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Dissipativity Based Stability Criterion for Aperiodic Sampled-data Systems subject to Time-delay ⋆

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Abstract: This extended abstract presents a dissipativity-based stability analysis of Linear Time Invariant (LTI) systems subjected to aperiodic sampling and time-varying delay. We provide a novel stability criterion which aids in making the trade-offs between maximum allowable sampling interval and delays while guaranteeing stability. Simulation results have been provided to demonstrate the effectiveness of the proposed criterion.

Keywords: Sampled-data system, dissipativity based stability, system with time-delay.

1. INTRODUCTION

The last few decades has seen a tremendous increase in research on sampled-data systems as well as their applications [Åström and Wittenmark (1996); Chen and Francis (1995); Hespanha et al. (2007)]. The stability analysis of such systems is an important research track that has been explored over the years and an overview of the different approaches that have been proposed can be found in [Hetel et al. (2017)]. The *Time-delay approach*, embeds the sampling phenomenon into a time-delay problem and generally consists of Lyapunov-Krasovskii Functional (LKF) based stability criteria [Mikheev et al. (1988); Fridman (2010); van de Wouw et al. (2010); Seuret (2012)]. Since sampled-data systems exhibit both continuous and discrete dynamics, the second approach, namely the *Hybrid systems* approach was developed [Naghshtabrizi et al. (2008); Nešić and Teel (2004)]. In the *Discrete-time* approach, stability criteria are obtained using system integration over sampling intervals and by convex embedding of the state transition matrix between sampling instants [Fujioka (2009); Cloosterman et al. (2010); van de Wouw et al. (2010)]. The third, *Input-output stability* approach treats the error induced by sampling as a perturbation to the continuous-time control system, which aids in employing classical robust control tools to analyse the stability of the system [Mirkin (2007); Fujioka (2009); Omran et al. (2013)]. The approach draws similarities with the input-output stability analysis of time-delay systems as provided in [Fridman and Shaked (2006); Kao and Rantzer (2007); Kao and Lincoln (2004)] for the case of continuous time-varying delay.

In this article, we focus on the input-output approach for stability analysis of aperiodic sampled-data systems subject to time-delay. A primary advantage of this approach is that it is intuitively simple to develop. The problem trickles down to the classical robust control framework that could possibly take into account various perturbations and non-linearities. However, several problems remain to be solved in the input-output framework. For example, the existing results provide only asymptotic stability criteria for sample-data systems. However, in numerous practical scenarios, it is desirable to at least have a measure of the system decay-rate so as to provide a basic performance. Furthermore, while the stability analysis of LTI systems in the presence of delay and sampling has been studied separately using the input-output approach, a result for sampled-data systems with time delay is yet to be provided. The extension towards this direction is challenging since directly embedding sampling as an additional delay leads to a discontinuous delay and the existing approaches in the input-output framework do not cover this scenario.

In this paper, we close the aforementioned gap for Linear Time Invariant (LTI) systems. The main contribution of this article is to provide an exponential stability criterion for systems with aperiodic sampling and time-delay. We

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provide a detailed analysis of sampling and delay, by taking into account the specific discontinuities in delay. The provided result will lay the foundation for a possible extension to the stability analysis of Non-linear sampled-data systems with delay.

2. PROBLEM STATEMENT

We consider the following LTI system:

$$\begin{aligned} &\dot{x}(t) = Ax(t) + Bu(t), \forall t \geq 0, \\ &x(0) = x_0, \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and A and B are matrices of appropriate dimensions. The time instant t_0^s specifies the first sampling instant, and belongs to the sampling sequence $\{t_k^s\}_{k\in\mathbb{N}}$ defined by

$$t_{k+1}^s - t_k^s = h_k, \forall k \in \mathbb{N}. \tag{2}$$

The possibly time-varying sequence of sampling intervals $\{h_k\}_{k\in\mathbb{N}}$ satisfying $h_k \in [\underline{h}, \bar{h}]$, with $\underline{h} > 0$, considers imperfections in sampling caused by jitter, data packet dropouts, etc. The actuation timing sequence $\{t_k^a\}_{k\in\mathbb{N}}$, implying the sequence of time-instants at which the control input $u(t)$ based on $x(t_k^s)$ will be implemented at the level of the actuator, is given by

