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The a-function for gauge theories

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Abstract

The a-function is a proposed quantity defined for quantum field theories which has a monotonic behaviour along renormalisation group flows, being related to the β-functions via a gradient flow equation involving a positive definite metric. We construct the a-function at four-loop order for a general gauge theory with fermions and scalars, using only one and two loop β-functions; we are then able to provide a stringent consistency check on the general three-loop gauge β-function. In the case of an $\mathcal{N} = 1$ supersymmetric gauge theory, we present a general condition on the chiral field anomalous dimension which guarantees an exact all-orders expression for the a-function; and we verify this up to fifth order (corresponding to the three-loop anomalous dimension).

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1 Introduction

It is natural to regard quantum field theories as points on a manifold with the couplings $\{g^I\}$ as co-ordinates, and with a natural flow determined by the β-functions $\beta^I(g)$. At fixed points the quantum field theory is scale-invariant and is expected to become a conformal field theory. It was suggested by Cardy [1] that there might be a four-dimensional generalisation of Zamolodchikov's c-theorem [2] in two dimensions, such that there is a function $a(g)$ which has monotonic behaviour under renormalisation-group (RG) flow (the strong a-theorem) or which is defined at fixed points such that $a_{\rm UV} - a_{\rm IR} > 0$ (the weak a-theorem). It soon became clear that the coefficient (which we shall denote $\frac{1}{4}A$) of the Gauss-Bonnet term in the trace of the energy-momentum tensor is the only natural candidate for the a-function. A proof of the weak a-theorem has been presented by Komargodski and Schwimmer [3] and further analysed and extended in Refs. [4, 5].

In other work, a perturbative version of the strong a-theorem has been derived [6] from Wess-Zumino consistency conditions for the response of the theory defined on curved spacetime, and with x-dependent couplings $g^I(x)$, to a Weyl rescaling of the metric [7]. This approach has been extended to other dimensions in Refs. [8, 9]. The essential result is that we can define a function $\tilde{A}$ by

$$\tilde{A} = A + W_I\beta^I , \tag{1.1}$$

where A is defined above and W_I is well-defined as an RG quantity on the theory extended as described above, such that $\tilde{A}$ satisfies the crucial equation

$$\partial_I\tilde{A} = T_{IJ}\beta^J , \tag{1.2}$$

for

$$T_{IJ} = G_{IJ} + 2\partial_{[I}W_{J]} + 2\tilde{\rho}_{[I}\cdot Q_{J]} \, . \tag{1.3}$$

Here $G_{IJ} = G_{JI}$, $\tilde{\rho}_I$ and Q_J may all be computed perturbatively within the theory extended to curved spacetime and x-dependent g^I; for weak couplings G_{IJ} can be shown to be positive definite in four dimensions (in six dimensions, G_{IJ} has recently been computed to be negative definite at leading order [10]). Eq. (1.2) implies

$$\mu\frac{d}{d\mu}\tilde{A} = \beta^I\frac{\partial}{\partial g^I}\tilde{A} = G_{IJ}\beta^I\beta^J \tag{1.4}$$

thus verifying the strong a-theorem so long as G_{IJ} is positive. Crucially Eq. (1.2) also imposes integrability conditions which constrain the form of the β-functions and are the focus of this paper. These conditions relate contributions to β-functions at different loop orders.

We should mention here that for theories with a global symmetry, β^I in these equations should be replaced by a B^I which is defined, for instance, in Ref. [6]; however it was shown in Ref. [11, 12] that the two quantities only begin to differ at three loops; and in Ref. [13]

that there is no difference to all orders in supersymmetric theories (see Ref. [14] for related results). Hence for our purposes in this article we may ignore the distinction.

The analysis of Ref. [6] was recently further extended in Ref. [15] (related results were also presented in Ref. [16]). Expressions for the three-loop Yukawa β-functions and for the three-loop contribution to the metric G_{IJ} were derived for a general fermion-scalar theory, and Eq. (1.2) was checked up to this order; indeed it was shown that Eq. (1.2) places strong constraints upon the form of the β-function at this order. The $\mathcal{N}=1$ supersymmetric Wess-Zumino model was also considered as a special case. Moreover, here it was possible to show that an exact formula for the a-function conjectured in Refs. [17–19] was valid at this order; in previous work [20] this had been checked at the level of the two-loop β-functions for a general, gauged $N=1$ supersymmetric theory (in the rest of this paper we shall describe this as a "two-loop check" although the corresponding a-function is actually a four-loop quantity). A sufficient condition on the chiral field anomalous dimension for this exact a-function to be viable was presented and shown (using the results of Ref. [21, 22]) to be satisfied up to three loops for the Wess-Zumino model.

Our goal in this article is to extend the work of Ref. [15] to the gauge case. In the non-supersymmetric case, we show that the two-loop Yukawa β-function for the most general renormalisable gauge theory coupled to fermions and scalars is compatible with Eq. (1.2); indeed by imposing Eq. (1.2) we are able to obtain the terms in $\tilde{A}$ containing scalar or Yukawa couplings at this order up to only three free parameters, without any further perturbative calculations–this approach has previously been applied in Ref. [23] to the case of the standard model. (Of course a gradient flow for purely scalar quantum field theories was postulated some time ago by Wallace and Zia [24].) As a bonus, we show that this expression for $\tilde{A}$ is also consistent with (and provides a stringent check upon) the general *three*-loop gauge β-function derived by Gracey, Jones and Pickering [25]. As a further check, we also compute $\tilde{A}$ using the dimensional reduction version of the Yukawa β-function and show that in the supersymmetric case, this reduces to the version of $\tilde{A}$ presented in Ref. [20] (and is hence compatible with Ref. [17–19]). Finally, again in the context of supersymmetry, we extend the condition on the chiral superfield anomalous dimension presented in Ref. [15] to the gauge case and show that it is satisfied at three loops. We also derive the (very non-trivial) constraints that this condition imposes upon the form of the three-loop anomalous dimension.

2 The non-supersymmetric case

We consider a general renormalisable gauge theory with a simple gauge group G and n_φ real scalars φ_a, n_ψ two-component Weyl fermions ψ_i, where $G \subset U(n_\psi) \cap O(n_\varphi)$. For a Yukawa interaction $\frac{1}{2}\psi_i^T C(Y_a)_{ij}\psi_j\,\varphi_a$+h.c. and a quartic scalar interaction $\frac{1}{4!}\lambda_{abcd}\varphi_a\varphi_b\varphi_c\varphi_d$ and with a gauge coupling g the basic couplings are then

$$g^I \equiv \{g, Y_a, \bar{Y}_a, \lambda_{abcd}\}\,, \quad {Y_a}^T = Y_a\,, \quad \bar{Y}_a = Y_a^*\,. \tag{2.1}$$

The (hermitian) gauge generators for the scalar and fermion fields are denoted respectively $t^\varphi_A = -t^{\varphi\,T}_A$ and t^ψ_A, $A = 1\ldots n_V$, where $n_V = \dim G$, and obey

$$[t_A, t_B] = i f_{ABC} t_C \,, \tag{2.2}$$

and gauge invariance requires $Y_a\, t^\psi_A + t^{\psi\,T}_A\, Y_a = t^\varphi_{A ab}\, Y_b$, $t^\varphi_{Aae}\lambda_{ebcd}\varphi_a\varphi_b\varphi_c\varphi_d = 0$. In order to simplify the form of our results, it is convenient to assemble the Yukawa couplings into a matrix

$$y_a = \begin{pmatrix} Y_a & 0 \\ 0 & \bar Y_a \end{pmatrix}, \qquad \hat y_a = \begin{pmatrix} \bar Y_a & 0 \\ 0 & Y_a \end{pmatrix} = \sigma_1 y_a \sigma_1 \,, \tag{2.3}$$

and

$$T_A = \begin{pmatrix} t^\psi_A & 0 \\ 0 & -t^{\psi *}_A \end{pmatrix}, \qquad \hat T_A = \sigma_1 T_A \sigma_1 = -T_A^{\,T} \,. \tag{2.4}$$

This corresponds to using the Majorana spinor $\Psi = \begin{pmatrix} \psi_i \\ -C^{-1}\bar\psi^{iT} \end{pmatrix}$.

