph. Then G being $\{R,\, S\}$–f–heavy implies that G is hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and S is one of the following:*
- P_4, P_5, P_6 (Chen, Wei and X. Zhang [12]),
- Z_1 (Bedrossian, Chen and Schelp [2]),
- B (G. Li, Wei and Gao [27]),
- N (Chen, Wei and X. Zhang [11]),
- Z_2, W (Ning and S. Zhang [30]).

Recently, motivated by the main result of [20], Li and Ning [23] introduced another type of heavy subgraphs. We say that an induced subgraph

H of G is *c–heavy* in G, if for every maximal clique C of H every non–trivial component of $H-C$ contains a vertex that is heavy in G. Graph G is said to be *H–c–heavy* if every induced subgraph of G isomorphic to H is c–heavy. For a family $\mathcal{H}$ of graphs, G is called *$\mathcal{H}$–c–heavy* if G is H–c–heavy for every $H \in \mathcal{H}$.

Observe that every graph is trivially $\{K_{1,3}, C_3, P_3\}$–c–heavy, since removal of a maximal clique from any of the three subgraphs results in a graph consisting of trivial components (or an empty graph). With that remark in mind, the authors of [23] extended Theorem 2 in the following way.

Theorem 8 (B. Li, Ning [23])**.** *Let S be a connected graph of order at least three and let G be a 2-connected claw–o–heavy graph. Then G being S–c–heavy implies that G is hamiltonian if and only if $S = P_4$, P_5, P_6, Z_1, Z_2, B, N or W.*

Similarly to implicit o–heavy and implicit f–heavy graphs, we can define *implicit H–c–heavy* and *implicit $\mathcal{H}$–c–heavy* graphs by replacing the degree condition in the definition of c–heavy graphs with implicit degree condition. In the light of the results presented so far, and noting that every implicit claw–f–heavy graph is implicit claw–heavy, it seems worthwhile to tackle the following problems.

Problem 1. Characterize all graphs S such that every 2–connected implicit claw–heavy and implicit S–o–heavy graph is hamiltonian.

Problem 2. Characterize all graphs S such that every 2–connected implicit claw–heavy and implicit S–f–heavy graph is hamiltonian.

Problem 3. Characterize all graphs S such that every 2–connected implicit claw–heavy and implicit S–c–heavy graph is hamiltonian.

As byproducts of the proof of our main result, we obtained the following partial answers to Problems 1– 3.

Theorem 9. *Let G be a 2–connected implicit claw-heavy graph. If G is implicit S–o–heavy for S being a subgraph of $K_2 \cup P_4$, then G is hamiltonian.*

Theorem 10. *Let G be a 2–connected implicit claw-heavy graph. If G is implicit S–f–heavy, with S being one of the graphs $K_1 \cup P_3$, $K_2 \cup P_3$, $K_1 \cup P_4$, $K_2 \cup P_4, P_4, Z_1$ and Z_2, then G is hamiltonian.*

Theorem 11. *Let G be a 2–connected implicit claw-heavy graph. If G is implicit S–c–heavy, with S being one of the graphs $K_1 \cup K_2$, $2K_1 \cup K_2$, $K_1 \cup 2K_2$, $K_2 \cup K_2$, $K_1 \cup P_3$, $K_2 \cup P_3$, $K_1 \cup P_4$, $K_2 \cup P_4, P_4, P_5$ and P_6 then G is hamiltonian.*

Clearly, for S being any of the graphs $K_1 \cup K_2$, $2K_1 \cup K_2$, $K_2 \cup K_2$ and $K_1 \cup 2K_2$, every graph is S–f–heavy. Observe also that each of the remaining subgraphs of $K_2 \cup P_4$ appear in each of Theorems 9–11. Hence, as corollaries from these theorems and Theorems 6–8, we get the following complete characterizations of heavy pairs of (not necessarily connected) subgraphs for hamiltonicity.

