Analytic signal in many dimensions

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Abstract

In this paper we extend analytic signal to the multidimensional case. First it is shown how to obtain separate phase-shifted components and how to combine them into instantaneous amplitude and phase. Secondly we define the proper hypercomplex analytic signal as a holomorphic hypercomplex function on the boundary of polydisk in the hypercomplex space. Next it is shown that the correct phase-shifted components can be obtained by positive frequency restriction of the Scheffers-Fourier transform based on the commutative and associative algebra generated by the set of elliptic hypercomplex numbers. Moreover we demonstrate that for $d > 2$ there is no corresponding Clifford-Fourier transform that allows to recover phase-shifted components correctly. Finally the euclidean-domain construction of instantaneous amplitude is extended to manifold and manifold-like graphs and point clouds.

1 Introduction

Analytic signal is analytic or holomorphic complex-valued function defined on the boundary of upper complex half-plane. The boundary of upper half-plane coincides with $\mathbb{R}$ and therefore analytic signal is given by the mapping $f_a : \mathbb{R} \to \mathbb{C}$. Starting from the middle of a previous century when in 1946 Denis Gabor [Gab46] proposed to use of analytic signal in signal analysis to study instantaneous amplitude and phase, analytic signal has found plenty of applications. The peculiarity of analytic signal was emphasized by [Vak96] where it was shown that only analytic signal meets suitable physical conditions for amplitude, phase and frequency. During the last few decades an interest emerged towards the studies of analytic signal in many dimensions motivated by the problems coming from the fields that range from the image/video processing to the multidimensional oscillating processes in physics like seismic, electromagnetic and gravitational waves. Mainly it was accepted that to generalize analytic signal properly to the case of several dimensions one should rely on algebraic construction which extends the ordinary complex numbers in a convenient manner. Such constructions are usually referred to as hypercomplex numbers [SKE]. Finally one should be able to construct a hypercomplex-valued analytic signal $f_h : \mathbb{R}^d \to \mathbb{S}$, where $\mathbb{S}$ represents some general hypercomplex algebraic system, that naturally extends all the desired properties to obtain instantaneous amplitude and phase.

A number of works addressed various issues related to the proper choice of hypercomplex number system, definition of hypercomplex Fourier transform and fractional Hilbert transforms for the sake of studying instantaneous amplitude and phase. Mainly these works were based on properties of various spaces such as $\mathbb{C}^d$, quaternions, Clifford algebras and Cayley-Dickson constructions. In the following we list just some of works dedicated to the studies of analytic signal in many dimensions. To the best of our knowledge, the first works on multidimensional analytic signal method arrived in the early 1990-s including the work of Ell [Ell92] on hypercomplex transforms, the work of Bülow on the generalization of analytic signal method to many dimensions [BS01] and the work of Felsberg and Sommer on monogenic signals [FS01]. Since then there was a huge number of works

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that studied various aspects of hypercomplex signals and their properties. Studies of Clifford-Fourier transform and transforms based on Caley-Dickson construction applied to the multidimensional analytic signal method include [SLB07], [HS11], [LBSE14], [HS13], [ELBS14], [ELBS14], [LBS08], [BS10], [AGSE07]. Partial Hilbert transforms were studied by [BS01], [YZ+08], [Zha14] and others. Recent studies on Clifford-Fourier transform include [BDSS06], [MH06], [DBDSS11], [DB12], [DBDSS11], [ES07], [BHHA08], [FBSC01] and many others. Monogenic signal method was recently studied and reviewed in the works [Ber14], [Bri17] and others. We do not aim to provide the complete list of references, however we hope that reader can find all the relevant references for example in the review works [HS16] and [BBRH13, LB17].

A hypercomplex analytic signal f_h is expected to extend all useful properties that we have in 1-D case. First of all we should be able to extract and generalize the instantaneous amplitude and phase to d dimensions. Second, the Fourier spectrum of complex analytic signal is supported only over positive frequencies, therefore we expect that hypercomplex Fourier transform will have its hypercomplex-valued spectrum to be supported only in some positive quadrant of hypercomplex space. Third the conjugated parts of complex analytic signal are related by Hilbert transform and we can expect that conjugated components in hypercomplex space should be related also by some combination of Hilbert transforms. And finally indeed the hypercomplex analytic signal should be defined as an extension of some hypercomplex holomorphic function of several hypercomplex variables defined to the boundary of some shape in hypercomplex space.

We address these issues in sequential order. First of all we start by considering the Fourier integral formula and show that Hilbert transform in 1-D is related to the modified Fourier integral formula. This fact allows us to define instantaneous amplitude, phase and frequency *without any reference to hypercomplex number systems and holomorphic functions.* We proceed by generalizing the modified Fourier integral formula to several dimensions and define all necessary phase-shifted components that we can assemble into instantaneous amplitude and phase. Second we address the question of existence of holomorphic functions of several hypercomplex variables. Following the work by [Sch93] it appears that commutative and associative hypercomplex algebra generated by the set of elliptic ($e_i^2 = -1$) generators is a suitable space for hypercomplex analytic signal to live in, we refer to such a hypercomplex algebra as *Scheffers space* and denote it by $\mathbb{S}_d$. Hypercomplex analytic signal therefore is defined as a holomorphic function on the boundary of the polydisk/upper half-plane in certain hypercomplex space that we refer to as *total Scheffers space* and denote it by $\mathbb{S}^d$. Then we observe the validity of Cauchy integral formula for the functions $\mathbb{S}^d \to \mathbb{S}_d$ that is calculated over a hypersurface inside the polydisk in $\mathbb{S}^d$ and deduce the corresponding fractional Hilbert transforms that relate the hypercomplex conjugated components. Finally it appears that Fourier transform with values in Scheffers space is supported only over non-negative frequencies.

The paper is organised in the following way. In Section 2, we give general description of analytic signal, establish modified Fourier integral formula and relate it to the Hilbert transform. In Section 3, it is shown how to obtain phase-shifted components in several dimensions and combine them to define the instantaneous amplitude, phase and frequency. Then, in Section 4 we describe general theory of hypercomplex holomorphic functions of several hypercomplex variables and introduce Cauchy integral formula and Hilbert transform in polydisk. Next, in Section 5, we review the commutative hypercomplex Scheffers-Fourier transform, which is based on Sheffers algebra, and show how to obtain phase-shifted replicas of the original function by the restriction of the Scheffers-Fourier spectrum only to the positive frequencies and conclude by proving the Bedrosian's theorem for hypercomplex analytic signal. In Section 6, the basic facts about hypercomplex Clifford algebras of type $\mathcal{C}l_{0,q}(\mathbb{R})$ are given and it is demonstrated that positive frequency restriction of the class of hypercomplex Clifford-Fourier transforms *does not* provide correct phase-shifted components correctly for $d > 2$. The paper is concluded by Section 7 where possible extension to the definition of instantaneous amplitude of the oscillating function on a manifold is given and a numerical example of construction of phase-shifted components over a patch of discrete manifold-like graph is shown.

2 In one dimension

Let us assume that we have some oscillating function $f : \mathbb{R} \to \mathbb{R}$ and we want to obtain its envelope function, that "forgets" its local oscillatory behavior and instantaneous phase, that shows how this oscillatory behavior evolves. In one dimension one gets analytic signal by combining the original signal with its phase shifted version. The phase-shifted version is given by the Hilbert transform

$\tilde{f}$ of the original function f

$$\tilde{f} = H\left[f\right](x) := \text{p.v.}\frac{1}{\pi}\int\limits_{\mathbb{R}} \frac{f(y)}{x-y}\,\mathrm{d}y := \lim_{\varepsilon\to 0}\frac{1}{\pi}\int\limits_{|x-y|>\varepsilon}\frac{f(y)}{x-y}\,\mathrm{d}y, \quad x\in\mathbb{R}. \tag{1}$$

The analytic signal $f_a : \mathbb{R} \to \mathbb{C}$ is obtained by combining f with $\tilde{f}$, i.e. $f_a = f + i\tilde{f}$, where $i^2 = -1$. This definition comes from the fact that analytic signal is the restriction of holomorphic (analytic) function in the upper half-plane of $\mathbb{C}$ to its boundary $\mathbb{R} \subset \mathbb{C}$, i.e. f and $\tilde{f}$ are in fact harmonic conjugates on the boundary of upper complex half-plane. Basic theory of Hilbert transform is concisely given in Appendix A. We will need several standard definitions.

Definition 2.1. The Fourier transform of a function $f : \mathbb{R} \to \mathbb{C}$ is defined as

$$\begin{aligned}\hat{f}(\omega) = F[f](\omega) &:= \int\limits_{\mathbb{R}} f(x)e^{-i\omega x}\,\mathrm{d}x,\\ f(x) = F^{-1}[\hat{f}](x) &:= \frac{1}{2\pi}\int\limits_{\mathbb{R}}\hat{f}(\omega)e^{i\omega x}\,\mathrm{d}\omega.\end{aligned} \tag{2}$$

Definition 2.2. The sign function is defined as

$$\operatorname{sign}(x) := \begin{cases}-1 & \text{if } x<0,\\ 0 & \text{if } x=0,\\ 1 & \text{if } x>0.\end{cases}$$

It turns out that $F[f_a](\omega)$ is supported only on $[0,\infty)$ and therefore spectrum of f_a does not have negative frequency components, that frequently are redundant for applications. This is due to the following relation.

Lemma 2.3. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and continuously differentiable, then

$$F[\tilde{f}](\omega) = \int\limits_{-\infty}^{\infty} H[f](x)\,e^{-i\omega x}\,\mathrm{d}x = -i\operatorname{sign}(\omega)F[f](\omega) \tag{3}$$

Appropriate combination of the function f and its Hilbert transform $\tilde{f}$ provide us with all the necessary information about the envelope function and instantaneous phase and frequency.

Definition 2.4 (Instantaneous amplitude, phase and frequency)**.**

- The absolute value of complex valued analytic signal is called *instantaneous amplitude* or *envelope* $a(x)$

$$a(x) = |f_a(x)| = \sqrt{f(x)^2 + \tilde{f}(x)^2} \tag{4}$$

- The argument $\phi(x)$ of analytic signal $f_a(x)$ is called *instantaneous phase*

$$\phi(x) = \arg[f_a](x) = \arctan\left(\frac{\tilde{f}(x)}{f(x)}\right) \tag{5}$$

- The *instantaneous frequency* $\nu(x)$ is defined as derivative of instantaneous phase

$$\nu(x) = \frac{\mathrm{d}\phi}{\mathrm{d}x} \tag{6}$$

Remark 2.5. Frequently analytic signal $f_a(x)$ is defined in terms of its Fourier transform by discarding the negative frequency components

$$\hat{f}_a(\omega) = (1 + \operatorname{sign}(\omega))\,\hat{f}(\omega) \tag{7}$$

Quite complementary perspective on the relationship between f and $\tilde{f}$ may be obtained by observing that

$$H\left[\cos(\omega x' + \phi)\right](x) = \sin(\omega x + \phi), \tag{8}$$

therefore Hilbert transform phase-shifts the cosine by $\pi/2$ and we obtain sine. This will be the basic requirement that we will like to preserve for multidimensional case.

Next we give definitions of sine and cosine transforms that will appear handy in defining the phase-shifted components.

Definition 2.6. The Fourier cosine and sine transforms of continuous and absolutely integrable $f : \mathbb{R} \to \mathbb{R}$ are defined by

$$F_c[f](\omega) = \int_{-\infty}^{\infty} f(x)\cos(\omega x)\ \mathrm{d}x, \tag{9}$$

$$F_s[f](\omega) = \int_{-\infty}^{\infty} f(x)\sin(\omega x)\ \mathrm{d}x. \tag{10}$$

In the following we will rely on the Fourier integral formula that follows directly from the definition of Fourier transform.

Lemma 2.7 (Fourier integral formula)**.** Any continuous and absolutely integrable on its domain f, may be represented as

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(x')\cos\left[(x-x')\omega\right]\ \mathrm{d}x'\,\mathrm{d}\omega. \tag{11}$$

Proof. From the definition of Fourier transform we can write the decomposition

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x')e^{(x-x')\omega}\ \mathrm{d}x'\,\mathrm{d}\omega. \tag{12}$$

Taking the real part of both sides and observing that cosine is even in ω we arrive to the result. $\square$

In the following theorem we deduce the modified Fourier integral formula and relate it to the Hilbert transform.

Theorem 2.8. *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and absolutely integrable and let $H[f]$ be the Hilbert transform of f. Then we have*

$$H[f](x) = \frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(x')\sin\left[(x-x')\omega\right]\ \mathrm{d}x'\ \mathrm{d}\omega. \tag{13}$$

Proof. Let us denote for convenience the left-hand side of (13) by $\tilde{f}$ and right-hand side by f_1. In the proof we treat the generalized function $\delta(x)$ as an infinitely localized measure or a Gaussian function in the limit of zero variance. Under this assumption, we can write the following cumulative distribution function

$$H(x) = \int_{-\infty}^{x} \delta(s)\ \mathrm{d}s,$$

which is the Heaviside step function $H(\omega) = \frac{1}{2}(1 + \mathrm{sign}(\omega))$, and, in particular, we have

$$H(0) = \int_{-\infty}^{0} \delta(s)\ \mathrm{d}s = \int_{0}^{\infty} \delta(s)\ \mathrm{d}s = \frac{1}{2}. \tag{14}$$

Then we write the Fourier transform of the right hand side of (13) as

$$
\begin{aligned}
F[f_1](\omega') =& \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\int_{0}^{\infty} \int_{-\infty}^{\infty} f(x') \sin\left[\omega(x - x')\right] \mathrm{d}x' \, \mathrm{d}\omega \right) \; e^{-i\omega' x} \; \mathrm{d}x \\
&= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \int_{0}^{\infty} \int_{-\infty}^{\infty} f(x') \left[e^{i\omega(x-x')} - e^{-i\omega(x-x')} \right] e^{-i\omega' x} \; \mathrm{d}x' \, \mathrm{d}\omega \, \mathrm{d}x \\
&= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \int_{0}^{\infty} F[f](\omega) e^{i(\omega - \omega')x} \; \mathrm{d}\omega \, \mathrm{d}x - \frac{1}{2\pi i} \int_{-\infty}^{\infty} \int_{0}^{\infty} F[f](-\omega) e^{i(-\omega-\omega')x} \; \mathrm{d}\omega \, \mathrm{d}x \\
&= -i \int_{0}^{\infty} F[f](\omega) \delta(\omega - \omega') \; \mathrm{d}\omega + i \int_{0}^{\infty} F[f](-\omega) \delta(-\omega - \omega') \; \mathrm{d}\omega.
\end{aligned}
\tag{15}
$$

In the derivations above we used the relation $\int_{-\infty}^{\infty} e^{i(\omega-\omega')x} \; \mathrm{d}x = 2\pi\delta(\omega - \omega')$. Now we see, by taking into account (14), that

$$
\begin{aligned}
\int_{0}^{\infty} F[f](\omega)\delta(\omega - \omega') \; \mathrm{d}\omega = H(\omega')F[f](\omega'), \\
\int_{0}^{\infty} F[f](-\omega)\delta(-\omega - \omega') \; \mathrm{d}\omega = H(-\omega')F[f](\omega').
\end{aligned}
$$

Finally we see from the last line in (15) that

$$
F[f_1](\omega) = \begin{cases} -iF[f](\omega), & \text{if } \omega > 0, \\ -\frac{i}{2}F[f](\omega) + \frac{i}{2}F[f](-\omega), & \text{if } \omega = 0, \\ iF[f](\omega), & \text{if } \omega < 0, \end{cases} \tag{16}
$$

which is exactly in accordance with (3) and therefore $f_1 = \tilde{f}$. □

Now by expanding sine of difference in the integrand of (13), we can write

$$
\begin{aligned}
f_1(x) &= \frac{1}{\pi} \int_{0}^{\infty} \int_{-\infty}^{\infty} f(x') \left[\sin(x\omega) \cos(x'\omega) - \cos(x\omega) \sin(x'\omega) \right] \mathrm{d}x' \; \mathrm{d}\omega \\
&= \frac{1}{\pi} \int_{0}^{\infty} \left(\int_{-\infty}^{\infty} f(x') \cos(x'\omega) \, \mathrm{d}x' \right) \sin(x\omega) \; \mathrm{d}\omega \\
&\quad - \frac{1}{\pi} \int_{0}^{\infty} \left(\int_{-\infty}^{\infty} f(x') \sin(x'\omega) \, \mathrm{d}x' \right) \cos(x\omega) \; \mathrm{d}\omega.
\end{aligned}
\tag{17}
$$

For the sake of brevity we introduce the following notation

$$\begin{aligned}
\alpha_0 &= \alpha_0(x,\omega) := \cos(\omega x)\,,\\
\alpha_1 &= \alpha_1(x,\omega) := \sin(\omega x)\,,\\
\alpha^0 &= \alpha^0(\omega) := \int_{-\infty}^{\infty} f(x')\cos(x'\omega)\ \mathrm{d}x'\\
\alpha^1 &= \alpha^1(\omega) := \int_{-\infty}^{\infty} f(x')\sin(x'\omega)\ \mathrm{d}x',\\
\hat{\alpha}^0 &= \hat{\alpha}^0(x) := \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\cos(\omega x)\ \mathrm{d}\omega,\\
\hat{\alpha}^1 &= \hat{\alpha}^1(x) := \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\sin(\omega x)\ \mathrm{d}\omega.
\end{aligned} \tag{18}$$

For the integration over ω of a pair of functions $a(\omega)$ and $b(x,\omega)$ we will write

$$\begin{aligned}
\langle a, b\rangle_+ &:= \frac{1}{\pi}\int_{0}^{\infty} a(\omega)b(x,\omega)\ \mathrm{d}\omega,\\
\langle a, b\rangle &:= \frac{1}{2\pi}\int_{-\infty}^{\infty} a(\omega)b(x,\omega)\ \mathrm{d}\omega.
\end{aligned} \tag{19}$$

Observation 2.9. Equipped with the notation introduced in (18) and (19) and using Fourier integral formula (11) we can expand $f_0 = f(x)$ in the following form

$$f_0 = \langle\alpha^0, \alpha_0\rangle_+ + \langle\alpha^1, \alpha_1\rangle_+. \tag{20}$$

Similarly using the results of Theorem 2.8 we can write for $f_1 = \tilde{f}$

$$f_1 = \langle\alpha^0, \alpha_1\rangle_+ - \langle\alpha^1, \alpha_0\rangle_+. \tag{21}$$

The above observation shows that the original signal $f(x)$ is obtained by first projecting onto sine and cosine harmonics and then reconstructed by using the same harmonics correspondingly. Hilbert transform of $f(x)$, or phase-shifted version of $f(x)$, is obtained differently though. First we project initial function f on sine and cosine harmonics to obtain α^1 and α^0, however then for reconstruction the phase-shifted harmonics are used, i.e. α_0 and α_1.