$$t_k^a = t_k^s + \tau_k, t_k^a \leq t_{k+1}^a, \forall k \in \mathbb{N} \tag{3}$$

where $\tau_k \in [\underline{\tau}, \bar{\tau}]$ represents the time-varying delay between sampling and actuations instants. Without loss of generality, we consider that the first actuation instance occurs at time $t_0^a = \bar{\tau} + \bar{h}$, while the first sampling instant occurs at $t_0^s = t_0^a - \tau_0$. Based on the sequences (2) and (3), we have the control input $u(t)$ in (1) satisfying the state-feedback and zero-order hold strategies given by

$$u(t) = \begin{cases} 0, & \forall t \in [0, t_0^a), \\ Kx(t_k^s), & \forall t \in [t_k^a, t_{k+1}^a), k \in \mathbb{N}, \end{cases} \tag{4}$$

where $K \in \mathbb{R}^{n\times n}$ is the controller gain. The objective of this paper is to analyse the stability of the system defined by (1), (2), (3) and (4) using input-output stability analysis approach.

3. MAIN RESULTS

3.1 System Reformulation

In this section, we reformulate the sampled-data system model to include the effects of sampling and delay as perturbations to the system defined by (1)-(4). We have, for all $t \in [t_k^a, t_{k+1}^a)$, $k \in \mathbb{N}$,

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ &= Ax(t) + BKx(t) - BKx(t) + BKx(t_k^s), \\ &= A_{cl}x(t) + B_{cl}e(t), \end{aligned} \tag{5}$$

where $A_{cl} = A + BK$, $B_{cl} = BK$, and

$$e(t) = x(t_k^s) - x(t), \forall t \in [t_k^a, t_{k+1}^a), k \in \mathbb{N} \tag{6}$$

represents the error induced by sampling and delay. By choosing an auxiliary output $y(t) = \dot{x}(t)$, we can reformulate system (1)-(4) as follows:

$$\begin{aligned} &\dot{x}(t) = A_{cl}x(t) + B_{cl}e(t), \forall t \geq t_0^a \\ &y(t) = \dot{x}(t) = A_{cl}x(t) + B_{cl}e(t), \\ &e(t) = x(t_k^s) - x(t) = -\int_{t_k^s}^{t} y(s)ds, \forall t \in [t_k^a, t_{k+1}^a), k \in \mathbb{N}. \end{aligned} \tag{7}$$

For $t \in [0, t_0^a)$, the system (1) is in open-loop and therefore can be remodelled as

$$\begin{aligned} &\dot{x}(t) = Ax(t), \\ &y(t) = \dot{x}(t) = Ax(t), \\ &e(t) = 0. \end{aligned} \tag{8}$$

3.2 Stability criterion

By exploiting the properties of the error $e(t)$ and using arguments inspired by the dissipativity theory, we derive the following result.

Theorem 1. The system defined by (1)-(4) or (7) and (8) is exponentially stable with a decay rate $\alpha/2$ if there exist symmetric positive definite matrices P, R_1, and R_2 such that the following linear matrix inequalities are feasible,

$$\begin{bmatrix} Q + C_{cl}^T(\gamma_1^2 R_1 + \gamma_2^2 R_2)C_{cl} & PB_{cl} + C_{cl}^T(\gamma_1^2 R_1 + \gamma_2^2 R_2)D_{cl} \\ \star & D_{cl}^T(\gamma_1^2 R_1 + \gamma_2^2 R_2)D_{cl} - (R_1 + R_2) \end{bmatrix} < 0, \tag{9}$$

and

$$\begin{bmatrix} Q + \gamma_2^2 C_{cl}^T R_2 C_{cl} & PB_{cl} + \gamma_2^2 C_{cl}^T R_2 D_{cl} \\ \star & \gamma_2^2 D_{cl}^T R_2 D_{cl} - R_2 \end{bmatrix} < 0, \tag{10}$$

where

$$Q = A_{cl}^T P + PA_{cl} + \alpha P, \tag{11}$$

and

$$\begin{aligned} &\gamma_1^2 = (\bar{h} + \bar{\tau})^2, \\ &\gamma_2^2 = (\bar{h} + \bar{\tau})^2 e^{\alpha(\bar{h}+\bar{\tau})}. \end{aligned} \tag{12}$$