We should mention here that in our present calculations we have ignored potential parity violating counterterms (i.e. containing ϵ-tensors). The analysis of Ref. [6] was recently extended [28] to the case of theories with chiral anomalies, including the possibility of parity violating anomalies. It would be interesting to carry out the detailed computations necessary to exemplify the general conclusions of Ref. [28].

The one- and two-loop gauge β-functions are given by

$$\begin{aligned}
\beta_g &= -\beta_0 g^3 - \beta_1 g^5 - g^3\,\frac{1}{2n_V}\mathrm{tr}[C^\psi\,\hat y_a y_a] \,, \\
\beta_0 &= \tfrac13 (11 C_G - 2R^\psi - \tfrac12 R^\varphi) \,, \\
\beta_1 &= \tfrac13 C_G(34 C_G - 10 R^\psi - R^\varphi) - \frac{1}{n_V}\mathrm{tr}[(C^\psi)^2] - \frac{4}{n_V}\mathrm{tr}[(C^\varphi)^2] \,,
\end{aligned} \tag{2.5}$$

where

$$\begin{gathered}
\mathrm{tr}[t^\psi_A t^\psi_B] = R^\psi \delta_{AB}, \qquad \mathrm{tr}[t^\varphi_A t^\varphi_B] = R^\varphi \delta_{AB}, \\
C^\varphi = t^\varphi_A t^\varphi_A, \qquad C^\psi = T_A T_A \,, \quad \hat C^\psi = C^{\psi\,T} \,,
\end{gathered} \tag{2.6}$$

with T_A as defined in Eq. (2.4). We follow Ref. [15] in removing factors of $1/16\pi^2$ which arise at each loop order by redefining

$$\lambda_{abcd} \to 16\pi^2 \lambda_{abcd} \,, \quad Y_a \to 4\pi Y_a, \quad g \to 4\pi g \,. \tag{2.7}$$

The one-loop Yukawa β-function is given by

$$\beta^{(1)}_{y\,a} = 2\, y_b \hat y_a y_b + \tfrac12 [y_b \hat y_b - 6 g^2 \hat C^\psi] y_a + \tfrac12 y_a [\hat y_b y_b - 6 g^2 C^\psi] + \tfrac12\,\mathrm{tr}[y_a \hat y_b] y_b \,, \tag{2.8}$$

and the one-loop scalar β-function is given by

$$\begin{aligned}
\tfrac{1}{4!}\,\beta^{(1)}_{\lambda\,abcd}\,\varphi_a\varphi_b\varphi_c\varphi_d = \big(&\tfrac18\,\lambda_{abef}\lambda_{cdef} + \tfrac32\, g^4 (t^\varphi_A t^\varphi_B)_{ab}(t^\varphi_A t^\varphi_B)_{cd} - \tfrac12\,\mathrm{tr}[y_a \hat y_b y_c \hat y_d] \\
&- \tfrac12\,\lambda_{abce} g^2 (C^\varphi)_{ed} + \tfrac{1}{12}\,\lambda_{abce}\mathrm{tr}[y_e \hat y_d]\big)\varphi_a\varphi_b\varphi_c\varphi_d \,.
\end{aligned} \tag{2.9}$$

The leading terms in the metric G_{IJ} in Eq. (1.2) may be written as [6]

$$\mathrm{d}s^2 = G_{IJ}\mathrm{d}g^I\mathrm{d}g^J = 2n_V\frac{1}{g^2}(1+\sigma g^2)(\mathrm{d}g)^2 + \tfrac{1}{6}\,\mathrm{tr}[\mathrm{d}\hat{y}_a\mathrm{d}y_a] + \tfrac{1}{144}\,\mathrm{d}\lambda_{abcd}\mathrm{d}\lambda_{abcd}\,, \tag{2.10}$$

where σ is given (using dimensional regularisation, DREG) by [6,20]

$$\sigma = \tfrac{1}{6}\left(102C_G - 20R^\psi - 7R^\varphi\right)\,. \tag{2.11}$$

We emphasise here that y and $\hat{y}$ are not independent; and furthermore, the result of a trace is unchanged by interchanging y and $\hat{y}$.

The lowest-order contributions to $\tilde{A}$ are given implicitly in Ref. [20] as

$$\begin{aligned}
\tilde{A}^{(2)} &= -\,n_V\beta_0 g^2,\\
\tilde{A}^{(3)} &= -\,\tfrac{1}{2}n_V g^4(\beta_1+\sigma\,\beta_0) - \tfrac{1}{2}g^2\mathrm{tr}[y_a\hat{y}_a\hat{C}^\psi]\\
&\quad + \tfrac{1}{24}\left(\mathrm{tr}[y_a\hat{y}_a y_b\hat{y}_b] + 2\,\mathrm{tr}[y_a\hat{y}_b y_a\hat{y}_b] + \mathrm{tr}[y_a\hat{y}_b]\,\mathrm{tr}[y_a\hat{y}_b]\right).
\end{aligned} \tag{2.12}$$

They can easily be checked to satisfy Eq. (1.2) with Eqs. (2.5), (2.8).

To proceed to the next order, we shall need the two-loop Yukawa β-function in addition to the one-loop scalar β-function in Eq. (2.9). The two-loop β-function is given in general by Refs. [26,27] in the form

$$\begin{aligned}
\beta^{(2)}_{y\,a} = &\sum_{\alpha=1}^{7} c_\alpha (G^y_\alpha)_a + \sum_{\alpha=8}^{19} c_\alpha[(G^y_\alpha)_a + (\hat{G}^y_\alpha)_a)\\
&+ c_{20}g^4(\hat{C}^\psi y_a + y_a C^\psi) + c_{21}g^4\hat{C}^\psi y_a C^\psi\\
&+ c_{22}g^4[(\hat{C}^\psi)^2 y_a + y_a(C^\psi)^2] + c_{23}g^4(C^\varphi)_{ab}(\hat{C}^\psi y_b + y_b C^\psi)\\
&+ \big(\mathrm{tr}[c_{24}\,y_a\hat{y}_b y_c\hat{y}_c + c_{25}\,y_a\hat{y}_c y_b\hat{y}_c + c_{26}\,g^2\hat{C}^\psi y_a\hat{y}_b]\\
&\qquad + c_{27}\,g^4(C^{\varphi 2})_{ab} + c_{28}g^4(C^\varphi)_{ab} + c_{29}\,\lambda_{acde}\lambda_{bcde}\big)y_b\,.
\end{aligned} \tag{2.13}$$

The contributions G^y_α are depicted in Table (1); $\hat{G}^y_\alpha$ is the transpose of G^y_α. A solid or open box represents g^2C^ψ or g^2C^φ respectively. A box with a letter "A" represents the gauge generator gT_A. Note that for each of G^y_α, there is an alternation between "hatted" and "unhatted" y matrices, as can be seen in Eq. (2.13) for G^y_α, $\alpha = 20,\ldots 28$. To give a couple of examples, G^y_4 represents

$$(G^y_4)_a = \lambda_{abcd}\,y_b\hat{y}_c y_d\,, \tag{2.14}$$

and G^y_{19} corresponds to

$$(G^y_{19})_a = g^2 y_b\,C^\psi\,\hat{y}_a y_b, \qquad (\hat{G}^y_{19})_a = g^2 y_b\hat{y}_a\,\hat{C}^\psi\,y_b\,. \tag{2.15}$$

G^y_1	G^y_2	G^y_3	G^y_4
G^y_5	G^y_6	G^y_7	G^y_8
G^y_9	G^y_{10}	G^y_{11}	G^y_{12}
G^y_{13}	G^y_{14}	A A G^y_{15}	G^y_{16}
G^y_{17}	G^y_{18}	G^y_{19}	