Observation 2.10. The Fourier transform of a general complex-valued function $f : \mathbb{R} \to \mathbb{C}$, with $f(x) = f_0(x) + if_1(x)$, in terms of projection on phase-shifted harmonics may be written as

$$\hat{f}(\omega) = F[f](\omega) = \alpha^0(\omega) - i\alpha^1(\omega), \tag{22}$$

which follows if we apply Euler's formula $e^{ix} = \cos x + i \sin x$ to the definition (2). The inverse transform may be written as

$$F^{-1}[\hat{f}](x) = \hat{\alpha}^0(x) + i\hat{\alpha}^1(x). \tag{23}$$

The original function may be recovered by

$$f(x) = F^{-1}[\hat{f}](x) = \langle\alpha^0, \alpha_0\rangle + \langle\alpha^1, \alpha_1\rangle + i\left(\langle\alpha^0, \alpha_1\rangle - \langle\alpha^1, \alpha_0\rangle\right). \tag{24}$$

A variety of approaches for generalization of the analytic signal method to many dimensions has been employed. For example quaternionic-valued generalization of analytic signal has been realized [BS01] and works well in case $d = 2$. Later on in the Example 6.4 we show why and how the quaternionic based approach is consistent with presented theory. As a next step we are going to generalize formulas (20) and (21) to multidimensional setting, first, without relying on any additional hypercomplex structure, while later in Sections 4 and 5 we choose a convenient hypercomplex algebra and Fourier transform.

3 In many dimensions

In several dimensions we are puzzled with the same questions on what are the adequate definitions of instantaneous and phase? Again we start by considering some function $f : \mathbb{R}^d \to \mathbb{R}$ that supposedly has some local and global oscillatory behavior. We start by extending the Fourier integral formula to many dimensions.

For convenience and brevity we introduce the following notation. For a given binary $\{0,1\}$-vector $\boldsymbol{j} \in \{0,1\}^d$ we define the functions $\alpha_{\boldsymbol{j}} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ by

$$\alpha_{\boldsymbol{j}} = \alpha_{\boldsymbol{j}}(\boldsymbol{x}, \boldsymbol{\omega}) := \prod_{l=1}^{d} \cos\left(\omega_l x_l - j_l \frac{\pi}{2}\right). \tag{25}$$

Theorem 3.1. *Any continuous and absolutely integrable over its domain $f(\boldsymbol{x})$ may be decomposed as*

$$\begin{aligned} f(x_1, ..., x_d) = \frac{1}{\pi^d} \int_0^\infty \int_{-\infty}^\infty ... \int_0^\infty \int_{-\infty}^\infty f(x_1', x_2', ..., x_d') \cos\left[(x_1 - x_1')\omega_1\right] \cdot ... \\ \cdot \cos\left[(x_d - x_d')\omega_d\right] \, \mathrm{d}x_1' \, \mathrm{d}\omega_1 ... \mathrm{d}x_d' \, \mathrm{d}\omega_d. \end{aligned} \tag{26}$$

or shortly, by employing the notation (25),

$$f(\boldsymbol{x}) = \frac{1}{\pi^d} \int_{\boldsymbol{0}}^{\boldsymbol{\infty}} \int_{\boldsymbol{-\infty}}^{\boldsymbol{\infty}} f(\boldsymbol{x'}) \alpha_{\boldsymbol{0}} \left(\boldsymbol{x} - \boldsymbol{x'}, \boldsymbol{\omega}\right) \, \mathrm{d}\boldsymbol{x'} \, \mathrm{d}\boldsymbol{\omega} \tag{27}$$

with $\boldsymbol{0}$ *being the all-zeros vector.*

Proof. Proof is the same as for 1-D case. □

Similarly to the result of Theorem 2.8, next we define the phase-shifted copies of the original signal.

Definition 3.2. The phase-shifted version $f_{\boldsymbol{j}}(\boldsymbol{x})$, in the direction $\boldsymbol{j} \in \{0,1\}^d$, of the function $f(\boldsymbol{x})$ is given by

$$f_{\boldsymbol{j}}(\boldsymbol{x}) = \frac{1}{\pi^d} \int_{\boldsymbol{0}}^{\boldsymbol{\infty}} \int_{\boldsymbol{-\infty}}^{\boldsymbol{\infty}} f(\boldsymbol{x'}) \alpha_{\boldsymbol{j}} \left(\boldsymbol{x} - \boldsymbol{x'}, \boldsymbol{\omega}\right) \, \mathrm{d}\boldsymbol{x'} \, \mathrm{d}\boldsymbol{\omega}. \tag{28}$$

Next we give the definitions of instantaneous amplitude, phase and frequency.

Definition 3.3.

- The square root of sum of all phase-shifted copies of signal is called *instantaneous amplitude* or *envelope* $a(\boldsymbol{x})$
$$a(\boldsymbol{x}) := \sqrt{\sum_{\boldsymbol{j} \in \{0,1\}^d} f_{\boldsymbol{j}}(\boldsymbol{x})^2} \tag{29}$$
- The *instantaneous phase* $\phi_{\boldsymbol{j}}(\boldsymbol{x})$ in the direction $\boldsymbol{j} \in \{0,1\}^d$ is defined as
$$\phi_{\boldsymbol{j}}(\boldsymbol{x}) := \arctan\left(\frac{f_{\boldsymbol{j}}(\boldsymbol{x})}{f(\boldsymbol{x})}\right) \tag{30}$$
- The *instantaneous frequency* $\nu_{\boldsymbol{j}}(\boldsymbol{x})$ in the direction $\boldsymbol{j} \in \{0,1\}^d$ is defined as partial derivatives of the corresponding instantaneous phase in the directions given by $\boldsymbol{j}$
$$\nu_{\boldsymbol{j}}(\boldsymbol{x}) = \partial^{\boldsymbol{j}} \phi_{\boldsymbol{j}} := \partial_1^{j_1} \circ \partial_2^{j_2} \circ ... \circ \partial_d^{j_d} \left(\phi_{\boldsymbol{j}}\right)(\boldsymbol{x}), \tag{31}$$
where we used multi-index notation for derivatives, $\partial_i^1 = \frac{\partial}{\partial x_i}$ and $\partial_i^0 = \mathrm{id}$. Please note, that no Einstein summation convention was employed in (31).

We can generalize Theorem 2.8 that connects phase-shifts $f_{\boldsymbol{j}}$ with the corresponding multidimensional Hilbert transform. However first we define Hilbert transform in several dimensions.

Definition 3.4. The Hilbert transform in the direction $\boldsymbol{j} \in \{0,1\}^d$ is defined by

$$H_{\boldsymbol{j}}[f](\boldsymbol{x}) = \text{p.v.}\frac{1}{\pi^{|\boldsymbol{j}|}} \int\limits_{\mathbb{R}^{|\boldsymbol{j}|}} \frac{f(\boldsymbol{y})}{(\boldsymbol{x}-\boldsymbol{y})^{\boldsymbol{j}}} \, \mathrm{d}\boldsymbol{y}^{\boldsymbol{j}}, \tag{32}$$

where $|\boldsymbol{j}|$ gives the number of 1-s in $\boldsymbol{j}$, $(\boldsymbol{x}-\boldsymbol{y})^{\boldsymbol{j}} = \prod_{i=1}^{d}(x_i - y_i)^{j_i}$ and $\mathrm{d}\boldsymbol{y}^{\boldsymbol{j}} = \mathrm{d}y_1^{j_1} \ldots \mathrm{d}y_d^{j_d}$, i.e. integration is performed only over the variables indicated by the vector $\boldsymbol{j}$.

Theorem 3.5. *For a continuous $f(\boldsymbol{x})$ that is also absolutely integrable in its domain, we have*

$$f_{\boldsymbol{j}} = H_{\boldsymbol{j}} f. \tag{33}$$

Proof. proof follows the lines of the proof of Theorem 2.8. □

One important property of instantaneous amplitude is that it represents intrinsic property of function f not dependent on the choice of coordinate system.

Observation 3.6. Let us suppose we are given a non-degenerate linear transformation $A : \mathbb{R}^d \to \mathbb{R}^d$, $\det A \neq 0$. Transformation A maps old coordinate system to a new one by $\boldsymbol{x'} = A\boldsymbol{x}$. First it follows from (32) that Hilbert transform is invariant with respect to A

$$H_{\boldsymbol{j}}\left[f(A\boldsymbol{x})\right](\boldsymbol{y}) = H_{\boldsymbol{j}}\left[f(\boldsymbol{x})\right](A\boldsymbol{y}), \tag{34}$$

and therefore for $f'(\boldsymbol{x}) = f(A\boldsymbol{x})$ we will have from (29) that $a'(\boldsymbol{x}) = a\,(A\boldsymbol{x})$.

It will be useful to express the phase-shifted in the direction $\boldsymbol{j}$ function as a combination of harmonics $\alpha_{\boldsymbol{j}}$ and corresponding projection coefficients $\alpha^{\boldsymbol{i}}$. We define projection coefficients similarly to (18),

$$\begin{aligned}
\alpha^{\boldsymbol{j}} &= \alpha^{\boldsymbol{j}}(\boldsymbol{\omega}) := \int\limits_{-\infty}^{\infty} f(\boldsymbol{x})\alpha_{\boldsymbol{j}}\left(\boldsymbol{x},\boldsymbol{\omega}\right) \mathrm{d}\boldsymbol{x}, \\
\hat{\alpha}^{\boldsymbol{j}} &= \hat{\alpha}^{\boldsymbol{j}}(\boldsymbol{x}) := \frac{1}{(2\pi)^d} \int\limits_{-\infty}^{\infty} \hat{f}(\boldsymbol{\omega})\alpha_{\boldsymbol{j}}\left(\boldsymbol{x},\boldsymbol{\omega}\right) \mathrm{d}\boldsymbol{\omega}, \\
\hat{\alpha}_{+}^{\boldsymbol{j}} &= \hat{\alpha}_{+}^{\boldsymbol{j}}(\boldsymbol{x}) := \frac{1}{\pi^d} \int\limits_{\boldsymbol{0}}^{\infty} \hat{f}(\boldsymbol{\omega})\alpha_{\boldsymbol{j}}\left(\boldsymbol{x},\boldsymbol{\omega}\right) \mathrm{d}\boldsymbol{\omega}
\end{aligned} \tag{35}$$

and brackets, similarly to (19), as

$$\begin{aligned}
\langle a, b\rangle_{+} &= \langle a, b\rangle_{+}(\boldsymbol{x}) := \frac{1}{\pi^d} \int\limits_{\boldsymbol{0}}^{\infty} a(\boldsymbol{\omega})b(\boldsymbol{x},\boldsymbol{\omega}) \, \mathrm{d}\boldsymbol{\omega}, \\
\langle a, b\rangle &= \langle a, b\rangle(\boldsymbol{x}) := \frac{1}{(2\pi)^d} \int\limits_{-\infty}^{\infty} a(\boldsymbol{\omega})b(\boldsymbol{x},\boldsymbol{\omega}) \, \mathrm{d}\boldsymbol{\omega},
\end{aligned} \tag{36}$$

for two functions $a(\boldsymbol{\omega}) = a(\omega_1, \ldots, \omega_d)$ and $b(\boldsymbol{x},\boldsymbol{\omega}) = b(x_1, \ldots, x_d, \omega_1, \ldots, \omega_d)$. It is useful to observe how phase-shifted functions $f_{\boldsymbol{j}}$ could be obtained from $\alpha^{\boldsymbol{j}}$.

Theorem 3.7. *For a given function $f : \mathbb{R}^d \to \mathbb{R}$, one obtains its phase-shifted in the direction $\boldsymbol{j} \in \{0,1\}^d$ copy $f_{\boldsymbol{j}}$, by*

$$f_{\boldsymbol{j}} = \sum_{\boldsymbol{i} \in \{0,1\}^d} (-1)^{|(\boldsymbol{i}\oplus\boldsymbol{j})\ominus\boldsymbol{i}|} \langle \alpha^{\boldsymbol{i}\oplus\boldsymbol{j}}, \alpha_{\boldsymbol{i}}\rangle_{+}, \tag{37}$$

where $\oplus$ is a binary exclusive OR operation acting elementwise on its arguments and $\ominus$ is defined as following: $1 \ominus 0 = 1$ and the result is 0 otherwise.

Proof. The proof follows immediately after we will expand all the cosines and sines of difference in (28) and make all the substitutions from (25) and (35). ☐

The sign before each term in (37) is selected by the following rule. We have "$-$ " sign before the term in the sum when the corresponding component of superscript, i.e. $\boldsymbol{i} \oplus \boldsymbol{j}$, inside the brackets is 1 and correspondent component of subscript, i.e. $\boldsymbol{i}$, is 0 and " $+$ " sign in any other case. For example in 2-D case we will have for component f_{01}

$$f_{01} = -\langle \alpha^{01}, \alpha_{00} \rangle_+ + \langle \alpha^{00}, \alpha_{01} \rangle_+ - \langle \alpha^{11}, \alpha_{10} \rangle_+ + \langle \alpha^{10}, \alpha_{11} \rangle_+. \tag{38}$$

Remark 3.8. Before moving to the theory of hypercomplex holomorphic functions and analytic signals, it will be useful to note that from the definition (28) there will be in total 2^d different components $f_{\boldsymbol{j}}$. It hints us that highly likely the dimensionality of hypercomplex algebra in which analytic signal should take it's values should be also 2^d.

4 Holomorphic functions of several hypercomplex variables

4.1 Commutative hypercomplex algebra

The development of hypercomplex algebraic systems started with the works of Gauss, Hamilton [Ham44], Clifford [Cli71], Cockle [Coc49] and many others. Generally a hypercomplex variable w is given by

$$w = \sum_{i=0}^{n} w_i e_i, \tag{39}$$

with coefficients $w_i \in \mathbb{R}$. If otherwise not stated explicitly we always suppose $w_i \in \mathbb{R}$. Product for the units e_i is defined by the rule

$$e_i e_j = \sum_{s=0}^{n} \gamma_{ij}^s e_s, \tag{40}$$

where $\gamma_{ij}^s \in \mathbb{R}$ are the structure coefficients. Only algebras with unital element or module are of interest to us, i.e. those having element $\epsilon = 1$ such that $\epsilon x = x\epsilon = x$. The unital element have expansion

$$\epsilon = \sum_{i=0}^{n} \epsilon_i e_i.$$

For simplicity we will always assume that the unital element $\epsilon = e_0 = 1$.