Corollary 1. *Let R and S be graphs other than P_3 and $3K_1$, and let G be a 2-connected graph. Then G being $\{R, S\}$–o–heavy implies G is hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and S is an induced subgraph of P_5, W, N or $K_2 \cup P_4$.*

Corollary 2. *Let R and S be graphs other than P_3 and $3K_1$, and let G be a 2-connected graph. Then G being $\{R, S\}$–f–heavy implies G is hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and S is one of $P_4, P_5, P_6, Z_1, Z_2, B, N, W$, $K_1 \cup P_3$, $K_2 \cup P_3$, $K_1 \cup P_4$ and $K_2 \cup P_4$.*

Corollary 3. *Let S be a graph of order at least three other than P_3 and $3K_1$, and let G be a 2-connected graph, claw–o–heavy graph. Then G being S-c-heavy implies G is hamiltonian if and only if S is one of P_4, P_5, P_6, Z_1, Z_2, B, N, W, $K_1 \cup K_2$, $2K_1 \cup K_2$, $K_1 \cup 2K_2$, $K_2 \cup K_2$, $K_1 \cup P_3$, $K_2 \cup P_3$, $K_1 \cup P_4$ and $K_2 \cup P_4$.*

We note that the assumption of the graph S being of order at least three in Corollary 3 is necessary, since every graph is trivially $\{K_1, 2K_1, K_2\}$–c–heavy.

For triples of forbidden subgraphs there are also many results. The following are two well–known results of this type (graphs D and H, called *deer* and *hourglass*, respectively, are represented on Fig. 1).

Theorem 12 (Broersma and Veldman [5]; Brousek [6]). *Let G be a 2-connected graph. If G is $\{K_{1,3}, P_7, D\}$–free, then G is hamiltonian.*

Theorem 13 (Faudree, Ryjáček and Schiermeyer [16]; Brousek [6]). *Let G be a 2-connected graph. If G is $\{K_{1,3}, P_7, H\}$–free, then G is hamiltonian.*

Note that the pair $\{K_{1,3}, P_6\}$ that is present in Theorem 2 is missing in Theorem 6. A construction of a 2–connected, claw–free and P_6–o–heavy graph that is not hamiltonian can be found in [25] [1]. Since every P_6–o–heavy graph is also implicit $\{P_7, D\}$–o–heavy, it is clear that Theorems 12 and 13 can not be improved by imposing the condition of implicit o–heaviness on all of their forbidden subgraphs. However, a slightly stronger implicit degree sum conditions are sufficient to ensure hamiltonicity. Our main result is the following.

[1] Nevertheless, the condition of P_6–o–heaviness can be replaced with other degree conditions on paths P_6 to ensure hamiltonicity of 2-connected claw–o–heavy graphs. We refer an interested reader to [24] for details.

Theorem 14. *Let G be a 2-connected, implicit claw–heavy graph of order n such that in every path $v_1v_2v_3v_4v_5v_6v_7$ induced in G at least one of the following conditions is satisfied:*

(1) $id(v_4) \geq n/2$, or

(2) $id(v_i) + id(v_j) \geq n$ for some $i \in \{1, 2\}$, $j \in \{6, 7\}$.

If

(1) in every induced D of G with the set of vertices $\{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$ and the set of edges $\{u_1u_2, u_2u_3, u_3u_4, u_3u_5, u_4u_5, u_5u_6, u_6u_7\}$ at least one of the following conditions is satisfied: (a) $id(u_4) \geq n/2$, or (b) $id(u_i) + id(u_j) \geq n$ for some $i \in \{1, 2, 4\}$, $j \in \{6, 7\}$, or

(2) in every induced H of G with the set of vertices $\{u_1, u_2, u_3, u_4, u_5\}$ and the set of edges $\{u_1u_2, u_2u_3, u_1u_3, u_3u_4, u_3u_5, u_4u_5\}$ at least one of the following conditions is satisfied: (a) both u_1 and u_2 are implicit heavy, or (b) $id(u_i) + id(u_j) \geq n$ for some $i \in \{1, 2\}$, $j \in \{4, 5\}$,

then G is hamiltonian.