Table 1: The diagrams contributing to $\beta_{y\,a}^{(2)}$

We present here the results for the coefficients evaluated using standard dimensional regularisation, DREG [26, 27]:

$$\begin{gathered}
c_1 = 2, \quad c_2 = -1, \quad c_3 = -2, \quad c_4 = -2, \quad c_5 = -12, \\
c_6 = 6, \quad c_7 = 0, \quad c_8 = -\tfrac{1}{8}, \quad c_9 = 0, \quad c_{10} = -\tfrac{3}{8}, \quad c_{11} = -1, \\
c_{12} = 0, \quad c_{13} = -\tfrac{7}{4}, \quad c_{14} = -\tfrac{1}{4}, \quad c_{15} = 6, \quad c_{16} = \tfrac{9}{2}, \quad c_{17} = 0, \\
c_{18} = 3, \quad c_{19} = 5, \quad c_{20} = -\tfrac{1}{12}\left(194C_G - 20R^{\psi} - 11R^{\varphi}\right), \\
c_{21} = 0, \quad c_{22} = -\tfrac{3}{2}, \quad c_{23} = 6, \quad c_{24} = -\tfrac{3}{4}, \quad c_{25} = -\tfrac{1}{2}, \\
c_{26} = \tfrac{5}{2}, \quad c_{27} = -\tfrac{21}{2}, \quad c_{28} = \tfrac{1}{12}(147C_G - 12R^{\psi} - 3R^{\varphi}), \quad c_{29} = \tfrac{1}{12}.
\end{gathered} \tag{2.16}$$

There are 33 coefficients altogether (counting c_{20} and c_{28} as three each). We do however have the freedom to redefine the couplings, corresponding to a change in renormalisation scheme; at this order we may consider

$$
\begin{aligned}
\delta y_a &= \mu_1\, y_b \hat{y}_a y_b + \mu_2\, (y_b \hat{y}_b y_a + y_a \hat{y}_b y_b) + \mu_3\, \mathrm{tr}[y_a \hat{y}_b] y_b \\
&\quad + \mu_4\, g^2 (\hat{C}^\psi y_a + y_a C^\psi) + \mu_5\, g^2 C^\phi_{ab} y_b\,, \\
\delta g &= (\nu_1\, C_G + \nu_2\, R^\psi + \nu_3\, R^\phi) g^3\,.
\end{aligned} \tag{2.17}
$$

This results in a change in the β-function

$$
\delta \beta^{(2)}_{y\,a} = \left(\beta^{(1)}_y \cdot \frac{\partial}{\partial y} + \beta^{(1)}_g \frac{\partial}{\partial g} \right) \delta y_a - \left(\delta y \cdot \frac{\partial}{\partial y} + \delta g \, \frac{\partial}{\partial g} \right) \beta^{(1)}_{y\,a}\,. \tag{2.18}
$$

Using Eqs. (2.5), (2.8) this leads to

$$
\begin{gathered}
\delta c_2 = \mu_1 - 4\mu_3, \quad \delta c_6 = -4\mu_5, \quad \delta c_7 = -\mu_5, \quad \delta c_9 = -\mu_1 + 4\mu_2, \\
\delta c_{10} = \mu_2 - \mu_3, \quad \delta c_{11} = \mu_1 - 4\mu_2, \quad \delta c_{13} = -6\mu_2 - \mu_4, \quad \delta c_{14} = -6\mu_2 - \mu_4, \\
\delta c_{16} = -\mu_5, \quad \delta c_{19} = -6\mu_1 - 4\mu_4, \quad \delta c_{24} = -2\mu_2 + 2\mu_3, \quad \delta c_{25} = -\mu_1 + 4\mu_3,
\end{gathered}
$$
$$
\begin{aligned}
\delta c_{26} &= \ - 12\mu_3 - 2\mu_4, \\
\delta c_{20} &= 2C_G \left(3\nu_1 - \tfrac{11}{3}\mu_4\right) + 2R^\psi \left(3\nu_2 + \tfrac{2}{3}\mu_4\right) + R^\phi \left(6\nu_3 + \tfrac{1}{3}\mu_4\right), \\
\delta c_{28} &= \ - \tfrac{1}{3} \left(22C_G - 4R^\psi - R^\phi\right) \mu_5.
\end{aligned} \tag{2.19}
$$

We observe that the redefinitions corresponding to μ_{1-4} are not all independent; for instance we may remove μ_4 by redefining

$$
\begin{aligned}
&\mu_1 \to \mu_1 - \tfrac{2}{3}\mu_4, \quad \mu_2 \to \mu_2 - \tfrac{1}{6}\mu_4, \quad \mu_3 \to \mu_3 - \tfrac{1}{6}\mu_4, \\
&\nu_1 \to \nu_1 + \tfrac{11}{9}\mu_4, \quad \nu_2 \to \nu_2 - \tfrac{2}{9}\mu_4, \quad \nu_3 \to \nu_3 - \tfrac{1}{18}\mu_4,
\end{aligned} \tag{2.20}
$$

This is a general consequence of the form of the redefinition given by Eq. (2.18), which implies that a redefinition

$$
\delta y_a = \beta^{(1)}_{y\,a}, \quad \delta g = \beta^{(1)}_g \tag{2.21}
$$

has no effect on $\beta^{(2)}_{y\,a}$; however μ_5 yields an independent redefinition due to the fact that there happens to be no corresponding $C^\phi_{ab} y_b$ term in $\beta^{(1)}_{y\,a}$. It then follows that μ_{1-5} and ν_{1-3} yield only 7 independent redefinitions; we therefore have $33 - 7 = 26$ independent coefficients in the two-loop β-function. Under the change Eq. (2.17)

$$
\delta \tilde{A}^{(3)} = -\delta g \, \frac{\partial}{\partial g} \tilde{A}^{(2)}\,, \tag{2.22}
$$

which corresponds to taking $\delta\sigma = 4\left(\nu_1\, C_G + \nu_2\, R^\psi + \nu_3\, R^\phi\right)$ in Eq. (2.10).

Applying Eq. (1.2), we require $\tilde{A}^{(4)}$ to satisfy

$$
\begin{aligned}
\mathrm{d}_y \tilde{A}^{(4)} &= \mathrm{d}y \cdot T^{(3)}_{yy} \cdot \beta^{(1)}_y + t_1\, g \mathrm{tr}[\mathrm{d}\hat{y}_a y_a C^\psi]\, \beta^{(1)}_g + t_2\, g (C^\varphi)_{ab} \mathrm{tr}[\mathrm{d}\hat{y}_a y_b]\, \beta^{(1)}_g \\
&\quad + \tfrac{1}{12}\, \mathrm{tr}[\mathrm{d}\hat{y}_a \beta^{(2)}_{y\,a}]\,, \\
\mathrm{d}_\lambda \tilde{A}^{(4)} &= \tfrac{1}{144}\, \mathrm{d}\lambda_{abcd}\, \beta^{(1)}_\lambda{}_{abcd}\,,
\end{aligned} \tag{2.23}
$$

G_1^T	G_2^T	G_3^T	G_4^T
G_5^T	G_6^T	G_7^T	G_8^T
	G_9^T	G_{10}^T	

Table 2: Contributions to $\mathrm{d}y \cdot T_{yy}^{(3)} \cdot \mathrm{d}'y$

where $\mathrm{d}_y = \mathrm{d}y \cdot \frac{\partial}{\partial y}$, etc, the lower-order metric contributions were read off from Eq. (2.10) and we write

$$\mathrm{d}y \cdot T_{yy}^{(3)} \cdot \mathrm{d}'y = \sum_{\alpha=1}^{10} T_\alpha G_\alpha^T \,, \tag{2.24}$$

The contributions to $\mathrm{d}y \cdot T_{yy}^{(3)} \cdot \mathrm{d}'y$ at this order are depicted in Table (2). Here a diamond represents $\mathrm{d}'y$ and a cross $\mathrm{d}y$. As an example, G_1^T represents

$$G_1^T = \mathrm{tr}[\mathrm{d}\hat{y}_a y_b \hat{y}_b \mathrm{d}' y_a] \,. \tag{2.25}$$

$T_{yy}^{(3)}$ is symmetric up to the order at which we are working. The β-functions $\beta_g^{(1)}$, $\beta_y^{(1)}$ in Eq. (2.23) are given in Eqs. (2.5), (2.8). There are no “off-diagonal” fermion-scalar contributions to this order. We parameterise $\tilde{A}^{(4)}$ as

$$\tilde{A}^{(4)} = \sum_{\alpha=1}^{28} A_\alpha G_\alpha^A + \mathrm{O}(g^6) \,, \tag{2.26}$$

where the different contributions G_α^A are depicted in Table (3), with a similar notation to Table (2). We have included G_{28}^A as a reflection of the general freedom to redefine

$$\tilde{A} \to \tilde{A} + g_{IJ} \beta^I \beta^J \,, \tag{2.27}$$

G_1^A	G_2^A	G_3^A	G_4^A
G_5^A	G_6^A	G_7^A	G_8^A
G_9^A	G_{10}^A	G_{11}^A	G_{12}^A
G_{13}^A	G_{14}^A	G_{15}^A	G_{16}^A
	A A		
G_{17}^A	G_{18}^A	G_{19}^A	G_{20}^A
B B			
A A			
G_{21}^A	G_{22}^A	G_{23}^A	G_{24}^A
			$(1)y$
			$(1)y$
G_{25}^A	G_{26}^A	G_{27}^A	G_{28}^A

