Interval estimation for second-order delay differential equations with delayed measurements and uncertainties

Tatiana Kharkovskaia, Denis Efimov, Emilia Fridman, Andrey Polyakov, Jean-Pierre Richard

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***Abstract*—The interval estimation design is studied for a second-order delay differential equation with position delayed measurements, uncertain input and initial conditions. The proposed method contains two consecutive interval observers. The first one estimates the interval of admissible values for the position without delay for each instant of time using new delay-dependent conditions on positivity. Then derived interval estimates of the position are used to design the second observer estimating an interval of admissible values for the velocity of the considered dynamical system. The results are illustrated by numerical experiments for an example.**

I. INTRODUCTION

Delays appear in many control systems at different levels of conception: at the state dynamics, in the control channel, or in the measurement signals. Sensors, actuators, and communication networks that are involved in the feedback loops usually introduce delays. Arising in differential equations, they may cause instability and oscillation of the solutions of the considered system [1], [2]. For such models, which are infinite-dimensional in contrast with ordinary differential equations (ODEs), the analysis and design are much more complicated and require specially developed concepts and algorithms [3]. For instance, the observability and methods for estimation for delayed systems with unknown inputs and nonlinearities are considered in [4], [5], application of an algebraic approach for observer design in LPV time-delay systems is presented in [6], an estimation problem for positive systems with time-varying unknown delays is studied in [7]. Due to presence of uncertain initial conditions and disturbances, the exact estimation may be impossible, and that is why there exists another popular solution to use interval observers for uncertain systems, which provides at each instant of time a set of admissible values (an interval) for the state whose size is consistent with the system uncertainty and obtained measurements [8], [9], [10], [11], [12], [13].

In this work a simple benchmark problem is investigated of an unstable second order delayed system with delayed measurements (*e.g.* a delayed model for motion of a single mass point)

$$\ddot{x}(t) = -ax(t-\tau) + f(t),\ t \geq 0,$$
$$y(t) = x(t-\theta) + \nu(t),\ \tau \neq \theta,$$

where $x(t) \in \mathbb{R}$ is the position, $y(t) \in \mathbb{R}$ is the measured output signal, $f : \mathbb{R}_+ \to \mathbb{R}$ and $\nu : \mathbb{R}_+ \to \mathbb{R}$ are state perturbation and measurement noise (unknown bounded signals), $\tau > 0$ and $\theta > 0$ are the known delays. Our goal is to design an interval observer for this system, but the main issue is that beside stability conditions, to construct an interval observer it is necessary to check that the estimation error dynamics possess the positivity property. The existing solutions in the field [14], [15], [16], [17] are based on the delay-independent conditions of positivity from [18], [19]. Some results on interval observer design for uncertain time-varying delay can be found in [7], [15]. In [20] the delay-dependent conditions on positivity are introduced for the case with equal delay in the state and in the output, that may correspond to a delay-free system with delayed measurements. Therefore, the obstacle for an interval observer design is that for the considered system, by its form, the existing conditions on stability and positivity are not met. To this end, in this paper the measurement delays are supposed to be different from state delays. Using the theory of non-oscillatory solutions for second-order functional differential equations [21], [22], the new conditions on positivity of estimation error dynamics are developed and, furthermore, a full-state interval estimation technique for considered class of systems is designed, which consists in consequent two interval observers. The first one generates the interval estimates for the position, and the second interval observer estimates the system velocity. The efficiency of proposed technique is shown by a numerical example.

The outline of this paper is as follows. After preliminaries in Section II, and an introduction of the considered time delay system properties in Section III, the interval observer design is given in Section IV. The results of the motivation example simulation are presented in Section V.