Proof: We provide as follows, a sketch of the proof. The result is inspired by the frequency domain criteria provided by [Fujioka (2009)] for sampled-data systems without delay and further extended in [Omran et al. (2016)]. The proof is based on the existence of a storage function $V : \mathbb{R}^n \to \mathbb{R}^+$, $V(x) = x^T P x$, and a supply function $\mathcal{S}\big(t, y(t), e(t)\big) : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, such that the exponential dissipativity inequality

$$\dot{V}(x(t)) + \alpha V(x(t)) \leq e^{-\alpha\tau(t)}\mathcal{S}\big(t, y(t), e(t)\big), \forall t \geq t_0^a, \tag{13}$$

with $\tau(t) = t - t_0^a$, is satisfied. For the supply function given by

$$\begin{aligned} \mathcal{S}\big(\theta, y(\theta), e(\theta)\big) &= e^T(\theta)\big(R_1 + e^{\alpha\tau(\theta)}R_2\big)e(\theta) \\ &\quad - y^T(\theta)\big(\gamma_1^2 R_1 + e^{\alpha\tau(\theta)}\gamma_2^2 R_2\big)y(\theta), \end{aligned} \tag{14}$$

with $R_1 = R_1^T > 0$, $R_2 = R_2^T > 0$, and γ_1, γ_2 given by (12), it can be shown that the inequalities

$$\int_0^t \mathcal{S}\big(\theta, y(\theta), e(\theta)\big)d\theta \leq 0, \forall t \geq 0, \tag{15}$$

$$-\int_0^{t_0^a} \mathcal{S}\big(\theta, y(\theta), e(\theta)\big)d\theta \leq \eta V(x(t_0^a)), \tag{16}$$

hold with

$$\eta = \frac{\bar{h} + \bar{\tau}}{eigmin(P)} \max_{\theta\in[0,t_0^a]} \left\{ eigmax\left[(e^{A\tau(\theta)})^T A^T (\gamma_1^2 R_1 + e^{\alpha\tau(\theta)}\gamma_2^2 R_2) A (e^{A\tau(\theta)}) \right] \right\}. \tag{17}$$

Using simple manipulations, (13), (15) and (16) imply

$$V(x(t)) \leq \mathcal{C}e^{-\alpha t}V(x(t_0^a)), \forall t \geq t_0^a, \tag{18}$$

for some $\mathcal{C} > 0$. To conclude, the set of LMI conditions (9) and (10) are sufficient for the dissipation inequality (13) to

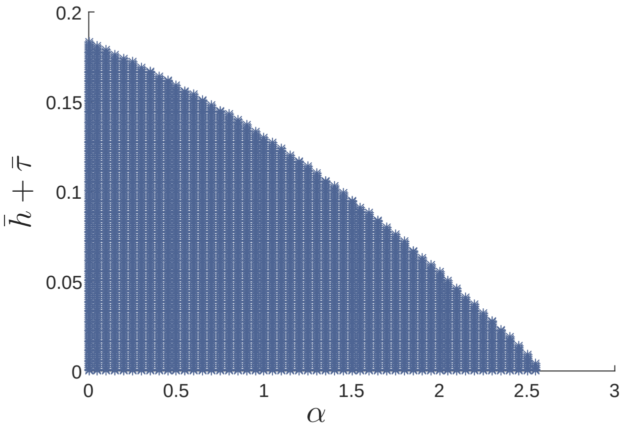


Fig. 1. Trade-off between α and $\bar{h}+\bar{\tau}$ for system (19), satisfying Theorem 1