The units e_i, $i \neq 0$, could be subdivided into three groups [CBC+08].

1. e_i is *elliptic unit* if $e_i^2 = -e_0$
2. e_i is *parabolic unit* if $e_i^2 = 0$
3. e_i is *hyperbolic unit* if $e_i^2 = e_0$.

Example 4.1. General properties of some abstract hypercomplex algebra $\mathbb{S}$ that is defined by its structure constants γ_{ij}^s may be quite complicated. We will focus mainly on the commutative and associative algebras constructed from the set of elliptic-type generators. For example suppose we have two elliptic generators $e_1^2 = -1$ and $e_2 = -1$, then one can construct the simplest associative and commutative elliptic algebra as $\mathbb{S}_2 := \text{span}_{\mathbb{R}} \{e_0 \equiv 1, e_1, e_2, e_3 \equiv e_1 e_2\}$. The algebra have $2^2 = 4$ units. For $\mathbb{S}_2$ we have the following multiplication table

e_0	e_1	e_2	e_3
e_1	$-e_0$	e_3	$-e_2$
e_2	e_3	$-e_0$	$-e_1$
e_3	$-e_2$	$-e_1$	e_0

One could assemble various algebras depending on the problem. All of the above unit systems have their own applications. For example, numbers with elliptic unit relate group of rotations and translations of 2-dimensional Euclidean space to the complex numbers along with their central role in harmonic analysis. Algebras with parabolic units may represent Galileo's transformation and finally algebra with hyperbolic units is used to represent Lorentz group in special relativity

[CBC+08]. In this work we mainly focus on the concepts of instantaneous amplitude, phase and frequency of some oscillating process. Elliptic units are of great interest to those studying the oscillating processes due to the famous Euler's formula. We briefly remind how it comes. Taylor series of exponential is given by

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

We could have various Euler's formulae. In elliptic case when $z = e_i x$, $e_i^2 = -1$, we can write from the above Taylor's expansion

$$e^{e_i x} = \cos(x) + e_i \sin(x). \tag{41}$$

When $e_i^2 = 0$ we have

$$e^{e_i x} = 1 + e_i x. \tag{42}$$

And finally when $e_i^2 = 1$ we get

$$e^{e_i x} = \cosh(x) + e_i \sinh(x). \tag{43}$$

It is due to the relation (41) that one relies on the complex exponential for analysis of oscillating processes. One may well expect that to obtain amplitude/frequency information, one will need a Fourier transform based on the algebra containing some *elliptic* numbers.

It was a question whether the chosen hypercomplex algebra for multidimensional analytic signal should be commutative, anticommutative, associative or neither. Whether it is allowed to have zero divisors or not. In Section 6 we show that commutative and associative algebra not only suffices our goals but also is essentially a *necessary* condition to define Fourier transform coherently with (37). Based on these general considerations we define the simplest associative and commutative algebra for a set of elliptic units. We call it *elliptic Scheffers*[1] *algebra* and denote it by $\mathbb{S}_d$.

Definition 4.2. The elliptic Scheffers algebra $\mathbb{S}_d$ over a field $\mathbb{R}$ is an algebra of dimension 2^d with unit $\epsilon = e_0 = 1$ and generators $\{e_1, \ldots, e_d\}$ satisfying the conditions $e_i^2 = -1$, $e_i e_j = e_j e_i$, $i, j = 1, \ldots, d$. The basis of the algebra $\mathbb{S}_d$ consists of the elements of the form $e_0 = 1$, $e_\beta = e_{\beta_1} e_{\beta_2} \ldots e_{\beta_d}$, $\beta = (\beta_1, \ldots, \beta_s), \beta_1 < \beta_2 < \cdots < \beta_s$, $1 \le s \le d$. Each element $w \in \mathbb{S}_d$ has the form

$$w = \sum_{\beta=0}^{2^d-1} w_\beta e_\beta = w_0 \epsilon + \sum_{s=1}^{d} \sum_{\beta_1 < \cdots < \beta_s} w_\beta e_\beta, \quad w_\beta \in \mathbb{R}. \tag{44}$$

If otherwise not stated explicitly we will consider mainly algebras over $\mathbb{R}$. As an illustration to the above definition we can write any $w \in \mathbb{S}_2$ in the form

$$w = w_0 + w_1 e_1 + w_2 e_2 + w_{12} e_1 e_2. \tag{45}$$

Remark 4.3. We will employ three notations for indices of independent units and their components. The dimension of Scheffers algebra is 2^d therefore there always will be 2^d indexes however one can choose between labelings. First notation is given by natural numbering of hypercomplex units. For example for $\mathbb{S}_2$ we will have the following generators $\{e_0 \equiv 1, e_1, e_2, e_3 \equiv e_1 e_2\}$. Second notation is given by the set of indices in subscript, i.e. for $\mathbb{S}_2$ we have the generators $\{e_0 \equiv 1, e_1, e_2, e_3 \equiv e_{1,2}\}$. Third notation uses binary representation $\boldsymbol{j} \in \{0,1\}^d$. For $\mathbb{S}_2$ example generators will be labeled as $\{e_{00} \equiv 1, e_{10}, e_{01}, e_{11}\}$. To map between naturally ordered and binary representations we use index function $\mathrm{ind}(\cdot)$, for example in case of $\mathbb{S}_2$ we have $\mathrm{ind}(11) = \{1,2\} = 3$ as well as we have $\mathrm{ind}(3) = 11 \in \{0,1\}^2$ or $\mathrm{ind}(1,2) = 3 \in [0, 1, ..., 2^d - 1]$. Even though the function $\mathrm{ind}(\cdot)$ is overloaded, in practice it is clear how to apply it.

Remark 4.4. The space $\mathbb{S}_d$ is a Banach space, with norm defined by

$$\|w\|^2 = \sum_\beta |w_\beta|^2. \tag{46}$$

[1] Due to the work of G. W. Scheffers [Sch93]

One can check that $\mathbb{S}_d$ is a unital commutative ring. The main difference of the elliptic Scheffers algebra from the algebra of complex numbers is that factor law does not hold in general, i.e. if we have vanishing product of some non zero $a, b \in \mathbb{S}_d$, it does not necessarily follow that either a or b vanish. In case $ab = 0,\ a, b \neq 0$, a and b are called zero divisors. *Zero divisors are not invertible.* However what is of actual importance for us is that $\mathbb{S}_d$ has subspaces spanned by elements $\{e_0, e_i\}$, each of which has the structure of the field of complex numbers and therefore factor law holds inside these subspaces. We will need this observation to define properly the Cauchy integral formula later [Ped97].

Definition 4.5. The space $\mathbb{S}(i),\ i = 1, ..., d$, is defined by

$$\mathbb{S}(i) := \{a + be_i | a, b \in \mathbb{R}\}. \tag{47}$$

As in complex analysis we define *open disk, polydisk* and *upper half-plane*.

Definition 4.6. The disk $D_i(w, r) \subset \mathbb{S}(i)$ of radius r centered at $w \in \mathbb{S}(i)$ is defined as

$$D_i(w, r) = \{x \in \mathbb{S}(i) : |x_\beta - w_\beta| < r, \beta \in \{0, i\}\}. \tag{48}$$

Definition 4.7. The upper half-plane $\mathbb{S}_+(i)$ is given by

$$\mathbb{S}_+(i) = \{a + be_i | a, b \in \mathbb{R}; b > 0\}. \tag{49}$$

Finally we combine these spaces to provide the overall domains for the hypercomplex functions.

Definition 4.8. The *total Scheffers space* $\mathbb{S}^d$ is a direct sum of d subalgebras $\mathbb{S}(i)$

$$\mathbb{S}^d = \bigoplus_{i=1}^{d} \mathbb{S}(i). \tag{50}$$

Definition 4.9. The *upper Scheffers space* $\mathbb{S}^d_+$ is a direct sum of d subalgebras $\mathbb{S}_+(i)$

$$\mathbb{S}^d_+ = \bigoplus_{i=1}^{d} \mathbb{S}_+(i). \tag{51}$$

Definition 4.10. For $\boldsymbol{j} \in \{0, 1\}^d$ we define $\boldsymbol{j}$-polydisk $D_{\boldsymbol{j}} \subset \mathbb{S}^d$ as a product of $|\boldsymbol{j}|$ disks $D_i(w^i, r^i)$

$$D_{\boldsymbol{j}}(w, r) = \prod_{i \in \mathrm{ind}(\boldsymbol{j})} D_i(w^i, r^i) \tag{52}$$

and we will write for d-polydisk $D_d(w, r) = \prod_{i=1}^{d} D_i(w^i, r^i)$ when we take product of all d disks. To avoid possible confusion we write explicitly d-polydisk D_d and 1-disk D_d to differentiate between a polydisk and a disk.

Observation 4.11. The boundary of $\mathbb{S}^d_+$ coincides with $\mathbb{R}^d$, i.e.

$$\partial \mathbb{S}^d_+ = \overline{\mathbb{S}^d_+} \setminus \mathbb{S}^d_+ \simeq \mathbb{R}^d. \tag{53}$$

While each $\mathbb{S}(i)$ is equivalent to the complex plane $\mathbb{C}$, the combination $\mathbb{S}(i) \oplus \mathbb{S}(j)$ is not equivalent to $\mathbb{C}^2$ because imaginary units have different labels – in a sense $\mathbb{S}(i) \oplus \mathbb{S}(j)$ contains more information than $\mathbb{C}^2$.

4.2 Holomorphic (analytic) functions of hypercomplex variable

The scope of this section is to define hypercomplex holomorphic functions of type $f : \mathbb{S}^d \to \mathbb{S}_d$. Indeed we are interested in rather restricted case of mappings between hypercomplex spaces which finally will be consistent with our definition of analytic signal as a function on the boundary of the polydisc in $\mathbb{S}^d$ in full correspondence with 1-dimensional analytic signal. There are some similarities and differences with theory of functions of several complex variables. We refer the reader to the book [Kra01] for details on several complex variables and to the concise review [GG07] to refresh basic facts on the complex analysis in memory. We start by providing several equivalent definitions of a holomorphic function of a complex variable by following Krantz [Kra01]. Then we define holomorphic function of several hypercomplex variables in a pretty similar fashion. Starting from the very basics will be instructive for the hypercomplex case.

The derivative of a complex-valued function $f : \mathbb{C} \to \mathbb{C}$ is defined as a limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}. \tag{54}$$

If the limit exists one says that f is *complex differentiable* at point z_0.

Definition 4.12. If $f : \Omega \to \mathbb{C}$, defined on some open subset $\Omega \subseteq \mathbb{C}$, is complex differentiable at every point $z_0 \in \Omega$, then f is called *holomorphic* on Ω.

Complex differentiability means that the derivative at a point does not depend on the way sequence approaches the point. If one writes two limits, one approaching in the direction parallel to the real axis and one parallel to the imaginary axis, then after equating the corresponding real and imaginary terms one obtains the Cauchy-Riemann equations and therefore equivalent definition of a holomorphic function.

Definition 4.13. A function $f : \Omega \to \mathbb{C}$, explicitly written as a combination of two real-valued functions u and v as $f(x + iy) = u(x, y) + iv(x, y)$, is called *holomorphic* in some open domain $\Omega \subseteq \mathbb{C}$ if u and v satisfy the Cauchy-Riemann equations

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y}, \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}. \end{aligned} \tag{55}$$

A holomorphic function of a complex variable $z \in \mathbb{C}$ can be compactly introduced using the derivatives with respect to z and $\bar{z}$

$$\begin{aligned} \frac{\partial}{\partial z} &= \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \\ \frac{\partial}{\partial \bar{z}} &= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right). \end{aligned} \tag{56}$$

Definition 4.14. A function $f : \Omega \to \mathbb{C}$ is called *holomorphic* in some open domain $\Omega \in \mathbb{C}$ if for each $z \in \Omega$

$$\frac{\partial f}{\partial \bar{z}} = 0, \tag{57}$$

which means that holomorphic function is a proper function of a single complex variable z and not of a conjugated variable $\bar{z}$.

Contour integration in the complex space is yet another viewpoint on the holomorphic functions. Complex integration plays the central role in our work due to our reliance on Hilbert transform that is defined as a limiting case of contour integral over the boundary of an open disk (see Appendix A).

Definition 4.15. A continuous function $f : \Omega \to \mathbb{C}$ is called *holomorphic* in some open domain $\Omega \in \mathbb{C}$ if there is an $r = r(w) > 0$ such that $\overline{D(w, r)} \subseteq \Omega$ and

$$f(z) = \frac{1}{2\pi i} \oint_{|\zeta - w| = r} \frac{f(\zeta)}{\zeta - z}\, \mathrm{d}\zeta \tag{58}$$

for all $z \in D(w, r)$.

Remark 4.16. Structure of $\mathbb{C}$ as a field is important for the definition of the integral in (58). Only because every element of a field has a multiplicative inverse, we can write the integration of the kernel $\frac{1}{z}$. However in imaginary situation when the contour of integration passes through some points where $(\zeta - z)$ is not invertible, we will be unable to write the Cauchy formula (58).

Finally we can justify the word "analytic" in the title of the paper by giving the following definition. Note that analytic and holomorphic means essentially the same.

Definition 4.17. A function $f : \Omega \to \mathbb{C}$ is called *analytic* (holomorphic) in some open domain $\Omega \in \mathbb{C}$ if for each $z_0 \in \Omega$ there is an $r = r(z_0) > 0$ such that $D(z_0, r) \subseteq \Omega$ and f can be written as an absolutely and uniformly convergent power series

$$f(z) = \sum_k a_k \left(z - z_0\right)^k \tag{59}$$

for all $z \in D(z_0, r)$.

Having introduced basic notions of holomorphic functions now we are ready to define holomorphic function in the hypercomplex space. Theory of holomorphic functions was generalized to the functions of commutative hypercomplex variables by G.W. Scheffers in 1893 [Sch93], then his work was extended in 1928 by P.W. Ketchum [Ket28] and later unified by V.S. Vladimirov and I.V. Volovich [VV84a], [VV84b] to the theory of superdifferentiable functions in superspace with commuting and anti-commuting parts. It is important to distinct that in this work we present different Cauchy formula from one given by Ketchum and Vladimirov. Their approach is based on complexification (i.e. $w_\beta \in \mathbb{C}$) of the underlying hypercomplex algebra while we are working in the hypercomplex algebra over reals. We briefly describe their formula in Appendix B.

First by following [Sch93] we give definition of a holomorphic function in some general hypercomplex space Γ that is defined by its structure constants γ^s_{ik}, see (40). This definition generalizes ordinary Cauchy-Riemann equations and follows from the same argument that hypercomplex derivative at a point should be independent on the way one approaches the point in a hypercomplex space. Finally we will restrict attention to the functions $\mathbb{S}(i) \to \mathbb{S}_d$ and will see that we are consistent with our previous derivations.

Definition 4.18 ([Sch93])**.** A function $f : \Gamma \to \Gamma,\ w \mapsto f(w)$ is *holomorphic* in an open domain $\Omega \subseteq \Gamma$ if for each point $w \in \Omega$ it satisfies the generalized Cauchy-Riemann-Scheffers equations

$$\frac{\partial f_s}{\partial w_k} = \sum_{i=0}^{2^d-1} \gamma^s_{ik} \frac{\partial f_i}{\partial w_0}, \ \ s, k = 0, \ldots, 2^d - 1. \tag{60}$$

General equations could be simplified if we consider the functions that have only subsets of $\Gamma = \mathbb{S}_d$ as their domain, in our case we consider functions $f : \mathbb{S}(i) \to \mathbb{S}_d$ that map plane $\mathbb{S}(i)$ in $\mathbb{S}_d$ to the total space $\mathbb{S}_d$. If we restrict our domain only to the plane $\mathbb{S}(i)$ and insert values of structure constants for $\mathbb{S}_d$ we readily arrive to simple Cauchy-Riemann-Scheffers equations (see for example eqq. (65) and (66) for 2-D case). We will arrive to the explicit equations in a while. Mappings from restricted domains, i.e. in our case $\mathbb{S}(i) \subseteq \mathbb{S}_d$, are described in details in [Ket28].

