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Optimal Lower Barrier on Modified Surplus Process

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We obtain the optimal pair of initial surplus and barrier level under the minimum ruin probability constraint. We consider the lower barrier model on the modified surplus process under the reinsurance arrangement. We examine the defective distribution function of the time to ruin $T_{u,k}$ with lower barrier k and initial surplus u which is suggested by Nie et al.[2]. We aim to take this approach one step further by proposing optimal reinsurance. We calculate the optimal reinsurance criteria as the released capital, expected profit and expected utility for different times, loading factors and weights of the criteria. In decision making process, we use the Technique for Order of Preference by Similarity to Ideal Solution method (TOPSIS) with Mahalanobis distance. We analyse the robustness of the results with sensitivity analysis.

Keywords: Reinsurance, Ruin probability, Lower barrier model, TOPSIS, Mahalanobis distance

1. Introduction

The classical risk process is based on initial surplus, premiums and claims. It is assumed that the surplus process starts with an initial level u and continues according to two opposing cash flows: the premium income per unit of time and the aggregate claim amount up to time t, $S(t)$. The insurer's surplus (or risk) process, $\{U(t)\}_{t\geq 0}$, is defined by

$$U(t) = u + ct - S(t).$$

The aggregate claim amount up to time t, $S(t)$, is

$$S(t) = \sum_{i=1}^{N(t)} X_i,$$

where $N(t)$ denotes the number of claims that occur in the fixed time interval $[0, t]$. The individual claim amounts are modelled as independent and identically distributed ($i.i.d.$) random variables $\{X_i\}_{i=1}^{\infty}$ with distribution function $F(x) = \Pr(X_1 \leq x)$ such that $F(0) = 0$ and X_i is the amount of the i^{th} claim. We use the notation f and m_k to represent the density function and k^{th} moment of X_1, respectively, and it is assumed that $c > E[N]\ m_1$.

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The probability of ruin in continuous time for infinite time (ultimate ruin probability) is denoted by $\psi(u)$ where

$$\psi(u) = Pr(U(t) < 0 \quad \text{for} \quad \text{some} \quad t > 0).$$

$\psi(u)$ is the probability that the insurer's surplus falls below zero at some time in the future, that is claims outgo exceed the initial surplus plus premium income. In order to verify the validity of the classical risk model, some assumptions are made. Hence, it is usually assumed that the premium income is greater than the expected aggregate claim amount per unit of time ($c > \lambda\mu_1$). Otherwise, $\psi(u) = 1$ for all $u > 0$.

Moreover, the probability of ruin in continuous and the finite time is defined as

$$\psi(u,t) = \Pr\big(U(s) < 0 \qquad \text{for some s}, \qquad 0 < s \leq t\big),$$

where $\psi(u,t)$ is the probability that the insurer's surplus falls below zero in the finite time interval $(0,t]$.

The classical risk models assume that the premium income has a constant rate and the aggregate claim amount has a compound Poisson process. Although the literature on the ruin probability on the classical risk model shows a variety of approaches, in the last few years there has been a growing interest on minimizing the ruin probability or maximizing the survival probability of the insurance company.

Most of the literature on minimizing the ruin probability is based on reinsurance arrangements. In traditional reinsurance arrangements; excess of loss reinsurance and proportional reinsurance, the reinsurance premium is calculated according to the reinsurance level and the reinsurance loading factor.

In a reinsurance arrangement, the optimal level and the type of reinsurance can be determined under a constraint such as ruin probability or risk measure.

We mainly point out recent studies which discuss the optimal reinsurance by using reinsurance arrangement under the ruin probability constraint [1], [2],[3], [4],[5],[6], [7] and [8].

The literature review shows that most of the studies focus on optimal reinsurance based only on a single constraint and the optimal reinsurance strategy change under different constraints. Therefore, Karageyik and Dickson [9] develop optimal reinsurance criteria which consider three quantities that affect the optimal reinsurance level: released capital, expected profit, and expected utility of resulting wealth. They aim to find the pair of initial surplus and reinsurance level which minimises the finite time ruin probability and maximises the output of these three quantities as well by using the classical risk model assumptions. Karageyik and Dickson [9] consider the classical risk model under a reinsurance arrangement either excess of loss or proportional and use the translated gamma process to approximate the compound Poisson process.

Nie et al. [8] propose a different kind of reinsurance arrangement in which the reinsurer's payments are bounded above by a fixed level which is related to lower barrier model. According to Nie et al. [8], whenever the surplus falls between 0 and this fixed level, the reinsurance company makes an additional payment in such a way that the surplus process returns to the fixed level. The reinsurance premium is also calculated according to these capital injections. The optimal initial surplus and the fixed reinsurance level are calculated so that the ultimate ruin probability is minimised.

In this study, we aim to take Nie et al.[8]'s approach one step further by proposing optimal reinsurance in the lower barrier model which is directly related to the insurer's surplus process. This modified surplus process does not depend on the traditional reinsurance arrangements. Moreover, this surplus process is neither based on the individual nor the aggregate claim process, but based only on the insurer's surplus. The lower barrier model is interpreted as when the insurer's surplus falls below a certain barrier level, the reinsurance company makes a capital injection to close the gap and raises the surplus back to the required level. The reinsurance premium is calculated according to this capital injection amount. Moreover, the distribution function of the time to ruin is used to obtain the finite time ruin probability on lower barrier model.

We focus on the lower barrier model where we reformulate for the finite time ruin probability under the reinsurance arrangement. We examine the calculation of the defective distribution function of $T_{u,k}$, the time to ruin for the process with the lower barrier, with initial surplus u and lower barrier k. We have also aimed to demonstrate how to take reinsurance premium into account for the lower barrier model.

We obtain the optimal initial surplus and corresponding barrier level according to released capital, expected profit and expected utility under the fixed ruin probability constraint. In reinsurance decision process, we use TOPSIS method with Mahalanobis distance. We have obtained and compared the optimal initial surplus and barrier level for the combinations of different loading factors, time horizons and different scenarios. To our knowledge, this is the first study dealing with determining the optimal barrier level under the ruin probability constraint.

This paper is organised as follows. Section 2 introduces the assumptions of the lower barrier model on modified surplus process. We consider the lower barrier model for infinite and finite time and give formulas for the calculation of the ruin probability.

In Section 3, we show the calculation of reinsurance premium on lower barrier model. In Section 4, we consider the optimal reinsurance criteria under the assumption of the ruin probability to be fixed at a minimum level. We use the optimal reinsurance criteria: released capital, expected profit and expected utility from the insurers' point of view. In Section 5, we give some information about TOPSIS method with Mahalanobis distance. In Section 6, we analyse the optimal barrier levels. We give some numerical examples for the compound Poisson process with individual claim amounts which have exponential distribution. We calculate the optimal initial surplus and barrier level under TOPSIS method and compare the optimal pair of initial surplus and barrier level for different time horizons and different loading factors. In Section 7, we present the sensitivity analysis of the calculation on the optimal barrier level. In Section 8, we discuss the results and conclude the paper.

2. Lower Barrier Model on Modified surplus process

In the case of lower barrier model, a barrier k is determined where $0 \leq k \leq u$. In this modified surplus process, each time the surplus drops below this barrier level k but not below zero, an amount in the manner restore the insurer's surplus back to level k is paid by the reinsurer. When a claim which causes the insurer's surplus fall from a level above k to a level below zero occurs, the reinsurance company cannot make a payment and the ruin occurs. This reinsurance arrangement differs radically from the traditional type of reinsurance arrangements discussed in many actuarial papers. The excess of loss reinsurance and proportional reinsurance arrangements depend on the individual or aggregate claims. However this reinsurance arrangement does not depend on neither the individual claims nor the aggregate claims. It is related to the surplus process falls below the lower barrier model where $0 \leq k \leq u$ [10]. A realization of the modified surplus process with lower barrier k is given in Figure 1 [10].