Note that the conditions imposed on paths of order seven in Theorem 14 are satisfied in particular by implicit P_7–f–heavy and implicit P_7–c–heavy graphs. Similarly, the conditions imposed on induced deers are satisfied by implicit D–f–heavy graphs and implicit D–c–heavy graphs, and the conditions imposed on hourglasses are satisfied by implicit H–c–heavy graphs, implicit H–f–heavy graphs and implicit H–o–heavy graphs. Hence, Theorem 14 implies the following new results.

Corollary 4. *Let G be a 2-connected, implicit claw–heavy graph. If G is*

- *implicit $\{P_7, D\}$–c–heavy or implicit $\{P_7, H\}$–c–heavy, or*
- *implicit P_7–f–heavy and implicit D–c–heavy, or*
- *implicit P_7–f–heavy and implicit H–c–heavy, or*
- *implicit P_7–f–heavy and implicit H–o–heavy, or*
- *implicit P_7–c–heavy and implicit H–o–heavy, or*
- *implicit P_7–c–heavy and implicit H–f–heavy,*

then G is hamiltonian.

Some previously known results, including recent extensions of Theorem 12 and Theorem 13, can also be deduced from Theorem 14.

Corollary 5 (Huang [21])**.** *Let G be a 2–connected, implicit claw–heavy graph. If G is P_6–free, then G is hamiltonian.*

Corollary 6 (Broersma, Ryjáček and Schiermeyer [4]). *Let G be a 2–connected, claw–f–heavy graph. If G is $\{P_7, D\}$–free or $\{P_7, H\}$–free, then G is hamiltonian.*

Corollary 7 (Cai and H. Li [8]). *Let G be a 2–connected, implicit claw–f–heavy graph. If G is $\{P_7, D\}$–free or $\{P_7, H\}$–free, then G is hamiltonian.*

Corollary 8 (Ning [29]). *Let G be a 2–connected, claw–f–heavy graph. If G is $\{P_7, D\}$–f–heavy or $\{P_7, H\}$–f–heavy, then G is hamiltonian.*

Corollary 9 (Huang [22]). *Let G be a 2-connected, claw–f–heavy graph. If G is implicit $\{P_7, D\}$–f–heavy or implicit $\{P_7, H\}$–f–heavy, then G is hamiltonian.*

Corollary 10 (Cai and Zhang [9]). *Let G be a 2-connected, implicit claw–heavy graph. If G is implicit $\{P_7, D\}$–f–heavy or implicit $\{P_7, H\}$–f–heavy, then G is hamiltonian.*

The rest of the paper is organized as follows. In Section 2 we define some auxiliary notions and present lemmas used throughout the proof. The proof of Theorems 9, 10, 11 and 14 is presented in Section 3.

2 Preliminaries

In this section we present three lemmas that will be used throughout the proofs of our main results. They make use of the notion of an *implicit heavy cycle*, which is a cycle that contains all implicit heavy vertices of a graph. For a vertex $v \in V(G)$ lying on a cycle C with a given orientation, we denote by v^+ its successor on C and by v^- its predecessor. For a set $A \subset V(C)$ the sets A^+ and A^- are defined analogously, i.e., $A^+ = \{v^+ : v \in A\}$ and $A^- = \{v^- : v \in A\}$. We write xCy for the path from $x \in V(C)$ to $y \in V(C)$ following the orientation of C, and $x\overline{C}y$ denotes the path from x to y opposite to the direction of C. Similar notation is used for paths.

The next lemma is implicit in [28].

