Asymptotic analysis of subspace-based data-driven residual for fault detection with uncertain reference

Eva Viefhues, Michael Döhler, Falk Hille, Laurent Mevel

Asymptotic analysis of subspace-based data-driven residual for fault detection with uncertain reference ⋆

Eva Viefhues [*] **Michael Döhler** [**] **Falk Hille** [*] **Laurent Mevel** [**]

[*] *Federal Institute for Materials Research and Testing (BAM), Division 7.2 Buildings and Structures, Unter den Eichen 87, 12205 Berlin, Germany (e-mails: {eva.viefhues, falk.hille}@bam.de).*
[**] *Univ. Rennes, Inria, IFSTTAR, Campus de Beaulieu, 35042 Rennes, France (e-mails: {michael.doehler, laurent.mevel}@inria.fr)*

Abstract: The local asymptotic approach is promising for vibration-based fault diagnosis when associated to a subspace-based residual function and efficient hypothesis testing tools. It has the ability of detecting small changes in some chosen system parameters. In the residual function, the left null space of the observability matrix associated to a reference model is confronted to the Hankel matrix of output covariances estimated from test data. When this left null space is not perfectly known from a model, it should be replaced by an estimate from data to avoid model errors in the residual computation. In this paper, the asymptotic distribution of the resulting data-driven residual is analyzed and its covariance is estimated, which includes also the covariance related to the reference null space estimate. The importance of including the covariance of the reference null space estimate is shown in a numerical study.

Keywords: Fault detection, uncertainty in reference, residual evaluation, statistical tests, vibration measurement

1. INTRODUCTION

Vibration-based structural health monitoring (SHM) of civil or mechanical structures is based on the fact that the dynamical behavior of a structure is affected by damages (Farrar and Worden, 2007). The detection of such damages is a fundamental task in this context. Damages can be modeled as changes in the parameters of the underlying mechanical system, inducing small changes in the eigenstructure (eigenvalues and eigenvectors) of a linear system. A particular difficulty for SHM is caused by the absence of known system inputs, since the structural excitation is usually only ambient and not measurable, leading to an output-only monitoring problem.

Among the many model-based or data-driven methods for damage detection (Carden and Fanning, 2004; Fan and Qiao, 2011), methods based on direct model-data matching are particularly appealing for an automated damage diagnosis, where current measurement data are directly confronted to a reference. For instance, such methods include non-parametric change detection based on novelty detection (Worden et al., 2000) or whiteness tests on Kalman filter innovations (Bernal, 2013). Another method within this category, the local asymptotic approach to change detection (Benveniste et al., 1987), has the ability of focusing the detection of small changes in some chosen system parameters. Associated to a subspace-based residual function (Basseville et al., 2000) and efficient hypothesis testing tools, this method has led to successful applications in the field of vibration monitoring, e.g. in (Jhinaoui et al., 2012; Döhler and Mevel, 2013b; Döhler et al., 2014a), including fault isolation and estimation (Döhler et al., 2016; Bhuyan et al., 2017). Note that the considered changes in the system parameters affect the observed linear system in a non-additive way. The present method particularly deals with these non-additive faults.

In this framework, the subspace-based residual function is built on the estimated Hankel matrix of output covariances, which is computed on current measurement data, and the left null space of the Hankel matrix associated to the nominal reference model. Currently, the statistical evaluation of the residual takes into account the covariance related to the Hankel matrix estimate, while the left null space associated to the nominal model is assumed to be perfectly known. However, in practice, this left null space is often estimated, namely from measurement data in the nominal state. The conception of such a data-driven residual avoids bias due to model errors in a nominal model. Note that this residual is then tested for changes in deterministic system parameters, such as stiffness parameters from a finite element model.

In this paper, this more realistic case is considered where the residual is built using the null space *estimate* from data in the nominal state. Uncertainty in data-driven residuals was treated e.g. in (Dong et al., 2012) in the context of null space-based *additive* fault detection, where nominal parameter estimates and the resulting residual depend

⋆ The support from DAAD/German Federal Ministry of Education and Research and the French Ministries of Foreign Affairs (MAEDI) and Education (MENESR) in the PROCOPE program is gratefully acknowledged.

linearly on the noise contained in the measurement data. In our work, *non-additive* faults are considered, and the noise properties of the residual are related less directly to the noise in the measurement data. Instead, they are analyzed asymptotically through a central limit theorem. We analyze the impact of the uncertainty related to the null space estimate in the nominal state on the statistical distribution of the residual. The developed test then takes into account the uncertainties related to measurements of both the nominal state (for the residual setup) and to the current state (for the residual evaluation), while the former has not been considered in previous works. Since in practice the nominal state is never perfectly known, the developed test closes a theoretical gap for real-world applications. The resulting test performance is evaluated with a numerical example.

This paper is organized as follows. In Section 2, the background of the change detection approach is recalled. In Section 3, the data-driven residual is introduced and its asymptotic distribution is analyzed for the computation of a respective test statistic. Finally, a numerical example is given in Section 4.

2. BACKGROUND

2.1 Vibration modelling

The vibration behavior of a mechanical structure can be described by a linear time-invariant dynamical system

$$\mathbf{M}\ddot{z}(t) + \mathbf{C}\dot{z}(t) + \mathbf{K}z(t) = \nu(t), \tag{1}$$

where t denotes continuous time, $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K} \in \mathbb{R}^{m\times m}$ are the mass, damping and stiffness matrices. Vector $z(t) \in \mathbb{R}^m$ contains the displacements of the m degrees of freedom of the structure. $\nu(t)$ is an external force, which is usually unknown for civil structures and modeled as white noise.