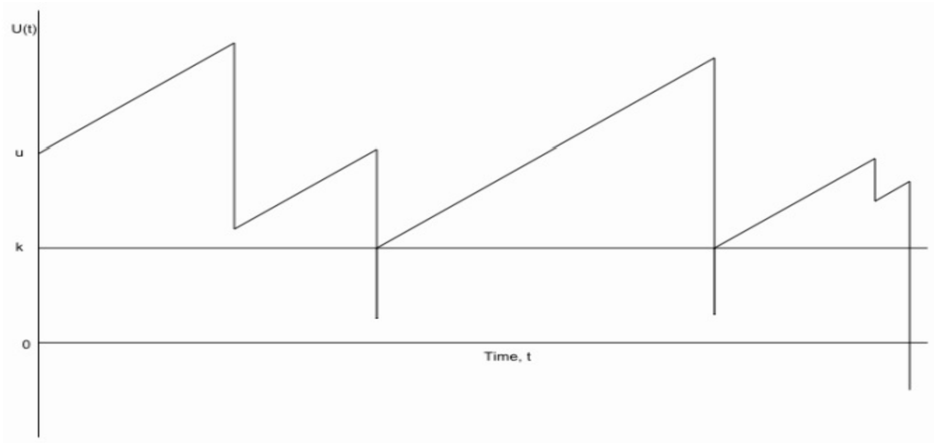

Figure 1.: A realization of the modified surplus process with lower barrier k

Let the initial surplus is u, and the time of ruin is T_u which is denoted by

$$T_u = \inf\{t : U(t) < 0\},$$

where $U(t)$ is the surplus process at time t. For all $t > 0,\ \ T_u = \infty$ if $U(t) \geq 0$ and hence $\psi(u) = \Pr(T_u < \infty)$ [11]. When ruin occurs, the insurer's deficit at ruin or severity of ruin is, at most y, denoted as

$$G(u, y) = \Pr\left(T_u < \infty \quad \text{and} \quad U(T_u) \geq -y\right),$$

and it yields that

$$\lim_{y\to\infty} G(u, y) = \psi(u).$$

Therefore, the defective distribution with (defective) density is

$$g(u, y) = \frac{\partial}{\partial y} G(u, y).$$

Dickson [11] states $G(u, y)$ under the exponential individual claims as

$$\begin{aligned} G(u, y) &= \frac{\lambda}{\alpha c} e^{-(\alpha-\lambda c)u}(1 - e^{-\alpha y}), \\ &= \psi(u)(1 - e^{-\alpha y}). \end{aligned} \tag{1}$$

Note that $\frac{\lambda}{\alpha c}\, e^{-(\alpha-\lambda c)u}$ represents the exact ultimate ruin probability $\psi(u)$. The finding is quite surprising and suggests that the distribution of the severity of ruin is identical to the distribution of the individual claim amount. However, this approach is only convenient for the exponential case because of the memoryless property of the exponential distribution. [11].

2.1. *Lower Barrier Model for Infinite Time*

Nie et al. [8] suggest a formula for the calculation of the ultimate ruin probability under the lower barrier model. The ultimate ruin probability for the modified surplus process with the lower barrier at k is denoted by $\psi_k(u)$. This probability must be treated according to two cases $u = k$ and $u > k$. Nie et al. [8] obtains a formula for $\psi_k(u)$ as when $u = k$

$$\psi_k(k) = \int_0^k g(0,y)\psi_k(k)dy + \int_k^\infty g(0,y)dy,$$

where $g(u,y)$ and $G(u,y)$ are the density and distribution functions of the deficit at ruin in a classical risk model, respectively. This formula depends on the condition on the amount of the first drop below lower level k. Hence, it can be written as

$$\psi_k(k) = \frac{\psi(0) - G(0,k)}{1 - G(0,k)}.$$

The survival probability of the classical risk process and modified risk process are defined as $\bar{\psi}(u) = 1 - \psi(u)$ and $\bar{\psi}_k(u) = 1 - \psi_k(u)$, respectively. When $u > k \geq 0$, the ruin probability is obtained by conditioning on the amount of the first drop below level k as

$$\begin{aligned}\psi_k(u) &= \psi(u-k) - G(u-k,k)\ (1 - \psi_k(k)) \\ &= \psi(u-k) - G(u-k,k)\ \frac{1-\psi(0)}{1-G(0,k)}.\end{aligned} \quad (2)$$

2.2. *Lower Barrier Model for Finite Time*

Nie et al. [12] extend the previous analysis according to time horizons. They suggest an explicit formula for the finite time ruin probability on lower barrier model. Let $W_{u,k}(t)$ be the defective distribution function of the time to ruin $T_{u,k}$ with lower barrier k and initial surplus u and defined as

$$W_{u,k}(t) = \Pr[T_{u,k} \leq t].$$

Nie et al. [12] start with the case when u is equal to k. Under the classical risk model, let $G(u,.)$ be the defective distribution function of the deficit at ruin. The probability that ruin occurs on the n^{th} occasion, $n = 1, 2, 3, \cdots$ that the surplus falls below k is

$$G(0,k)^{n-1}(\psi(0) - G(0,k)). \quad (3)$$

When ruin occurs on the n^{th} occasion that the surplus drops below k, the ruin time is defined as the sum of n random variables [10]. The distribution function of the first $n-1$ independent and identically random variables, D, is defined as

$$D(t) = \Pr[\tau \leq t] = \Pr[T_0 \leq t | T_0 < \infty \text{ and } Y \leq k] = \frac{W(0,k,t)}{G(0,k)},$$

where Y is the deficit at ruin, $W(u, y, t)$ is the joint distribution function of the deficit at ruin and the time to ruin given initial surplus u. Further, τ_L denotes the waiting time between the $(n-1)^{th}$ and n^{th} drop, and has a distribution function as

$$\begin{aligned} D_L(t) &= \Pr[\tau_L \le t], \\ &= \Pr[T_0 \le T | T_0 < \infty \ \text{ and } \ Y > k] = \frac{\psi(0,t) - W(0,k,t)}{\psi(0) - G(0,k)}. \end{aligned}$$

Thus, the defective distribution function of $T_{k,k}$ i.e. finite time ruin probability for k initial surplus and k barrier level is obtained as

$$W_{k,k}(t) = \Pr(T_{k,k} \le t) = \sum_{n=1}^{\infty} G(0,k)^{n-1} \big(\psi(0) - G(0,k)\big) D^{(n-1)*} * D_L(t), \qquad (4)$$

where $D^{(n-1)*} * D_L(t)$ shows the convolution of the $(n-1)$ fold convolution of the function D with the function D_L.

According to Nie et al. [12], when $u > k$, there are two possible situations where ruin happens: the first situation is that ruin happens at the first time when the surplus process drops below level k whereas the second situation is that ruin does not happen at the first time the surplus process drops below level k, and the surplus restarts from k and ruin subsequently occurs.

In the first situation, the probability of the ruin happens at the first time the surplus process drops below level k is $\psi(u-k) - G(u-k, k)$. The waiting time for such an event is denoted by τ_1 and its distribution function, D_1, is

$$\begin{aligned} D_1(t) &= \Pr(\tau_1 \le t), \\ &= \Pr[T_{u-k} \le t | T_{u-k} < \infty \ \text{ and } \ Y_{u-k} > k] \\ &= \frac{\psi(u-k,t) - W(u-k,k,t)}{\psi(u-k) - G(u-k,k)}. \end{aligned}$$

In the second situation, the probability of the ruin does not occur at the first time which the surplus drops below k is defined as $G(u-k, k)$. The waiting time until the surplus first drops below k without ruin occurring is presented by τ_2. The distribution function of τ_2, $D_2(t)$, is stated as

$$\begin{aligned} D_2(t) &= \Pr(\tau_2 \le t), \\ &= \Pr[T_{u-k} \le t | T_{u-k} < \infty \ \text{ and } \ Y_{u-k} \le k], \\ &= \frac{W(u-k,k,t)}{G(u-k,k)}. \end{aligned}$$

Thus, the defective distribution function of $T_{u,k}$ is shown as

$$\begin{aligned} W_{u,k}(t) &= \Pr[T_{u,k} \le t], \\ &= \big[\psi(u-k) - G(u-k,k)\big] D_{u-k,1} + G(u-k,k) D_2 * W_{k,k}(t). \end{aligned} \qquad (5)$$

where $D_2 * W_{k,k}(t)$ denotes the convolution of the function D_2 with the function $W_{k,k}$ which is obtained from Eq.4.