Lemma 1 (Li, Ning and Cai [28]). *Every 2-connected graph contains an implicit heavy cycle.*

A cycle C is called *nonextendable* if there is no cycle longer than C in G containing all vertices of C. We use $E^*(G)$ to denote the set $\{xy : xy \in E(G) \text{ or } id(x) + id(y) \geq n\}$.

Lemma 2 (Huang [21]). *Let G be a 2-connected graph on $n \geq 3$ vertices and C be a nonextendable cycle of G of length at most $n - 1$. If P is an xy–path in G such that $V(C) \subset V(P)$, then $xy \notin E^*(G)$.*

3 Proofs of Theorems 9–11 and 14

For a proof by contradiction suppose that a graph G satisfying the assumptions of any of the Theorems 9, 10, 11 or 14 is not hamiltonian. Then G is a 2-connected implicit claw-heavy graph. By Lemma 1, there is an implicit heavy cycle in G. Let C be a longest implicit heavy cycle in G and give C an orientation. From the assumption of 2-connectivity of G it follows that there is a path P connecting two vertices x_1, $x_2 \in V(C)$ internally disjoint with C such that $|V(P)| \geq 3$. Let $P = x_1u_1u_2...u_rx_2$ be such a path of minimum length. Note that this implies that P is induced unless $x_1x_2 \in E(G)$. The following four claims, as readers can see, also appeared in [9, 21, 22] since they are basic properties of a longest implicit heavy cycle. We also use them to start our proof.

Claim 1. *$u_kx_i^+ \notin E^*(G)$ and $u_kx_i^- \notin E^*(G)$ for every $k \in \{1, 2, \ldots, r\}$ and $i \in \{1, 2\}$.*

Proof. Since $P_1 = x_1^+Cx_1Pu_k$ and $P_2 = x_1^-\overline{C}x_1Pu_k$ are paths such that $V(C) \subset V(P_1)$ and $V(C) \subset V(P_2)$, $u_kx_1^+ \notin E^*(G)$ and $u_kx_1^- \notin E^*(G)$ by Lemma 2. Similarly, $u_kx_2^+ \notin E^*(G)$ and $u_kx_2^- \notin E^*(G)$. □

Claim 2. *$x_1^-x_1^+ \in E^*(G)$ and $x_2^-x_2^+ \in E^*(G)$.*

Proof. If $x_1^-x_1^+ \notin E(G)$, then the set $\{x_1, x_1^-, x_1^+, u_1\}$ induces a claw. By Claim 1, we have $id(u_1) + id(x_1^-) < n$ and $id(u_1) + id(x_1^+) < n$. Since G is implicit claw–heavy, this implies that $id(x_1^-) + id(x_1^+) \geq n$. Thus, $x_1^-x_1^+ \in E^*(G)$. Similarly, $x_2^-x_2^+ \in E^*(G)$. □

Claim 3. *$x_1^-x_2^- \notin E^*(G)$ and $x_1^+x_2^+ \notin E^*(G)$.*

Proof. Observe that the paths $P_1 = x_1^-\overline{C}x_2\overline{P}x_1Cx_2^-$ and $P_2 = x_1^+Cx_2\overline{P}x_1\overline{C}x_2^+$ are paths such that $V(C) \subset V(P_1)$ and $V(C) \subset V(P_2)$. Thus, the Claim follows from Lemma 2. □

Claim 4. *$x_1^-x_1^+ \in E(G)$ or $x_2^-x_2^+ \in E(G)$.*

Proof. Suppose to the contrary that $x_1^-x_1^+ \notin E(G)$ and $x_2^-x_2^+ \notin E(G)$. Then $id(x_1^-) + id(x_1^+) \geq n$ and $id(x_2^-) + id(x_2^+) \geq n$ by Claim 2. Thus, $id(x_1^-) + id(x_2^-) \geq n$ or $id(x_1^+) + id(x_2^+) \geq n$, contradicting Claim 3. □

By Claim 4, without loss of generality, we assume that $x_1^-x_1^+ \in E(G)$. The following two claims were proved in [9], here we omit their proofs.