Table 3: Contributions to $A^{(4)}$ in the non-supersymmetric case

together with a related redefinition of T_{IJ}; see Ref. [15] for further details. The purely g-dependent contributions to $\tilde{A}^{(4)}$ of course cannot be determined from Eq. (2.23). Eq. (2.23) entails the system of equations

$$\begin{aligned}
6A_4 = \tfrac{1}{6}c_1, \quad 2A_{11} + 4A_{28} = \tfrac{1}{2}T_8 + \tfrac{1}{6}c_2 = 2T_6 + 2T_5 + \tfrac{1}{6}c_{25} = 2T_4 + \tfrac{1}{2}T_7,\\
2A_{10} + 8A_{28} = 2T_7 + \tfrac{1}{6}c_3 = T_8, \quad 4A_3 = \tfrac{1}{6}c_4, \quad 2A_{17} = 2T_{10} + \tfrac{1}{6}c_5 = \tfrac{1}{6}c_6,\\
4A_{20} = \tfrac{1}{2}T_{10} + \tfrac{1}{6}c_7, \quad 4A_6 + 2A_{28} = \tfrac{1}{2}(T_2 + T_3) + \tfrac{1}{3}c_8 = T_1,\\
2A_9 + 8A_{28} = 2T_2 + 2T_3 + \tfrac{1}{3}c_9 = T_7 + \tfrac{1}{2}T_8 + \tfrac{1}{3}c_{11} = 2T_1 + \tfrac{1}{2}T_8,\\
2A_7 + 2A_{28} = \tfrac{1}{2}(T_2 + T_3) + \tfrac{1}{3}c_{10} = \tfrac{1}{2}T_1 + T_4 = T_6 + T_5 + \tfrac{1}{6}c_{24},\\
2A_{14} - 12A_{28} = -3T_1 + \tfrac{1}{2}T_9 + \tfrac{1}{3}c_{12} = -\,3T_2 - 3T_3 + \tfrac{1}{3}c_{14},\\
4A_{13} - 24A_{28} = -3(T_1 + T_2 + T_3) + \tfrac{1}{2}T_9 + \tfrac{1}{3}c_{13}, \quad 4A_{18} = \tfrac{1}{3}c_{15}, \quad 2A_{15} = \tfrac{1}{3}c_{16} = T_{10} + \tfrac{1}{3}c_{17},\\
2A_{16} - 48A_{28} = 2T_9 - 3T_8 + \tfrac{1}{3}c_{18} = -\,6T_7 - 3T_8 + \tfrac{1}{3}c_{19}, \quad 2A_{26} = \tfrac{1}{3}c_{20},\\
36A_{28} + 2A_{23} = -3T_9 + \tfrac{1}{6}c_{21}, \quad 36A_{28} + 2A_{22} = -\,3T_9 + \tfrac{1}{3}c_{22}, \quad 2A_{25} = -6T_{10} + \tfrac{1}{3}c_{23},\\
2A_{19} - 12A_{28} = -6T_6 - 6T_5 + \tfrac{1}{6}c_{26} = -\,6T_4 + \tfrac{1}{2}T_9,\\
2A_{24} = \tfrac{1}{6}c_{27}, \quad 2A_{27} = \tfrac{1}{6}c_{28}, \quad 2A_2 = \tfrac{1}{6}c_{29},\\
6A_5 + 3A_{28} = \tfrac{1}{2}(T_1 + T_2 + T_3), \quad 6A_8 + \tfrac{3}{2}A_{28} = \tfrac{1}{2}(T_4 + T_6 + T_5)
\end{aligned} \tag{2.28}$$

where the c_α are given in Eq. (2.16). (The coefficients A_1, A_{12} and A_{21} are determined immediately by the second of Eqs. (2.23), so we simply list their values later in Eqs. (2.31).) Solving Eqs. (2.28), we find the conditions

$$\begin{aligned}
T_2 + T_3 = 2T_1 - \tfrac{2}{3}c_8, \quad T_4 = \tfrac{1}{2}T_1 - \tfrac{1}{3}c_8 + \tfrac{1}{3}c_{10},\\
T_5 + T_6 = T_1 - \tfrac{1}{6}c_{24} - \tfrac{1}{3}c_8 + \tfrac{1}{3}c_{10}, \quad T_7 = 2T_1 - \tfrac{1}{3}c_{11},\\
T_8 = 4T_1 - \tfrac{8}{3}c_8 + \tfrac{2}{3}c_9, \quad T_9 = -\,6T_1 + 2c_{24} + \tfrac{1}{3}c_{26}, \quad T_{10} = \tfrac{1}{12}(c_6 - c_5),
\end{aligned} \tag{2.29}$$

together with conditions on the β-function coefficients

$$\begin{aligned}
c_2 = -\tfrac{1}{4}c_3 - c_9 + 4c_{10}, \quad c_{11} = \tfrac{1}{4}c_3 + 4c_8 - c_9,\\
c_5 - c_6 = -4(c_{16} - c_{17}), \quad c_{14} = \tfrac{3}{8}c_3 - \tfrac{3}{2}c_9 + c_{12} - \tfrac{1}{4}(c_{18} - c_{19})\\
c_{24} = \tfrac{1}{8}c_3 + 2c_8 - \tfrac{1}{2}c_9 + \tfrac{1}{2}c_{25}, \quad c_{26} = -\tfrac{1}{2}(c_{18} - c_{19}) - 3c_{25}.
\end{aligned} \tag{2.30}$$

The conditions on T_{1-6} in Eq. (2.29) were already derived in Ref. [15]. Reassuringly, the conditions Eq. (2.30) are satisfied by the coefficients in Eq. (2.16), and also by the redefinitions in Eq. (2.19). These six constraints in principle leave only 19 of the 25 independent coefficients in the two-loop β-function to be determined by perturbative computation.

It turns out that Eq. (2.23) is sufficient to determine the Yukawa or λ-dependent part of $\tilde{A}^{(4)}$ up to three free parameters; here are the results for the case of dimensional regularisation:

$$
\begin{aligned}
&A_1 = \tfrac{1}{144}, \quad A_2 = \tfrac{1}{144}, \quad A_3 = -\tfrac{1}{12}, \quad A_4 = \tfrac{1}{18}, \quad A_5 = \tfrac{1}{144}, \quad A_6 = A_{11} = 0, \\
&A_7 = -\tfrac{1}{24}, \quad A_8 = -\tfrac{1}{288}, \quad A_9 = \tfrac{1}{12}, \quad A_{10} = \tfrac{1}{6}, \quad A_{12} = -\tfrac{1}{24}, \quad A_{13} = -\tfrac{7}{24}, \\
&A_{14} = -\tfrac{1}{6}, \quad A_{15} = \tfrac{3}{4}, \quad A_{16} = -\tfrac{2}{3}, \quad A_{17} = \tfrac{1}{2}, \quad A_{18} = \tfrac{1}{2}, \quad A_{19} = \tfrac{1}{12}, \\
&A_{20} = \tfrac{3}{16}, \quad A_{21} = \tfrac{1}{4}, \quad A_{22} = \tfrac{3}{4}, \quad A_{23} = 1, \quad A_{24} = -\tfrac{7}{8}, \quad A_{25} = -\tfrac{7}{2}, \\
&A_{26} = -\tfrac{1}{72}(194C_G - 20R^\psi - 11R^\varphi) - t_1\beta_0, \\
&A_{27} = \tfrac{1}{144}(147C_G - 12R^\psi - 3R^\varphi) - t_2\beta_0,
\end{aligned} \tag{2.31}
$$

where β_0 is given in Eq. (2.5). Since A_6 only appears in Eq. (2.28) in the combination $4A_6 + 2A_{28}$, we have set $A_6 = 0$ in line with Ref. [15]. We note that under the redefinitions in Eq. (2.17),

$$
\delta\tilde{A}^{(4)} = -\left(\delta y \cdot \frac{\partial}{\partial y} + \delta g \, \frac{\partial}{\partial g}\right) \tilde{A}^{(3)} + \mathrm{O}(g^6)\,. \tag{2.32}
$$

Moreover the effect of these redefinitions on the metric coefficients in Eq. (2.23) (as parametrised in Eq. (2.24)) may easily be computed using Eq. (2.10) as

$$
\begin{aligned}
\delta T_1 = \delta T_2 = &\delta T_3 = -\tfrac{2}{3}\mu_2, \\
\delta T_4 = \delta T_5 = &\delta T_6 = -\tfrac{1}{3}\mu_3, \\
\delta T_7 = &\tfrac{1}{2}\delta T_8 = -\tfrac{1}{3}\mu_1, \\
\delta T_9 = -\tfrac{2}{3}\mu_4, &\quad \delta T_{10} = -\tfrac{1}{3}\mu_5, \\
\delta t_1 = -\tfrac{4}{3}\mu_4, &\quad \delta t_2 = -\tfrac{2}{3}\mu_5.
\end{aligned} \tag{2.33}
$$

Using Eq. (2.19)), these results are easily seen to agree with Eq. (2.29).