2.3. *Exponential Case on Lower Barrier Model*

Nie et al. [12] show how the defective distribution function of $T_{u,k}$ is calculated under the assumption of exponential claim size. The exponential distribution with parameter $\alpha > 0$ has a density function

$$f(x) = \alpha e^{-\alpha x}.$$

As a result of memorylessness property of the exponential distribution, under the classical risk model, $W(u, y, t)$ can be obtained as

$$W(u, y, t) = \psi(u, t)(1 - e^{-\alpha y})$$

which gives

$$W(0, k, t) = \psi(0, t)(1 - e^{-\alpha k})$$

Consider the first situation when $u = k$. $D(t)$ and $D_L(t)$ are respectively given as

$$D(t) = D_L(t) = \frac{\psi(0, t)}{\psi(0)}.$$

Hence, Eq.4 gives

$$W_{k,k}(t) = \sum_{n=1}^{\infty}(1 - e^{-\alpha k})^{n-1} e^{-\alpha k} \psi^{n*}(0, t), \tag{6}$$

where ψ^{n*} represents the n-fold convolution of the function ψ. Next, consider the situation of $u > k$. It is straightforward to find $D_1(t)$ and $D_2(t)$ as

$$D_1(t) = D_2(t) = \frac{\psi(u - k, t)}{\psi(u - k)}.$$

The probability of ruin at time t with a deficit of at most y at ruin, in other words the defective distribution function of $T_{u,k}$, the time to ruin, for the process with the lower barrier, with initial surplus u and lower barrier k is

$$\begin{aligned} W_{u,k}(t) &= [\psi(u - k) - G(u - k, k)]D_{u-k,1} + G(u - k, k)D_{u-k,2} * W_{k,k}(t) \\ &= \psi(u - k, t)e^{-\alpha k} + \psi(u - k, t)(1 - e^{-\alpha k}) * W_{k,k}(t) \end{aligned} \tag{7}$$

where $W_{k,k}(t)$ is calculated from Eq.(6).

It is obvious that $W_{u,k}(t)$ can be derived from the fundamental functions of $\psi(u)$, $\psi(u, t)$ and $\psi(0, t)$.

2.3.1. *Calculation of $\psi(u)$*

In calculation of $\psi(u)$, the exact ruin probability is used as

$$\psi(u) = \psi(0)\exp\{-Ru\}, \tag{8}$$

where $R = \alpha - \frac{\lambda}{c}$ with $\psi(0) = (1/(1+\theta))$.

2.3.2. *Calculation of $\psi(u,t)$*

We aim to calculate the exact finite time ruin probability with initial surplus u at time t. Asmussen [13] suggests an explicit formula which is valid just under the assumption that claim amounts have an exponential distribution. Asmussen [14] presents the finite time ruin probability formula when the individual claim amounts are distributed exponentially with $\beta = 1$ and the premium rate per unit time is equal to 1 $(c = 1)$. Hence, the ruin probability for finite time is calculated as

$$\psi(u,T) = \lambda\exp\{-(1-\lambda)u\} - \frac{1}{\pi}\int_0^{\pi}\frac{f_1(x)f_2(x)}{f_3(x)}dx, \tag{9}$$

where

$$f_1(x) = \lambda\exp\left\{2\sqrt{\lambda T}\cos(x) - (1+\lambda)T + u(\sqrt{\lambda}\cos(x) - 1)\right\},$$

$$f_2(x) = \cos\left(u\sqrt{\lambda}\sin(x)\right) - \cos\left(u\sqrt{\lambda}\sin(x) + 2x\right),$$

and

$$f_3(x) = 1 + \lambda - 2\sqrt{\lambda}\cos(x).$$

An important implication of this method is to remove the restriction on the parameter of individual claims distribution and premium rate. When $\beta \neq 1$, the following equation is applied [13].

$$\psi_{\lambda,\beta}(u,T) = \psi_{\frac{\lambda}{\beta},1}(\beta u, \beta T), \tag{10}$$

and the following equation is valid when $c \neq 1$, [15]

$$\psi_{\lambda,c}(u,T) = \psi_{\frac{\lambda}{c},1}(u, cT). \tag{11}$$

2.3.3. *Calculation of $\psi(0,t)$*

When initial surplus equals to zero, $u = 0$, most of the approximations are not valid. The formula for the exact survival probability is

$$1 - \psi(0,t) = \int_0^{ct} F(x,t)dx \tag{12}$$

where $F(x,t) = Pr(S(t) < x)$, and $S(t)$ has a compound Poisson distribution. This formula is often called Seal's formula but originating from Prabhu [16]. See Asmussen [13] for the proof of this theorem.

2.3.4. Calculation of $W_{k,k}(t)$

According to Eq.6, it yields that

$$W_{k,k}(t) = \sum_{n=1}^{\infty}(1-e^{-\alpha k})^{n-1}e^{-\alpha k}\psi^{n*}(0,t). \tag{13}$$

$W_{k,k}(t)$ can be calculated by using the Beekman's convolution formula [17] which is given below.

$$\psi(u) = 1 - \sum_{n=0}^{\infty} H^{*n}(u)p(1-p)^n. \tag{14}$$

This equation is obtained by using Panjer's recursive formula in many practical situations. This formula requires discretization of $H(\cdot)$. In such circumstances, Beekman's formula yields approximated ruin probabilities.

Obviously, there is a relationship between $\psi(u)$ and $W_{k,k}(t)$. Therefore, we aim to find a relationship between $\psi^{n*}(0,t)$ and $H^{*n}(u)$. Thus, we assume that individual claims have an exponential distribution with $\alpha = 1$, and number of claims have a Poisson distribution with $\lambda = 500$. We adjust the starting point of the lower limit of summation and n^{th} power of the expression in calculation of $W_{k,k}(t)$. The algorithm is designed by using **R** and Wolfram Mathematica programming languages.

3. Calculation of Reinsurance Premium on Lower Barrier Model

Nie et al. [8] define the reinsurance premium as the expected total claim amount for the reinsurer up to the time of ruin. Nie et al. [8] assume that the aggregate amount needed to restore the modified surplus process to k up to time t, given initial surplus u, be $S_{u,k}(t)$. Thus, $S_{u,k} = S_{u,k}(T_{u,k})$ is the expected total claim amount for the reinsurer up to the time of ruin. It is started with the case $u = k$, $E(S_{u,k})$ is calculated by using the idea of Pafumi [18] as given below.

When $u = k$, the expected total claim amounts by the reinsurer is obtained as

$$\begin{aligned} E(S_{k,k}) &= \int_0^k (y + E(S_{k,k}))g(0,y)dy, \\ &= \frac{\int_0^k yg(0,y)dy}{1-G(0,k)}. \end{aligned} \tag{15}$$

When $u > k$, $E(S_{u,k})$ is calculated as

$$E(S_{u,k}) = \int_0^k (y + E(S_{k,k}))g(0, y)dy,$$
$$= \int_0^k y\ g(u - k, y)dy + E(S_{k,k})G(u - k, k). \quad (16)$$

3.1. *Calculation of Insurer's Net Premium Income Per Unit Time*

In this study, we assume that the insurer's net premium income per unit time is calculated by the difference between the premium that the insurer receives to cover the risk and reinsurance premium to cover its liability. The expected value premium principle is employed in the calculation of the net premium income.

Insurance loading factor is denoted by θ, and reinsurance loading factor is denoted by ξ. Thus, there is a decrease in the insurance premium due to the reinsurance arrangement.The net of reinsurance premium income after paying the reinsurance premium is calculated as

$$\text{Net of Reinsurance Premium} = c_{net} = (1 + \theta)\ E[X]\ E[N] - (1 + \xi)\ E(S_{u,k})$$

where $E[S_{u,k}]$ denotes the reinsurance premium, by initial surplus u and barrier level k. The net of reinsurance premium is denoted by c_{net}.

As a consequence of the reinsurance arrangement, net of reinsurance premium income equals to multiplying the net liability of insurance company by a net of loading factor, θ_{net}. Net of loading factor θ_{net} is identified the insurance's loading factor after the reinsurance arrangement.

$$c_{net} = (1 + \theta_{net})\big(E[S] - E(S_{u,k})\big),$$

$$\frac{c_{net}}{\big(E[S] - E(S_{u,k})\big)} = (1 + \theta_{net}),$$

$$\theta_{net} = \frac{c_{net}}{\big(E[S] - E(S_{u,k})\big)} - 1. \quad (17)$$

It should be noted that, in the rest of the calculation, θ_{net} is used instead of θ as the insurance loading factor. Hence, the effect of reinsurance can be incorporated into the finite time ruin probability on lower barrier model.