II. PRELIMINARIES

- $\mathbb{R}$ is the Euclidean space ($\mathbb{R}_+ = \{\tau \in \mathbb{R} : \tau \geq 0\}$), $\mathcal{C}_\tau^n = C([-\tau, 0], \mathbb{R}^n)$ is the set of continuous maps from $[-\tau, 0]$ into $\mathbb{R}^n$ for $n \geq 1$; $\mathcal{C}_{\tau+}^n = \{y \in \mathcal{C}_\tau^n :\ y(s) \in \mathbb{R}_+^n,\ s \in [-\tau, 0]\}$;
- x_t is an element of $\mathcal{C}_\tau^n$ defined as $x_t(s) = x(t+s)$ for all $s \in [-\tau, 0]$;
- $|x|$ denotes the absolute value of $x \in \mathbb{R}$, $||x||_2$ is the Euclidean norm of a vector $x \in \mathbb{R}^n$, $||\varphi|| = \sup_{t\in[-\tau,0]} ||\varphi(t)||_2$ for $\varphi \in \mathcal{C}_\tau^n$;
- for a measurable and locally essentially bounded input $u : \mathbb{R}_+ \to \mathbb{R}^p$ the symbol $||u||_{[t_0,t_1)}$ denotes its L_∞ norm $||u||_{[t_0,t_1)} = ess\sup\{||u(t)||_2, t \in [t_0, t_1)\}$, the set of all such inputs $u \in \mathbb{R}^p$ with the property $||u||_{[0,+\infty)} < \infty$ will be denoted as $\mathcal{L}_\infty^p$;
- I_n denote the identity matrix of dimensions $n \times n$;
- $a\,\mathcal{R}\,b$ corresponds to an elementwise relation $\mathcal{R} \in \{<, >, \leq, \geq\}$ (a and b are vectors or matrices): for example $a < b$ (vectors) means $\forall i : a_i < b_i$; for $\phi, \varphi \in \mathcal{C}_\tau$ the relation $\phi\,\mathcal{R}\,\varphi$ has to be understood elementwise for all domain of definition of the functions, *i.e.* $\phi(s)\,\mathcal{R}\,\varphi(s)$ for all $s \in [-\tau, 0]$.

T. Kharkovskaia is with Department of Control Systems and Informatics, Saint Petersburg State University of Information Technologies Mechanics and Optics (ITMO), 49 Kronverkskiy av., 197101 Saint Petersburg, Russia.

D. Efimov, A. Polyakov and J.-P. Richard are at Inria, Univ. Lille, Centrale Lille, CNRS, UMR 9189 - CRIStAL, F-59000 Lille, France

E. Fridman is with School of Electrical Engineering, Tel-Aviv University, Tel-Aviv 69978, Israel.

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A. Interval relations

Given a matrix $A \in \mathbb{R}^{m\times n}$, define $A^+ = \max\{0, A\}$, $A^- = A^+ - A$ (similarly for vectors) and denote the matrix of absolute values of all elements by $|A| = A^+ + A^-$.

Lemma 1. [23] *Let $A \in \mathbb{R}^{m\times n}$ be a constant matrix and $x \in \mathbb{R}^n$ be a vector variable and $\underline{x} \le x \le \overline{x}$ for some $\underline{x}, \overline{x} \in \mathbb{R}^n$, then*

$$A^+\underline{x} - A^-\overline{x} \le Ax \le A^+\overline{x} - A^-\underline{x}. \tag{1}$$

B. Delay-independent positivity

Consider a time-invariant linear system with time-varying delay:

$$\dot{x}(t) = A_0x(t) + A_1x(t-\tau(t)) + Bf(t),\ t \in [0, +\infty), \tag{2}$$

$$x(h) = \phi(h) \text{ for } -\overline{\tau} \le h \le 0,\ \phi \in \mathcal{C}^n_{\overline{\tau}}, \tag{3}$$

where $x(t) \in \mathbb{R}^n$, $x_t \in \mathcal{C}^n_{\overline{\tau}}$ is the state function; $\tau : \mathbb{R}_+ \to [-\overline{\tau}, 0]$ is the time-varying delay, a Lebesgue measurable function of time, $\overline{\tau} \in \mathbb{R}_+$ is the maximum delay; $f \in \mathcal{L}^m_\infty$ is the input; the constant matrices A_0, A_1 and B have appropriate dimensions; $\phi : (-\overline{\tau}, 0] \to \mathbb{R}^n$ is a Borel measurable bounded function of initial conditions.

Definition 1. [21] Function $x : \mathbb{R} \to \mathbb{R}^n$, which is locally absolutely continuous on $[0,\infty)$, is called a solution of problem (2), (3) if it satisfies (2) for almost all $t \in [0,\infty)$ and equality (3) for $t \le 0$.

The matrix A_0 is called Metzler if all its off-diagonal elements are nonnegative. The system (2) is called *positive* if for $\phi \in \mathcal{C}^n_{\overline{\tau}+}$ it has the corresponding solution $x(t) \ge 0$ for all $t \ge 0$.

Lemma 2. [18], [19] *The system (2) is positive iff A_0 is Metzler, $A_1 \ge 0$ and $Bf(t) \ge 0$ for all $t \ge 0$. A positive system (2) is asymptotically stable with $f(t) \equiv 0$ for all $t \in \mathbb{R}_+$ iff there are $p, q \in \mathbb{R}^n_+(p > 0$ and $q > 0)$ such that*

$$p^T[A_0 + A_1] + q^T = 0.$$

Under conditions of the above lemma the system has bounded solutions for $f \in \mathcal{L}^m_\infty$ [1]. Note that for linear time-invariant systems the conditions of positive invariance of polyhedral sets have been similarly given in [24], as well as conditions of asymptotic stability in the nonlinear case have been considered in [25], [26], [27].