Remark 4.19. To avoid confusion the comment on the relationship between the spaces $\mathbb{S}(i)$, $\mathbb{S}^d$ and $\mathbb{S}_d$ will be useful. While we can embed trivially $\mathbb{S}(i) \hookrightarrow \mathbb{S}^d$ as well as $\mathbb{S}(i) \hookrightarrow \mathbb{S}_d$, one may tend to think that it will also be natural to embed $\mathbb{S}^d \hookrightarrow \mathbb{S}_d$. Last embedding always exists because dimensionality of $\mathbb{S}_d$ is 2^d and is non smaller that the dimensionality of $\mathbb{S}^d$ which is $2d$. However this embedding will not be trivial. For example consider an element $\boldsymbol{x} \in \mathbb{R}^d \subseteq \mathbb{S}^d$. In $\mathbb{S}_d$ there is only one unital element, while in $\mathbb{S}^d$ there are d of them, because $\mathbb{S}^d := \bigoplus_{i=1}^d \mathbb{S}(i)$, therefore it is impossible to "match" the unital elements of $\mathbb{S}_d$ and $\mathbb{S}^d$ in a trivial way. Therefore it is better to think about $\mathbb{S}^d$ and $\mathbb{S}_d$ as being two different spaces that however "share" planes $\mathbb{S}(i)$.

From now on we study the functions $f : \mathbb{S}^d \to \mathbb{S}_d$. A function f could be explicitly expressed as a function of several hypercomplex variables $f(z_1, ..., z_d) \equiv f(x_1 + e_1 y_1, ..., x_d + e_d y_d)$ with $z_k \in \mathbb{S}(k)$. A function f is called *hypercomplex differentiable in the variable* z_k if the following limit exists

$$\frac{\partial f}{\partial z_k}(z_1, ..., z_0, ..., z_d) := \lim_{z \to z_0} \frac{f(z_1, ..., z, ..., z_d) - f(z_1, ..., z_0, ..., z_d)}{z - z_0}. \tag{61}$$

Definition 4.20. If $f : \Omega \to \mathbb{S}_d$, defined on some open subset $\Omega \subseteq \mathbb{S}^d$, is hypercomplex differentiable in each variable separately at every point $z_0 \in \Omega$, then f is called *holomorphic* on Ω.

One obtains the generalized Cauchy-Riemann equations using the same argument as in complex analysis. We provide explicit detailed example for a function $f : \mathbb{S}^2 \to \mathbb{S}_2$, which is also easy to generalize for any $d > 2$.

Example 4.21. Let us consider a function $f : \mathbb{S}^2 \to \mathbb{S}_2$. It depends on 4 real variables (x_1, y_1, x_2, y_2) because $f(z_1, z_2) \equiv f(x_1 + e_1 y_1, x_2 + e_2 y_2)$. Moreover in $\mathbb{S}_2$ we can expand the value of f componentwise

$$\begin{aligned} f(z_1, z_2) \equiv f\left(x_1 + e_1 y_1, x_2 + e_2 y_2\right) = {} & f_0\left(x_1 + e_1 y_1, x_2 + e_2 y_2\right) \\ & + e_1 f_1(x_1 + e_1 y_1, x_2 + e_2 y_2) \\ & + e_2 f_2(x_1 + e_1 y_1, x_2 + e_2 y_2) \\ & + e_1 e_2 f_{12}(x_1 + e_1 y_1, x_2 + e_2 y_2). \end{aligned} \tag{62}$$

First we calculate the derivative in the "real" direction of the plane $\mathbb{S}(1)$ as

$$\lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x + e_1 y_1, z_2) - f(z_1, z_2)}{\Delta x} = \frac{\partial f_0}{\partial x_1} + e_1 \frac{\partial f_1}{\partial x_1} + e_2 \frac{\partial f_2}{\partial x_1} + e_1 e_2 \frac{\partial f_{12}}{\partial x_1}. \tag{63}$$

Then calculate the derivative in the orthogonal direction e_1, note that $1/e_1 = e_1/e_1^2 = -e_1$,

$$\lim_{\Delta y \to 0} \frac{f(x_1 + e_1 y_1 + e_1 \Delta y, z_2) - f(z_1, z_2)}{e_1 \Delta y} = -e_1 \frac{\partial f_0}{\partial y_1} + \frac{\partial f_1}{\partial y_1} - e_1 e_2 \frac{\partial f_2}{\partial y_1} + e_2 \frac{\partial f_{12}}{\partial y_1}. \tag{64}$$

Equating the corresponding components gives us the Cauchy-Riemann-Scheffers equations for the variable z_1

$$\begin{aligned} \frac{\partial f_0}{\partial x_1} &= \frac{\partial f_1}{\partial y_1}, \\ \frac{\partial f_1}{\partial x_1} &= -\frac{\partial f_0}{\partial y_1}, \\ \frac{\partial f_2}{\partial x_1} &= \frac{\partial f_{12}}{\partial y_1}, \\ \frac{\partial f_{12}}{\partial x_1} &= -\frac{\partial f_2}{\partial y_1}. \end{aligned} \tag{65}$$

Similarly we get 4 equations for z_2

$$\begin{aligned} \frac{\partial f_0}{\partial x_2} &= \frac{\partial f_1}{\partial y_2}, \\ \frac{\partial f_1}{\partial x_2} &= -\frac{\partial f_0}{\partial y_2}, \\ \frac{\partial f_2}{\partial x_2} &= \frac{\partial f_{12}}{\partial y_2}, \\ \frac{\partial f_{12}}{\partial x_2} &= -\frac{\partial f_2}{\partial y_2}. \end{aligned} \tag{66}$$

These 8 equations provide necessary and sufficient conditions for a function $f(z_1, z_2)$ to be holomorphic. It comes immediately after one observes that all three available directions e_0, e_1, e_2 are mutually orthogonal in the euclidean representation of $\mathbb{S}_2$ and by linearity of the derivative operator.

Let us now express the conditions on holomorphic function using the derivative with respect to conjugated variable, where the conjugation of $z_j = a + e_j b$ is defined by $\bar{z}_j = a - e_j b$, $j = 1, ..., d$. Derivative operators are defined similar to (56) by

$$\begin{aligned} \frac{\partial}{\partial z_j} &= \frac{1}{2}\left(\frac{\partial}{\partial x_j} - e_j \frac{\partial}{\partial y_j}\right), \\ \frac{\partial}{\partial \bar{z}_j} &= \frac{1}{2}\left(\frac{\partial}{\partial x_j} + e_j \frac{\partial}{\partial y_j}\right). \end{aligned} \tag{67}$$

Definition 4.22. A function $f : \Omega \to \mathbb{S}_d$, defined on some open subset $\Omega \subseteq \mathbb{S}^d$, is *holomorphic* on Ω if

$$\frac{\partial f}{\partial \bar{z}_j} = 0, \text{for } j = 1, ..., d, \tag{68}$$

at every point $z_0 \in \Omega$.

Again we give an example for $f:\mathbb{S}^2\to\mathbb{S}_2$ which is easy to generalize to any dimension.

Example 4.23. The derivative with respect to $\bar{z}_1$ of $f:\mathbb{S}^2\to\mathbb{S}_2$ is given by

$$\begin{aligned}\frac{\partial f}{\partial \bar{z}_1}=\frac{1}{2}\left(\frac{\partial f}{\partial x_1}+e_1\frac{\partial f}{\partial y_1}\right)=\frac{1}{2}\Bigg(&\frac{\partial f_0}{\partial x_1}+e_1\frac{\partial f_1}{\partial x_1}+e_2\frac{\partial f_2}{\partial x_1}+e_1e_2\frac{\partial f_{12}}{\partial x_1}\\ &+e_1\frac{\partial f_0}{\partial y_1}-\frac{\partial f_1}{\partial y_1}+e_1e_2\frac{\partial f_2}{\partial y_1}-e_2\frac{\partial f_{12}}{\partial y_1}\Bigg)\end{aligned}\tag{69}$$

Equating the components in front of each e_i to zero we get equations (65).

We are ready to introduce the Cauchy integral formula for the functions $f:\mathbb{S}^d\to\mathbb{S}_d$. The concept of Riemann contour integration as well as Cauchy's integral theorem are well defined in the *unital real, commutative and associative* algebras [Ped97]. Some care will be taken to define Cauchy integral formula because some elements of $\mathbb{S}_d$ are not invertible. To introduce the concept of integral in the space $\mathbb{S}_d$ it is natural to assume (cf. (44)) that

$$\mathrm{d}w=\sum_{\beta=0}^{2^d-1}e_\beta\,\mathrm{d}w_\beta.\tag{70}$$

Therefore we understand the integral $\int f(w)\,\mathrm{d}w$ of $\mathbb{S}_d$-valued function $f(w)$ of the variable $w\in\mathbb{S}_d$ as integral of differential form $\sum_\beta f(w)e_\beta\,\mathrm{d}w_\beta$ along a curve in $\mathbb{R}^{2^d}$. As it was pointed out in the Remark 4.16, presence of the inverse function in Cauchy formula implies that inverse of the function $(z-\zeta)$ does not pass through zero divisors in $\mathbb{S}_d$. It is true that $\mathbb{S}_d$ contains some zero divisors. For example $(e_1-e_2)(e_1+e_2)=0$. However if we restrict our attention to the functions $\mathbb{S}(i)\to\mathbb{S}_d$ then it is easy to see that each nonzero $z\in\mathbb{S}(i),\ i=1,...,d$ is invertible simply because each $\mathbb{S}(i)$ is a field. Cauchy formula for $f:\mathbb{S}(i)\to\mathbb{S}_d$ for a disk $D_i\subseteq\mathbb{S}(i)$ will have the form

$$\frac{1}{2\pi e_i}\int\limits_{\partial D_i}\frac{f(\zeta)}{\zeta-z}\,\mathrm{d}\zeta_i=\begin{cases}f(z) & \text{if } z\in D_i,\\ 0 & \text{if } z\in\mathbb{S}(i)\setminus\overline{D}_i.\end{cases}\tag{71}$$

Therefore we can also construct the multidimensional Cauchy integral formula for holomorphic functions $f:\mathbb{S}^d\to\mathbb{S}_d$ without any hesitation. General Cauchy formula for a simple open polydisk $D_{\boldsymbol{j}}:=\prod_{j\in\mathrm{ind}(\boldsymbol{j})}D_j\subset\mathbb{S}^d$ is thus given by

$$\frac{1}{(2\pi)^{|\boldsymbol{j}|}e_{\boldsymbol{j}}}\int\limits_{\partial D_{\boldsymbol{j}}}\frac{f(\zeta_{\boldsymbol{j}})}{(\boldsymbol{\zeta}-\boldsymbol{z})^{\boldsymbol{j}}}\,\mathrm{d}\boldsymbol{\zeta}^{\boldsymbol{j}}=\begin{cases}f(z) & \text{if } z\in D_{\boldsymbol{j}},\\ 0 & \text{if } z\in\mathbb{S}^d\setminus\overline{D}_{\boldsymbol{j}}.\end{cases}\tag{72}$$

This motivates the following equivalent definition of a holomorphic function.

Definition 4.24. Let function $f:\Omega\to\mathbb{S}_d$, defined on some open subset $\Omega\subseteq\mathbb{S}^d$, be continuous in each variable separately and locally bounded. The function f is said to be *holomorphic* in Ω if there is an $r=r(w)>0$ such that d-polydisk $\overline{D_d(w,r)}\subseteq\Omega$ and

$$f(z_1,...,z_d)=\frac{1}{(2\pi)^d e_1\cdot...\cdot e_d}\oint\limits_{|\zeta_1-w_1|=r}\cdots\oint\limits_{|\zeta_d-w_d|=r}\frac{f(\zeta_1,...,\zeta_d)}{(\zeta_1-z_1)\cdot...\cdot(\zeta_d-z_d)}\,\mathrm{d}\zeta_1...\mathrm{d}\zeta_d\tag{73}$$

for all $z\in D_d(w,r)$.

Analytic hypercomplex function therefore will be defined as following.

Definition 4.25. A function $f:\Omega\to\mathbb{S}_d$ is called *analytic* (holomorphic) in some open domain $\Omega\subseteq\mathbb{S}^d$ if for each $w\in\Omega$ there is an $r=r(w)>0$ such that $D_d(w,r)\subseteq\Omega$ and f can be written as an absolutely and uniformly convergent power series for all $\boldsymbol{z}\in D_d(w,r)$

$$f(\boldsymbol{z})=\sum_{i_1,...,i_d=0}^{\infty}a_{i_1,...,i_d}\,(z_1-w_1)^{i_1}\cdot...\cdot(z_d-w_d)^{i_d}\tag{74}$$

with coefficients

$$a_{i_1,...,i_d}=\frac{1}{i_1!\cdot...\cdot i_d!}\left(\frac{\partial}{\partial z_1}\right)^{i_1}...\left(\frac{\partial}{\partial z_d}\right)^{i_d}f(z).\tag{75}$$

In case of a single complex variable Hilbert transform of a function provides the relationship between conjugated parts on the boundary of a unit disk. Unit disk then can be mapped to the upper complex half-plane and we arrive to the usual definition of the Hilbert transform on the real line. In case of polydisk in a hypercomplex space $\mathbb{S}^d$ we proceed similarly. First we define Hilbert transform on the boundary of unit polydisk and then consider all biholomorphic mappings from the polydisk to the uppers Scheffers hyperplane $\mathbb{S}^d_+$.

Definition 4.26. The Hilbert transform $\mathring{H}_{\boldsymbol{j}}[f]$ for a unit polydisk $D_{\boldsymbol{j}}(0,1)$ of a function $f : \partial D_{\boldsymbol{j}}(0,1) \to \mathbb{R}$ is given by

$$\mathring{H}_{\boldsymbol{j}}[f]\left(e^{e_1\theta_1}, ..., e^{e_d\theta_d}\right) := \frac{1}{(4\pi)^{|\boldsymbol{j}|}} \int_{-\boldsymbol{\pi}}^{\boldsymbol{\pi}} f\left(e^{e_i t_i} | i \in \operatorname{ind} \boldsymbol{j}\right) \prod_{i \in \operatorname{ind} \boldsymbol{j}} \cot\left(\frac{\theta_i - t_i}{2}\right) \mathrm{d}\boldsymbol{t}^{\boldsymbol{j}}, \tag{76}$$

where we integrate f only over the variables $z_i \equiv e^{e_i t_i}$ indicated in $\boldsymbol{j} \in \{0,1\}^d$.

To construct a holomorphic hypercomplex function from the real-valued function defined on the boundary of polydisk we can find the corresponding hypercomplex conjugated component by $f_{\boldsymbol{j}} = \mathring{H}_{\boldsymbol{j}}[f]$.

In the above definition we used fractional Hilbert transforms to find hypercomplex conjugates. The relationship with one-variable Hilbert transform is the following. In complex analysis each holomorphic function on the boundary of the unit disk has the form $f_a = f + iH[f]$, i.e. Hilbert transform relates two real-valued functions. In case of several hypercomplex variables the situation is slightly different. The $\mathbb{S}_d$-valued function may be written for each $j = 1, ..., d$ in the form $f = f_1 + e_j f_2$, however now f_1 and f_2 are not real-valued functions. Functions f_1 and f_2 have values lying in the span $\{e_\beta | \beta \text{ does not contain } j\}$. Therefore Hilbert transform in the variable z_j relates hypercomplex conjugates within the plane $\mathbb{S}(j)$ by $f_2 = H_j[f_1]$.

Next we simply map the boundary of a polydisk to the boundary of Scheffers upper half-plane. All biholomorphic mappings [GG07] from the unit polydisk to the upper Scheffers half-plane, i.e. $D_d(0,1) \to \mathbb{S}^d_+$, have the form

$$(w_1, ..., w_d) \mapsto \left(\frac{\bar{a}_1 w_1 - e^{e_1\theta_1} a_1}{w_1 - e^{e_1\theta_1}}, ..., \frac{\bar{a}_d w_d - e^{e_d\theta_d} a_d}{w_d - e^{e_d\theta_d}}\right), \text{ where } a_i \in D_i(0,1),\ \theta_i \in \mathbb{R}. \tag{77}$$

By picking up one such mapping we easily define the Hilbert transform $H_{\boldsymbol{j}}$ on the boundary of Scheffers upper half-plane $\mathbb{S}^d_+$ and get (32) similarly as it was done in Appendix A. Hypercomplex analytic signal is defined to be any holomorphic function $f : \mathbb{S}^d_+ \to \mathbb{S}_d$ on the boundary $\partial\mathbb{S}^d_+ \simeq \mathbb{R}^d$.