Observed at r sensor positions (e.g. by displacement, velocity or acceleration sensors) at discrete time instants $t = k\tau$ (with sampling rate $1/\tau$) and allowing measurement noise, system (1) can be transformed to the discrete-time stochastic state space system model (Juang, 1994)

$$\begin{cases} x_{k+1} = Ax_k + w_k \\ y_k = Cx_k + v_k, \end{cases} \tag{2}$$

with the state vector $x_k = \left[z(k\tau)^T \;\; \dot{z}(k\tau)^T\right]^T \in \mathbb{R}^n$, the measured outputs $y_k \in \mathbb{R}^r$, the state transition matrix

$$A = \exp\left(\begin{bmatrix} 0 & I \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C}\end{bmatrix}\tau\right) \in \mathbb{R}^{n\times n} \tag{3}$$

and the observation matrix

$$C = \left[L_d - L_a\mathbf{M}^{-1}\mathbf{K} \quad L_v - L_a\mathbf{M}^{-1}\mathbf{C}\right] \in \mathbb{R}^{r\times n}, \tag{4}$$

where $n = 2m$ is the model order and $L_d, L_v, L_c \in \{0,1\}^{r\times m}$ are selection matrices indicating the positions of displacement, velocity or acceleration sensors at the degrees of freedom of the structure, respectively. The state noise v_k and output noise w_k are unmeasured, and assumed to be stationary, centered and having finite fourth moments. System (2) is assumed to be stable.

2.2 Change detection methodology

Let the monitored system (1), and respectively (2), be characterized by a (deterministic) parameter vector θ, whose nominal value is θ_0. This parameter is chosen for the particular monitoring problem at hand, and it consists e.g. of the vibration modes, or stiffness or mass parameters of the different structural elements.

A subspace-based residual function was introduced in (Basseville et al., 2000) for the detection of changes from θ_0 using a set of measurements $\mathcal{Y}_N = \{y_1, y_2 \ldots, y_N\}$ of the current system, as follows. Let $G = \mathbf{E}(x_{k+1}y_k^T)$ be the cross-covariance between the states and outputs, let $R_i = \mathbf{E}(y_{k+i}y_k^T) = CA^{i-1}G$ be the output covariances (Van Overschee and De Moor, 1996) and

$$\mathcal{H} \stackrel{\text{def}}{=} \begin{bmatrix} R_1 & R_2 & \ldots & R_q \\ R_2 & R_3 & \ldots & R_{q+1} \\ \vdots & \vdots & \ddots & \vdots \\ R_{p+1} & R_{p+2} & \ldots & R_{p+q} \end{bmatrix} \stackrel{\text{def}}{=} \text{Hank}(R_i) \in \mathbb{R}^{(p+1)r\times qr}$$

is the block Hankel matrix of output covariances, where $\min\{pr, qr\} > n$ with usually $p + 1 = q$. This matrix has the well-known factorization property $\mathcal{H} = \mathcal{OC}$ into observability and controllability matrix (of rank n). Let $\mathcal{O}(\theta_0)$ be the observability matrix in the nominal state, corresponding to some system parameter θ_0, and let $S(\theta_0)$ be its left null space matrix. Then, assuming that parameter θ_0 correctly represents the system in the nominal state, the characteristic property of the a system in the nominal state writes as

$$S(\theta_0)^T\mathcal{H} = 0.$$

From the current measurements $\mathcal{Y}_N$, a consistent estimate

$$\widehat{\mathcal{H}} = \text{Hank}(\widehat{R}_i), \quad \widehat{R}_i = \frac{1}{N}\sum_{k=1}^{N} y_{k+i}y_k^T \tag{5}$$

is obtained from the estimated output covariances $\widehat{R}_i$, and the respective residual vector is defined as

$$\zeta(\theta_0, \mathcal{Y}_N) \stackrel{\text{def}}{=} \sqrt{N}\,\text{vec}(S(\theta_0)^T\widehat{\mathcal{H}}). \tag{6}$$

It has the property

$$\mathbf{E}_\theta(\zeta(\theta_0, \mathcal{Y}_N)) = 0 \quad \text{iff} \quad \theta = \theta_0, \tag{7}$$

where $\mathbf{E}_\theta$ denotes the expectation when $\mathcal{Y}_N$ is measured when the system parameter is θ.

The residual is evaluated with the local asymptotic approach to change detection (Benveniste et al., 1987), assuming the close hypotheses

$$\begin{array}{lll} \mathbf{H}_0: & \theta = \theta_0 & \text{(nominal reference system)}, \\ \mathbf{H}_1: & \theta = \theta_0 + \delta/\sqrt{N} & \text{(faulty system)}, \end{array} \tag{8}$$

where vector δ is unknown but fixed. Then, the residual (6) satisfies the central limit theorem (CLT), ensuring that

$$\zeta(\theta_0, \mathcal{Y}_N) \xrightarrow{d} \begin{cases} \mathcal{N}(0, \Sigma) & \text{under } \mathbf{H}_0 \\ \mathcal{N}(\mathcal{J}\delta, \Sigma) & \text{under } \mathbf{H}_1 \end{cases} \tag{9}$$

for $N \to \infty$, where $\mathcal{J}$ is the asymptotic sensitivity with respect to parameter θ (evaluated at θ_0), Σ is the covariance of the residual, and "d" denotes convergence in distribution. It is assumed that $\mathcal{J}$ has full column rank and Σ is positive definite. Following (9), our change detection problem corresponds to detecting changes in the mean of an asymptotic Gaussian variable. The generalized likelihood ratio test leads to the test statistic (Basseville, 1997)

$$t = \zeta^T\Sigma^{-1}\mathcal{J}\left(\mathcal{J}^T\Sigma^{-1}\mathcal{J}\right)^{-1}\mathcal{J}^T\Sigma^{-1}\zeta, \tag{10}$$

which follows asymptotically a χ^2 distribution with $\dim(\theta)$ degrees of freedom and the non-centrality parameter