C. Representation of the solution for delay differential equations

Since $\overline{\tau}$ is the maximum delay for $\tau(t)$ define a bounded set $\mathcal{T} = \{s \in (0, \overline{\tau}] : s \le \tau(s)\}$, then the non-zero initial value problem (2), (3) can be rewritten to have zero initial conditions for $t < 0$ and with the same solution $x(t)$ for all $t \ge 0$:

$$\dot{x}(t) = A_0x(t) + A_1x(t-\tau(t)) + Bf(t) + f^*(t),\ t \in [0, +\infty), \tag{4}$$

$$x(h) = 0 \text{ for } h < 0,$$

where

$$f^*(t) = \begin{cases} A_1\phi(t-\tau(t)) & t \in \mathcal{T} \\ 0 & \text{otherwise} \end{cases}.$$

In addition to the problem (2), (3), where x, f and ϕ are vector signals or functions, we will consider a problem where the solution is a $n \times n$ matrix function. For example, the $n \times n$ matrix function $C(t,s) = X(t)X^{-1}(s)$, where $X(t)$ satisfies a homogeneous initial value problem

$$\dot{X}(t) = A_0X(t) + A_1X(t-\tau(t)),\ t \ge s, \tag{5}$$

$$X(\theta) = 0 \text{ for } \theta < s,\ X(s) = I_n$$

for each $s \ge 0$, is called the Cauchy matrix of (5). By construction $C(t,s) = 0$ for $0 \le t < s$. Using this Cauchy matrix $C(t,0)$, a unique solution of non-homogeneous system (4) will take the form

$$x(t) = C(t,0)x(0) + \int_0^t C(t,s)\left(Bf(s) + f^*(s)\right) ds, \tag{6}$$

which is also the solution for the representation (2) with the initial conditions (3).

Based on this idea, for non-zero initial function in [20] the conditions on delay-dependent positivity are introduced by verifying an additional constrain on the first interval $t \in [0, \overline{\tau}]$:

Lemma 3. [20] *The system (2) with $Bf(t) \ge 0$ for all $t \ge 0$, $x(0) \in \mathbb{R}^n_+$, with a Metzler matrix A_0, $A_1 \ge 0$ and $0 \le (A_0)_{i,i} \le e(A_1)_{i,i} < (A_0)_{i,i} + \overline{\tau}^{-1}$ for all $i = 1, \dots, n$, has the corresponding solution $x(t) \ge 0$ for all $t \ge 0$ provided that*

$$Bf(t) \ge -f^*(t) \quad \forall t \in [0, \overline{\tau}].$$

D. Conditions on positivity of a second order system

Following the result of [22], consider the second-order delay differential equation with an input signal:

$$\ddot{x}(t) + a(t)x(t-\tau) - b(t)x(t-\theta) = f(t) \quad t \in [0, +\infty), \tag{7}$$

$$x(h) = \phi(h) \quad \text{for } h \le 0,\ \dot{x}(0) \in \mathbb{R}$$

with constant delays $\tau, \theta \ge 0$ and nonnegative functions $a, b, f \in \mathcal{L}_\infty$ and $\phi \in \mathcal{C}_{\max\{\tau,\theta\}}$. The corresponding homogeneous equation is considered as

$$\ddot{x}(t) + a(t)x(t-\tau) - b(t)x(t-\theta) = 0 \quad t \in [0, +\infty). \tag{8}$$

For a signal $q \in \mathcal{L}_\infty$ denote further the following short hands

$$q_* = \underset{t\ge 0}{\text{essinf}}\, q(t),\ q^* = \underset{t\ge 0}{\text{esssup}}\, q(t).$$

Theorem 1. [22] *Assume that $0 \le \tau < \theta$ and there exists $\varepsilon > 0$ such that the inequalities*

$$\varepsilon \le \{a(t) - b(t)\} \le \frac{1}{4}b_*^2(\theta-\tau)^2, \quad \forall t \in [0, +\infty) \tag{9}$$

and

$$\frac{1}{\sqrt{a^*}exp\left\{\frac{b_*(\theta-\tau)^2}{4}\right\}} arctan \frac{b_*(\theta-\tau)}{2\sqrt{a^*}exp\left\{\frac{b_*(\theta-\tau)^2}{4}\right\}} > \theta - \tau \tag{10}$$

are fulfilled. Then

(1) the elements $C_{11}(t,s)$ and $C_{12}(t,s)$ of the Cauchy matrix

$$C(t,s) = \begin{bmatrix} C_{11}(t,s) & C_{12}(t,s) \\ C_{21}(t,s) & C_{22}(t,s) \end{bmatrix} \in \mathbb{R}^{2\times 2}$$

of (8) are nonnegative for $0 \le s < t < +\infty$;