Definition 4.27. The *Scheffers hypercomplex analytic signal* $f_S : \mathbb{R}^d \to \mathbb{S}_d$ is defined on the boundary of upper Scheffers space $\partial\mathbb{S}^d_+ \simeq \mathbb{R}^d$ from the real valued signal $f : \mathbb{R}^d \to \mathbb{R}$ by

$$f_S(\boldsymbol{x}) = \sum_{\boldsymbol{j} \in \{0,1\}^d} e_{\boldsymbol{j}} H_{\boldsymbol{j}}[f](\boldsymbol{x}). \tag{78}$$

In one-varialbe complex analysis the Riemann mapping theorem states that for any simply connected open subset $U \subseteq \mathbb{C}$ there exists biholomorphic mapping of U to the open unit disk in $\mathbb{C}$. However in case of several complex variables $d \geq 2$ this result does not hold anymore [Kra01]. Poincaré proved that in any dimension $d \geq 2$ in case of d several complex variables the ball is not biholomorphic to the polydisk. Even though the proof of Poincaré theorem in case of mappings $\mathbb{S}^d \to \mathbb{S}_d$ is out of scope of this paper this conjecture has important consequences. We rely mainly on the d-polydisk $D_d \subset \mathbb{S}^d$ as a domain of the hypercomplex holomorphic function. If there is no biholomorphic mapping from polydisk in $\mathbb{S}^d$ to the ball in $\mathbb{S}^d$ then on their boundaries holomorphic functions will be quite different as well, i.e. it will be impossible to map holomorphic function defined as a limit on the boundary of polydisk to the holomorphic function defined on the boundary of a ball. There is one simple argument on why probably there is no biholomorphic mapping from boundary of open polydisk to the boundary of a ball. The boundary of polydisk is given by torus, while the boundary of a ball is given by hypersphere. Torus and sphere are not homeomorphic for $d \geq 2$. Therefore it is hard to expect existence of a biholomorphic mapping between the two topologically non-similar domains.

Analytic signal was defined as an extension of some holomorphic function inside polydisk to its boundary that is a torus. The extension of the mapping (77) to the boundary provides us with the correspondence of points on torus T^d with points on $\mathbb{R}^d$. There are other mappings that one can use to relate points on some compact shape with the points of $\mathbb{R}^d$. On the other hand the shape that one choses to represent $\mathbb{R}^d$ could also be used for convenient parametrization of signal. For example in complex analysis the phase of a signal is given by the angle on the boundary of unit disk. Here we advocate the use of torus T^d as a natural domain not only to define analytic signal but also to parametrize signal's phase.

Suppose instead we defined analytic signal on a unit sphere in $\mathbb{S}^d$. In this case if there is no biholomorphic mapping between ball and polydisk in $\mathbb{S}^d$ the definition of analytic signal as a limiting case of holomorphic function on the boundary of the ball will differ from the polydisk case. Very probable that we will not be able to employ fractional Hilbert transforms $H_{\boldsymbol{j}}$ as simple relationships between conjugated components because there are no "selected" directions on the sphere. On the other hand parametrization by a point on the sphere in $\mathbb{S}^d$ could provide an alternative definition for the phase of a signal in $\mathbb{R}^d$. Sphere and torus both locally look similar to $\mathbb{R}^d$. However torus is given by the product $T^d = S^1 \times ... \times S^1$ and $\mathbb{R}^d$ is given by the product $\mathbb{R}^d = \mathbb{R}^1 \times ... \times \mathbb{R}^1$, therefore there is a natural way to assign circle to each direction in $\mathbb{R}^d$. In contrast hypersphere is a simple object that cannot be decomposed. Later in Observation 5.6 we use parametrization on torus to describe certain class of signals.

5 Commutative hypercomplex Fourier transform

In this section we define Scheffers-Fourier transform. Let us say we are working in some appropriately defined Schwartz space [Khr12] $\mathcal{S}_d(\mathbb{R}^d, \mathbb{S}_d)$ of rapidly decreasing $\mathbb{S}_d$-valued functions. Fourier transform $F : \mathcal{S}_d \to \mathcal{S}_d$ is an automorphism on $\mathcal{S}_d$. After defining the Fourier transform we show that analytic signal defined above has it's support only in positive quadrant in frequency space, see Remark 2.5. In other words we are able to recover the phase-shifted functions $f_{\boldsymbol{j}}$ by restricting spectrum of Scheffers-Fourier transform of real-valued function to the positive quadrant in frequency space. We start by defining the hypercomplex Fourier transform in terms of phase-shifted harmonics.

Definition 5.1. The *Scheffers-Fourier transform* of a function $f : \mathbb{R}^d \to \mathbb{S}_d$ is defined as

$$\hat{f}(\boldsymbol{\omega}) = \alpha^{0\ldots0} - \sum_{i=1}^{d} e_i \alpha^{0\ldots1(i)\ldots0} + \sum_{i<j} e_i e_j \alpha^{0\ldots1(i,j)\ldots0} - \sum_{i<j<k} e_i e_j e_k \alpha^{0\ldots1(i,j,k)\ldots0} + \ldots, \tag{79}$$

where $0\ldots1(i,j,k)\ldots0$ means that there are 1-s on the i-th, j-th and k-th positions of binary string. The inverse is given by

$$f(\boldsymbol{x}) = \hat{\alpha}^{0\ldots0} + \sum_{i=1}^{d} e_i \hat{\alpha}^{0\ldots1(i)\ldots0} + \sum_{i<j} e_i e_j \hat{\alpha}^{0\ldots1(i,j)\ldots0} + \sum_{i<j<k} e_i e_j e_k \hat{\alpha}^{0\ldots1(i,j,k)\ldots0} + \ldots. \tag{80}$$

This definition is equivalent (by applying Euler's formula) to the expected canonical form of hypercomplex Fourier transform

$$\hat{f}(\omega_1, \ldots, \omega_d) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, x_2, ..., x_d) e^{-e_1\omega_1 x_1} \cdot ... \cdot e^{-e_d\omega_d x_d} \,\mathrm{d}x_1 \ldots \mathrm{d}x_d, \tag{81}$$

$$f(x_1, x_2, ..., x_d) = \frac{1}{(2\pi)^d} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \hat{f}(\omega_1, \ldots, \omega_d) e^{e_1\omega_1 x_1} \cdot ... \cdot e^{e_d\omega_d x_d} \,\mathrm{d}x_1 \ldots \mathrm{d}x_d. \tag{82}$$

The inverse Fourier transform (80) may be rewritten using (36) as

$$f(\boldsymbol{x}) = \langle \hat{f}(\boldsymbol{\omega}), \alpha_{0\ldots0} \rangle + \sum_i e_i \langle \hat{f}(\boldsymbol{\omega}), \alpha_{0\ldots1(i)\ldots0} \rangle + \sum_{i<j} e_i e_j \langle \hat{f}(\boldsymbol{\omega}), \alpha_{0\ldots1(i,j)\ldots0} \rangle + \ldots \tag{83}$$

Next we observe that phase-shifted functions $f_{\boldsymbol{j}}$ are easily recovered from the restriction of Fourier transform to only positive frequencies.

Theorem 5.2. *The function $f_h : \mathbb{R}^d \to \mathbb{S}_d$ defined by the real-valued function $f : \mathbb{R}^d \to \mathbb{R}$ as*

$$f_h(\boldsymbol{x}) = \langle \hat{f}(\boldsymbol{\omega}), \alpha_{0\ldots 0}\rangle_+ + \sum_i e_i \langle \hat{f}(\boldsymbol{\omega}), \alpha_{0\ldots 1(i)\ldots 0}\rangle_+ + \sum_{i<j} e_i e_j \langle \hat{f}(\boldsymbol{\omega}), \alpha_{0\ldots 1(i,j)\ldots 0}\rangle_+ + \ldots \tag{84}$$

has as components the corresponding phase-shifted functions $f_{\boldsymbol{j}}$, i.e.

$$f_h(\boldsymbol{x}) = f(\boldsymbol{x}) + \sum_i e_i f_{0\ldots 1(i)\ldots 0}(\boldsymbol{x}) + \sum_{i<j} e_i e_j f_{0\ldots 1(i,j)\ldots 0}(\boldsymbol{x}) + \sum_{i<j<k} e_i e_j e_k f_{0\ldots 1(i,j,k)\ldots 0}(\boldsymbol{x}) + \ldots, \tag{85}$$

and therefore $f_h = f_S$.

Proof. For a proof we simply put each term from the sum in (79) inside (84) and observe finally that components in (85) are defined by (37). □

Observation 5.3. The Scheffers-Fourier transform of f_h is given by

$$\hat{f}_h(\boldsymbol{\omega}) = \prod_{i=1}^{d} (1 + \operatorname{sign}(\omega_i))\, \hat{f}(\omega_1, \ldots, \omega_d). \tag{86}$$

In the following we proceed to extend Bedrosian's theorem that states Hilbert transform of the product of a low-pass and a high-pass functions with non-overlapping spectra is given by the product of the low-pass function by the Hilbert transform of the high-pass function. We rely on the original work [Bed62]. In the following $\operatorname{ind}(\boldsymbol{j}) = \{i | j_i = 1\}$, the set of positions of 1-s in the vector $\boldsymbol{j} \in \{0,1\}^d$ and $\overline{\operatorname{ind}}(\boldsymbol{j}) = \{i | j_i = 0\}$ is the complementary set.

Lemma 5.4. For a product of exponentials for a given $\boldsymbol{j} \in \{0,1\}^d$

$$h(\boldsymbol{x}) = \prod_{i \in \operatorname{ind}(\boldsymbol{j})} e^{e_i \omega_i x_i} \tag{87}$$

we have identity

$$H_{\boldsymbol{j}}[h](\boldsymbol{x}) = \prod_{i \in \operatorname{ind}(\boldsymbol{j})} e_i \operatorname{sign}(\omega_i)\, e^{e_i \omega_i x_i} \tag{88}$$

Proof. The result follows directly from the 1-dimensional case by succesive application of Hilbert transform. □

Theorem 5.5 (Bedrosian)**.** *Let $f, g : \mathbb{R}^d \to \mathbb{S}_d$ and let us given some $\boldsymbol{j} \in \{0,1\}^d$ and suppose that for each $k \in \operatorname{ind}(\boldsymbol{j})$ we have corresponding $a_k > 0$ such that*

$$\hat{f}_k(\omega_k; x_1, \ldots, x_{k-1}, x_{k+1}, \ldots x_d) = \int_{\mathbb{R}} f(x_1, \ldots, x_k, \ldots, x_d) e^{-e_k \omega_k x_k}\, \mathrm{d}x_k = 0 \text{ for all } |\omega_k| > a_k \tag{89}$$

and

$$\hat{g}_k(\omega_k; x_1, \ldots, x_{k-1}, x_{k+1}, \ldots x_d) = \int_{\mathbb{R}} g(x_1, \ldots, x_k, \ldots, x_d) e^{-e_k \omega_k x_k}\, \mathrm{d}x_k = 0 \text{ for all } |\omega_k| \le a_k. \tag{90}$$

Then we have the identity

$$H_{\boldsymbol{j}}[f \cdot g] = f \cdot H_{\boldsymbol{j}}[g]. \tag{91}$$

Proof. First we give the proof for 1-dimensional case [Bed62] and then just extend it to our case. We can write the product of two functions $a(x)$ and $b(x)$ in Fourier domain as

$$a(x)b(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{a}(u)\hat{b}(v) e^{i(u+v)x}\, \mathrm{d}u\, \mathrm{d}v \tag{92}$$

and

$$H[a \cdot b](x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{a}(u)\hat{b}(v) i \operatorname{sign}(u+v) e^{i(u+v)x}\, \mathrm{d}u\, \mathrm{d}v. \tag{93}$$

If we assume that for some $a > 0$ we have $\hat{a}(\omega) = 0$ for $|\omega| > a$ and $\hat{b}(\omega) = 0$ for $|\omega| \leq a$, then the product $\hat{a}(\omega)\hat{b}(\omega)$ will be non-vanishing only on two semi-infinite stripes on the plane $\{(u,v) : |u| < a, |v| > a\}$. So for this integration region the value of the integral (93) will not change if we replace

$$\operatorname{sign}(u+v) \to \operatorname{sign}(v).$$

Then we will have

$$\begin{aligned} H[a \cdot b](x) &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{a}(u)\hat{b}(v) i \operatorname{sign}(v) e^{i(u+v)x} \, \mathrm{d}u \, \mathrm{d}v \\ &= a(x) \frac{1}{2\pi} \int_{\mathbb{R}} \hat{b}(v) i \operatorname{sign}(v) e^{ivx} \, \mathrm{d}v. \end{aligned} \tag{94}$$

But we know from Lemma 5.4 that

$$H[b](x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{b}(v) H[e^{ivx'}](x) \, \mathrm{d}v = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{b}(v) i \operatorname{sign}(v) e^{ivx} \, \mathrm{d}v. \tag{95}$$

So finally we have

$$H[a \cdot b] = a \cdot H[b]. \tag{96}$$

To prove the final result (91), we succesively apply 1-dimensional steps taken in (94), (95) and the result of Lemma 5.4

$$\begin{aligned} H_{\boldsymbol{j}}[f \cdot g](\boldsymbol{x}) = \frac{1}{(2\pi)^{2|\boldsymbol{j}|}} \int_{\mathbb{R}^{|\boldsymbol{j}|}} \int_{\mathbb{R}^{|\boldsymbol{j}|}} & \hat{f}_{\boldsymbol{j}}(\boldsymbol{u}_{\mathrm{ind}(\boldsymbol{j})}; \boldsymbol{x}_{\overline{\mathrm{ind}}(\boldsymbol{j})}) \hat{g}_{\boldsymbol{j}}(\boldsymbol{v}_{\mathrm{ind}(\boldsymbol{j})}; \boldsymbol{x}_{\overline{\mathrm{ind}}(\boldsymbol{j})}) \\ & \cdot \prod_{i \in \mathrm{ind}(\boldsymbol{j})} e_i \operatorname{sign}(u_i + v_i) e^{e_i (u_i + v_i) x_i} \, \mathrm{d}\boldsymbol{u}^{\boldsymbol{j}} \, \mathrm{d}\boldsymbol{v}^{\boldsymbol{j}}, \end{aligned} \tag{97}$$

where we denoted by $\hat{f}_{\boldsymbol{j}}(\boldsymbol{u}_{\mathrm{ind}(\boldsymbol{j})}; \boldsymbol{x}_{\overline{\mathrm{ind}}(\boldsymbol{j})})$ and $\hat{g}_{\boldsymbol{j}}(\boldsymbol{v}_{\mathrm{ind}(\boldsymbol{j})}; \boldsymbol{x}_{\overline{\mathrm{ind}}(\boldsymbol{j})})$ the Fourier transforms of f and g over the directions indicated by $\boldsymbol{j}$. Thus $\hat{f}_{\boldsymbol{j}}(\boldsymbol{u}_{\mathrm{ind}(\boldsymbol{j})}; \boldsymbol{x}_{\overline{\mathrm{ind}}(\boldsymbol{j})})$ has as its arguments frequency variables indexed by $\mathrm{ind}(\boldsymbol{j})$ and the remaining spatial variables indexed by the complementary set $\overline{\mathrm{ind}}(\boldsymbol{j})$. Finally by repeating the steps taken in (94), (95) we see that $H_{\boldsymbol{j}}[f \cdot g] = f \cdot H_{\boldsymbol{j}}[g]$. □

Observation 5.6. Suppose we have two real-valued functions $A(\boldsymbol{x})$ and $B(\boldsymbol{x})$ in $\mathbb{R}^d$ that satisfy conditions (89) and (90) respectively for $\boldsymbol{j} = \boldsymbol{1}$. Additionally suppose that $B(\boldsymbol{x}) = \alpha_{\boldsymbol{0}}(\boldsymbol{x}, \boldsymbol{\omega_0})$ for some choice of frequencies $\boldsymbol{\omega_0}$. Then we can write the Scheffers analytic signal of the product function $C(\boldsymbol{x}) = A(\boldsymbol{x}) \cdot B(\boldsymbol{x})$ in compact form

$$C_h(\boldsymbol{x}) = A(\boldsymbol{x}) e^{e_1 \omega_1 x_1} \cdot \dots \cdot e^{e_d \omega_d x_d}. \tag{98}$$

Supposing more generally that $B(\boldsymbol{x})$ is of the form

$$B(\boldsymbol{x}) = \prod_{l=1}^{d} \cos(\phi_l(\boldsymbol{x})) \tag{99}$$

and satisfies condition (90), we can write it's analytic signal $C_h(\boldsymbol{x})$ then as

$$C_h(\boldsymbol{x}) = A(\boldsymbol{x}) e^{e_1 \phi_l(\boldsymbol{x})} \cdot \dots \cdot e^{e_d \phi_l(\boldsymbol{x})}. \tag{100}$$

For a 1-dimensional complex-valued function we interpret it's multiplication by a complex number as rotation and scaling of a function in the complex plane. Here we have similar geometric interpretation of narrowband hypercomplex analytic signal having the form (100). First of all, each exponential $e^{e_i \phi_i}$ represent phase rotations in the corresponding hypercomplex subspace $\mathbb{S}(i)$. Therefore for $C_h(\boldsymbol{x})$ in (100) we have a combination of d separate rotations, i.e. the overall rotation will correspond to a point on the product of d circles $S^1 \times \dots \times S^1$. The product of circles forms a d-dimensional torus T^d (see also discussion in the end of Section 4.2). The total phase of some hypercomplex analytic signal (100) therefore corresponds to some continuous map on torus $\gamma : \mathbb{R}^d \to T^d$. Corresponding amplitude function $A(\boldsymbol{x})$ is attached to the points on the torus T^d.