$\delta^T F \delta$, where $F = \mathcal{J}^T \Sigma^{-1} \mathcal{J}$ is the Fisher information matrix. To decide between $\mathbf{H}_0$ and $\mathbf{H}_1$, the test variable t is compared to a threshold, which is set up such that the probability of false alarms is below some chosen level. In theory this choice can be made according to the χ^2 distribution of t from the reference system.

3. FAULT DETECTION WITH DATA-DRIVEN RESIDUAL

3.1 Motivation and residual definition

For real SHM problems, the matrices $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ in model (1) are often either unknown, or, if they are known from finite element modelling, they are only approximations of the monitored structure. Then, matrix $S(\theta_0)$ in the residual definition (6) would either be unknown or possibly inaccurate when based on the model, leading to bias in (7) in the latter case and consequently to a non-centrality parameter of the test statistic (10) even in the reference state. While such an analytical model maybe unknown or too inaccurate for the residual definition, model (2) can represent the structural vibration behavior accurately when using estimates of A and C from measurement data. Note that matrices $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ in model (1) cannot be estimated from output-only measurements.

This motivates the replacement of $S(\theta_0)$ by the left null space matrix S_0 of $\mathcal{H}^{(0)}$ in the reference state, so that (7) can indeed be fulfilled in the reference state, independently of the definition of θ_0. Since in practice only an estimate $\widehat{S}_0$ is available, this leads us to the definition of the data-driven residual function as

$$\tilde{\zeta}(\widehat{S}_0, \mathcal{Y}_N) \stackrel{\text{def}}{=} \sqrt{N}\, \text{vec}(\widehat{S}_0^T \widehat{\mathcal{H}}), \tag{11}$$

where $\widehat{S}_0$ is estimated from measurements, e.g. using the singular value decomposition (SVD) of $\widehat{\mathcal{H}}^{(0)}$ in the reference state,

$$\widehat{\mathcal{H}}^{(0)} = [U_1 \ U_2] \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} \tag{12}$$

with $\widehat{S}_0 = U_2 \in \mathbb{R}^{t \times s}$ and $U_1 \in \mathbb{R}^{t \times n}$, where $t = (p+1)r$ and $s = (p+1)r - n$. Assume that output data of length N has been used for the estimate $\widehat{\mathcal{H}}^{(0)}$ and thus for $\widehat{S}_0$.

Note that with the definition of the data-driven residual, the residual computation becomes independent of the chosen system parametrization θ. In this case, the reference property analogous to (7) holds when the reference state is described by $\widehat{S}_0$, even if the value of reference parameter θ_0 is imprecise. Then, the parameter θ_0 only intervenes in the residual evaluation through sensitivity $\mathcal{J}$. Hence, errors in $\mathcal{J}$ only impact the non-centrality parameter of the test statistic in the faulty system, but not in the reference system when $\widehat{S}_0$ is accurate, while errors in $S(\theta_0)$ could already lead to a non-centrality parameter even in the reference system and hence to false alarms.

While the previous residual (6) depends only on one random variable, namely $\widehat{\mathcal{H}}$, the data-driven residual is considered as a function of both random variables $\widehat{S}_0$ and $\widehat{\mathcal{H}}$ in the appropriate joint probability space. This has clearly an impact on the asymptotic distribution of the data-driven residual, and in particular modifies the asymptotic residual covariance compared to (9). The analysis of its distribution and the computation of the respective test statistic is the main problem in the remainder of this paper. In the following it is shown that the asymptotic distribution is also Gaussian, and the asymptotic covariance is evaluated in detail. Then the respective test statistic can be developed for $\tilde{\zeta}(\widehat{S}_0, \mathcal{Y}_N)$ analogously to (10).

3.2 Asymptotic normality of data-driven residual

The asymptotic distribution of $\tilde{\zeta}(\widehat{S}_0, \mathcal{Y}_N)$ relies on the asymptotic normality of $\widehat{\mathcal{H}}^{(0)}$ and $\widehat{\mathcal{H}}$.

Proposition 1. Let the estimates $\widehat{\mathcal{H}}^{(0)}$ in the reference state (corresponding to θ_0) and $\widehat{\mathcal{H}}$ in the current state (corresponding to θ) be computed on N data samples, as in (5). Then the CLTs

$$\sqrt{N}\text{vec}(\widehat{\mathcal{H}}^{(0)} - \mathcal{H}^{(0)}) \xrightarrow{d} \mathcal{N}(0, \Sigma_{\mathcal{H}}),$$

$$\sqrt{N}\text{vec}(\widehat{\mathcal{H}} - \mathcal{H}^{(0)}) \xrightarrow{d} \begin{cases} \mathcal{N}(0, \Sigma_{\mathcal{H}}) & \text{under } \mathbf{H}_0 \\ \mathcal{N}(\mathcal{J}_{\mathcal{H}}\, \delta, \Sigma_{\mathcal{H}}) & \text{under } \mathbf{H}_1 \end{cases}$$

hold, where δ is defined in (8), $\mathcal{J}_{\mathcal{H}}$ is the asymptotic sensitivity of $\text{vec}(\mathcal{H})$ with respect to parameter θ (evaluated at θ_0), and $\Sigma_{\mathcal{H}}$ is the asymptotic covariance in both cases.