Claim 5 (Cai and Zhang [9])**.** *$x_ix_{3-i}^- \notin E^*(G)$ and $x_ix_{3-i}^+ \notin E^*(G)$ for $i \in \{1, 2\}$.*

By Claim 5, there is a vertex in $x_i^+Cx_{3-i}^-$ not adjacent to x_i in G for $i = 1, 2$. Let y_i be the first vertex in $x_i^+Cx_{3-i}^-$ not adjacent to x_i in G for $i = 1, 2$. Let u be any vertex of P other than x_1 and x_2 and let z_i be an arbitrary vertex in $x_i^+Cy_i$ for $i = 1, 2$.

Claim 6 (Cai and Zhang [9])**.** *uz_1, uz_2, z_1x_2, z_2x_1, $z_1z_2 \notin E^*(G)$.*

The proof splits now into subcases, depending on the conditions satisfied by G.

Case 1. G is implicit $K_2 \cup P_4$–o–heavy or implicit $K_2 \cup P_4$–f–heavy.

By Claim 6, we have that both sets $\{y_1^-, y_1, u_r, x_2, y_2^-, y_2\}$ and $\{y_2^-, y_2, u_1, x_1, y_1^-, y_1\}$ induce a graph isomorphic to $K_2 \cup P_4$ in G.

Assume that G is implicit $K_2 \cup P_4$–f–heavy. Since none of the vertices u_1 and u_r belongs to C, both these vertices are implicit light. This implies that both y_2^- and y_1^- are implicit heavy, contradicting Claim 6. This contradiction proves the part of Theorem 10 regarding implicit $K_2 \cup P_4$–f–heavy graphs. By taking induced subgraphs from $\{y_1^-, y_1, u_r, x_2, y_2^-, y_2\}$ and $\{y_2^-, y_2, u_1, x_1, y_1^-, y_1\}$ corresponding to $K_1 \cup P_4$, P_4, $K_1 \cup P_3$ and $K_2 \cup P_3$, we get the same contradiction which can also prove the part of Theorem 10 regarding implicit $K_1 \cup P_4$–f–heavy graphs, implicit P_4–f–heavy graphs, implicit $K_1 \cup P_3$–f–heavy graphs and implicit $K_2 \cup P_3$–f–heavy graphs, respectively.

Consider now the case when G is implicit $K_2 \cup P_4$–o–heavy. Then there is a pair of nonadjacent vertices in both $\{y_1^-, y_1, u_r, x_2, y_2^-, y_2\}$ and $\{y_2^-, y_2, u_1, x_1, y_1^-, y_1\}$ which have implicit degree sum not less than n. Let us focus on the set $\{y_1^-, y_1, u_r, x_2, y_2^-, y_2\}$. Since $uz_1, z_1x_2, z_1z_2 \notin E^*(G)$ by Claim 6, it follows that the pair of nonadjacent vertices with implicit degree sum at least n belongs to the set $\{u_r, x_2, y_2^-, y_2\}$. Since $uz_2 \notin E^*(G)$ by Claim 6, we have $id(x_2) + id(y_2) \geq n$. Now by $id(x_1) + id(y_1) + id(x_2) + id(y_2) \geq 2n$, we have $id(x_1) + id(y_2) \geq n$ or $id(x_2) + id(y_1) \geq n$, which contradicts Claim 6. This contradiction proves the part of Theorem 9 regarding implicit $K_2 \cup P_4$–o–heavy graphs, and the left part regarding implicit S–o–heavy graphs for any proper subgraph S of $K_2 \cup P_4$ is implied by the validity of theorem for $K_2 \cup P_4$. Thus, the proof of Theorem 9 is completed.

Case 2. G is implicit S–f–heavy for S being one of Z_1 and Z_2.