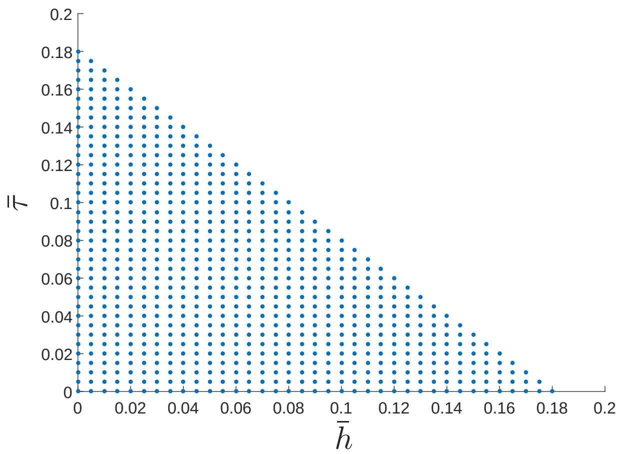


Fig. 2. Feasible values of $\bar{h}$ and $\bar{\tau}$ for system (19) with $\alpha = 0.001$, satisfying Theorem 1

hold, and are obtained by standard matrix manipulations and convexity arguments.

3.3 Numerical example

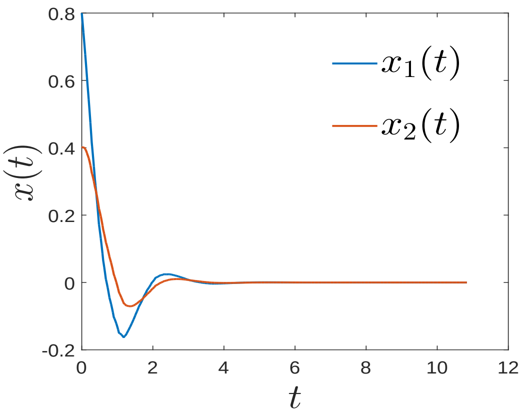
Consider the system (1)-(4) characterized by the parameters [Zhang (2001)]

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0.6 \end{bmatrix}, K = -\begin{bmatrix} 1 & 6 \end{bmatrix}. \tag{19}$$

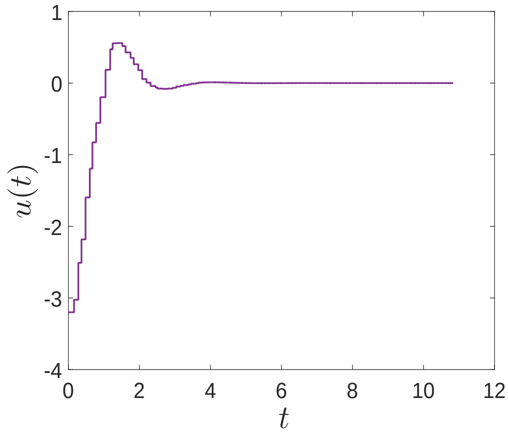
By virtue of Theorem 1, we can compute the maximum allowable values of $\bar{h}+\bar{\tau}$ with respect to α. The trade-off region thus obtained is shown in Figure 1. For example, considering $\alpha = 0.001$, we obtain the stability domain shown in Figure 2. Choosing the following parameters in accordance with the stability region,

$$\begin{aligned} \bar{h} &= 0.1, \\ \bar{\tau} &= 0.06, \end{aligned} \tag{20}$$

along with an initial condition $x_0 = [0.8, 0.4]^T$, we can see from Figure 3 that the system is indeed exponentially stable.



(a) States $x_1(t)$ and $x_2(t)$



(b) Control input $u(t)$

Fig. 3. Evolution of system (19) with parameters (20).

4. CONCLUSION

In this paper we have considered the stability analysis of aperiodic sampled-data systems subjected to time-varying delay. A novel stability criterion based on the dissipativity theory has been provided. The criterion is obtained by designing a suitable supply function that satisfies a general dissipativity condition for stability. The effectiveness of the obtained result has been corroborated through simulation results.

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