Proof. The asymptotic normality follows directly from the asymptotic normality of the output covariance estimates $\widehat{R}_i$ (Hannan, 1970). The asymptotic mean follows from the close hypotheses (8) in the local approach, and the asymptotic covariances are the same in the reference and current states in the local approach, as shown analogously in (Benveniste et al., 1987). □

This result is used to show asymptotic normality of the data-driven residual $\tilde{\zeta}(\widehat{S}_0, \mathcal{Y}_N)$.

Theorem 2. It holds

$$\tilde{\zeta}(\widehat{S}_0, \mathcal{Y}_N) \xrightarrow{d} \begin{cases} \mathcal{N}(0, \tilde{\Sigma}) & \text{under } \mathbf{H}_0 \\ \mathcal{N}(\mathcal{J}\, \delta, \tilde{\Sigma}) & \text{under } \mathbf{H}_1 \end{cases} \tag{13}$$

where δ is defined in (8), $\mathcal{J}$ is the sensitivity analogous as in (9) when replacing $S(\theta_0)$ by S_0 in its computation, and $\tilde{\Sigma}$ is the asymptotic covariance of the data-driven residual that will be detailed later on.

Proof. Since $\widehat{\mathcal{H}}^{(0)}$ and $\widehat{\mathcal{H}}$ are computed on different datasets in the reference and in the current states, respectively, they can be considered as (asymptotically) independent, since the linear system is stable. Thus they are asymptotically jointly normal distributed with

$$\sqrt{N}\left(\begin{bmatrix} \text{vec}(\widehat{\mathcal{H}}^{(0)}) \\ \text{vec}(\widehat{\mathcal{H}}) \end{bmatrix} - \begin{bmatrix} \text{vec}(\mathcal{H}^{(0)}) \\ \text{vec}(\mathcal{H}^{(0)}) \end{bmatrix}\right) \xrightarrow{d} \mathcal{N}\left(\begin{bmatrix} 0 \\ \mathcal{J}_{\mathcal{H}}\delta \end{bmatrix}, \begin{bmatrix} \Sigma_{\mathcal{H}} & 0 \\ 0 & \Sigma_{\mathcal{H}} \end{bmatrix}\right). \tag{14}$$

Since $\text{vec}(\widehat{S}_0^T \widehat{\mathcal{H}})$ in the data-driven residual $\tilde{\zeta}(\widehat{S}_0, \mathcal{Y}_N)$ can be written as a function of

$$\hat{h}_N \stackrel{\text{def}}{=} \begin{bmatrix} \text{vec}(\widehat{\mathcal{H}}^{(0)}) \\ \text{vec}(\widehat{\mathcal{H}}) \end{bmatrix},$$

the delta method (Casella and Berger, 2002) can be applied, and from (14) the CLT (13) follows, where

$$\tilde{\Sigma} = \tilde{\mathcal{J}} \begin{bmatrix} \Sigma_{\mathcal{H}} & 0 \\ 0 & \Sigma_{\mathcal{H}} \end{bmatrix} \tilde{\mathcal{J}}^T, \ \tilde{\mathcal{J}} = \frac{\partial \text{vec}(S_0^T \mathcal{H})}{\partial h}, \ h = \begin{bmatrix} \text{vec}(\mathcal{H}^{(0)}) \\ \text{vec}(\mathcal{H}) \end{bmatrix}. \tag{15}$$

The asymptotic sensitivity yields

$$\tilde{\mathcal{J}}\begin{bmatrix}0\\ \mathcal{J}_{\mathcal{H}}\end{bmatrix}=\frac{\partial\text{vec}(S_0^T\mathcal{H})}{\partial\text{vec}(\mathcal{H})}\frac{\partial\text{vec}(\mathcal{H})}{\partial\text{vec}(\theta)},$$

which is analogous to $\mathcal{J}$ in (9) when replacing $S(\theta_0)$ by S_0. The asymptotic covariance $\tilde{\Sigma}$ is the same in both reference and faulty states due to the close hypotheses definition, as in (9). $\square$

3.3 Asymptotic covariance

The sensitivity matrix $\tilde{\mathcal{J}}$ needs to be obtained for the evaluation of the asymptotic covariance $\tilde{\Sigma}$ of the data-driven residual. The required sensitivity is obtained through first-order perturbations $\Delta(\cdot)$ of the data-driven residual, which is convenient for asymptotical Gaussian variables, as e.g. in (Mellinger et al., 2016). For vector-valued functions $\hat{Y}=f(\hat{X})$ of some estimate $\hat{X}$ of X, a first-order Taylor approximation yields $f(\hat{X})\approx f(X)+\mathcal{J}_{Y,X}(\hat{X}-X)$, or simply $\Delta Y\approx\mathcal{J}_{Y,X}\Delta X$, where $\mathcal{J}_{Y,X}$ is the derivative of f, and $\Delta X=\hat{X}-X$ for $\hat{X}$ close to X.