(2) the Cauchy function $C_{12}(t,s)$ of (8) satisfies the exponential estimate

$$|C_{12}(t,s)| \le Ne^{-\alpha(t-s)}\ \forall 0 \le s \le t < +\infty$$

for some $N > 0$, $\alpha > 0$ and the integral estimate

$$\underset{t\ge 0}{sup} \int_0^t C_{12}(t,s)ds \le \frac{1}{\varepsilon};$$

(3) if there exists $lim_{t\to\infty}\{a(t)-b(t)\}=k$, then equality

$$\lim_{t\to\infty}\int_0^t C_{12}(t,s)ds=\frac{1}{k}$$

is fulfilled.

According to this theorem, the solution $x(t)$ of the equation (7) being written in the form (6) has nonnegative elements $C_{11}(t,s)$ and $C_{12}(t,s)$ of the Cauchy matrix function, and with nonnegative $f(t)$ and the proper choice of parameters $a(t)$, $b(t)$ and delays τ, θ the system (7) will have a bounded solution. The Theorem 1 also establishes the exponential estimate for the Cauchy function $C_{12}(t,s)$ and its convergence rate. This result concerns the exponential stability of (8), which are based on the maximum principles for the second order delay differential equation (8) [22]. Using the representation of solutions (6) (see also (14) below), we can see that this principle is reduced to positivity of the Cauchy matrix element $C_{11}(t,s)$ and $C_{12}(t,s)$ for nonnegative initial conditions.

Proposition 1. *Let $0<\tau<\theta$ in the system (7) and the conditions (9), (10) of the Theorem 1 hold. If $f(t)+f^*(t)\geq 0$ for all $t\geq 0$, $\dot{x}(0)\in\mathbb{R}_+$ and $\phi(h)\geq 0$ for all $h\in[-\theta,0]$, where*

$$f^*(t)=\begin{cases}-a(t)\phi(t-\tau)+b(t)\phi(t-\theta) & t\in[0,\tau]\\ b(t)\phi(t-\theta) & t\in(\tau,\theta]\\ 0 & t>\theta\end{cases},$$

then the corresponding solution satisfies $x(t)\geq 0$ for all $t\geq 0$.

III. Problem statement

The main object of study in this note is the delay differential equation of the second order, which represents an unstable delayed model of motion of a single mass point:

$$\ddot{x}(t)=-a(t)x(t-\tau)+f(t),\ t\in[0,+\infty) \quad (11)$$

where $x(t)\in\mathbb{R}$ is the position of the point, $\dot{x}(t)\in\mathbb{R}$ and $\ddot{x}(t)\in\mathbb{R}$ are the velocity and the acceleration of the point motion, respectively; $\tau\geq 0$ is a known constant state delay, $a(t)\in\mathbb{R}_+$ with $a\in\mathcal{L}_\infty$ is the parameter function, $f\in\mathcal{L}_\infty$ is the external input. The equation (11) can be presented in a time-varying version of (2) with the corresponding matrices:

$$A_0=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix},\ A_1(t)=\begin{bmatrix}0 & 0\\ -a(t) & 0\end{bmatrix},\ B=\begin{bmatrix}0\\ 1\end{bmatrix}. \quad (12)$$

The initial conditions for (11) are considered as a scalar function only for position:

$$x(h)=\phi(h)\text{ for }-\tau\leq h\leq 0,\ \phi\in\mathcal{C}_\tau;\ \dot{x}(0)\in\mathbb{R}, \quad (13)$$

then the solution (6) can be rewritten to describe only the position of (11) as follows:

$$\begin{aligned}x(t) &= C_{11}(t,0)x(0)+C_{12}(t,0)\dot{x}(0)\\ &\quad+\int_0^t C_{12}(t,s)\left(f(s)+f^*(s)\right)ds,\end{aligned} \quad (14)$$

where

$$C(t,s)=\begin{bmatrix}C_{11}(t,s) & C_{12}(t,s)\\ C_{21}(t,s) & C_{22}(t,s)\end{bmatrix}\in\mathbb{R}^{2\times 2}$$

with the initial condition $C(0,0)=I_2$, and

$$f^*(t)=\begin{cases}-a(t)\phi(t-\tau) & t\in[0,\tau]\\ 0 & t>\tau\end{cases}.$$