It is remarkable that no knowledge of the "metric" coefficients T_α is required to determine the A_α in this fashion; of course the t_i in Eq. (2.31), which define the "off-diagonal" fermion-gauge metric in Eq. (2.23), could be determined by a perturbative calculation if required, as was accomplished for the fermion-scalar case in Ref. [15]. The results in Eq. (2.31) will be used in Sect. 3 in a check of the three-loop β_g.

In Ref. [15] the extension to three loops was accomplished by first inferring the three-loop Yukawa β-function for a chiral fermion-scalar theory, using the three-loop results derived in Ref. [30] for the standard model, combined with the results for the supersymmetric Wess-Zumino model. Such an approach will not work in the gauged case, unfortunately; the results of Ref. [30] are only for the $SU(3)$ colour gauge group, which of course is not sufficient to determine how the three-loop Yukawa β-function depends on a general gauge coupling.

3 The three-loop gauge β-function

The three-loop gauge β-function was computed in Ref. [25] for a general gauge theory coupled to fermions and scalars. In this section we shall show that our result for $\tilde{A}^{(4)}$ is compatible with this result via Eq. (1.2). In fact, our result for $\tilde{A}^{(4)}$ determines the 16 terms in $\beta_g^{(3)}$ with Yukawa couplings up to 4 (see later) unknown parameters. It is rather striking that the two-loop calculation of $\beta_y^{(2)}$ (and the one-loop result $\beta_\lambda^{(1)}$) have thereby provided so much information on a three-loop RG quantity. This is an example of the "$3-2-1$" phenomenon noted in Refs. [23, 31]; namely that the gauge-gauge, fermion-fermion and scalar-scalar contributions to the metric G_{IJ} start at successive loop orders.

In our notation, $\beta_g^{(3)}$ can be written

$$\begin{aligned}\tfrac{1}{g}\beta_g^{(3)} = \tfrac{1}{16n_V}\big(&-\tfrac{2}{3}G^A_{12} + G^A_{13} + 3G^A_{14} + 4G^A_{15} + 12G^A_{16} - 8G^A_{17} + 8G^A_{18}\\ &+ 7G^A_{19} - G^A_{20} + 8G^A_{21} - 10G^A_{22} - 2G^A_{23}\\ &- 28G^A_{24} - 64G^A_{25} - 48C_G G^A_{26} + 18C_G G^A_{27} + \mathrm{O}(g^6)\big)\,,\end{aligned} \tag{3.1}$$

where the G^A_α are implicitly defined in Table (3). The purely g-dependent terms are not determined in this analysis. It is then easy to show, using Eqs. (2.26), (2.31), that we can write

$$g\frac{\partial}{\partial g}\tilde{A}^{(4)} = T^{(1)}_{gg}\beta_g^{(3)} + T^{(2)}_{gg}\beta_g^{(2)} + T^{(3)}_{gg}\beta_g^{(1)} + T^{(3)}_{gy}\cdot\beta_y^{(1)}\,, \tag{3.2}$$

in the form,

$$\begin{aligned}g\frac{\partial}{\partial g}\tilde{A}^{(4)} = {}& 2n_V\,\tfrac{1}{g}\beta_g^{(3)} + \tfrac{1}{3}gn_V\left(102C_G - 20R^\psi - 7R^\varphi\right)\beta_g^{(2)} + T^{(3)}_{gg}\beta_g^{(1)}\\ &- g^2\tfrac{17}{12}\,\mathrm{tr}[\hat{y}_a C^\varphi \beta^{(1)}_{y\,a}] + g^2 (C^\varphi)_{ab}\,\mathrm{tr}[\hat{y}_a\beta^{(1)}_{y\,b}]\,,\end{aligned} \tag{3.3}$$

where $\beta_g^{(1)}, \beta_g^{(2)}$ are given in Eq. (2.5). We notice that $T^{(2)}_{gg}$ agrees with the result for σ in Eq. (2.11). $T^{(3)}_{gg}$ takes the form

$$T^{(3)}_{gg} = g\left(-\tfrac{10}{3} + 4t_1\right)\mathrm{tr}[\hat{y}_a y_a C^\psi] + g\left(-\tfrac{1}{2} + 4t_2\right)(C^\varphi)_{ab}\mathrm{tr}[y_a\hat{y}_b] + \mathrm{O}(g^3)\,. \tag{3.4}$$

Unfortunately we have no means of disentangling the separate purely g-dependent contributions in $\tilde{A}^4$ and in $T^{(3)}_{gg}\beta_g^{(1)}$, without a three-loop calculation; but all the Yukawa or λ dependent contributions match exactly. If

$$t_1 = -\tfrac{17}{12}\,, \qquad t_2 = 1\,, \tag{3.5}$$

then we would have T_{IJ} symmetric at this order; but as demonstrated in Ref. [15], at three loops T_{IJ} is not symmetric even for a pure fermion-scalar theory for a general renormalisation scheme.

Had we not known $\beta_g^{(3)}$ then it would have been determined by Eq. (3.2) up to the four parameters consisting of the two coefficients in $T^{(2)}_{gy}$ and the two coefficients in $T^{(3)}_{gg}$ (the values quoted for these quantities in Eqs. (3.3), (3.4) of course deriving from our current knowledge of $\beta_g^{(3)}$).

4 The supersymmetric case

Here the analysis is extended to a general $\mathcal{N}=1$ supersymmetric gauge theory, which may in principle be obtained from the general non-supersymmetric theory discussed in Sect. 2 by an appropriate choice of fields and couplings. Such a theory can of course be rewritten in terms of n_C chiral and corresponding conjugate anti-chiral superfields, and indeed perturbative computations are enormously simplified through the use of this formalism; moreover, in the light of the non-renormalisation theorem and the NSVZ formula [32, 33] for the exact gauge β-function, the renormalisation of the theory is essentially entirely determined by the chiral superfield anomalous dimension γ (at least in a suitable renormalisation scheme). In this section we shall therefore start anew using results derived using superfield methods. Nevertheless, in Sect 5 we show that (at least up two loops) the results obtained using the two approaches match, as indeed they must.

The crucial new feature in the supersymmetric context is the existence of a proposed exact formula for the a-function [17–19]. This exact form was verified up to two loops in Ref. [20] for a general supersymmetric gauge theory, and up to three loops [15] in the case of the Wess-Zumino model. Moreover in Ref. [15] a sufficient condition on γ to guarantee the validity of this exact result was found and shown to be satisfied up to three loops; related considerations appear in Refs. [18, 19], see later for a discussion. In this section we shall generalise this condition to the gauged case and check that it is satisfied up to three loops, using the results of Ref. [22].