4. Optimal Reinsurance Criteria on Lower Barrier Model

Karageyik and Dickson [9] examine the optimal reinsurance criteria under the translated gamma process approximation on the classical risk process. In this study, we calculate the values of these three criteria on lower barrier model on modified surplus process.

As is the case with classical risk model under the translated gamma process approximation, differences on pairs of insurance and reinsurance loading factors or time horizons cause significant changes in values of these criteria. Different from the traditional reinsurance arrangements, on modified surplus process, the ruin probability is directly related to the surplus process instead of the insurer's individual or aggregate claims.

4.1. *Released Capital*

Released capital is defined as the difference between the largest and the required initial surplus of each alternative which makes the ruin probability equal to certain level. We assume that the insurer's ruin probability on lower barrier model is fixed at a minimum level 0.01.

In the case of no-reinsurance, the required initial surplus which makes the insurer's ruin probability equal to 0.01 is defined as *the Largest Initial Surplus* and denoted by u_L. In reinsurance case, it is not possible to get a ruin probability such as 0.01 if the initial surplus is below a certain level. The insurer's surplus on modified surplus process must start with this adequate initial surplus called as *Smallest Initial Surplus* and denoted by u_S [9].

For each barrier level, there is a unique initial surplus which satisfies 0.01 ruin probability. The difference between the largest initial surplus and this value is called as savings amount or gains by choosing this reinsurance arrangement. Figure 2 shows a set of 407 pairs of initial surplus and lower barrier level based on exponentially distributed individual claim amounts on lower barrier model.

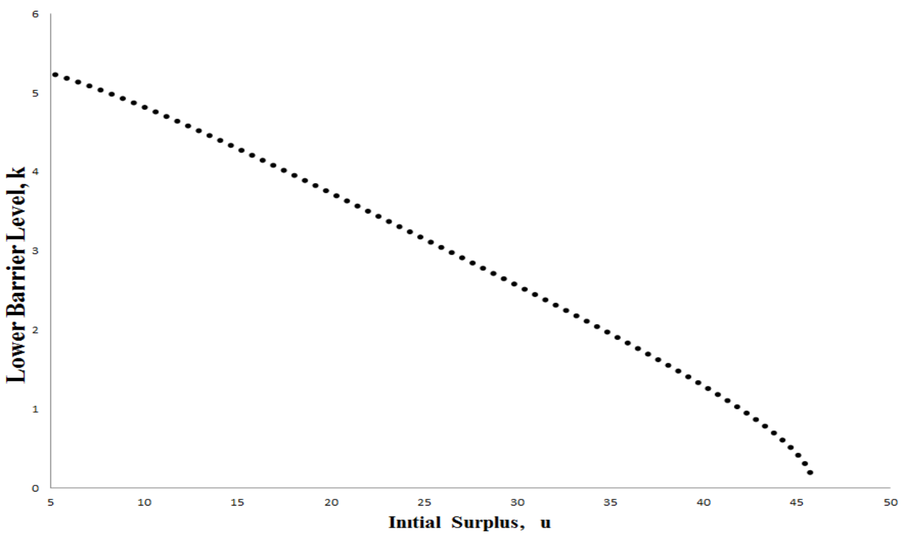

Figure 2.: The relation between the initial surplus and lower barrier level

$\psi(u, k, t)$ denotes the finite time ruin probability with initial surplus u and lower barrier k at time t. The pair of initial surplus and lower barrier level is represented as follows.

$$\psi(u_1, k_1, t) = 0.01,$$
$$\vdots \qquad \vdots$$
$$\psi(u_n, k_n, t) = 0.01.$$

where k_i denotes the lower barrier level and u_i denotes the initial surplus corresponding to each lower barrier level for $i = 1, 2, \cdots, n$.

Due to the relationship between the initial surplus and lower barrier level, it can be observed that when the initial surplus increases, the corresponding barrier level decreases. A set of initial surplus starts with the smallest initial surplus and ends with the largest initial surplus. At the beginning of each set of alternatives, the barrier level is very close to the smallest initial surplus. However it should be noted that the constraint $u > k$ must be hold. Released capital is maximum at the smallest initial surplus level whereas it is minimum in fact close to zero, at the largest initial surplus. The same pattern is observed under the translated gamma process on the classical risk model.

4.2. *Expected Profit*

Insurance profit, $P(t)$, is calculated as the difference between the net insurance premium income $\Pi_{net}(t)$ and the insurer's expected total claim at time t, $E[S_I(t)]$, and defined as

$$P(t) = \Pi_{net}(t) - E[S_I(t)].$$

Under the lower barrier model on modified surplus process, according to the expected premium principle with insurance loading factor θ and reinsurance loading factor ξ, insurer's expected profit up to time t is defined as

$$P(t) = \big((1+\theta)\ E[S] - (1+\xi)\ E[S_{u,k}]\big) - E[S_I], \tag{18}$$

where $E[S]$ denotes the aggregate claim amount and $E[S_{u,k}]$ is the expected total claim amount paid by the reinsurer up to the time of ruin. The net of expected claim amount paid by the insurer is represented by $E[S_I]$. Hence, it is assumed that

$$E[S] = E[S_{u,k}] + E[S_I].$$

Contrary to the classical risk model, the expected profit depends not only on the lower barrier level but also its corresponding initial surplus.

A decrease in the lower barrier level causes a decrease in the transferred risk to the reinsurance company and in the reinsurance premium . Therefore, the net insurance premium income increases and eventually the expected profit of the insurance company increases.

The results show that under the fixed minimum ruin probability constraint on lower barrier model, there is a positive correlation between the lower barrier level and the expected profit of the insurance company up to time t.

4.3. *Expected Utility*

The exponential utility function, $\boldsymbol{u}$ is defined as

$$\boldsymbol{u}(x) = 1 - \exp(-Bx), \qquad B > 0$$

where B is the parameter of the utility function. The insurer's wealth at the end of t, $U(t)$, is defined from the insurer's point of view as

$$U(t) = u + \Pi_{net} - S_I(t).$$

Therefore, the net of insurance liability, $S_I(t)$, on lower barrier model is calculated as the difference between the expected total aggregate claim and the expected total claim amounts for the reinsurer up to time of ruin. To avoid the confusion of the notations, initial surplus is represented by u_0 instead of u. The expected utility of insurance's wealth at time t is denoted as

$$\begin{aligned} E[\boldsymbol{u}(U(t))] &= E\big[\boldsymbol{u}\ (u_0 + \Pi_{net} - S_I(t))\big]\ , \\ &= E\big[1 - \exp\big(- B(u_0 + \Pi_{net} - S_I(t))\big)\big], \\ &= 1 - \big[E[\exp(-B\ u_0)]\ E\big[\exp(-B\ \Pi_{net})\big]\ E\big[\exp\big(B\ S_I(t)\big)\big]\big]. \end{aligned} \tag{19}$$

In this way, we obtain the expected utility of the insurer's wealth on lower barrier model. As it is clear from Eq.19, the expected utility criterion depends on both the lower barrier and the corresponding initial surplus.

The calculation shows that the expected utility of the insurer's wealth increases exponentially as the lower barrier level decreases. The slope gradient of expected utility mainly depends on the risk averse coefficient, i.e. parameter of the exponential utility function. In this study, we assume that the parameter of the exponential utility function is equal to 0.02 as in Karageyik and Dickson [9]. In sensitivity analysis, we examine the effect of the optimal levels according to the different values of the exponential utility parameter.

In order to obtain the optimal initial surplus and lower barrier level, we use the TOPSIS method with Mahalanobis distance.

5. The Technique for Order of Preference by Similarity to ideal Solution (TOPSIS)

TOPSIS is the acronym for "Technique for Order of Preference by Similarity to ideal Solution"and it is suggested by Hwang and Yoon [19] to determine the best alternative based on the concepts of a compromised solution. The compromised solution can be regarded as choosing the solution with the shortest Euclidean distance from the ideal solution and the farthest Euclidean distance from the negative ideal solution. The ideal solution is defined as the one which maximises the benefit criteria and minimises the cost criteria.