Hence, the goal of this section is to determine $\tilde{\mathcal{J}}$ through the relationship

$$\text{vec}(\Delta(S_0^T\mathcal{H}))=\tilde{\mathcal{J}}\begin{bmatrix}\text{vec}(\Delta\mathcal{H}^{(0)})\\ \text{vec}(\Delta\mathcal{H})\end{bmatrix}. \quad (16)$$

Using the relation $\text{vec}(AXB)=(B^T\otimes A)\text{vec}(X)$, where $\otimes$ is the Kronecker product, it holds

$$\begin{aligned}\text{vec}(\Delta(S_0^T\mathcal{H}))&=\text{vec}(\Delta S_0^T\mathcal{H})+\text{vec}(S_0^T\Delta\mathcal{H})\\&=\mathcal{J}_{\tilde{\zeta},S_0}\,\text{vec}(\Delta S_0)+\mathcal{J}_{\tilde{\zeta},\mathcal{H}}\text{vec}(\Delta\mathcal{H}),\end{aligned} \quad (17)$$

with $\mathcal{J}_{\tilde{\zeta},S_0}=(\mathcal{H}^T\otimes I_s)\mathcal{P}_{t,s}$, $\mathcal{J}_{\tilde{\zeta},\mathcal{H}}=I_{qr}\otimes S_0^T$ and $\mathcal{P}_{a,b}$ is a permutation matrix such that $\text{vec}(X^T)=\mathcal{P}_{a,b}\text{vec}(X)$ for a matrix $X\in\mathbb{R}^{a\times b}$ (Döhler and Mevel, 2013a). Furthermore, a perturbation of S_0 yields the relation $\text{vec}(\Delta S_0)=\mathcal{J}_{S,\mathcal{H}^{(0)}}\text{vec}(\Delta\mathcal{H}^{(0)})$ as follows. First, $\Delta\mathcal{H}^{(0)}$ is propagated to the column space U_1 in SVD (12) by (Liu et al., 2008)

$$\Delta U_1=U_1R+U_2U_2^T\Delta\mathcal{H}^{(0)}\ V_1D_1^{-1},$$

where R is a matrix that will be canceled in the following, and the expected values of all singular values in D_1 are distinct from zero. In the vectorized form it follows

$$\begin{aligned}\text{vec}(\Delta U_1)&=(I_n\otimes U_1)\ \text{vec}(R)\\&\quad+(D_1^{-1}V_1^T\otimes\ U_2U_2^T)\ \text{vec}(\Delta\mathcal{H}^{(0)}).\end{aligned} \quad (18)$$

This perturbation is now propagated to the left null space $S_0=U_2$. From $U_1^TU_2=0$ it follows $\Delta U_1^TU_2+U_1^T\Delta U_2=0$,

$$(I_s\otimes U_1^T)\ \text{vec}(\Delta U_2)=-(U_2^T\otimes I_n)\ \mathcal{P}_{t,n}\ \text{vec}(\Delta U_1). \quad (19)$$

Considering $U_2^TU_2=I$ and thus $\Delta(U_2^TU_2)=0$, it follows $\Delta U_2^TU_2+U_2^T\Delta U_2=0$ and

$$\mathcal{P}_{s,s}(I_s\otimes U_2^T)\,\text{vec}(\Delta U_2)+(I_s\otimes U_2^T)\,\text{vec}(\Delta U_2)=0. \quad (20)$$

From (19) and (20), a particular solution for $\text{vec}(\Delta U_2)$ follows as

$$\begin{aligned}\text{vec}(\Delta U_2)&=-(I_s\otimes U_1)(U_2^T\otimes I_n)\ \mathcal{P}_{t,n}\ \text{vec}(\Delta U_1)\\&=-\mathcal{P}_{s,t}(U_1\otimes U_2^T)\ \text{vec}(\Delta U_1),\end{aligned} \quad (21)$$

using properties of the permutation matrix $\mathcal{P}$ (Döhler and Mevel, 2013a). Since $S_0=U_2$, and substituting (18) into (21) finally leads to

$$\text{vec}(\Delta S_0)=\mathcal{J}_{S_0,\mathcal{H}^{(0)}}\text{vec}(\Delta\mathcal{H}^{(0)})$$

where $\mathcal{J}_{S_0,\mathcal{H}^{(0)}}=-\mathcal{P}_{s,t}(U_1D_1^{-1}V_1^T\otimes S_0^T)$. Then, in (17) it holds $\mathcal{J}_{\tilde{\zeta},S_0}\,\text{vec}(\Delta S_0)=\mathcal{J}_{\tilde{\zeta},\mathcal{H}^{(0)}}\text{vec}(\Delta\mathcal{H}^{(0)})$, where

$$\begin{aligned}\mathcal{J}_{\tilde{\zeta},\mathcal{H}^{(0)}}&=\mathcal{J}_{\tilde{\zeta},S_0}\mathcal{J}_{S_0,\mathcal{H}^{(0)}}\\&=(\mathcal{H}^T\otimes I_s)\ \mathcal{P}_{t,s}(-\mathcal{P}_{s,t})(U_1D_1^{-1}V_1^T\otimes S_0^T)\\&=-\mathcal{H}^TU_1D_1^{-1}V_1^T\otimes S_0^T\\&=-V_1V_1^T\otimes S_0^T\end{aligned}$$

since the asymptotic covariance can be evaluated in the reference state at $\mathcal{H}=\mathcal{H}^{(0)}$. Then it follows from (17) the desired sensitivity $\tilde{\mathcal{J}}$ in (16)