The couplings g^I are now given by $g^I = \{g, Y^{ijk}, \bar{Y}_{ijk}\}$ with $\bar{Y}_{ijk} = (Y^{ijk})^*$. The supersymmetric Yukawa β-functions are expressible in terms of the anomalous dimension matrix $\gamma_i{}^j$ in the form

$$\beta_Y = Y * \gamma\,, \qquad \beta_{\bar{Y}} = \gamma * \bar{Y}\,, \tag{4.1}$$

where for arbitrary $\omega_i{}^j$ we define

$$\begin{aligned}(Y*\omega)^{ijk} &\equiv Y^{ljk}\omega_l{}^i + Y^{ilk}\omega_l{}^j + Y^{ijl}\omega_l{}^k\,,\\ (\omega*\bar{Y})_{ijk} &\equiv \omega_i{}^l\,\bar{Y}_{ljk} + \omega_j{}^l\,\bar{Y}_{ilk} + \omega_k{}^l\,\bar{Y}_{ijl}\,.\end{aligned} \tag{4.2}$$

We also introduce a scalar product for Yukawa couplings[3]

$$Y\circ\bar{Y} = \bar{Y}\circ Y = \tfrac{1}{6}\,Y^{ijk}\bar{Y}_{ijk}\,, \tag{4.3}$$

and it is further useful to define

$$(\bar{Y}Y)_i{}^j = \tfrac{1}{2}\,\bar{Y}_{ikl}Y^{jkl} \quad\Rightarrow\quad Y\circ(\omega*\bar{Y}) = (Y*\omega)\circ\bar{Y} = \mathrm{tr}\big((\bar{Y}Y)\,\omega\big)\,. \tag{4.4}$$

The gauge β-function is assumed to have the form

$$\beta_g = f(g)\,\tilde{\beta}_g\,, \qquad \tilde{\beta}_g = Q - 2n_V^{-1}\,\mathrm{tr}[\gamma\,C_R]\,, \quad f(g) = g^3 + \mathrm{O}(g^5)\,, \tag{4.5}$$

[3]The normalisation here is different from [15].

where, with R_A the gauge group generators,

$$Q = T_R - 3C_G, \quad T_R\delta_{AB} = \mathrm{tr}(R_A R_B), \quad C_G\delta_{AB} = f_{ACD}f_{BCD}, \quad (C_R)_i{}^j = (R_A R_A)_i{}^j\,, \tag{4.6}$$

and n_V is the dimension of the gauge group. For gauge invariance we must have

$$Y * R_A = 0\,, \qquad R_A * \bar{Y} = 0\,. \tag{4.7}$$

Under a change $g \to g'(g) = g + \mathrm{O}(g^3)$ then in Eq. (4.5)

$$f(g) \to f'(g') = \frac{\partial g'}{\partial g}\, f(g)\,, \qquad \gamma(g) \to \gamma'(g') = \gamma(g)\,, \tag{4.8}$$

assuming g' is independent of $Y, \bar{Y}$. For an infinitesimal change $\delta f = f\,\partial_g \delta g - \delta g\,\partial_g f$ and $\delta\gamma = -\delta g\,\partial_g \gamma$. The NSVZ form for the β-function is obtained if

$$f(g) = \frac{g^3}{1 - 2\,C_G\,g^2}\,. \tag{4.9}$$

The resulting expression for β_g originally appeared (for the special case of no chiral superfields) in Ref. [29], and was subsequently generalised, using instanton calculus, in Ref. [32]. (See also Ref. [33].) We note here that this result (called the NSVZ form of β_g) is only valid in a specific renormalisation scheme, which we correspondingly term the NSVZ scheme. The exact expression generalises one and two-loop results obtained in Refs. [34–36]. These results were computed using the dimensional reduction (DRED) scheme; though in any case, the DRED and NSVZ schemes only part company at three loops [39].

The one and two-loop results for γ are given by [37,38]

$$\begin{aligned} \gamma^{(1)} &= P\,, \\ \gamma^{(2)} &= \,-\,S_1 - 2g^2\,C_R P + 2g^4\,Q\,C_R\,, \end{aligned} \tag{4.10}$$

where P and S_1 are defined by

$$\begin{aligned} P_i{}^j &= (\bar{Y}Y)_i{}^j - 2g^2 (C_R)_i{}^j\,, \\ S_{1i}{}^j &= \bar{Y}_{ikn} Y^{jmn} P_m{}^k\,. \end{aligned} \tag{4.11}$$

We use here the notation and conventions of Ref. [22].

In the supersymmetric theory Eq. (1.2) is assumed to now take the form

$$\begin{aligned} \mathrm{d}_Y \tilde{A} &= \tfrac{1}{2}\,\mathrm{d}Y \circ T_{Y\bar{Y}} \circ \beta_{\bar{Y}} + \mathrm{d}Y \circ T_{Yg}\,\tilde{\beta}_g\,, \\ \mathrm{d}_g \tilde{A} &= \mathrm{d}g\big(T_{gg}\,\tilde{\beta}_g + T_{gY} \circ \beta_Y + T_{g\bar{Y}} \circ \beta_{\bar{Y}}\big)\,, \end{aligned} \tag{4.12}$$

(with a similar equation for $\mathrm{d}_{\bar{Y}} \tilde{A}$). We have written the RHS in terms of $\tilde{\beta}_g$, effectively absorbing the factor $f(g)$ in Eq. (4.5) into T_{Yg} and T_{gg}. We omit potential β_Y terms in the first of Eqs (4.12) since are not necessary to the order we shall consider. For $\mathcal{N}=1$ supersymmetric theories there is, at critical points with vanishing β-functions, an exact expression for a [17] in terms of the anomalous dimension matrix γ or alternatively the R-charge $R=\frac{2}{3}(1+\gamma)$. Introducing terms linear in β-functions there is a corresponding expression which is valid away from critical points and this can then be shown to satisfy many of the properties associated with the a-theorem [18], [19]. For the theory considered here, with n_C chiral scalar multiplets, these results take the form

$$\tilde{A} = \tfrac{1}{12}(n_C + 9\,n_V) - \tfrac{1}{2}\,\mathrm{tr}(\gamma^2) + \tfrac{1}{3}\,\mathrm{tr}(\gamma^3) + \Lambda\circ\beta_{\bar{Y}} + n_V\,\lambda\,\tilde{\beta}_g + \beta_Y\circ H\circ\beta_{\bar{Y}}\,, \tag{4.13}$$

where $\tilde{\beta}_g$ is given by Eq. (4.5) and we require

$$\Lambda\circ\beta_{\bar{Y}} = \beta_Y\circ\bar{\Lambda}\,. \tag{4.14}$$

For the remainder of this section we omit for simplicity the term involving H in Eq. (4.13); but return to it in Sect. 5. In Refs. [18] and [19] Λ,λ are Lagrange multipliers enforcing constraints on the R-charges. At lowest order the result for Λ and also the metric G obtained in Ref. [18] are equivalent, up to matters of definition and normalisation, with those obtained here. The general form for $\tilde{A}$ given by Eq. (4.13) was verified up to two-loop order (for the anomalous dimension) in Ref. [20]. Λ may be constrained by imposing Eq. (4.12). Then

$$\begin{aligned}\mathrm{d}_Y\tilde{A} = {}& \mathrm{tr}\big[\mathrm{d}_Y\gamma\,\big((\bar{Y}\Lambda) - 2\,\lambda\,C_R - \gamma + \gamma^2\big)\big]\\ &+ (\mathrm{d}_Y\Lambda)\circ\beta_{\bar{Y}} + n_V\,\mathrm{d}_Y\lambda\,\tilde{\beta}_g\,.\end{aligned} \tag{4.15}$$

We also have

$$\begin{aligned}\mathrm{d}_g\tilde{A} = {}& \mathrm{tr}\big[\mathrm{d}_g\gamma\,\big((\bar{Y}\Lambda) - 2\,\lambda\,C_R - \gamma + \gamma^2\big)\big]\\ &+ (\mathrm{d}_g\Lambda)\circ\beta_{\bar{Y}} + n_V\,\mathrm{d}_g\lambda\,\tilde{\beta}_g\,.\end{aligned} \tag{4.16}$$