Suppose first that G is implicit Z_1–f–heavy. Then, since the vertex u_1 is implicit light by the choice of C and the set $\{x_1^-, x_1^+, x_1, u_1\}$ induces Z_1, both vertices x_1^- and x_1^+ are implicit heavy. Now it follows from Claim 3 that both x_2^- and x_2^+ are implicit light. Then $x_2^-x_2^+ \in E(G)$, by Claim 2. But now the set $\{x_2^-, x_2^+, x_2, u_r\}$ induces Z_1. A contradiction. Thus, G is Z_2–f–heavy.

Suppose that $r \geq 2$ or $r = 1$ and $x_1x_2 \notin E(G)$. Then one of the sets $\{x_1^-, x_1^+, x_1, u_1, u_2\}$ or $\{x_1^-, x_1^+, x_1, u_1, x_2\}$ induces Z_2. Similarly to the previous paragraph, this implies that both x_1^- and x_1^+ are implicit heavy, and in consequence x_2^- and x_2^+ are implicit light vertices forming an edge in G. But then either $\{x_2^-, x_2^+, x_2, u_r, u_{r-1}\}$ or $\{x_2^-, x_2^+, x_2, u_r, x_1\}$ also induces a Z_2, a contradiction.

Thus, $r = 1$ and $x_1x_2 \in E(G)$. But now both sets $\{u_1, x_2, x_1, y_1^-, y_1\}$ and $\{u_1, x_1, x_2, y_2^-, y_2\}$ induce Z_2, implying that both y_1^- and y_2^- are implicit heavy. This contradicts Claim 6. Together with Case 1, this contradiction completes the proof of Theorem 10.

Case 3. G is implicit $K_1 \cup P_3$–c–heavy.

Claim 7. *x_1 and x_2 are implicit heavy.*

Proof. By Claim 6, we have that both sets $\{x_1^+, x_2, y_2^-, y_2\}$ and $\{x_2^+, x_1, y_1^-, y_1\}$ induce a graph isomorphic to $K_1 \cup P_3$ in G. Since G is implicit $K_1 \cup P_3$–c–heavy and the independent vertex of $K_1 \cup P_3$ is a maximal clique, there is an implicit heavy vertex in both sets $\{x_2, y_2^-, y_2\}$ and $\{x_1, y_1^-, y_1\}$. If y_1 or y_1^- is implicit heavy, then none of the vertices of $\{x_2, y_2^-, y_2\}$ can be implicit heavy by Claim 6, a contradiction. Hence, x_1 is implicit heavy. Similarly, x_2 is also implicit heavy. □

Claim 8. *$x_2^-x_2^+ \in E(G)$.*

Proof. By Claim 5 and Claim 7, we have that x_2^- and x_2^+ are implicit light. Since G is implicit claw-heavy, $x_2^-x_2^+ \in E(G)$. □

By Claim 3, there is a vertex in $x_i^+Cx_{3-i}^-$ not adjacent to x_i^- in G for $i = 1, 2$. Let w_i be the first vertex in $x_i^+Cx_{3-i}^-$ not adjacent to x_i^- in G for $i = 1, 2$. Note that $w_i \neq x_i^+$.

Claim 9. *$uw_i^- \notin E(G)$ and $uw_i \notin E(G)$.*

Proof. Suppose that $uw_1^- \in E(G)$. By Claim 1, we have that $w_1^- \neq x_1^+$. Then $C' = x_1Puw_1^-Cx_1^-w_1^{--}\bar{C}x_1$ is a cycle such that $V(C) \subset V(C')$, a contradiction. Hence, $uw_1^- \notin E(G)$. We also have that $uw_1 \notin E(G)$; otherwise, $C'' = x_1Puw_1Cx_1^-w_1^-\bar{C}x_1$ is a cycle such that $V(C) \subset V(C'')$, a contradiction. By symmetry, we have that $uw_2^- \notin E(G)$ and $uw_2 \notin E(G)$. □