$$\begin{aligned}\tilde{\mathcal{J}}&=\begin{bmatrix}\mathcal{J}_{\tilde{\zeta},\mathcal{H}^{(0)}} & \mathcal{J}_{\tilde{\zeta},\mathcal{H}}\end{bmatrix}\\&=\begin{bmatrix}-V_1V_1^T\otimes S_0^T & I_{qr}\otimes S_0^T\end{bmatrix},\end{aligned}$$

and finally the asymptotic covariance of the data-driven residual from (15)

$$\tilde{\Sigma}=\Sigma_1+\Sigma_2, \quad (22)$$

where $\Sigma_1=\mathcal{J}_{\tilde{\zeta},\mathcal{H}^{(0)}}\Sigma_{\mathcal{H}}\mathcal{J}_{\tilde{\zeta},\mathcal{H}^{(0)}}^T$, $\Sigma_2=\mathcal{J}_{\tilde{\zeta},\mathcal{H}}\Sigma_{\mathcal{H}}\mathcal{J}_{\tilde{\zeta},\mathcal{H}}^T$. Note that Σ_1 is related to the uncertainty in $\widehat{S}_0$, and Σ_2 to the current test data.

3.4 Covariance estimation

A consistent estimate of the asymptotic covariance $\tilde{\Sigma}$ in (22) can be obtained by computing $\widehat{\Sigma}_{\mathcal{H}}$ as the sample covariance over several datasets in the reference state (see e.g. (Döhler et al., 2014b)), and by using consistent estimates $\widehat{S}_0$ and V_1 computed from $\widehat{\mathcal{H}}^{(0)}$ in (12). However, note that this assumes the same number of samples N for the estimate $\widehat{\mathcal{H}}^{(0)}$ (and $\widehat{S}_0$) in the reference state and for $\widehat{\mathcal{H}}$ in the current state in (14).

In general, the residual covariance $\tilde{\Sigma}\approx\text{cov}(\sqrt{N}\text{vec}(\widehat{S}_0^T\widehat{\mathcal{H}}))$ satisfies analogously to (14) and (22)

$$\begin{aligned}\text{cov}(\sqrt{N}\text{vec}(\widehat{S}_0^T\widehat{\mathcal{H}}))&\approx\mathcal{J}_{\tilde{\zeta},\mathcal{H}^{(0)}}\text{cov}(\sqrt{N}\text{vec}(\widehat{\mathcal{H}}^{(0)}))\mathcal{J}_{\tilde{\zeta},\mathcal{H}^{(0)}}^T\\&\quad+\mathcal{J}_{\tilde{\zeta},\mathcal{H}}\text{cov}(\sqrt{N}\text{vec}(\widehat{\mathcal{H}}))\mathcal{J}_{\tilde{\zeta},\mathcal{H}}^T\end{aligned} \quad (23)$$

When the number of samples used in the reference state for the estimates $\widehat{\mathcal{H}}^{(0)}$ and $\widehat{S}_0$ is not the same as the number of samples in the current state for $\widehat{\mathcal{H}}$, the asymptotic covariance expressions need be renormalized with respect to the number of samples. Let N be the number of samples for estimate $\widehat{\mathcal{H}}$ and M the number of samples for $\widehat{\mathcal{H}}^{(0)}$, and assume that $\frac{N}{M}$ is a constant in the asymptotic analysis. Thanks to (14), we can approximate $\text{cov}(\sqrt{N}\text{vec}(\widehat{\mathcal{H}}))\approx\Sigma_{\mathcal{H}}$ and $\text{cov}(\sqrt{M}\text{vec}(\widehat{\mathcal{H}}^{(0)}))\approx\Sigma_{\mathcal{H}}$, thus $\text{cov}(\sqrt{N}\text{vec}(\widehat{\mathcal{H}}^{(0)}))\approx\frac{N}{M}\Sigma_{\mathcal{H}}$. Then, it follows from (23) for the respective covariance estimates

$$\widehat{\tilde{\Sigma}}\approx\tfrac{N}{M}\widehat{\Sigma}_1+\widehat{\Sigma}_2. \quad (24)$$

Hence, when the number of samples M to compute the reference matrix $\widehat{S}_0$ is large with respect to the number of samples N for computing $\widehat{\mathcal{H}}$ from the current test data, the residual covariance depends more strongly on $\widehat{\mathcal{H}}$ (related to Σ_2), which is the more uncertain part in the residual computation in this case. Thus, the covariance contribution Σ_1 related to the reference matrix gets weaker, which is reflected in (24) where $\frac{N}{M}$ is small in this case.

On the other side, when M is small compared to N, the contribution to the residual covariance of the uncertainty related to the reference matrix $\widehat{S}_0$ is large compared to the uncertainty of $\widehat{\mathcal{H}}$ computed from the test data. Then, $\frac{N}{M}$ is large and indeed the contribution of Σ_1 in (24) is larger.

3.5 Computation of test statistic

In the previous sections, the asymptotic distribution of the data-driven residual $\tilde{\zeta}(\widehat{S}_0, \mathcal{Y}_N)$ has been characterized, with the CLT in (13) and the estimation of the respective covariance $\tilde{\Sigma}$ in Section 3.4. The residual sensitivity $\mathcal{J}$ has been detailed in previous works, e.g. in (Balmès et al., 2008; Döhler et al., 2014b), and is estimated analogously when replacing $S(\theta_0)$ by $\widehat{S}_0$. Finally, the GLR test for hypotheses (8) based on (13) writes

$$\tilde{t} = \tilde{\zeta}^T \tilde{\Sigma}^{-1} \mathcal{J} \left(\mathcal{J}^T \tilde{\Sigma}^{-1} \mathcal{J} \right)^{-1} \mathcal{J}^T \tilde{\Sigma}^{-1} \tilde{\zeta} \qquad (25)$$

analogously to (10), and is consistently computed when replacing $\mathcal{J}$ and $\tilde{\Sigma}$ by their respective estimates.