Hence if Λ, λ are required to obey

$$(\bar{Y}\Lambda) - 2\,\lambda\,C_R = \gamma - \gamma^2 + \Theta\circ\beta_{\bar{Y}} + \theta\,\tilde{\beta}_g\,, \tag{4.17}$$

where making the indices explicit $\Theta\circ\mathrm{d}\bar{Y} \to \Theta_i{}^{j,klm}\mathrm{d}\bar{Y}_{klm}$ and $\theta\to\theta_i{}^j$, Eq. (4.13) then satisfies Eq. (4.12) if we take

$$\begin{aligned}\tfrac{1}{2}\,\mathrm{d}Y\circ T_{Y\bar{Y}}\circ\mathrm{d}\bar{Y} &= \mathrm{tr}\big[\mathrm{d}_Y\gamma\;\Theta\circ\mathrm{d}\bar{Y}\big] + \mathrm{d}_Y\Lambda\circ\mathrm{d}\bar{Y}\,,\\ \mathrm{d}Y\circ T_{Yg} &= \mathrm{tr}[\mathrm{d}_Y\gamma\,\theta] + n_V\,\mathrm{d}_Y\lambda\,,\\ \mathrm{d}g\,T_{gg} &= \mathrm{tr}\,[\mathrm{d}_g\gamma\,\theta] + n_V\,\mathrm{d}_g\lambda\,,\\ \mathrm{d}g\,T_{g\bar{Y}}\circ d\bar{Y} &= \mathrm{tr}\,\big[\mathrm{d}_g\gamma\,\Theta\circ\mathrm{d}\bar{Y}\big] + \mathrm{d}_g\Lambda\circ\mathrm{d}\bar{Y}\,.\end{aligned} \tag{4.18}$$

Here $T_{gY}=0$. However from Eq. (4.14)

$$\mathrm{d}_g\Lambda\circ\beta_{\bar{Y}} - \beta_Y\circ\mathrm{d}_g\bar{\Lambda} = \mathrm{tr}\big[\mathrm{d}_g\gamma\big((\bar{\Lambda}Y) - (\bar{Y}\Lambda)\big)\big]\,, \tag{4.19}$$

which may be used to write Eq. (4.18) in equivalent forms with non-zero T_{gY}.

A related result to Eq. (4.17), with effectively $\Theta, \theta = 0$, is contained in Ref. [18] and also discussed in Ref. [19]. For supersymmetric theories, satisfying Eq. (4.17) is consequently essentially equivalent to requiring Eq. (4.12), although terms involving Θ are necessary at higher orders. However, the work of Refs. [18, 19] implies that at least in the pure gauge case, there may be renormalisation schemes in which θ may be set to zero. It is striking that only minor modifications to the condition proposed in Ref. [15] are required for the extension to the gauged case.

The condition (4.17) does not fully determine λ, θ since we have the freedom

$$\lambda \sim \lambda + \mu\, \tilde{\beta}_g\,, \qquad \theta \sim \theta - 2\,\mu\, C_R\,, \tag{4.20}$$

for arbitrary μ. There is also a similar freedom in Λ, Θ.

At lowest order Θ, θ do not contribute so that (4.17) becomes

$$(\bar{Y}\Lambda^{(1)}) - 2\,\lambda^{(1)}\, C_R = \gamma^{(1)}\,, \tag{4.21}$$

and we may simply take from Eqs. (4.10), (4.11)

$$\Lambda^{(1)} = Y\,, \qquad \lambda^{(1)} = g^2\,, \tag{4.22}$$

from which

$$\tfrac{1}{2}\,\mathrm{d}Y \circ T_{Y\bar{Y}}{}^{(1)} \circ \mathrm{d}\bar{Y} = \mathrm{d}Y \circ \mathrm{d}\bar{Y}\,, \qquad T_{gg}{}^{(1)} = 2\, n_V\, g\,. \tag{4.23}$$

At the next order we require

$$(\bar{Y}\Lambda^{(2)}) - 2\,\lambda^{(2)} C_R = \gamma^{(2)} - \gamma^{(1)2} + \Theta^{(1)} \circ \beta_{\bar{Y}}{}^{(1)} + \theta^{(1)}\, Q\,, \tag{4.24}$$

since $\tilde{\beta}_g{}^{(1)} = Q$, with Q as defined in Eq. (4.6). We may parameterise $\Lambda^{(2)}$ and $\Theta^{(1)}$ by

$$\Lambda^{(2)} = \tilde{\Lambda}\, Y * P\,, \qquad \Theta^{(1)} \circ \mathrm{d}\bar{Y} = \tilde{\Theta}\,(\mathrm{d}\bar{Y}\, Y)\,, \tag{4.25}$$

since $(\bar{Y}\, Y * P) = S_1 + (\bar{Y}Y)\, P$ and $(\beta_{\bar{Y}}{}^{(1)} Y) = (P * \bar{Y}\, Y) = S_1 + P\,(\bar{Y}Y)$ with S_1 as in Eq. (4.11). We then find Eq. (4.24) requires, since $(\bar{Y}Y)C_R = C_R(\bar{Y}Y)$,

$$\tilde{\Lambda} - \tilde{\Theta} = -1\,, \tag{4.26}$$

and

$$-2\,\lambda^{(2)} C_R - \theta^{(1)}\, Q = 2\, g^4\, Q C_R\,. \tag{4.27}$$

Hence

$$\lambda^{(2)} = \tilde{\lambda}\, g^4 Q\,, \qquad \theta^{(1)} = \tilde{\theta}\, g^4 C_R\,, \tag{4.28}$$

with

$$2\,\tilde{\lambda} + \tilde{\theta} = -2\,. \tag{4.29}$$

G^{Λ}_1 G^{Λ}_2 G^{Λ}_3 G^{Λ}_4

G^{Λ}_5 G^{Λ}_6 G^{Λ}_7

G^{Λ}_8 G^{Λ}_9

Table 4: Contributions to $\Lambda^{(3)} \circ \mathrm{d}\bar{Y}$

The freedom of choosing $\tilde{\lambda}, \theta$ while satisfying Eq. (4.29) is a reflection of Eq. (4.20). From Eq. (4.18)

$$T_{gg}{}^{(2)} = 4\tilde{\lambda}\, n_V g^3 Q\,. \tag{4.30}$$

As a consequence of (4.20) $\tilde{\lambda}$ is arbitrary. The computation in Ref. [20] (specialising the DRED version of Eq. (2.11) to the supersymmetric case; and adjusting for the differing definition of the "gg" metric) for $T_{gg}{}^{(2)}$ fixes

$$\tilde{\lambda} = -\tfrac{5}{2}\,, \tag{4.31}$$

in this scheme.

At third order we require now in order to satisfy Eq. (4.17)

$$\begin{aligned}(\bar{Y}\Lambda^{(3)}) - 2\,\lambda^{(3)} C_R = \gamma^{(3)} - \gamma^{(2)}\gamma^{(1)} - \gamma^{(1)}\gamma^{(2)} + \Theta^{(2)} \circ \beta_{\bar{Y}}{}^{(1)} + \theta^{(2)} Q \\ + \Theta^{(1)} \circ \beta_{\bar{Y}}{}^{(2)} + \theta^{(1)} \tilde{\beta}_g{}^{(2)},\end{aligned} \tag{4.32}$$

where we write

$$\begin{aligned}\Lambda^{(3)} \circ \mathrm{d}\bar{Y} = {} & \sum_{\alpha=1}^{9} \Lambda_\alpha G^{\Lambda}_\alpha \\ & + g^2(\Lambda_{10} Q + \Lambda_{11} C_G)\,(Y * P) \circ \mathrm{d}\bar{Y} \; + g^4(\Lambda_{12} Q + \Lambda_{13} C_G)\,(Y * C_R) \circ \mathrm{d}\bar{Y}\,,\end{aligned} \tag{4.33}$$

G_1^Θ G_2^Θ G_3^Θ G_4^Θ

G_5^Θ G_6^Θ

Table 5: Contributions to $\Theta^{(2)} \circ \mathrm{d}\bar{Y}$

where P is defined in Eq. (4.11), and the other distinct terms contributing to $\Lambda^{(3)} \circ \mathrm{d}\bar{Y}$ are shown diagrammatically in Table (4). Here a "blob" represents an insertion of the one-loop anomalous dimension. The 3-point vertices alternate between Y and $\bar{Y}$. As an example, G_6^Λ represents a contribution

$$\Lambda^{(3)} \circ \mathrm{d}\bar{Y} = g^2\, Y^{ikl} P_k{}^m (C_R)_l{}^n \mathrm{d}\bar{Y}_{imn}\,, \tag{4.34}$$

and Eq. (4.4) then implies a contribution to $(\bar{Y}\Lambda^{(3)})$ of the form

$$(\bar{Y}\Lambda^{(3)})_i{}^j = g^2\big(\bar{Y}_{imn} Y^{jkl} P_k{}^m (C_R)_l{}^n + S_{1i}{}^k (C_R)_k{}^j + S_{2i}{}^k P_k{}^j\big)\,. \tag{4.35}$$