Furthermore, we consider that the position of (11) is available for measurements with some known constant delay θ, $\theta\geq\tau$:

$$y(t)=x(t-\theta)+\nu(t), \quad (15)$$

where $y(t)\in\mathbb{R}$ is an output with measurement noise $\nu\in\mathcal{L}_\infty$. It is worth stressing that since $\theta\geq\tau$, the initial conditions for (11), $x(h)=\phi(h)$, should be defined for $h\in[-\theta,0]$. To continue the analysis with this data, we also need to introduce the following hypothesis:

Assumption 1. *There exist known functions $\underline{\phi},\overline{\phi}\in\mathcal{C}_\theta$ such that $\underline{\phi}(h)\leq\phi(h)\leq\overline{\phi}(h)$ for all $h\in[-\theta,0]$, and $\underline{\dot{x}}_0\leq\dot{x}(0)\leq\overline{\dot{x}}_0$ for some known $\underline{\dot{x}}_0,\overline{\dot{x}}_0\in\mathbb{R}$.*

The assumption about a known set $[\underline{\phi},\overline{\phi}]$ for the initial conditions ϕ is standard for the interval or set-membership estimation theory [15], [8], [9], [10], [11]. We will assume that the parameter a is known and the instant values of the signals $f(t)$ and $\nu(t)$ are unavailable:

Assumption 2. *There exist known signals $\underline{f},\overline{f}\in\mathcal{L}_\infty$ and a constant $\nu_0>0$ such that $\underline{f}(t)\leq f(t)\leq\overline{f}(t)$ and $|\nu(t)|\leq\nu_0$ for all $t\geq 0$.*

Therefore, the uncertain inputs $f(t)$ and $\nu(t)$ in (11) and (15) belong to the known intervals $[\underline{f}(t),\overline{f}(t)]$ and $[-\nu_0,\nu_0]$ respectively for all $t\geq 0$.

The goal is to design an interval observer for (11), (15)

$$\begin{aligned}&\dot{\xi}(t)=F[\xi_t,\ \underline{f}(t),\ \overline{f}(t),\ \nu_0,\ y(t)],\ \xi_t\in\mathcal{C}_\theta^s,\ s>0,\\ &\begin{bmatrix}\underline{x}(t)\\ \underline{\dot{x}}(t)\end{bmatrix}=\underline{G}[\xi_t,\ \underline{f}(t),\ \overline{f}(t),\ \nu_0,\ y(t)],\\ &\begin{bmatrix}\overline{x}(t)\\ \overline{\dot{x}}(t)\end{bmatrix}=\overline{G}[\xi_t,\ \underline{f}(t),\ \overline{f}(t),\ \nu_0,\ y(t)]\end{aligned}$$

such that for all $t\geq 0$

$$\begin{aligned}\underline{x}(t)\leq x(t)\leq\overline{x}(t),\\ \underline{\dot{x}}(t)\leq\dot{x}(t)\leq\overline{\dot{x}}(t)\end{aligned}$$

provided that assumptions 1 and 2 are satisfied and $x-\underline{x}$, $\overline{x}-x$, $\dot{x}-\underline{\dot{x}}$, $\overline{\dot{x}}-\dot{x}\in\mathcal{L}_\infty$. A similar problem has been studied in [14], [15], [28].

IV. Interval observer design

In this section we will present two steps to design interval observers for the system (11) and (15). First, using the result of Theorem 1 an observer will be given for interval estimation of the position $x(t)$. Second, another interval observer will be designed to obtain the interval inclusion for $\dot{x}(t)$.

A. The first observer for the position

The corresponding homogeneous equation

$$\ddot{x}(t)=-a(t)x(t-\tau),\ t\in[0,+\infty)$$

for (11) has unbounded solutions in case of a constant parameter a [22], which means that (11) is unstable. For the output $y(t)$ given in (15) the observer for (11) can be constructed as follows:

$$\ddot{x}=-a(t)x(t-\tau)+\ell(t)x(t-\theta)+f(t)-\ell(t)y(t)+\ell(t)\nu(t), \quad (16)$$

or in the state space form:

$$\begin{aligned}&\dot{x}_1=x_2,\\ &\dot{x}_2=-a(t)x_1(t-\tau)+\ell(t)x_1(t-\theta)+f(t)-\ell(t)y(t)+\ell(t)\nu(t),\end{aligned}$$

where $\ell \in \mathcal{L}_\infty$ is an observer gain to be designed.