4. NUMERICAL APPLICATION

The developed method is applied to an eight mass-spring-damper system with masses $m_1 = m_3 = m_5 = m_7 = 1, m_2 = m_4 = m_6 = m_8 = 2$ and stiffnesses $k_1 = k_3 = k_5 = k_7 = 200, k_2 = k_4 = k_6 = k_8 = 100$ (Fig. 1). Classical damping is defined such that all modes have a damping ratio of 2%. A fault is simulated by decreasing stiffness of spring 3 by 1.5%. The considered system parameter θ is chosen as the stiffnesses of the 8 springs. Thus the χ^2-distributed test statistic has 8 degrees of freedom, and an expected value of 8 in the reference state.

Acceleration data are simulated from random white noise excitation at four sensor positions sampled at 20 Hz, with added white measurement noise having 5% standard deviation of the signals. The null space $\hat{S}_0$ and the residual sensitivity and covariance are computed in the reference state.

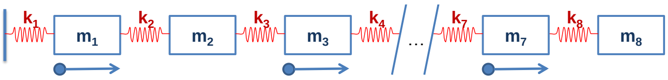

Fig. 1. Mass-spring chain with four sensors.

In a first example, the problems of the model-based residual are illustrated under model errors. In this case, $S(\theta_0)$ is computed as the null space of the observability matrix $\mathcal{O}(\theta_0)$ that is computed from the matrices A and C in (3)–(4) from the model of the structure. A model error is introduced by changing the stiffness parameters arbitrarily between 0.2 and 5%. Note that in practice, the parameters of a finite element model may have quite significant errors, while it is still desired to detect small changes in the system. 100 datasets are generated in both reference and faulty states of length N = 100,000, and the histograms of the respective test statistics are shown without and with model error in the computation of $S(\theta_0)$ in Figure 2. The distribution of the test statistic in the reference state is well centered around the theoretical value when the model is perfect, and the distribution in the faulty state is well separated (left). However, under the considered model errors, the distributions in the reference state is far away from the theoretical mean, and the distributions of the reference and faulty states are inverted (right), so the fault cannot be detected anymore.

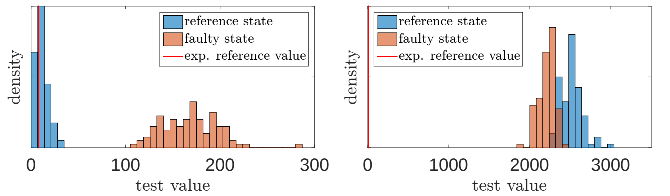

Fig. 2. Histograms of model-based test statistic without (left) and with (right) model error.

This motivates the use of the data-driven residual to reduce the effect of model errors. In this case, $S(\theta_0)$ is replaced by the estimated $\widehat{S}_0$. First, this replacement is made in the conventional test (10), where the new data-driven residual (11) is used, but the covariance computation from the conventional test is used, not taking into account the uncertainty related to $\widehat{S}_0$ (i.e. $\Sigma_1 = 0$ in (24)). Finally, the covariance of the data-driven residual is correctly evaluated and the new test (25) is computed to demonstrate the influence of the new covariance computation when the reference null space is estimated. Fig. 3 shows the respective histograms of the test statistic for $N = M = 100,000$ in the reference and faulty states. As it can be seen, the distributions of the test statistics using the data-driven residual are clearly separated for the reference and faulty states. When not considering the uncertainty related to the null space estimate $\widehat{S}_0$ in Fig. 3 (left) in the conventional test, the mean of the test statistic in the reference state deviates significantly from its theoretical value of 8. This is due to the fact that the reference null space estimate has not converged yet. With the new test computation, the uncertainty of the reference null space estimate is taken into account, and the distribution of the test statistic in the reference state is close to the theoretical one.

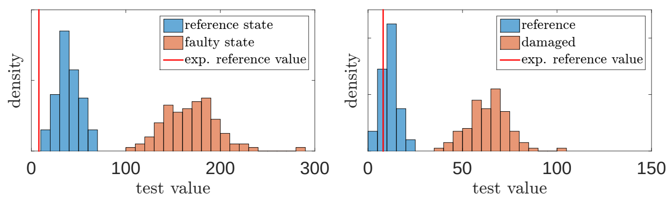

Fig. 3. Histograms of test statistics with data-driven residual. Test with conventional covariance computation ($\Sigma_1 = 0$, left) and new test (right).