6 Why Clifford-Fourier transform doesn't fit for d>2?

There were propositions to apply Clifford-Fourier transform to study instantaneous amplitude, phase and frequency. The aim of this section is to show that *there is no* Clifford-Fourier transform in the canonical form (81), which lead us to the phase-shifted functions f_j as components of Clifford algebra valued function with it's Clifford-Fourier spectrum having support in positive quadrant. It is interesting to note that Bülow and Sommer in the early work [BS01] mention slightly the commutative hypercomplex numbers and associated Fourier transform, however later studies were mainly focused on the non-commutative hypercomplex systems.

First we make a brief introduction to the Clifford algebras. A Clifford algebra is a unital associative algebra that is generated by a vector space V over some field K, where V is equipped with a quadratic form $Q : V \to K$. If the dimension of V over K is d and $\{e_1, \dots, e_d\}$ is an orthogonal basis of (V, Q), then $\mathcal{C}l(V, Q)$ is a free vector space with a basis

$$\{e_{i_1} e_{i_2} \dots e_{i_k} | 1 \le i_1 < i_2 < \dots < i_k \le d \text{ and } 0 \le k \le d\}. \tag{101}$$

The element e_0 is defined as the multiplicative identity element. Due to the fact that V is equipped with quadratic form we have an orthogonal basis

$$(e_i, e_j) = 0 \text{ for } i \neq j, \text{ and } (e_i, e_i) = Q(e_i) \tag{102}$$

where $(\cdot, \cdot)$ denotes the symmetric bilinear form associated to Q. The fundamental Clifford identity implies that Clifford algebra is anticommutative

$$e_i e_j = -e_j e_i \text{ for } i \neq j, \text{ and } (e_i, e_i) = Q(e_i), \tag{103}$$

which makes multiplication of the elements of Clifford algebra quite simple, we can put elements of a Clifford algebra in standard order simply by a number of swaps of neighbouring elements. Every nondegenerate form Q can be written in standard diagonal form:

$$Q(v) = v_1^2 + \dots + v_p^2 - v_{p+1}^2 - \dots - v_{p+q}^2, \tag{104}$$

with $d = p + q$. The pair of integers (p, q) is called the signature of the quadratic form. The corresponding Clifford algebra is then denoted as $\mathcal{C}l_{p,q}(\mathbb{R})$. For our purposes of analytic signal construction (see discussion on elliptic units in Section 4.1) we will consider just Clifford algebras $\mathcal{C}l_{0,q}(\mathbb{R})$ over the field of real numbers $\mathbb{R}$, with the quadratic form

$$Q(e_i) = -1 \text{ for } 1 \le i \le d.$$

Each element $x \in \mathcal{C}l_{0,d}(\mathbb{R})$ of Clifford algebra may be thought as a linear combination of its basis elements

$$x = x_0 1 + \sum_i x_i e_i + \sum_{i<j} x_{ij} e_i e_j + \sum_{i<j<k} x_{ijk} e_i e_j e_k + \dots, \tag{105}$$

with real coefficients and for which we also have identities $e_i e_j = -e_j e_i$ for $i \neq j$, and $e_i^2 = -1$.

First it will be handy to define the set-valued function Π that returns all permuted products of its arguments. For example for 3 arguments it acts as

$$\Pi(a, b, c) = \{a \cdot b \cdot c, a \cdot c \cdot b, b \cdot a \cdot c, c \cdot a \cdot b\ , b \cdot c \cdot a, c \cdot b \cdot a\}. \tag{106}$$

For d arguments Π acts analogously.

A particular Clifford-Fourier transform is then defined as

$$\hat{f}^C(\omega_1, \dots, \omega_d) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \rho \left(f(x_1, x_2, ..., x_d), e^{-e_1 \omega_1 x_1}, \dots, e^{-e_d \omega_d x_d} \right) \mathrm{d}x_1 \dots \mathrm{d}x_d, \tag{107}$$

where $\rho\left(f(x_1, x_2, ..., x_d), e^{-e_1 \omega_1 x_1}, \dots, e^{-e_d \omega_d x_d}\right) \in \Pi\left(f(x_1, x_2, ..., x_d), e^{-e_1 \omega_1 x_1}, \dots, e^{-e_d \omega_d x_d}\right)$.

This type of Clifford-Fourier transform has the inverse transform given by

$$f(x_1, \dots, x_d) = \frac{1}{(2\pi)^d} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \rho' \left(\hat{f}^C(\omega_1, \omega_2, ..., \omega_d), e^{e_1 \omega_1 x_1}, \dots, e^{e_d \omega_d x_d} \right) \mathrm{d}x_1 \dots \mathrm{d}x_d, \tag{108}$$

for some $\rho'\left(\hat{f}^C(\omega_1,\omega_2,...,\omega_d), e^{e_1\omega_1 x_1},\dots,e^{e_d\omega_d x_d}\right) \in \Pi\left(\hat{f}^C(\omega_1,\omega_2,...,\omega_d), e^{e_1\omega_1 x_1},\dots,e^{e_d\omega_d x_d}\right)$. We can expand the exponentials in (107) and (108) to write down the general Clifford-Fourier transform in terms of functions $\alpha^{\boldsymbol{k}}$ and $\hat{\alpha}^{\boldsymbol{k}}$

$$\hat{f}^C = \rho_0 - \sum_{i=1}^d \rho_1(i) + \sum_{i<j} \rho_2(i,j) - \sum_{i<j<k} \rho_3(i,j,k) + \dots, \tag{109}$$

$$f = \rho'_0 + \sum_{i=1}^d \rho'_1(i) + \sum_{i<j} \rho'_2(i,j) + \sum_{i<j<k} \rho'_3(i,j,k) + \dots, \tag{110}$$

for some ρ_k and ρ'_k, where

$$\begin{aligned}
&\rho_0 = \alpha^{0\dots0},\\
&\rho_1(i) \in \Pi\left(e_i, \alpha^{0\dots1(i)\dots0}\right),\\
&\rho_2(i,j) \in \Pi\left(e_i, e_j, \alpha^{0\dots1(i,j)\dots0}\right),\\
&\rho_3(i,j,k) \in \Pi\left(e_i, e_j, e_k, \alpha^{0\dots1(i,j,k)\dots0}\right),\\
&\dots
\end{aligned} \tag{111}$$

and

$$\begin{aligned}
&\rho'_0 = \hat{\alpha}^{0\dots0},\\
&\rho'_1(i) \in \Pi\left(e_i, \hat{\alpha}^{0\dots1(i)\dots0}\right),\\
&\rho'_2(i,j) \in \Pi\left(e_i, e_j, \hat{\alpha}^{0\dots1(i,j)\dots0}\right),\\
&\rho'_3(i,j,k) \in \Pi\left(e_i, e_j, e_k, \hat{\alpha}^{0\dots1(i,j,k)\dots0}\right),\\
&\dots
\end{aligned} \tag{112}$$

We consider the definition of Clifford-Fourier "analytic" signal as summation of only positive frequency terms, like we did in (20), (21) and (84)

$$f_C = \rho'_{+0} + \sum_{i=1}^d \rho'_{+1}(i) + \sum_{i<j} \rho'_{+2}(i,j) + \sum_{i<j<k} \rho'_{+3}(i,j,k) + \dots, \tag{113}$$

with some ρ'_{+i}:

$$\begin{aligned}
&\rho'_{+0} = \hat{\alpha}_{C+}^{0\dots0},\\
&\rho'_{+1}(i) \in \Pi\left(e_i, \hat{\alpha}_{C+}^{0\dots1(i)\dots0}\right),\\
&\rho'_{+2}(i,j) \in \Pi\left(e_i, e_j, \hat{\alpha}_{C+}^{0\dots1(i,j)\dots0}\right),\\
&\rho'_{+3}(i,j,k) \in \Pi\left(e_i, e_j, e_k, \hat{\alpha}_{C+}^{0\dots1(i,j,k)\dots0}\right),\\
&\dots
\end{aligned} \tag{114}$$

where $\hat{\alpha}_{C+}^{\boldsymbol{j}}(\boldsymbol{x}) = \langle f^C(\boldsymbol{\omega}), \alpha_{\boldsymbol{j}}(\boldsymbol{x},\boldsymbol{\omega})\rangle_+$.

Theorem 6.1 (Non-existence result)**.** *The Clifford algebra valued "analytic" signal $f_C : \mathbb{R}^d \to \mathcal{C}l_{0,d}(\mathbb{R})$ of the original real valued function $f : \mathbb{R}^d \to \mathbb{R}$, defined as (113), in general* ***does not have*** *as components the corresponding phase-shifted functions $f_{\boldsymbol{j}}$, defined as (28), i.e. if we have*

$$f_C(\boldsymbol{x}) = f(\boldsymbol{x}) + \sum_i e_i \tilde{f}_{0\dots1(i)\dots0}(\boldsymbol{x}) + \sum_{i<j} e_i e_j \tilde{f}_{0\dots1(i,j)\dots0}(\boldsymbol{x}) + \sum_{i<j<k} e_i e_j e_k \tilde{f}_{0\dots1(i,j,k)\dots0}(\boldsymbol{x}) + \dots \tag{115}$$

then there exist $\boldsymbol{j} \in \{0,1\}^d$ such that $|\tilde{f}_{\boldsymbol{j}}| \neq |f_{\boldsymbol{j}}|$ for all $d > 2$. Therefore there is no Clifford-Fourier transform of the form (107) and (108) for $d > 2$ that will lead us to the phase-shifts $f_{0\dots1(i)\dots0}$ as components of e_i.

Proof. To prove this theorem it will suffice to demonstrate that we are not able to choose the proper ordering of multiplicative terms in (107) and (108) such that $\tilde{f}_{\boldsymbol{j}} = f_{\boldsymbol{j}}$ for all $\boldsymbol{j} \in \{0,1\}^d$.

Next we simply show that already for the elements of degree 1, i.e. in front of corresponding e_i, we already must have $|\tilde{f}_{0\ldots1(i)\ldots0}| \neq |f_{0\ldots1(i)\ldots0}|$ for some i. First let us write the Clifford-Fourier transform for elements of degree up to 1

$$\hat{f}^C(\boldsymbol{\omega}) = \alpha^{0\ldots0}(\boldsymbol{\omega}) - e_1\alpha^{10\ldots0}(\boldsymbol{\omega}) - e_2\alpha^{010\ldots\ldots0}(\boldsymbol{\omega}) - e_3\alpha^{0010\ldots\ldots0}(\boldsymbol{\omega}) - \cdots - e_d\alpha^{0\ldots01}(\boldsymbol{\omega}) + \ldots \quad (116)$$

For the elements of degree up to 1 we may not care about the order of multiplication because $f(\boldsymbol{x})$ is real valued. Then we show that there is no ordering of multiplicative terms in the inverse formula (108) to obtain the right signs of phase-shifted components in accordance with (37). For simplicity, but without loss of generality, we consider just the combinations of components corresponing to the first three elements e_1, e_2 and e_3. In this case potentially we have four candidates for the inverse transform

1. Zero elements are flipped:

$$\begin{aligned} c_0(\boldsymbol{x}) = \hat{\alpha}^{0\ldots0}(\boldsymbol{x}) + e_1\langle \hat{f}^C(\boldsymbol{\omega}), \alpha_{10\ldots0}(\boldsymbol{x},\boldsymbol{\omega})\rangle + e_2\langle \hat{f}^C(\boldsymbol{\omega}), \alpha_{010\ldots0}(\boldsymbol{x},\boldsymbol{\omega})\rangle + \\ + e_3\langle \hat{f}^C(\boldsymbol{\omega}), \alpha_{0010\ldots0}(\boldsymbol{x},\boldsymbol{\omega})\rangle + \ldots \end{aligned} \quad (117)$$

2. One element is flipped:

$$\begin{aligned} c_0(\boldsymbol{x}) = \hat{\alpha}^{0\ldots0}(\boldsymbol{x}) + \langle \hat{f}^C(\boldsymbol{\omega}), \alpha_{10\ldots0}(\boldsymbol{x},\boldsymbol{\omega})\rangle e_1 + e_2\langle \hat{f}^C(\boldsymbol{\omega}), \alpha_{010\ldots0}(\boldsymbol{x},\boldsymbol{\omega})\rangle + \\ + e_3\langle \hat{f}^C(\boldsymbol{\omega}), \alpha_{0010\ldots0}(\boldsymbol{x},\boldsymbol{\omega})\rangle + \ldots \end{aligned} \quad (118)$$

3. Two elements are flipped:

$$\begin{aligned} c_0(\boldsymbol{x}) = \hat{\alpha}^{0\ldots0}(\boldsymbol{x}) + \langle \hat{f}^C(\boldsymbol{\omega}), \alpha_{10\ldots0}(\boldsymbol{x},\boldsymbol{\omega})\rangle e_1 + \langle \hat{f}^C(\boldsymbol{\omega}), \alpha_{010\ldots0}(\boldsymbol{x},\boldsymbol{\omega})\rangle e_2 + \\ + e_3\langle \hat{f}^C(\boldsymbol{\omega}), \alpha_{0010\ldots0}(\boldsymbol{x},\boldsymbol{\omega})\rangle + \ldots \end{aligned} \quad (119)$$

4. Three elements are flipped:

$$\begin{aligned} c_0(\boldsymbol{x}) = \hat{\alpha}^{0\ldots0}(\boldsymbol{x}) + \langle \hat{f}^C(\boldsymbol{\omega}), \alpha_{10\ldots0}(\boldsymbol{x},\boldsymbol{\omega})\rangle e_1 + \langle \hat{f}^C(\boldsymbol{\omega}, \alpha_{010\ldots0}(\boldsymbol{x},\boldsymbol{\omega})\rangle e_2 + \\ + \langle \hat{f}^C(\boldsymbol{\omega}), \alpha_{0010\ldots0}(\boldsymbol{x},\boldsymbol{\omega})\rangle e_3 + \ldots \end{aligned} \quad (120)$$

Therefore for c_0 serving as inverse we will obtain the component of the degree 2 of the f_C by simple subtitution of each term of $\hat{f}^C(\boldsymbol{\omega})$ from (116) into (117), (118), (119) and (120). That will give us the following components after the restriction of integration only to positive frequencies.