Remark 1. Note that, as it has been explained in Section III, the equation (11) can be presented in the form (2) with time-varying matrices A_0, A_1 and B as in (12), and with the output in form (15) with different measurement delay $\theta \geq \tau$, then the matrix A_1 can be nonnegative only when $\theta = \tau$, in other cases it is always $A_1 < 0$, then the system (11) and (16) do not posses the delay-independent or delay-dependent positivity properties according to lemmas 2 and 3, respectively.

Let us consider the first observer for the position $x(t)$ of the system (16) in the form:

$$\begin{aligned} \ddot{x}^-(t) &= -a(t)x^-(t-\tau) + \ell(t)x^-(t-\theta) \\ &\quad + \underline{f}(t) - \ell(t)y(t) - \ell(t)\nu_0 - \varrho(t), \\ \ddot{x}^+(t) &= -a(t)x^+(t-\tau) + \ell(t)x^+(t-\theta) \\ &\quad + \overline{f}(t) - \ell(t)y(t) + \ell(t)\nu_0 + \varrho(t), \end{aligned} \tag{17}$$

where $x^-(t)$, $x^+(t) \in \mathbb{R}$ are the estimates for the position of motion (11) for $t \in [0, +\infty)$ with initial conditions

$$x^-(h) = \underline{\phi}(h),\ x^+(h) = \overline{\phi}(h) \quad \forall h \in [-\theta, 0], \\ \dot{x}^-(0) = \underline{\dot{x}}_0,\ \dot{x}^+(0) = \overline{\dot{x}}_0$$

from Assumption 1 and

$$\varrho(t) = \begin{cases} a(t)[\overline{\phi}(t-\tau) - \underline{\phi}(t-\tau)] & t \leq \tau \\ 0 & t > \tau \end{cases}.$$

Proposition 2. *Let the measurement delays satisfy the relation $\theta \geq \tau > 0$, and assumptions 1, 2 be satisfied. For the system (11) with initial conditions (13) and the observer (17) select the observer gain $\ell(t) \geq 0$ to satisfy the conditions (9), (10) of Theorem 1 with $b(t) = \ell(t)$ for all $t \geq 0$. Then its position satisfies the interval inclusion*

$$x^-(t) \leq x(t) \leq x^+(t)\ \forall t \in [0, +\infty) \tag{18}$$

and $x^+ - x, x - x^- \in \mathcal{L}_\infty$.

Using this observer it is possible to derive the interval estimates for the position $x(t)$ without delay, but the velocity is not yet estimated since the matrix $A_1 < 0$.

B. The second observer for the velocity

As mentioned above, the second order delay differential equation (11) can be presented in form of (2) with matrices (12). Let us consider the delayed term $-a(t)x(t-\tau)$ as a disturbance and rewrite (2) for this case:

$$\dot{x}(t) = Ax(t) + \rho\left(t, x(t), x(t-\tau), f(t)\right), \tag{19}$$

where $x(t) = [x_1(t)\ \ x_2(t)]^{\mathrm{T}}$ is the state vector, $x_1(t)$ is the estimated position by (18), $x_2(t)$ is a velocity of motion (11);

$$A = A_0 - KC = \begin{bmatrix} -k_1 & 1 \\ -k_2 & 0 \end{bmatrix},$$

$$\rho\left(t, x(t), x(t-\tau), f(t)\right) = \begin{bmatrix} k_1 x_1(t) \\ k_2 x_1(t) + f(t) - a(t)x_1(t-\tau) \end{bmatrix},$$

$K = [k_1\ k_2]^{\mathrm{T}}$ is a new observer gain, $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$.

Assumption 3. *There are $K \in \mathbb{R}^2$ and a Metzler matrix $D \in \mathbb{R}^{2\times 2}$ such that the matrices $A_0 - KC$ and D have the same eigenvalues and the pairs $(A_0 - KC, \chi_1)$ and (D, χ_2) are observable for some $\chi_1 \in \mathbb{R}^{1\times 2}$, $\chi_2 \in \mathbb{R}^{1\times 2}$.*