From Claim 1 and Claim 9 we have that $\{u, x_1^-, w_1^-, w_1\}$ induces a graph isomorphic to $K_1 \cup P_3$ in G. Since G is implicit $K_1 \cup P_3$–c–heavy, there is an implicit heavy vertex in the set $\{x_1^-, w_1^-, w_1\}$. By Claim 5 and Claim 7 we have that x_1^- is implicit light. If w_1^- is implicit heavy, then $w_1^- \neq x_1^+$ by Claim 5 and Claim 7. Thus $P_1 = w_1^-Cx_2^-x_2^+Cx_1^-w_1^{--}\bar{C}x_1Px_2$ is a path such that $V(C) \subset V(P_1)$ and $w_1^-x_2 \in E^*(G)$, contradicting Lemma 2. If w_1 is implicit heavy, then $P_2 = w_1Cx_2^-x_2^+Cx_1^-w_1^-\bar{C}x_1Px_2$ is a path such that $V(C) \subset V(P_2)$ and $w_1x_2 \in E^*(G)$, contradicting Lemma 2. Thus, the part of Theorem 11 regarding implicit $K_1 \cup P_3$–c–heavy graphs is finished by these contradictions. The validity of the remaining part of Theorem 11 will be completed in the following.

Case 4. G satisfies the assumptions of Theorem 14.

Claim 10. $x_1x_2 \in E(G)$.

Proof. Suppose that $x_1x_2 \notin E(G)$. By the choice of P, Claim 1 and Claim 6 we have that $P' = y_1y_1^- x_1u_1u_2 \dots u_rx_2y_2^- y_2$ is an induced P_{r+6}, where $r \geq 1$. Let $y_1y_1^- x_1u_1v_5v_6v_7$ be the path induced by the first seven vertices of P'. Since u_1 is implicit light, it follows from the assumptions of Theorem 14 that for some $a \in \{y_1, y_1^-\}$ and $b \in \{v_6, v_7\}$ the inequality $id(a) + id(b) \geq n$ holds. Since $b \in V(P) \cup \{x_2, y_2^-, y_2\}$, this contradicts Claim 6. □

We complete the proof by considering two cases, depending on the value of r. When $r \geq 2$, we can use the method of the proof of Case 2 in [9] completely, because the proof does not involve any heavy subgraphs other than the claw. Here we omit the proof and consider the case when $r = 1$.

Suppose that $r = 1$. Then the set $\{y_1,\ y_1^-,\ x_1,\ u_1,\ x_2,\ y_2^-,\ y_2\}$ induces a D. Since the vertex u_1 is implicit light, Claim 6 implies that G does not satisfy the conditions imposed on induced deers in Theorem 14. Hence, it satisfies the conditions imposed on H.

Observe that $\{x_1^-, x_1^+, x_1, u_1, x_2\}$ induces an H. Now it follows from Claim 5 and Claim 6 that both vertices x_1^- and x_1^+ are implicit heavy. Similarly as in Case 2, this implies that both x_2^- and x_2^+ are implicit light and $x_2^-x_2^+ \in E(G)$. But now the set $\{u_1, x_1, x_2, x_2^-, x_2^+\}$ induces an H. By Claim 5 and Claim 6, this contradicts the assumptions of Theorem 14. This final contradiction completes the proof of Theorem 14.

Observe that every 2–connected implicit–claw–heavy graph that is implicit S–c–heavy for S being one of $K_1 \cup K_2$, $2K_1 \cup K_2$, $K_1 \cup 2K_2$, $K_2 \cup K_2$, $K_2 \cup P_3$, $K_1 \cup P_4$, $K_2 \cup P_4$, P_4, P_5 and P_6 satisfies the assumptions of Theorem 14. Hence, together with Case 3, Case 4 completes also the proof of Theorem 11.

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