TOPSIS method is the most preferred decision technique, and according to most studies this method is also stated as the best alternative method among the multi attribute decision making (MADM) methods. The advantages of TOPSIS method are discussed in Kim et al. [20], Parkan and Wu [21], Zanakis et al. [22], Yeh [23] and Shih et al.[24].

The traditional TOPSIS method is based on Euclidean distance measure and it is assumed that there is no relationship between the attributes. This approach suffers from information overlap and either overestimates or underestimates the attributes which take slack information [25].

When attributes are dependent and influence each other, application on TOPSIS based on Euclidean distances can lead to inaccurate estimation of relative significances of the alternatives and cause the improper ranking results [26]. For this reason, another distance measure technique, Mahalanobis distance, is suggested instead of Euclidean distance in the TOPSIS method.

The Mahalanobis distance is also called quadratic distance and introduced by Mahalanobis [27]. For a multivariate vector $\mathbf{x} = (x_1, x_2, x_3, \ldots, x_N)^T$ from a group of observations with mean $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3, \ldots, \mu_N)^T$ and the covariance matrix Σ, the Mahalanobis distance is defined as follows

$$D_M(x) = \sqrt{(\mathbf{x} - \mu)^T \Sigma^{-1} (\mathbf{x} - \mu)}.$$

The Mahalanobis distance standardizes via the factor of inverse of the covariance matrix Σ^{-1}. This distance measure depends on the covariance between variables and it gives information about the similarities of an unknown sample to a known one. Mahalanobis distance can be calculated by using the correlation matrix of a variable in the case of different patterns are analyzed. Mahalanobis distance depends on the correlation of the variables so that Mahalanobis distance measure depends on the scale-invariant case [25].

When the attributes are independent, the weighted Mahalanobis distance and the weighted Euclidean distance will be equivalent [25].

References to the TOPSIS method with Mahalanobis distance can be found in Wang and Wang [25], Garca and Ibarra [28], Chang et al. [29], and Lahby et al. [30].

We suppose that there are m alternatives $A_1, A_2, \cdots, A_m$ and n decision attributes (criteria) $C_1, C_2, \cdots, C_n$. Let x_{ij} denote the attribute value of A_i on C_j for $i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, n$. The decision matrix consists of the element of X_{ij} and can be standardized as follow

$$r_{ij} = \frac{x_{ij}}{\sqrt{\sum\limits_{i=1}^{m} x_{ij}}}. \tag{20}$$

Let, the ideal solution and negative ideal solution (anti-ideal solution) be $\boldsymbol{S}^+$ and $\boldsymbol{S}^-$, respectively as in the case of TOPSIS and A_i denote the i^{th} alternative. Hence, the Mahalanobis distance from A_i to the ideal solution point is calculated as

$$d(r_i, S^+) = \sqrt{\{S_j^+ - r_{ij}\}^T \Omega^T \Sigma^{-1} \Omega \{S_j^+ - r_{ij}\}} \qquad i = 1, 2, \cdots, m\ .$$

Similarly, the Mahalanobis distance from A_i to the negative ideal solution point is calculated as

$$d(r_i, S^-) = \sqrt{\{S_j^- - r_{ij}\}^T \Omega^T \Sigma^{-1} \Omega \{S_j^- - r_{ij}\}} \qquad i = 1, 2, \cdots, m\ .$$

where ω is the weight vector such as $\omega = (\omega_1, \omega_2, \cdots, \omega_n)$ and Ω is defined as $\Omega = diag(\sqrt{w_1}, \sqrt{w_2}, \cdots, \sqrt{w_n})$.

The closeness of each alternative is given as

$$c_i = \frac{d(r_i, S^-)}{d(r_i, S^-) + d(r_i, S^+)} \qquad i = 1, 2, \cdots, m\ . \tag{21}$$

The results for each alternative are sorted according to the value of c_i. A higher c_i suggests that A_i is a better solution.

In the TOPSIS method, the covariance matrix can be calculated directly from the original data or the normalized data, see [26], [25] and [31].Therefore, we use both the covariance matrix directly from the original data or the vector-normalized data.

6. Analysis of Optimal Barrier Level

In this section, we illustrate the optimal barrier on lower barrier model with some numerical examples.The main assumptions of this approach are given as follows. The individual claim amount has an exponential distribution with probability density function $p(x) = e^{-x}$. For the number of claims per unit of time, we assume a Poisson distribution with parameter $\lambda = 500$. Four different loading combinations (θ, ξ), namely (0.1, 0.15), (0.1, 0.2), (0.1, 0.3) and (0.2, 0.3) are employed. For time horizons, $t = 0.1$, $t = 0.5$, $t = 1$, $t = 5$, $t = 10$ and $t = 20$ are used. Four different sets of weights; equal weights and three set of weights under which criterion is more important than the others are assumed. The sets of weights are given in Table 1.

Table 1.: The weights in the TOPSIS method

Set of Weights	Released Capital	Expected Profit	Expected Utility
Scenario 1	$0.\overline{3}$	$0.\overline{3}$	$0.\overline{3}$
Scenario 2	0.5	0.25	0.25
Scenario 3	0.25	0.5	0.25
Scenario 4	0.25	0.25	0.5

The algorithm for the exponential claims on lower barrier model can be summarised as follows.

Step 1: *Calculation of the finite time ruin probability on lower barrier model.*
Let $W_{u,k}(t)$ be the defective distribution function of the time to ruin $T_{u,k}$ with lower barrier k and initial surplus u and it is defined as

$$W_{u,k}(t) = e^{-\alpha k}\psi(u-k,t) + \sum_{n=1}^{\infty}(1-e^{-\alpha k})^n e^{-\alpha k}\psi^{n*}(0,t)\psi(u-k,t)$$

First, $\psi(u,t)$ is calculated by using the exact ruin probability formula which is given by Eq. 9 [13]. Second, $\psi(0,t)$ is calculated by using Prabhu's formula [16] which is given in Eq.12. Finally, $\psi^{n*}(0,t)$ is calculated by using Beekman's convolution formula Eq.14 [17].

Step 2: *Calculation of the largest initial surplus under the finite time ruin probability on lower barrier model which is fixed at 0.01 as a minimum level*
Largest initial surplus is calculated by using Eq. (7). The largest initial surplus which makes $W_{u,k}(t)$ formula equal to 0.01 without a lower barrier level is calculated. Hence, the largest initial surplus depends on θ and t. Assuming individual claims have an exponential distribution, the largest initial surpluses according to two different insurance loading factors $\theta = 0.1$ and $\theta = 0.2$ for various values of t are given in Table 2.
Step 3: *Calculation of the smallest initial surplus under the ruin probability which is fixed at 0.01 as a minimum level.*
The smallest initial surplus which makes the ruin probability equal to 0.01 in Eq.7 is calculated by using the one dimensional optimization technique. Then, the corresponding

Table 2.: Largest initial surplus, u_L, for the exponential claims

t	$\theta = 0.1$	$\theta = 0.2$
1	45.826	26.515
5	49.599	26.537
10	49.608	26.537
20	49.608	26.537
50	49.608	26.537
100	49.608	26.537
500	49.608	26.537
1000	49.608	26.537

lower barrier level k to the smallest initial surplus is calculated.

In calculation of the smallest initial surplus u_S, and the lower barrier level k, four different loading combinations $\theta = 0.1$ & $\xi = 0.15$, $\theta = 0.1$ & $\xi = 0.2$, $\theta = 0.1$ & $\xi = 0.3$ and $\theta = 0.2$ & $\xi = 0.3$ for various values of t are used. These loading factors are the same as in Dickson and Waters [32]. For the exponential claim amounts, the smallest initial surplus for various values of t and loading factors are given in Table 3. These smallest initial surplus are used as a starting point of the alternative set.