In Fig. 4 (top) the mean of the test statistic is shown similarly evaluating the data-driven residual with the conventional test (not taking into account the uncertainties related to $\widehat{S}_0$) and the new test, where different data lengths M for the computation of $\hat{S}_0$ are considered while the data length N of the test data is kept fixed at N = 100,000. It can be seen that the mean of the new test statistic is around the theoretical value of 8 already for quite short datasets of M = 50,000, while the mean of the conventional test deviates significantly from 8. The conventional

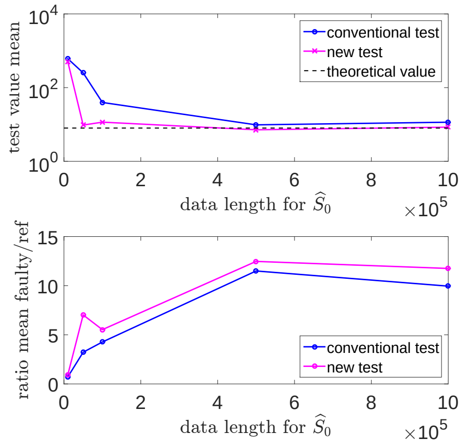

Fig. 4. Mean of test statistic (top) and test performance (bottom) for data-driven residual with conventional ($\Sigma_1 = 0$) and new covariance computation in dependence of data length in reference state.

test approaches this value as M increases, which leads to better estimates $\widehat{S}_0$ and thus less contribution of the respective covariance in (24). To illustrate both algorithms' performance in the faulty state, the ratios of test statistic means between the faulty and the reference state is shown in Fig. 4 (bottom) in the same setting. The ratio measures how far the mean of the test shifts in the faulty state. The new test performs better than the conventional test, in particular when few data is used in the reference state.

5. CONCLUSION

In this paper, a data-driven version of an established subspace-based residual function was analyzed within the asymptotic local approach to change detection, having applications in particular for vibration monitoring. The data-driven residual is convenient for realistic applications where a reference model may have errors or is unknown, and instead estimates from data in a reference state are used. It was shown that considering the uncertainty related to the estimation of the reference null space in the residual evaluation increases the performance of the associated fault detection tests.

REFERENCES

Balmès, E., Basseville, M., Mevel, L., Nasser, H., and Zhou, W. (2008). Statistical model-based damage localization: a combined subspace-based and substructuring approach. *Structural Control and Health Monitoring*, 15(6), 857–875.

Basseville, M. (1997). Information criteria for residual generation and fault detection and isolation. *Automatica*, 33(5), 783–803.

Basseville, M., Abdelghani, M., and Benveniste, A. (2000). Subspace-based fault detection algorithms for vibration monitoring. *Automatica*, 36(1), 101–109.

Benveniste, A., Basseville, M., and Moustakides, G. (1987). The asymptotic local approach to change detection and model validation. *IEEE Transactions on Automatic Control*, 32(7), 583–592.

Bernal, D. (2013). Kalman filter damage detection in the presence of changing process and measurement noise. *Mechanical Systems and Signal Processing*, 39(1-2), 361–371.

Bhuyan, M., Allahdadian, S., Döhler, M., Lecieux, Y., Mevel, L., Schoefs, F., and Ventura, C. (2017). Transfer matrices-based output-only statistical damage localization and quantification on a frame structure. In *Proc. 11th IWSHM*. Stanford, CA, USA.

Carden, E. and Fanning, P. (2004). Vibration based condition monitoring: a review. *Structural Health Monitoring*, 3(4), 355–377.

Casella, G. and Berger, R. (2002). *Statistical inference*. Duxbury Press, Pacific Grove, CA, USA.

Döhler, M., Hille, F., Mevel, L., and Rücker, W. (2014a). Structural health monitoring with statistical methods during progressive damage test of S101 Bridge. *Engineering Structures*, 69, 183–193.

Döhler, M. and Mevel, L. (2013a). Efficient multi-order uncertainty computation for stochastic subspace identification. *Mechanical Systems and Signal Processing*, 38(2), 346–366.

Döhler, M. and Mevel, L. (2013b). Subspace-based fault detection robust to changes in the noise covariances. *Automatica*, 49(9), 2734–2743.

Döhler, M., Mevel, L., and Hille, F. (2014b). Subspace-based damage detection under changes in the ambient excitation statistics. *Mechanical Systems and Signal Processing*, 45(1), 207–224.

Döhler, M., Mevel, L., and Zhang, Q. (2016). Fault detection, isolation and quantification from Gaussian residuals with application to structural damage diagnosis. *Annual Reviews in Control*, 42, 244–256.

Dong, J., Verhaegen, M., and Gustafsson, F. (2012). Robust fault detection with statistical uncertainty in identified parameters. *IEEE Transactions on Signal Processing*, 60(10), 5064–5076.

Fan, W. and Qiao, P. (2011). Vibration-based damage identification methods: a review and comparative study. *Structural Health Monitoring*, 10(1), 83–111.

Farrar, C. and Worden, K. (2007). An introduction to structural health monitoring. *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 365(1851), 303–315.

Hannan, E. (1970). *Multiple time series*. Wiley, New York.

Jhinaoui, A., Mevel, L., and Morlier, J. (2012). Subspace instability monitoring for linear periodically time-varying systems. In *Proc. 16th IFAC Symposium on System Identification (SYSID)*. Brussels, Belgium.

Juang, J.N. (1994). *Applied system identification*. Prentice Hall, Englewood Cliffs, NJ, USA.

Liu, J., Liu, X., and Ma, X. (2008). First-order perturbation analysis of singular vectors in singular value decomposition. *IEEE Transactions on Signal Processing*, 56(7), 3044–3049.

Mellinger, P., Döhler, M., and Mevel, L. (2016). Variance estimation of modal parameters from output-only and input/output subspace-based system identification. *Journal of Sound and Vibration*, 379, 1–27.

Van Overschee, P. and De Moor, B. (1996). *Subspace Identification for Linear Systems: Theory, Implementation, Applications*. Kluwer, Dordrecht, The Netherlands.

Worden, K., Manson, G., and Fieller, N. (2000). Damage detection using outlier analysis. *Journal of Sound and Vibration*, 229(3), 647–667.