1. For c_0 we have

$$\begin{aligned} \tilde{f}^0_{\boldsymbol{j}:|\boldsymbol{j}|=2} = & - e_2e_1\langle\alpha^{100\ldots0}, \alpha_{010\ldots0}\rangle_+ - e_1e_2\langle\alpha^{010\ldots0}, \alpha_{100\ldots0}\rangle_+ - \\ & - e_3e_1\langle\alpha^{100\ldots0}, \alpha_{0010\ldots0}\rangle_+ - e_1e_3\langle\alpha^{0010\ldots0}, \alpha_{100\ldots0}\rangle_+ - \\ & - e_3e_2\langle\alpha^{010\ldots0}, \alpha_{0010\ldots0}\rangle_+ - e_2e_3\langle\alpha^{0010\ldots0}, \alpha_{010\ldots0}\rangle_+ + \ldots \end{aligned}$$

2. For c_1 we have

$$\begin{aligned} \tilde{f}^1_{\boldsymbol{j}:|\boldsymbol{j}|=2} = & - e_2e_1\langle\alpha^{100\ldots0}, \alpha_{010\ldots0}\rangle_+ - e_2e_1\langle\alpha^{010\ldots0}, \alpha_{100\ldots0}\rangle_+ - \\ & - e_3e_1\langle\alpha^{100\ldots0}, \alpha_{0010\ldots0}\rangle_+ - e_3e_1\langle\alpha^{0010\ldots0}, \alpha_{100\ldots0}\rangle_+ - \\ & - e_3e_2\langle\alpha^{010\ldots0}, \alpha_{0010\ldots0}\rangle_+ - e_2e_3\langle\alpha^{0010\ldots0}, \alpha_{010\ldots0}\rangle_+ + \ldots \end{aligned}$$

3. For c_2 we have

$$\begin{aligned} \tilde{f}^2_{\boldsymbol{j}:|\boldsymbol{j}|=2} = & - e_1e_2\langle\alpha^{100\ldots0}, \alpha_{010\ldots0}\rangle_+ - e_2e_1\langle\alpha^{010\ldots0}, \alpha_{100\ldots0}\rangle_+ - \\ & - e_3e_1\langle\alpha^{100\ldots0}, \alpha_{0010\ldots0}\rangle_+ - e_3e_1\langle\alpha^{0010\ldots0}, \alpha_{100\ldots0}\rangle_+ - \\ & - e_3e_2\langle\alpha^{010\ldots0}, \alpha_{0010\ldots0}\rangle_+ - e_3e_2\langle\alpha^{0010\ldots0}, \alpha_{010\ldots0}\rangle_+ + \ldots \end{aligned}$$

4. For c_3 we have

$$\begin{aligned}\tilde{f}^{3}_{\boldsymbol{j}:|\boldsymbol{j}|=2} = & - e_1e_2\langle\alpha^{100\dots0},\alpha_{010\dots0}\rangle_+ - e_2e_1\langle\alpha^{010\dots0},\alpha_{100\dots0}\rangle_+ - \\ & - e_1e_3\langle\alpha^{100\dots0},\alpha_{0010\dots0}\rangle_+ - e_3e_1\langle\alpha^{0010\dots0},\alpha_{100\dots0}\rangle_+ - \\ & - e_2e_3\langle\alpha^{010\dots0},\alpha_{0010\dots0}\rangle_+ - e_3e_2\langle\alpha^{0010\dots0},\alpha_{010\dots0}\rangle_+ + \dots\end{aligned}$$

From the rule in (37) we know that with each flipping of 1 in the upper subscript to 0 at the same position in the lower subscript the sign of bracket changes. If we look at $\tilde{f}^{0}_{\boldsymbol{j}:|\boldsymbol{j}|=2}$, we see that terms that are components of e_1e_2 ,i.e. $\langle\alpha^{100\dots0},\alpha_{010\dots0}\rangle_+$ and $\langle\alpha^{010\dots0},\alpha_{100\dots0}\rangle_+$ sum up with opposite signs. The same is true for the components of e_1e_3 and e_2e_3 – they come all with opposite signs. Therefore case of c_0-type inverse formula is not in accordance with the rule (37). For the case c_1 we see that the components of e_1e_2 have the same sign and as well as components of e_1e_3, however components of e_2e_3 have opposite signs. Thus c_1 is not in accordance with (37). After checking the rule for c_2 and c_3 we see that we will always have different signs for some component. Therefore we are not able to order properly the terms in Clifford-Fourier transform and its inverse to correctly restore all the phase-shifted functions. □

Corollary 6.2. From the proof of Theorem 6.1 it follows that *essentially only* commutative hypercomplex algebra ($e_ie_j = e_je_i$) is a good candidate to provide us with the phase shifts (37) by using positive frequency restriction for Fourier transforms of the type (107) and (108). The word "essentially" above means that algebras with two hypercomplex units that anticommute are still allowed.

Remark 6.3 (Octonions, Sedenions and Cayley-Dickson construction)**.** There were propositions to use Cayley-Dickson algebras to define the Fourier transform (107), (108) and corresponding hypercomplex "analytic" signal [HS16], however as far as elements $\{e_i\}$ of these algebras are non-commutative we will not get the phase shifts as components of the corresponding hypercomplex analytic signal.

Example 6.4. We may wonder what happens in case $d = 2$. Is there any Clifford-Fourier transform that gives us the correct phase-shifted components? It seems that the answer should be - yes. And indeed the symmetric Clifford-Fourier transform does the job. Let $\mathbb{Q}$ be the ring of quaternions with the set of basis elements $\{1, i, j, k\}$. The ring of quaternions coincides with the Clifford algebra $\mathcal{C}l_{0,2}(\mathbb{R})$ that in turn has the set of basis elements $\{1, e_1, e_2, e_1e_2\}$ where $i = e_1, j = e_2, k = e_1e_2$. The symmetric Quaternionic-Fourier transform is defined as

$$\hat{f}^q(\omega_1,\omega_2) = \int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} e^{-e_1\omega_1x_1} f(x_1,x_2) e^{-e_2\omega_2x_2}\,\mathrm{d}x_1\,\mathrm{d}x_2, \tag{121}$$

$$f(x_1,x_2) = \frac{1}{(2\pi)^2}\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} e^{e_1\omega_1x_1}\hat{f}^q(\omega_1,\omega_2)e^{e_2\omega_2x_2}\,\mathrm{d}x_1\,\mathrm{d}x_2, \tag{122}$$

which we may rewrite by employing the notation (35) as

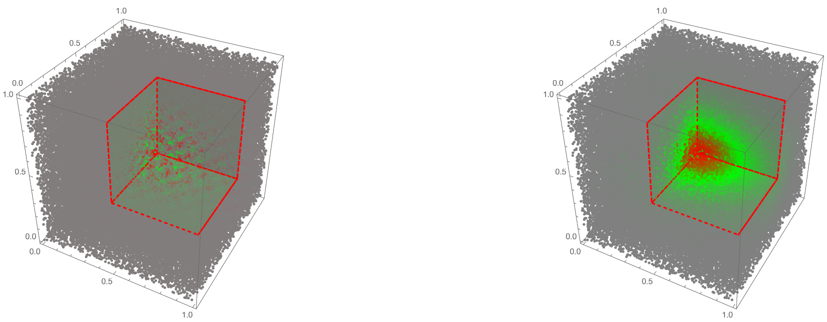
$$\hat{f}^q(\omega_1,\omega_2) = \alpha^{00} - e_1\alpha^{10} - e_2\alpha^{01} + e_1\alpha^{11}e_2, \tag{123}$$

$$f(x_1,x_2) = \langle\hat{f}^q,\alpha_{00}\rangle + e_1\langle\hat{f}^q,\alpha_{10}\rangle + \langle\hat{f}^q,\alpha_{01}\rangle e_2 + e_1\langle\hat{f}^q,\alpha_{11}\rangle e_2. \tag{124}$$

The quaternionic "analytic" signal is obtained by restriction of the quaternion Fourier transform to the positive frequencies as we did before

$$\begin{aligned}f^q_a = & \langle\alpha^{00},\alpha_{00}\rangle_+ + \langle\alpha^{10},\alpha_{10}\rangle_+ + \langle\alpha^{01},\alpha_{01}\rangle_+ + \langle\alpha^{11},\alpha_{11}\rangle_+ \\ & - e_1\langle\alpha^{10},\alpha_{00}\rangle_+ + e_1\langle\alpha^{00},\alpha_{10}\rangle_+ - e_1\langle\alpha^{11},\alpha_{01}\rangle_+ + e_1\langle\alpha^{01},\alpha_{11}\rangle_+ \\ & - \langle\alpha^{01},\alpha_{00}\rangle_+e_2 + \langle\alpha^{00},\alpha_{01}\rangle_+e_2 - \langle\alpha^{11},\alpha_{10}\rangle_+e_2 + \langle\alpha^{10},\alpha_{11}\rangle_+e_2 \\ & + e_1\langle\alpha^{11},\alpha_{00}\rangle_+e_2 + e_1\langle\alpha^{00},\alpha_{11}\rangle_+e_2 - e_1\langle\alpha^{10},\alpha_{01}\rangle_+e_2 - e_1\langle\alpha^{01},\alpha_{10}\rangle_+e_2.\end{aligned} \tag{125}$$

Finally we see that components of quaternionic analytic signal does provide us the correct phase-shifts given by (37).



(a) Oscillating function inside point-cloud (b) Amplitude

Figure 1: Point cloud of cube with a deleted octant is shown. (a) Illustration of oscillating process $f(x, y, z)$ that takes place inside the cube; (b) Instantaneous amplitude of the oscillating process.

7 Applications and extensions

7.1 Simple example in 3-D

As a direct illustration of how the construction of the envelope works, we provide an example in 3-D. The oscillating function that we analyse is given by the cosine wave modulated by a gaussian window:

$$\begin{aligned} f(x, y, z) &= e^{-10x^2-20y^2-20z^2} \cos(50x) \cos(40y) \cos(60z) \\ &= f_x(x) f_y(y) f_z(z). \end{aligned} \tag{126}$$

This particular toy example allows us to separate the variables when calculating the phase-shifted components $f_{\boldsymbol{j}}$ from (37). In three dimensions we will have in total $|\{0,1\}^3| = 8$ shifted functions.

Calculation of some $f_{\boldsymbol{j}}$ using (28) reduces to the calculation of forward and inverse cosine and sine transforms. For example for the f_{100} shifted component, we will have

$$f_{100}(x, y, z) = \frac{f_y(y) f_z(z)}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} f_x(x) \sin(\omega_1(x - x'))\, \mathrm{d}x'\, \mathrm{d}\omega_1. \tag{127}$$

Taking into account that f is even in each variable, after expanding the sine of difference, we see that we have to calculate only forward cosine and inverse sine transform. This double integral can be calculated into semi-analytical form by expressing it in terms of $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, \mathrm{d}t$ using the formula

$$\begin{aligned} &\frac{2}{\pi} \int\limits_0^{\infty} \sin(\omega x) \int\limits_0^{\infty} e^{-\alpha \tilde{x}^2} \cos(\omega_x \tilde{x}) \cos(\omega \tilde{x})\, \mathrm{d}\tilde{x}\, \mathrm{d}\omega \\ &= \frac{1}{2} i e^{-x(\omega_x i + \alpha x)} \left[\operatorname{erf}\left(\frac{\omega_x/2 - i\alpha x}{\sqrt{\alpha}} \right) - e^{2\omega_x i x} \operatorname{erf}\left(\frac{\omega_x/2 + i\alpha x}{\sqrt{\alpha}} \right) \right]. \end{aligned} \tag{128}$$

To illustrate the resulting envelope of signal (126), in Fig. 1 we vizualize a point cloud in a cube with a removed octant where each point is colour coded according to the signal value (the origin is shifted by 0.5). This method allows to visualize three dimensional signals over the inner faces of the deleted octant. In Fig. 1(a) the original $f(x, y, z)$ is plotted, while in Fig. 1(b) we plot the envelope function $a(x, y, z)$ obtained from (29) computed on the 8 phase-shifted functions obtained similarly to (127).

7.2 Analytic signal on space-time manifold

Originally 1-D analytic signal was developed to study time-varying signals. In this work we presented a way to construct hypercomplex analytic signal over some multidimensional euclidean domain that satisfies all the basic requirements for analytic signal to hold. It was implicitly assumed that $\boldsymbol{x} \in \mathbb{R}^d$ represents some space coordinate. However theory will indeed be valid if we allow signal f to be dependent also on time t, i.e. we will have the function $f : \mathbb{R}^{d+1} \to \mathbb{R}$. Recent examples where it could be used include studies of time-varying multidimensional oscillating processes, as for example [FCLB17], and studies of recently discovered gravitational waves[2]. Another example is provided by machine learning where the data is assumed to lie on some manifold that is embedded in multidimensional feature space [HNS07]. The data could again be time-varying and in this case again it will be useful to introduce separate time coordinate.

Suppose we are given a manifold M, that is Haussdorf and second countable space, with some smooth oscillating process $f : M \to \mathbb{R}$ and $f \in C^\infty(M)$. Basic facts on differential manifolds and topological spaces can be found for example in [Lee01] and [Mun18]. A cover of M is a collection $\mathcal{U}_I = \{U_\alpha\}_{\alpha \in I}$ of subsets of M whose union is M. For each U_α there is mapping $\varphi_\alpha : U_\alpha \to \mathbb{R}^{d+1}$. The local coordinate representation of f is given by $\hat{f}(x) = f \circ \varphi_\alpha^{-1}(x)$ and maps some open set of $\mathbb{R}^{d+1}$ to $\mathbb{R}$. Our final goal is to construct the global analytic signal $f_h : M \to \mathbb{S}_{d+1}$, obtain instantaneous amplitude $a : M \to \mathbb{R}$ and locally-defined collection of instantaneous phases $\phi_{\boldsymbol{j}} : M \to \mathbb{R}$.

The technique given next concerns only the construction of instantaneous amplitude, while the detailed treatment of proper construction of hypercomplex analytic signal on manifold we leave for future work. The idea is first to construct instantaneous amplitude locally on each open set $\phi_\alpha(U_\alpha)$ and then "glue" the resulting functions a_α consistently using some partition of unity to obtain the resulting instantaneous amplitude $a : M \to \mathbb{R}_{d+1}$. According to Observation 3.6 the instantaneous amplitude follows the transformation of the original oscillating function, therefore construction of instantaneous amplitude is insensitive to rotations of the neighboring patches.

First we division the function $f : M \to \mathbb{R}$ into a collection of functions $\{f_\alpha\}_{\alpha \in I}$ that is subordinate to the open cover $\mathcal{U}_I$ with $\operatorname{supp} f_\alpha \subseteq U_\alpha$, or in coordinate representation we will have the collection $\{\hat{f}_\alpha\}_{\alpha \in I}$ of functions $\mathbb{R}^{d+1} \to \mathbb{R}$. Then we construct instantaneous amplitude for each $\hat{f}_\alpha$ by applying any of considered methods and thus we get a collection of instantaneous amplitudes $\{\hat{a}_\alpha\}_{\alpha \in I}$ or equivalently $\{a_\alpha\}_{\alpha \in I}$. We will need the definition of partition of unity [Lee01] to extend the locally defined instantaneous amplitudes to the whole manifold M.

Definition 7.1. A partition of unity subordinate to the cover $\mathcal{U}_I$ is an indexed family $\{\psi_\alpha\}_{\alpha \in I}$ of continuous functions $\psi_\alpha : M \to \mathbb{R}$ with the following properties:

1. $0 \le \psi_\alpha(x) \le 1$ for all $\alpha \in I$ and all $x \in M$;
2. $\operatorname{supp} \psi_\alpha \subseteq U_\alpha$ for each $\alpha \in I$;
3. The family of supports $\{\operatorname{supp} \psi_\alpha\}_{\alpha \in I}$ is locally finite, meaning that every point has a neighbourhood that intersects $\operatorname{supp} \psi_\alpha$ for only finitely many values of α;
4. $\sum_{\alpha \in I} \psi_\alpha(x) = 1$ for all $x \in M$.

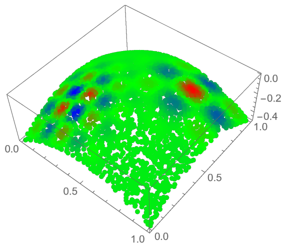
Finally we can define the global instantaneous amplitude subordinated to the cover $\mathcal{U}_I$ as follows.

Definition 7.2. The *global instantaneous amplitude* $a : M \to \mathbb{R}$ subordinate to the open cover $\mathcal{U}_I$ of M and partition of unity $\{\psi_\alpha\}_{\alpha \in I}$ is defined as

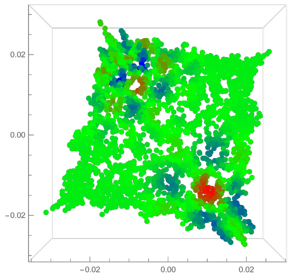
$$a(x) = \sum_{\alpha \in I} \psi_\alpha(x) a_\alpha(x), \text{ for every } x \in M. \tag{129}$$

A number of questions immediately arise because the described method relies on the arbitrarily chosen cover $\mathcal{U}_I$ of M. Is there a unique way to construct instantaneous amplitude and more generally hypercomplex analytic signal for signal over manifold and for what types of manidolds is it possible? How to select a cover $\{U_\alpha\}_{\alpha \in I}$ with maps $\{\phi_\alpha\}_{\alpha \in I}$ that gives the best approximation for analytic signal? For some particular cases with symmetries probably one could construct the instantaneous amplitude uniquely without resorting to the subdivision into subsets, what will be the proper definition of Fourier transform leading to the proper analytic signal and instantaneous amplitude in this case? We hope to address these questions in the future work, while in the next section we provide a discretized numerical example for instantaneous amplitude construction over a single patch.