Table 3.: The smallest initial surplus, u_S, for the exponential claims on lower barrier model

t	$\theta = 0.1$ & $\xi = 0.15$	$\theta = 0.1$ & $\xi = 0.2$	$\theta = 0.1$ & $\xi = 0.3$	$\theta = 0.2$ & $\xi = 0.3$
1	5.227	5.228	5.230	5.137
5	6.334	6.335	6.339	6.240
10	7.030	7.031	7.035	6.937
20	7.683	7.685	7.689	7.592
50	8.577	8.579	8.583	8.488
100	9.273	9.275	9.280	9.187
500	10.883	10.886	10.891	10.891
1000	11.580	11.583	11.589	11.508

Step 4: *Constitute the alternative set which consists of the pair of initial surplus and lower barrier level.*

We design a set of alternatives which consists of the pair of initial surplus and lower barrier level. We begin with the smallest initial surplus u_S and increase this value by 0.1 or 0.05 to the largest initial surplus u_L. We calculate the corresponding lower barrier level for each initial surplus. Hence we obtain an outcome set which represents all possible combinations of the lower barrier level and corresponding initial surplus. Each pair of this outcome set is denoted by (u_k, k). The set of u_k starts with the smallest initial surplus u_S and ends with the largest initial surplus u_L.

Step 5: *Calculation of the released capital, expected profit and expected utility according to the initial surplus and lower barrier level*

The set of (u_k, k) is used in the optimal reinsurance criteria. The released capital is calculated as $u_L - u_k$, expected profit is calculated by Eq. 18 and expected utility is calculated by Eq. 19.

In Table 4, we give the results for the three criteria when $\theta = 0.1$ and $\xi = 0.3$ on the lower barrier model. We extend this calculation for various time horizons and different loading factors. Finally, we have a large set of outcome under different assumptions. We can see that each criterion has a different pattern. When the initial surplus increases, its corresponding lower barrier level decreases in order to get a fixed ruin probability such as 0.01. When the initial surplus increases and the corresponding barrier level decreases, the released capital decreases. However, both of the expected profit and expected utility increase with different slopes. The reason is the difference between the variances of the expected profit and the expected utility.

Figure 3 shows the graphs of the three criteria when $\theta = 0.1$ & $\xi = 0.3$ for the exponential claims on lower barrier model. The x axis of this graph shows the number of alternative pair (u_k, k) while the results of three criteria appear on the y axis. We prefer to use a plot with three y-axes and one shared x-axis because of the different scales. While the main y-axis shows the released capital, second and third y-axes show the expected profit and expected utility, respectively. It is clearly seen that the released capital declines steadily to the level zero when the initial surplus closes to u_L. The expected profit increases with a decreasing slope. The first alternative begins with the maximum released capital but minimum expected profit and minimum expected utility. Conversely, the last alternative has the maximum expected profit and expected utility but minimum released capital.

Table 4.: Optimal reinsurance criteria when $\theta = 0.1$ and $\xi = 0.3$ for the vector-normalized covariance matrix on lower barrier model

t	Number of Alternative	Initial Surplus u_k	Lower Barrier Level k	Released Capital $u_L - u_k$	Expected Profit P	Expected Utility $E[\mathbf{u}[U(t)]]$
1	1	5.230	5.230	40.596	47.247	0.650
	2	5.330	5.223	40.496	47.276	0.651
	3	5.430	5.216	40.396	47.304	0.652
	4	5.530	5.208	40.296	47.331	0.653
	5	5.630	5.201	40.196	47.359	0.653
	⋮	⋮	⋮	⋮	⋮	⋮
	405	45.630	0.239	0.196	50.000	0.852
	406	45.730	0.190	0.096	50.000	0.853
	407	45.830	0.115	0.000	50.000	0.853
t	**Number of Alternative**	**Initial Surplus** u_k	**Lower Barrier Level** k	**Released Capital** $u_L - u_k$	**Expected Profit** P	**Expected Utility** $E[\mathbf{u}[U(t)]]$
5	1	6.339	6.339	43.260	235.452	0.657
	2	6.439	6.332	43.160	235.594	0.657
	3	6.539	6.326	43.060	235.735	0.658
	4	6.639	6.320	42.960	235.874	0.659
	5	6.739	6.314	42.860	236.013	0.660
	⋮		⋮	⋮	⋮	⋮
	431	49.339	1.240	0.260	249.983	0.863
	432	49.439	1.205	0.160	249.984	0.863
	433	49.539	1.167	0.060	249.985	0.863
t	**Number of Alternative**	**Initial Surplus** u_k	**Lower Barrier Level** k	**Released Capital** $u_L - u_k$	**Expected Profit** P	**Expected Utility** $E[\mathbf{u}[U(t)]]$
10	1	7.035	7.035	42.573	470.472	0.661
	2	7.135	7.029	42.473	470.757	0.662
	3	7.235	7.023	42.373	471.040	0.663
	4	7.335	7.018	42.273	471.320	0.664
	5	7.435	7.012	42.173	471.597	0.664
	⋮	⋮	⋮	⋮	⋮	⋮
	424	49.335	2.060	0.273	499.891	0.863
	425	49.435	2.032	0.173	499.895	0.863
	426	49.535	2.002	0.073	499.899	0.863
t	**Number of Alternative**	**Initial Surplus** u_k	**Lower Barrier Level** k	**Released Capital** $u_L - u_k$	**Expected Profit** P	**Expected Utility** $E[\mathbf{u}[U(t)]]$
50	1	8.583	8.583	41.026	2350.549	0.671
	2	8.683	8.578	40.926	2351.971	0.672
	3	8.783	8.573	40.826	2353.382	0.673
	4	8.883	8.568	40.726	2354.780	0.674
	5	8.983	8.562	40.626	2356.166	0.674
	⋮	⋮	⋮	⋮	⋮	⋮
	409	49.383	3.921	0.226	2498.188	0.863
	410	49.483	3.900	0.126	2498.218	0.863
	411	49.583	3.879	0.026	2498.247	0.863
t	**Number of Alternative**	**Initial Surplus** u_k	**Lower Barrier Level** k	**Released Capital** $u_L - u_k$	**Expected Profit** P	**Expected Utility** $E[\mathbf{u}[U(t)]]$
100	1	9.280	9.280	40.329	4700.567	0.676
	2	9.380	9.275	40.229	4703.405	0.676
	3	9.480	9.270	40.129	4706.219	0.677
	4	9.580	9.265	40.029	4709.009	0.678
	5	9.680	9.260	39.929	4711.776	0.679
	⋮	⋮	⋮	⋮	⋮	⋮
	402	49.380	4.752	0.229	4995.460	0.863
	403	49.480	4.733	0.129	4995.519	0.863
	404	49.580	4.714	0.029	4995.578	0.863
t	**Number of Alternative**	**Initial Surplus** u_k	**Lower Barrier Level** k	**Released Capital** $u_L - u_k$	**Expected Profit** P	**Expected Utility** $E[\mathbf{u}[U(t)]]$
1000	1	11.589	11.589	38.020	47000.630	0.690
	2	11.689	11.585	37.920	47028.680	0.691
	3	11.789	11.582	37.820	47056.510	0.692
	4	11.889	11.578	37.720	47084.110	0.693
	5	11.989	11.575	37.620	47111.490	0.693
	⋮	⋮	⋮	⋮	⋮	⋮
	379	49.389	7.442	0.220	49934.490	0.863
	380	49.489	7.427	0.120	49935.180	0.863
	381	49.589	7.411	0.020	49935.870	0.863