According to this assumption there is a nonsingular matrix $S \in \mathbb{R}^{2\times 2}$ such that $D = S(A_0 - KC)S^{-1}$ [13], and the new coordinates $z = Sx$ can be introduced transforming the system (19) to the form:

$$\dot{z}(t) = Dz(t) + S\rho\left(t, x(t), x(t-\tau), f(t)\right). \tag{20}$$

Using Lemma 1 we obtain that

$$\underline{\rho}(t) \leq \rho\left(t, x(t), x(t-\tau), f(t)\right) \leq \overline{\rho}(t),$$

where the functions $\underline{\rho}(t)$ and $\overline{\rho}(t)$ depend only on available information (the variables $x_1^-(t)$, $x_1^+(t)$, $x_1^-(t-\tau)$ and $x_1^+(t-\tau)$ are given by the first observer (17) for all $t \in [0, +\infty)$):

$$\underline{\rho}(t) = \begin{bmatrix} k_1 x_1^-(t) \\ k_2 x_1^-(t) + \underline{f}(t) - a(t)x_1^+(t-\tau) \end{bmatrix},$$

$$\overline{\rho}(t) = \begin{bmatrix} k_1 x_1^+(t) \\ k_2 x_1^+(t) + \overline{f}(t) - a(t)x_1^-(t-\tau) \end{bmatrix}.$$

Now, applying the results of [8], [29] two estimates $\underline{z}, \overline{z} \in \mathbb{R}^2$ can be calculated based on the available information on these intervals (the interval inclusion (18) for $x_1(t)$ without delay), such that

$$\underline{z}(t) \leq z(t) \leq \overline{z}(t) \quad \forall t \in [0, +\infty). \tag{21}$$

In other words, an interval observer can be designed for the transformed dynamics (20):

$$\begin{aligned} \dot{\underline{z}}(t) &= D\underline{z}(t) + S^+\underline{\rho}(t) - S^-\overline{\rho}(t), \\ \dot{\overline{z}}(t) &= D\overline{z}(t) + S^+\overline{\rho}(t) - S^-\underline{\rho}(t); \\ \underline{z}(h) &= S^+\underline{\Phi}(h) - S^-\overline{\Phi}(h), \\ \overline{z}(h) &= S^+\overline{\Phi}(h) - S^-\underline{\Phi}(h),\ \forall h \in [-\tau, 0]; \\ \underline{x}(t) &= (S^{-1})^+\underline{z}(t) - (S^{-1})^-\overline{z}(t), \\ \overline{x}(t) &= (S^{-1})^+\overline{z}(t) - (S^{-1})^-\underline{z}(t), \end{aligned} \tag{22}$$

where

$$\underline{\Phi}(h) = \begin{bmatrix} \underline{\phi}(h) \\ \begin{cases} 0 & h < 0 \\ \underline{\dot{x}}_0 & h = 0 \end{cases} \end{bmatrix},\ \overline{\Phi}(h) = \begin{bmatrix} \overline{\phi}(h) \\ \begin{cases} 0 & h < 0 \\ \overline{\dot{x}}_0 & h = 0 \end{cases} \end{bmatrix}$$

and the relations (1) are used to calculate the initial conditions for $\underline{z}(h)$, $\overline{z}(h)$ at $h \in [-\theta, 0]$ and the estimates $\underline{x}, \overline{x}$.

Proposition 3. *Let assumptions 1, 2 and 3 be satisfied. Then for the second order delay equation (11), presented in form (2), with initial conditions 3, and with the interval observer (22) the relations for the velocity*

$$\underline{x}_2(t) \leq x_2(t) \leq \overline{x}_2(t),\ x(t) \in \mathbb{R}^2,\ \forall t \in [0, +\infty) \tag{23}$$

are fulfilled provided that the conditions of Proposition 2 are verified.

V. Example

To show the efficiency of the proposed observers we consider the motivation example (11):

$$\ddot{x}(t) = -ax(t-\tau) + f(t),\ t \in [0, +\infty)$$

with the values of parameter $a = 2$, the internal delay $\tau = 0.2$, the perturbation $f(t) = 0.5(\cos(2t) + 0.3\cos(10t))$; the measurement delay $\theta = 0.5$ and the noise $\nu(t) = \nu_0\sin(60t)$ for $\nu_0 = 0.07$. The initial conditions (3) $x(h) = \phi(h)$ for $h \in [-\theta, 0]$, where $\phi(h) = 0.1\sin(\frac{1}{4\pi\theta}t)$, $\dot{x}(0) = 0$. The Assumption 2 is satisfied

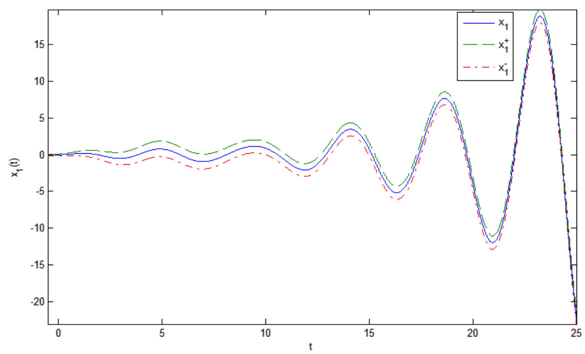

Figure 1. The result of simulation of observer (17) for the position $x(t)$ of (11).