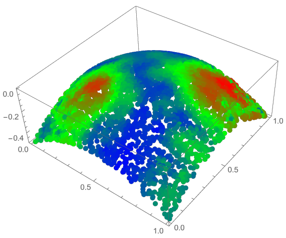
[2] What are Gravitational Waves?, LIGO



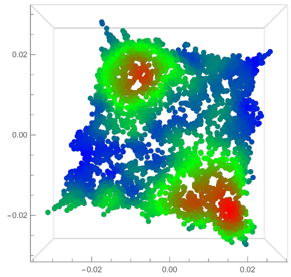
(a) Oscillating function on point-cloud



(b) Oscillating function in diffusion space



(c) Amplitude on point cloud



(d) Amplitude in diffusion space

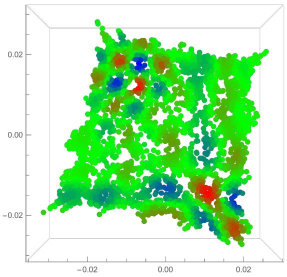
Figure 2: Point cloud and diffusion embedding of the corresponding knn-graph. Oscillation process is given by the sum of modulated gaussians: (a) original point cloud with signal over it; (b) embedding of the point cloud by the first two eigenvectors of normalized Laplacian; (c) envelope of the original signal; (d) envelope of the original signal in diffusion space.

7.3 Analytic signal on graph

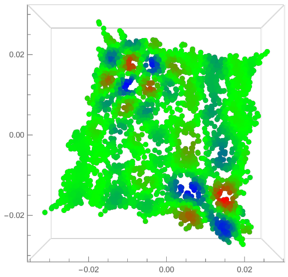
Our objective is to apply the multidimensional analytic signal method to study oscillating signals on graphs. Let us consider a graph $G = (V, E)$ consisting of the set of nodes V and the set of edges E. The topology of a graph can be captured by a variety of operators like adjacency operator or Laplacian [Chu97]. In machine learning scenario, data is frequently assumed to be noisy samples taken from some smooth manifold. Locally, the manifold is homeomorphic to some open ball in $\mathbb{R}^d$, so that we can easily apply the theory of multidimensional analytic signal on a patch of such a manifold. A now classical approach to embed a graph (or part of it) into $\mathbb{R}^d$, is to use the diffusion maps proposed by Coifman and Lafon [CL06, CLL+05].

Let us consider then the mapping $\phi : V \to \mathbb{R}^d$, for instance coming from the diffusion map [CL06]. Let us suppose also, that we have some oscillating process $f : V \to \mathbb{R}$ over the vertices of a graph. Our goal is to obtain some global information on this process, here instantaneous amplitude and phase. We illustrate the technique on a simple, yet edifying example. We consider a sum of two gaussians modulated by different frequencies that lie on a curved surface. The sampled patch is then given by a set of 4096 randomly picked points on the surface as displayed in Fig. 2(a). This resulting point cloud is inherently 2-dimensional although it is embedded in 3 dimensions. Then, to obtain an associated graph, we consider the $k = 20$ nearest neighbours graph to mesh the point cloud. Using the diffusion map given by the first two eigenvectors of the normalised Laplacian, to embed the patch in a 2-D Euclidean space, the resulting embedding of the corresponding k-nn graph with the same oscillating process is shown in Fig. 2 (b).

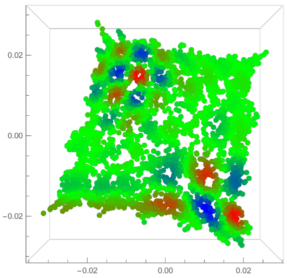
To calculate the discrete version of hypercomplex Fourier transform $\hat{f}(\boldsymbol{\omega})$, we applied the transformation (81) to the discrete signal of Fig. 2(b). Then corresponding Scheffers algebra-valued analytic signal $f_h(\boldsymbol{x})$ was constructed, by first restricting the spectrum $\hat{f}(\boldsymbol{\omega})$ to positive frequencies according to (86) and applying inverse Fourier transform (81). All computations were performed in the diffusion map domain. The signal domain was rescaled by a factor of 10 for convenience. Discrete version of the Fourier transform (81) was calculated for the first 50 frequencies with step 1. Then formula (86) was applied prior to the inverse discrete transform. In Fig. 2 (c) and (d), the envelope function of Eq. (29) is displayed on the original point cloud and in the diffusion space, respectively. Fig. 3 shows separate phase-shifted components $f_{\boldsymbol{j}}(\boldsymbol{x})$ for the four directions $\boldsymbol{j} = 00$, 10, 01 and 11 in the diffusion space.



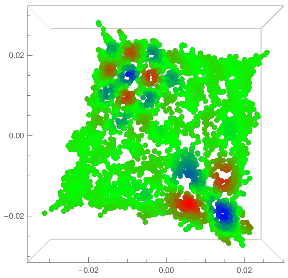
(a) f_{00} shifted component



(b) f_{10} shifted component



(a) f_{01} shifted component



(b) f_{11} shifted component

Figure 3: Phase-shifted components in the diffusion space.

8 Conclusion and future work

The presented hypercomplex analytic signal, aside being proper holomorphic function of several hypercomplex variables in total Scheffers space, extends all the desirable properties that we expect from 1-D analytic signals. There are several theoretical directions in which the presented work could be extended. First of all the proper extension of the construction of hypercomplex analytic signal to manifolds as it was initiated in Section 7.2 is needed. Second the proof of Poincaré theorem and discussion on general holomorphic functions over various domains in Shceffers space pose an interesting question. And finally the possibility of construction of Scheffers hypercomplex manifolds in analogy with complex and quaternionic manifolds [Kod06], [Ver05] pose another interesting direction. From practical point of view we hope to apply the presented theory to the studies of multidimensional oscillating processes such as seismic and gravitational waves in physics and to the studies in multidimensional data analysis and machine learning for global characterization of locally oscillating processes over manifold-like data points.

A Hilbert transform for unit disk and upper half-plane

In this Appendix we briefly describe the essence of the one dimensional Hilbert transform by following the lines of [Kra09]. The usual way to introduce Hilbert transform is by way of the Cauchy formula. If f is holomorphic on $D \subset \mathbb{C}$ and continuous on D up to the boundary ∂D, we can get the value of $f(z)$ from the values of f on the boundary ∂D by Cauchy formula

$$f(z) = \frac{1}{2\pi i} \int\limits_{\partial D} \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta, \quad z \in D. \tag{130}$$

We can express the Cauchy kernel,

$$\frac{1}{2\pi i} \frac{\mathrm{d}\zeta}{\zeta - z},$$

by taking $\zeta = e^{i\psi}$ and $z = re^{i\theta}$, as following

$$\begin{aligned}\frac{1}{2\pi i}\frac{\mathrm{d}\zeta}{\zeta - z} &= \frac{1}{2\pi}\frac{-ie^{-i\psi}\cdot ie^{i\psi}\,\mathrm{d}\psi}{e^{-i\psi}\left(e^{i\psi} - re^{i\theta}\right)} = \frac{1}{2\pi}\frac{\mathrm{d}\psi}{\left(1 - re^{i(\theta-\psi)}\right)} = \frac{1}{2\pi}\frac{1 - re^{-i(\theta-\psi)}}{\left|1 - re^{i(\theta-\psi)}\right|^2}\,\mathrm{d}\psi \\ &= \left(\frac{1}{2\pi}\frac{(1 - r\cos(\theta - \psi))}{\left|1 - re^{i(\theta-\psi)}\right|^2}\,\mathrm{d}\psi\right) + i\left(\frac{1}{2\pi}\frac{r\sin(\theta-\psi)}{\left|1 - re^{i(\theta-\psi)}\right|^2}\,\mathrm{d}\psi\right).\end{aligned} \tag{131}$$

If we subtract $\frac{1}{4\pi}\,\mathrm{d}\psi$ from the real part of the Cauchy kernel, we get the poisson kernel $P_r(\theta)$

$$\begin{aligned}\Re\left(\frac{1}{2\pi i}\frac{\mathrm{d}\zeta}{\zeta - z}\right) - \frac{1}{4\pi}\,\mathrm{d}\psi &= \frac{1}{2\pi}\left(\frac{1 - r\cos(\theta-\psi)}{\left|1 - re^{i(\theta-\psi)}\right|^2} - \frac{1}{2}\right)\mathrm{d}\psi \\ &= \frac{1}{2\pi}\left(\frac{\frac{1}{2} - \frac{1}{2}r^2}{1 - 2r\cos(\theta-\psi) + r^2}\right)\mathrm{d}\psi =: \frac{1}{2}P_r(\theta - \psi).\end{aligned} \tag{132}$$

Thus the real part is, up to a small correction, the Poisson kernel. The kernel that reproduces harmonic functions is the real part of the kernel that reproduces holomorphic functions.

We remind briefly the Poisson integral formula. If $D = \{z : |z| < 1\}$ is the open unit disc in $\mathbb{C}$ and ∂D is the boundary of D and there is continuous $g : \partial D \to \mathbb{R}$, then the function $u : D \to \mathbb{R}$, given by

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - \tau)g\left(e^{i\tau}\right)\,\mathrm{d}\tau, \quad 0 \le r < 1,$$

will be harmonic on D and has radial limit $r \to 1^-$ that agrees with g almost everywhere on ∂D.

Suppose now that we are given a real-valued function $f \in L^2(\partial D)$. Then we can use the Poisson integral formula to produce a function u on D such that $u = f$ on ∂D. Then we can find a harmonic conjugate $u^\dagger$ of u, such that $u^\dagger(0) = 0$ and $u + iu^\dagger$ is holomorphic on D. As a final goal we aim to produce a boundary function $f^\dagger$ for $u^\dagger$ and get the linear operator $f \mapsto f^\dagger$.

If we define a function h on D as

$$h(z) := \frac{1}{2\pi i}\int_{\partial D}\frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta, \quad z \in D,$$

then h will be holomorphic in D. From (132) we know that the real part of h is up to an additive constant equal to the Poisson integral u of f. Therefore $\Re(h)$ is harmonic on D and $\Im(h)$ is a harmonic conjugate of $\Re(h)$. Thus if h is continuous up to the boundary, then we can take $u^\dagger = \Im(h)$ and $f^\dagger(e^{i\theta}) = \lim_{r\to 1^-} u^\dagger(re^{i\theta})$.

For the imaginary part of the Cauchy kernel (131) under the limit $r \to 1^-$ we get

$$\begin{aligned}\frac{1}{2\pi}\frac{r\sin(\theta-\psi)}{\left|1 - re^{i(\theta-\psi)}\right|^2} &\xrightarrow{r\to 1^-} \frac{1}{2\pi}\frac{\sin(\theta-\psi)}{\left|1 - e^{i(\theta-\psi)}\right|^2} = \frac{\sin(\theta-\psi)}{4\pi\left(1 - \cos(\theta-\psi)\right)} = \frac{2\sin\left(\frac{\theta-\psi}{2}\right)\cos\left(\frac{\theta-\psi}{2}\right)}{8\pi\left(\cos\left(\frac{\theta-\psi}{2}\right)\right)^2} \\ &= \frac{1}{4\pi}\cot\left(\frac{\theta-\psi}{2}\right).\end{aligned}$$

Therefore we obtain the Hilbert transform $H : f \mapsto f^\dagger$ on the unit disk D as following

$$Hf(e^{i\theta}) = \frac{1}{4\pi}\int_{-\pi}^{\pi} f\left(e^{it}\right)\cot\left(\frac{\theta - t}{2}\right)\,\mathrm{d}t. \tag{133}$$

Taylor series expansion of the kernel in (133) gives us

$$\cot\left(\frac{\theta}{2}\right) = \frac{\cos\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} = \frac{1 - \frac{(\theta/2)^2}{2!} + \dots}{\frac{\theta}{2}\left(1 - \frac{(\theta/2)^2}{3!} + \dots\right)} = \frac{2}{\theta} + E(\theta),$$

where $E(\theta) = O(|\theta|)$ is a bounded continuous function. Finally for the Hilbert transform on ∂D we can write

$$Hf(e^{i\theta}) := \frac{1}{4\pi}\int\limits_{-\pi}^{\pi} f(e^{it})\cot\left(\frac{\theta-t}{2}\right)\,\mathrm{d}t = \frac{1}{4\pi}\int\limits_{-\pi}^{\pi} f(e^{it})\frac{2}{\theta-t}\,\mathrm{d}t + \frac{1}{4\pi}\int\limits_{-\pi}^{\pi} f(e^{it})E(\theta-t)\,\mathrm{d}t. \quad (134)$$

The first integral above is singular at $t = \theta$ and the second is bounded and easy to estimate. Usually we write for the kernel of Hilbert transform

$$\cot\left(\frac{\theta}{2}\right) \approx \frac{2}{\theta-t},$$

by simply ignoring the trivial error term. Finally in (134) we obtained the Hilbert transform for the boundary of unit disk. Hilbert transform for a unit disk gives us a way to obtain the harmonic conjugate for a periodic function f defined over ∂D. In the followign we briefly outline how to obtain harmonic conjugate for a function defined over $\mathbb{R} \subset \mathbb{C}$.

The unit disk D may be conformally mapped to the upper half-plane $U = \{\zeta \in \mathbb{C} : \Im(\zeta) > 0\}$ by the Möbius map [Kra09]

$$\begin{aligned} &c : D \to U, \\ &\zeta \mapsto i \cdot \frac{1-\zeta}{1+\zeta}. \end{aligned}$$

Since the conformal map of a harmonic function is harmonic, we can carry also Poisson kernel to the upper half-plane [T+48]. The Poisson integral equation for U will be

$$u(x+iy) = \int\limits_{-\infty}^{\infty} P(x-t, y)f(t)\,\mathrm{d}t, \quad y > 0,$$

with the kernel

$$P(x,y) = \frac{1}{\pi}\cdot\frac{y}{x^2+y^2}.$$

The harmonic conjugate function may be obtained from f by taking convolution with the conjugate kernel

$$Q(x,y) = \frac{1}{\pi}\cdot\frac{x}{x^2+y^2}.$$

The Cauchy kernel is related to P and Q by the relation $\frac{i}{\pi z} = P(x,y) + iQ(x,y)$. Finally we are able to construct the harmonic conjugate on the boudary $\partial U = \mathbb{R}$

$$u^{\dagger}(x,y) = \frac{1}{\pi}\int\limits_{-\infty}^{\infty} f(t)\frac{x-t}{(x-t)^2+y^2}\,\mathrm{d}t \xrightarrow{y\to 0^+} \frac{1}{\pi}\int\limits_{-\infty}^{\infty} f(t)\frac{1}{x-t}\,\mathrm{d}t. \quad (135)$$

B Cauchy formula by Ketchum and Vladimirov

We briefly describe the setting and the resulting Cauchy formula given in [Ket28] and [VV84b]. First let us take a bounded region $G \subseteq \mathbb{C}$ with piecewise smooth boundary ∂G. Second let us consider the mapping of G given by $l : w \mapsto a + bw$ with $a, b \in \mathbb{S}_d$ onto the plane $C = \{z : z = a + bw, w \in \mathbb{S}_d(\mathbb{C})\} \subset \mathbb{S}_d(\mathbb{C})$. G is mapped to L and ∂G is mapped to ∂L correspondingly. Suppose also that the function $f(z)$ is differentiable [VV84a] in some neighbourhood O of the closure $\bar{L} = L \cup \partial L$ and b is an invertible element of $\mathbb{S}_d(\mathbb{C})$. Then in the plane C we have the following Cauchy formula

$$\frac{1}{2\pi i}\int\limits_{\partial L} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = \begin{cases} f(z) & \text{if } z \in L, \\ 0 & \text{if } z \in C \setminus \bar{L}. \end{cases} \quad (136)$$

As it was outlined in Appendix A the Cauchy formula plays a vital role in the definition of Hilbert transform and one can directly extend the Hilbert transform and define it in the unit open disk (upper half-plane) of C.

Remark B.1 ([VV84b]). The Cauchy formula (136) holds for real Banach algebra $\mathbb{S}_d$ in which there exists an element $i^2 = -\epsilon$, where ϵ is the unit element in $\mathbb{S}_d$. The plane C consists of the elements

$$C = \{x : x = a + u\epsilon + vi,\ u, v \in \mathbb{R}\},\ a \in \mathbb{S}_d. \tag{137}$$

The Cauchy formula for several hypercomplex variables is written similarly [VV84b]. Now we have the region $L = L_1 \times \cdots \times L_d$, where $L_j = \{z_j = a_j + b_j w_j\}$ is compact in some open region. By the Hartogs's theorem we have that if the function of several variables $f(z_1, \ldots, z_d)$ is analytic with respect to each variable in $G_1 \times \cdots \times G_d$, then it is analytic in $G_1 \times \cdots \times G_d$ and we have the general Cauchy formula

$$f(z_1, ..., z_d) = \frac{1}{(2\pi i)^d} \int\limits_{\partial L_1} ... \int\limits_{\partial L_d} \frac{f(\zeta_1, ..., \zeta_d)}{(\zeta_1 - z_1) \cdot ... \cdot (\zeta_d - z_d)}\, \mathrm{d}\zeta_1 \ldots \zeta_d \tag{138}$$

for all $z \in L$.

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