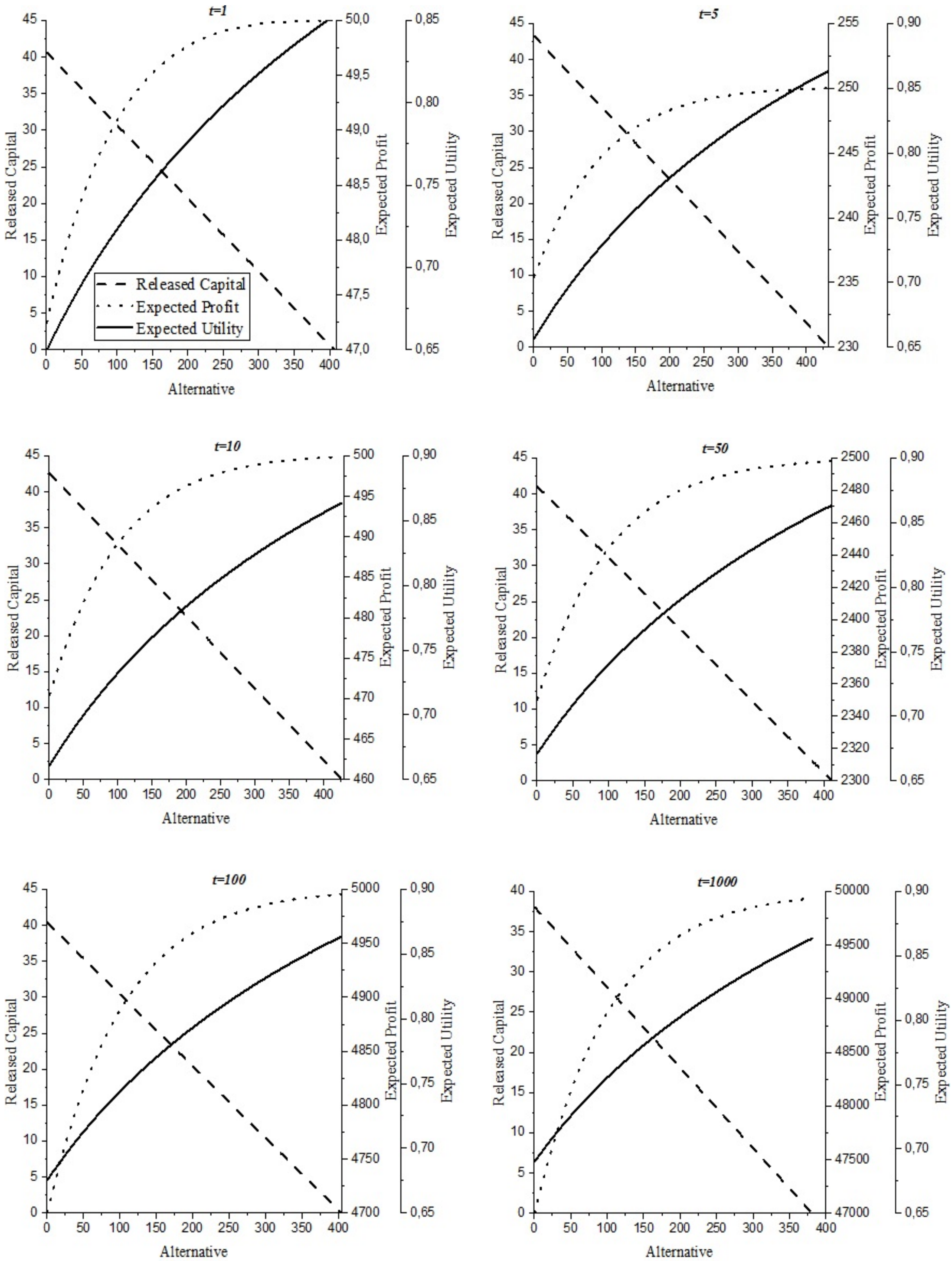

Figure 3.: The graph of optimal reinsurance criteria when $\theta = 0.1$ and $\xi = 0.3$ for the vector-normalized covariance matrix on lower barrier model

Step 6: *Deciding the optimal pair of initial surplus and lower barrier level under the TOPSIS method with Mahalanobis distance*

In order to decide the optimal pair of initial surplus and lower barrier level (u_k, k), we use the TOPSIS method with Mahalanobis distance. This method enables us to investigate the relationship between the criteria by using the vector-normalized covariance matrix and the original covariance matrix. Relative proximity to the ideal solution for each alternative is used to decide the optimal pairs (u_k, k).

Alternatives are not on the same scale as shown by Figure 3. In addition, the variance of the outcomes have an important role. The released capital criteria has the highest variance of outcomes. For the case of $t = 1$, $\theta = 0.1$ and $\xi = 0.3$, the variance of the released capital, expected profit and expected utility are 138.380, 0.526 and 0.003, respectively.

We use the vector-normalization technique to reduce the variance and convert criteria into a comparable form. In the covariance matrix of the vector-normalized outcomes, the values are smaller than the original outcomes.

Table 5 gives an example of the difference between the original covariance matrix and the vector-normalized covariance matrix for different times. We can see that the covariance matrix of the released capital, expected profit and expected utility outcomes in the original case has much higher variation than the others. It can be clearly seen from the covariance matrix, the relationship between the released capital and the pair of expected profit and expected utility is negative while the relationship between the expected profit and expected utility is positive. Moreover the variance of each criteria increases when the time horizon increases.

Efficiency of the criteria can change according to time and loading factors. The optimal pair (u_k, k) is obtained for eight different time horizons: t= 1, 5, 10, 20, 50, 100, 500, 1000 and four scenarios which are given in Table 1. It should also be noted that when $t > 20$ the expected utility criteria will not lose its efficiency and all outcomes of this criteria are not equal to 1 as in Karageyik and Dickson [9].

Table 5.: The covariance matrices for t=1 and t=10 for the original and vector-normalized data

t	Vector - normalized covariance matrix				Original covariance matrix			
		RC	EP	EU		RC	EP	EU
t=1	RC	0.0006182	-0.0000158	-0.0000906	RC	138.380	-7.457	-0.668
	EP	-0.0000158	0.0000005	0.0000025	EP	-7.457	0.526	0.039
	EU	-0.0000906	0.0000025	0.0000135	EU	-0.668	0.039	0.003
		RC	EP	EU		RC	EP	EU
t=10	RC	0.0005873	-0.0000162	-0.0000847	RC	151.585	-83.686	-0.695
	EP	-0.0000162	0.0000006	0.0000025	EP	-83.686	58.637	0.412
	EU	-0.0000847	0.0000025	0.0000125	EU	-0.695	0.412	0.003

Table 6.: The optimal pairs (u_k, k) on lower barrier model by using vector-normalized covariance matrix

Loading Factors	Time	Scenario 1-2-3		Scenario 4	
	t	**Optimal u_k**	**Optimal k**	**Optimal u_k**	**Optimal k**
	1	5.227	5.227	45.827	0.119
	5	6.334	6.334	49.534	1.168
θ=**0.1**	**10**	7.030	7.030	49.530	2.002
&	**20**	7.683	7.683	49.583	2.787
ξ=**0.15**	**50**	8.577	8.577	49.577	3.876
	100	9.273	9.273	49.573	4.710
	500	10.883	10.883	49.583	6.605
	1000	11.580	11.580	49.580	7.405
	t	**Optimal u_k**	**Optimal k**	**Optimal u_k**	**Optimal k**
	1	5.228	5.228	45.828	0.117
	5	6.335	6.335	49.535	1.168
θ=**0.1**	**10**	7.031	7.031	49.531	2.002
&	**20**	7.685	7.685	49.535	2.788
ξ=**0.2**	**50**	8.579	8.579	49.579	3.877
	100	9.275	9.275	49.575	4.711
	500	10.886	10.886	49.586	6.600
	1000	11.583	11.583	49.583	7.407
	t	**Optimal u_k**	**Optimal k**	**Optimal u_k**	**Optimal k**
	1	5.230	5.230	45.830	0.115
	5	6.339	6.339	49.539	1.167
θ=**0.1**	**10**	7.035	7.035	49.535	2.002
&	**20**	7.689	7.689	49.589	2.789
ξ=**0.3**	**50**	8.583	8.583	49.583	3.879
	100	9.280	9.280	49.580	4.714
	500	10.891	10.891	49.591	6.604
	1000	11.589	11.589	49.589	7.411
	t	**Optimal u_k**	**Optimal k**	**Optimal u_k**	**Optimal k**
	1	5.137	5.137	26.537	0.169
	5	6.240	6.240	25.640	1.885
θ=**0.2**	**10**	6.937	6.937	26.537	2.445
&	**20**	7.592	7.592	26.492	3.334
ξ=**0.3**	**50**	8.488	8.488	26.488	4.505
	100	9.187	9.187	26.487	5.400
	500	10.891	10.891	26.491	9.458
	1000	11.508	11.508	26.508	8.301

As can be seen from Table 6 and Table 7, the optimal barrier level increases as time horizon increases for all scenarios. Possible explanation is that when time horizons increases, the need for the reinsurance increases. Hence, the barrier level increases.

From the results that has been conducted, it is possible to conclude that the highest optimal initial surplus and lowest barrier level are obtained according to the original covariance matrix, whereas the lowest optimal initial surplus and highest barrier levels are obtained according to the vector-normalized covariance matrix. The reason is that the variance of three criteria is high and so the covariance matrix of the original data is higher than the other case. The optimal pairs which are obtained by using the covariance matrix of the original data, are equal to levels in which the expected profit and expected utility are maximum so that Scenarios 1,2,3 and 4 give the same optimal pairs.