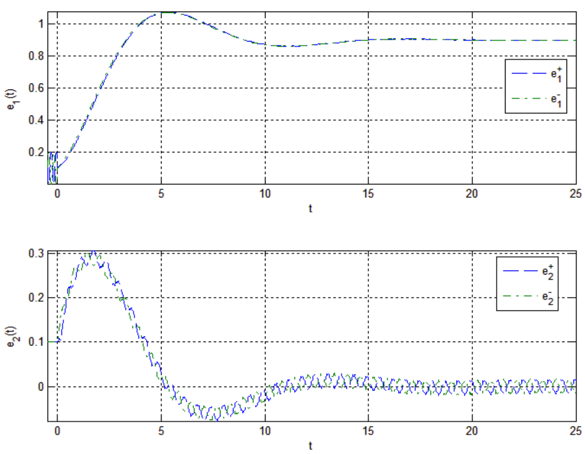

Figure 2. Errors of estimation $e_1(t)$, $e_2(t)$ of the first observer (17).

for $\underline{f}(t) = 0.5(\cos(2t) - 0.3)$, $\overline{f}(t) = 0.5(\cos(2t) + 0.3)$; and the bounds on initial conditions are given as $\underline{\phi}(h) = -0.1$, $\overline{\phi}(h) = 0.1$, $\overline{\dot{x}}_0 = -\underline{\dot{x}}_0 = 0.1$. For $\ell = 1.8$ the conditions (9), (10) are satisfied. For simulation the explicit Euler method with the step $T = 10^{-3}$ was used. The results of simulation of the first observer (17) for position $x(t)$ of (11) are shown in Fig. 1 for $t \in [0, 25]$. In Fig. 2 the errors of the estimation are presented: for position $e_1(t) \geq 0$ for $t \geq 0$; and for the velocity $e_2(t)$ is not positive (as it is supposed to be). For the second observer (22) the observer gain $K = [3\ 1]$ with the matrix $D = \mathrm{diag}[-2.618;\ -0.382]$ satisfies the Assumption 3 with the transformation matrix $S = \begin{bmatrix} -1.25 & 0.48 \\ 0.48 & -1.25 \end{bmatrix}$. The estimates derived for the velocity of (11) by the second observer (22) are shown at the Fig. 3.

VI. Conclusion

The problem of interval estimation for a second-order delay differential equation with position delayed measurements, uncertain input and initial conditions is studied in this work. The proposed approach consists in two consecutively connected interval observers. The first one estimates the set of admissible values for the position without delay using new delay-dependent conditions on positivity of a second order system. Then derived interval estimates of the position are used to design the second observer evaluating an interval of admissible values for the velocity of the considered dynamical system. The results are illustrated by numerical experiments for an example. The future directions of research will include generalization of this approach to a generic model of linear systems and control design procedure development.

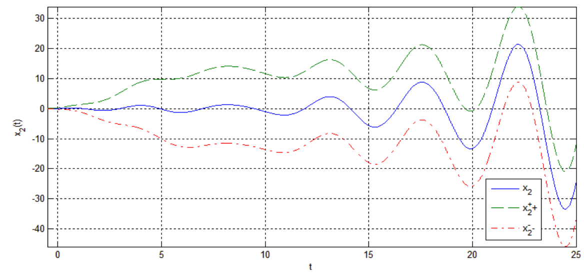

Figure 3. The result of simulation of observer (22) for the velocity $\dot{x}(t)$ of (11).

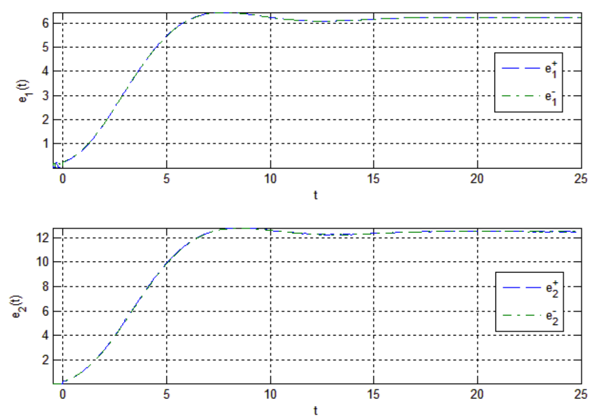

Figure 4. Errors of estimation $e_1(t)$, $e_2(t)$ of the second observer (22).

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