In Table 6 under Scenarios 1, 2, and 3, the optimal initial surplus are obtained as the smallest initial surplus where the released capital has its maximum value. The results show that the released capital has a higher effect than two criteria. In this case, the optimal barrier level equals to the optimal initial surplus because each term in the subtraction has three decimal places. However, it should be noted that $u > k$.

In Table 6 under Scenario 4, it is assumed that the more weight is given to the expected utility. In this case, the dominant effect of the released capital cannot be observed because of the weight of the criteria. In Scenario 3, although the weight of the released capital is not as significant as in Scenario 2, the effect of the released capital can be still observed. The reason is that the variance of the expected utility is higher than the expected profit under the vector-normalized case.

According to the original covariance matrix, the optimal levels are obtained as the largest initial surplus and the lowest barrier level for all scenarios.

In Table 6 under Scenario 4, it is assumed that more weight is given to the expected utility. According to the vector-normalized covariance matrix, the variance of the expected utility is higher than the expected profit, hence the effect of the expected utility on the optimal levels can be observed easily. The highest optimal initial surplus and the lowest initial barrier level are observed in the scenarios according to the original covariance matrix.

Table 7.: The optimal pairs (u_k, k) on lower barrier model by using original covariance matrix

Loading Factors	$\theta = 0.1$ & $\xi = 0.15$		$\theta = 0.1$,& $\xi = 0.2$		$\theta = 0.1$,& $\xi = 0.3$		$\theta = 0.2$,& $\xi = 0.3$	
Time	**Optimal** u_k	**Optimal** k	**Optimal** u_k	**Optimal** k	**Optimal** u_k	**Optimal** k	**Optimal** u_k	**Optimal** k
1	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169
5	49.534	1.168	49.335	1.168	49.539	1.167	25.640	1.885
10	49.539	2.002	49.531	2.002	49.535	2.002	26.537	2.445
20	49.583	2.787	49.685	2.788	49.589	2.789	26.492	3.334
50	49.577	3.876	49.579	3.877	49.582	3.879	26.488	4.505
100	49.573	4.710	49.575	4.711	49.580	4.714	26.487	5.400
500	49.583	6.605	49.586	6.600	49.591	6.604	26.508	8.301
1000	49.580	7.405	49.583	7.407	49.589	7.411	26.491	9.458

7. Sensitivity Analysis on Lower Barrier Model

In this section, we investigate the robustness of the results by changing one or more parameters. We examine the differences on the optimal levels by changing the weights of each criteria, the parameter of the utility function and also we analysis the maximum possible value of each criterion. In sensitivity analysis, an effective method is to compare the weights of criteria and observe the differences. The simpliest way to observe the changes in the optimal levels is to allow only one single criterion vary while the others are constant. We investigate two different approaches to measure the effect of the changes in the weights of the criteria. First, we assume that the weight of one criterion is increased from 0 to 1 step by step, while the other criteria have the same importance. We observe the differences on the optimal pair of initial surplus and lower barrier level according to the changes in the weights of the criteria. Second, we assume that the weight of one criterion is increased from 0 to 1 step by step while the other criteria have a relationship such as twofold of each other. Similarly, weights of other criteria are increased to find the optimal initial surplus and lower barrier level. Therefore, we can observe the general sensitivity trend under different assumptions.

An important implication of these findings is changing on the weight of each criteria causes a difference on the optimal levels. When the weight of the expected profit and expected utility are changed, the similar pattern can be observed. It should be noted that the increase in the released capital causes an increase in optimal barrier levels and a decrease in initial surplus. However, the increase in the weight of the expected profit or expected utility causes a decrease in optimal barrier levels and an increase in the initial surplus. The reason is that each criteria's maximum level is different.

The parameter of the exponential utility function has been chosen as 0.02. We test the effect of this parameter on the optimal pair of initial surplus and barrier level. For all combinations of time and loading factors, we find the same optimal pairs. The existence of these effect implies that the optimal pair of initial surplus and barrier level do not seem to be sensitive to the value of the parameter of the utility function.

We also use the same comparison technique with individual outcomes under each criterion as in Karageyik and Dickson[9]. In this context, the question under discussion measures the sensibility of the optimal pairs.

The values of the released capital, expected profit and expected utility regarding the optimal initial surplus and barrier levels are compared with their maximum possible values of each criteria to measure the sensibility of the optimal pairs. Table 8 presents the percentage of maximum possible values of each optimal pairs for vector-normalized covariance matrix.

Table 8.: Percentage of maximum possible value of each optimal pairs for vector-normalized covariance matrix

Premium Loadings Factor	Time	Scenario 1 - 2 - 3 Relased Capital(%)	Scenario 1 - 2 - 3 Expected Profit (%)	Scenario 1 - 2 - 3 Expected Utility (%)	Scenario 4 Relased Capital (%)	Scenario 4 Expected Profit (%)	Scenario 4 Expected Utility(%)
$\theta = 0.1$ & $\xi = 0.15$	**1**	100	97	77	0	100	100
	5	100	97	77	0	100	100
	10	100	97	78	0	100	100
	20	100	97	78	0	100	100
	50	100	97	79	0	100	100
	100	100	97	79	0	100	100
	500	100	97	80	0	100	100
	1000	100	97	81	0	100	100
$\theta = 0.1$ & $\xi = 0.2$	**1**	100	96	77	0	100	100
	5	100	96	77	0	100	100
	10	100	96	77	0	100	100
	20	100	96	78	0	100	100
	50	100	96	78	0	100	100
	100	100	96	79	0	100	100
	500	100	96	80	0	100	100
	1000	100	96	81	0	100	100
$\theta = 0.1$ & $\xi = 0.3$	**1**	100	94	76	0	100	100
	5	100	94	76	0	100	100
	10	100	94	77	0	100	100
	20	100	94	77	0	100	100
	50	100	94	78	0	100	100
	100	100	94	78	0	100	100
	500	100	94	79	0	100	100
	1000	100	94	80	0	100	100
$\theta = 0.2$ & $\xi = 0.3$	**1**	100	99	95	0	100	100
	5	100	99	98	0	100	100
	10	100	99	98	0	100	100
	20	100	99	96	0	100	100
	50	100	99	96	0	100	100
	100	100	99	96	0	100	100
	500	100	99	96	0	100	100
	1000	100	99	97	0	100	100

Particular attention is also paid to the sensibility of the optimal pair according to original covariance matrices. A closer look at the relationship between the outcomes indicates that the optimal levels obtained for the original covariance matrix mostly depend on the expected profit and the expected utility. We can see that the percentage of the released capital to maximum possible released capital is 0% whereas the percentage of the expected profit and expected utility to their maximum possible values are 100%.

8. Conclusions

In this study, we have focused on the lower barrier model where we reformulate for the finite time ruin probability under the reinsurance arrangement. We have examined the calculation of the defective distribution function of $T_{u,k}$, the time to ruin for the process with the lower barrier, with initial surplus u and lower barrier k.

We have given numerical analysis under the assumption of the aggregate claim process is a compound Poisson with individual claim amounts which have exponential distributions. We have calculated the optimal initial surplus and barrier level under the TOPSIS method with Mahalanobis distance. For different time horizons and different loading factors, we have obtained the optimal pairs. In addition, we have used the covariance matrix of the vector-normalized and the original outcomes to explain the dependency and effect of the high variance between the criteria.

We have discussed our results and have made some general comments about our approach according to different scenarios and set of weights. Moreover, we have measured the sensitivity of our approach on the vector-normalized covariance matrix and the original outcomes covariance matrix by comparing the results in the dominance weight set. We have addressed the sensitivity analysis not only on the weight of the criteria but also on the parameter of the exponential utility function. We have also considered the consequences of the maximum possible value of each criteria.

The originality of our solution lies in the fact that we have obtained the optimal initial surplus and barrier level according to three essential criteria: released capital, expected profit and expected utility under the ruin probability constraint. This study is a novel solution of the determining the optimal barrier. We have considered the consequences of adding the reinsurance premium to the finite time ruin probability formula and also we have achieved to obtain the finite time ruin probability according to the lower barrier model numerically.

This research is concerned with the TOPSIS method with Mahalanobis distance. However, the decision procedure can be applicable also to other decision making techniques. The findings are of direct practical relevance and the proposed method can be readily used in practice.

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