Here P, S_1 are given in Eq. (4.11) and S_2 is defined by

$$S_{2i}{}^j = \bar{Y}_{ikn} Y^{jmn} (C_R)_m{}^k\,. \tag{4.36}$$

Similarly we write

$$\begin{aligned}
\Theta^{(2)} \circ \mathrm{d}\bar{Y} &= \sum_{\alpha=1}^{6} \Theta_\alpha G_\alpha^\Theta + g^2\big(\Theta_7\, Q + \Theta_8\, C_G\big)\,(\mathrm{d}\bar{Y} Y\big)\,, \\
\lambda^{(3)} &= g^4 \tilde{\lambda}_1\, \mathrm{tr}[P C_R]/n_V + g^6\left(\tilde{\lambda}_2\, \mathrm{tr}[C_R^2]/n_V + \tilde{\lambda}_3\, Q^2 + \tilde{\lambda}_4\, Q C_G + \tilde{\lambda}_5\, C_G^2\right) \\
\theta^{(2)} &= g^4 \tilde{\theta}_1\, S_2 + g^4 \tilde{\theta}_2\, P C_R + g^6\big(\tilde{\theta}_3\, Q C_R + \tilde{\theta}_4\, C_G C_R + \tilde{\theta}_5\, C_R^2\big),
\end{aligned} \tag{4.37}$$

where the G_α^Θ are shown diagrammatically in Table (5). A term in S_1 is apparently possible in $\theta^{(2)}$ but is excluded since there is no contribution to $\gamma^{(3)}$ involving $g^2 Q S_1$. As a consequence of (4.20) the resulting equations depend only on $2\tilde{\lambda}_3 + \tilde{\theta}_3$, $2\tilde{\lambda}_4 + \theta_4$.

We expand the three-loop anomalous dimension as

$$\gamma^{(3)} = \sum_{\alpha=1}^{24} \gamma_\alpha G_\alpha^\gamma, \tag{4.38}$$

G_1^γ	G_2^γ	G_3^γ	G_4^γ
G_5^γ	G_6^γ	G_7^γ	
G_9^γ	G_{10}^γ		

Table 6: Contributions to $\gamma^{(3)}$

with

$$\begin{gathered} G_8^\gamma = g^2 S_1 C_R, \quad G_{11}^\gamma = g^4 P C_R^2, \quad G_{12}^\gamma = g^4 S_2 C_R, \quad G_{13}^\gamma = g^4 C_G S_2, \\ G_{14}^\gamma = g^4 C_G P C_R, \quad G_{15}^\gamma = g^4 Q P C_R, \quad G_{16}^\gamma = g^4 Q S_2, \quad G_{17}^\gamma = g^4 \mathrm{tr}[P C_R]/n_V \, C_R, \\ G_{18}^\gamma = g^6 C_R^3, \quad G_{19}^\gamma = g^6 C_G C_R^2, \quad G_{20}^\gamma = g^6 Q C_R^2, \quad G_{21}^\gamma = g^6 Q^2 C_R, \\ G_{22}^\gamma = g^6 Q C_G C_R, \quad G_{23}^\gamma = g^6 C_G^2 C_R, \quad G_{24}^\gamma = g^6 \mathrm{tr}[C_R^2]/n_V \, C_R, \end{gathered} \tag{4.39}$$

and with Q, P, $S_{1,2}$ as defined in Eqs. (4.6), (4.11), (4.36). The remainder of the distinct tensor contributions are depicted in diagrammatic form in Table (6). The basis for $\gamma^{(3)}$ is restricted by the absence of one particle reducible contributions such as P^3, $P^2 C_R$, $S_{1,2}P$, $PS_{1,2}$.

Using Eqs. (4.10), (4.25) in Eq. (4.32) leads to a large number of consistency equations which constrain $\gamma^{(3)}$. If $g = 0$ they reduce to

$$\begin{aligned} 2\Lambda_1 &= \gamma_1 + \Theta_3 + 2\Theta_4 = 2\Theta_1 + 2\Theta_2 \, , \\ 2\Lambda_2 &= 2\gamma_2 + 2\Theta_3 = 1 + 2\Theta_1 - \tilde{\Theta} \, , \\ 2\Lambda_3 &= \gamma_3 + 2\Theta_4 - \tilde{\Theta} = 1 + 2\Theta_2 + \Theta_3 \, , \\ 3\Lambda_4 &= \gamma_4 \, , \end{aligned} \tag{4.40}$$

which requires

$$\gamma_1 - 2\gamma_2 - \gamma_3 = -2 \, . \tag{4.41}$$

These results were obtained in Ref. [15]. The other special case is for $Y, \bar{Y} = 0$ when

$$\gamma^{(3)} = (\gamma_{18} - 2\gamma_{11})\, g^6 C_R^3 + \big((\gamma_{19} - 2\gamma_{14})\, C_G + (\gamma_{20} - 2\gamma_{15})\, Q\big) g^6 C_R^2 \\ + \big(\gamma_{21}\, Q^2 + \gamma_{22}\, Q C_G + \gamma_{23}\, C_G^2 + (\gamma_{24} - 2\gamma_{17})\, \mathrm{tr}[C_R^2]/n_V\big) g^6 C_R\,. \tag{4.42}$$

In this case applying Eq. (4.32) with $\Lambda, \Theta \to 0$ it is necessary to require the conditions

$$\gamma_{18} - 2\gamma_{11} = -16\,, \qquad \gamma_{19} - 2\gamma_{14} = 0\,, \tag{4.43}$$

as well as

$$\gamma_{20} - 2\gamma_{15} = \, -8 - \tilde{\theta}_5 + 2\tilde{\theta}_2\,, \quad \gamma_{21} = -2\tilde{\lambda}_3 - \tilde{\theta}_3\,, \quad \gamma_{22} = -2\tilde{\lambda}_4 - \tilde{\theta}_4\,, \\ \gamma_{23} = \, -2\tilde{\lambda}_5\,, \quad \gamma_{24} - 2\gamma_{17} = 4\tilde{\lambda}_1 - 2\tilde{\lambda}_3 - 4\tilde{\theta}\,. \tag{4.44}$$

The relations in Eq. (4.41) were obtained in Refs. [18, 19].

For the general case Eq. (4.32) implies additional relations which further constrain γ_α. From terms which start at $\mathrm{O}((Y\bar{Y})^2)$ and using Eq. (4.40)

$$2\Lambda_5 = \gamma_5 + \Theta_6 + 4\Theta_4 - 2\tilde{\Theta} = 4 + 2\Theta_5 - 2\tilde{\Theta}\,, \\ \Lambda_6 = \gamma_6 + \Theta_6 = 0\,, \quad 2\Lambda_7 = \gamma_7 = \Theta_6\,, \\ \Lambda_6 + 4\Lambda_3 = \gamma_8 + 2\Theta_5 - 2\tilde{\Theta}\,, \\ \Lambda_{10} = \Theta_7\,, \quad \Lambda_{11} = \Theta_8\,, \tag{4.45}$$

which then entail, using Eq. (4.40) to eliminate Λ_3,

$$\gamma_8 + \gamma_5 - \gamma_6 = 4 + 2\gamma_3\,, \quad \gamma_6 + \gamma_7 = 0\,. \tag{4.46}$$

The remaining conditions arise from terms at $\mathrm{O}(Y\bar{Y})$ which become, using Eq. (4.45) to eliminate Λ_5,

$$2\Lambda_8 = \gamma_9 = \gamma_{11} - 8\,, \\ \Lambda_9 = \gamma_{10}\,, \quad 2\Lambda_9 + 4\Lambda_7 = \gamma_{12}\,, \\ \Lambda_{12} = \gamma_{16} + 2\tilde{\Theta} + \tilde{\theta}_1\,, \\ \Lambda_{12} + 2\Lambda_{10} = \gamma_{15} - 4 + 2\Theta_7 + 2\tilde{\Theta} + \tilde{\theta}_2\,, \\ \Lambda_{13} = \gamma_{13}\,, \quad \Lambda_{13} y + 2\Lambda_{11} = \gamma_{14} + 2\Theta_8\,, \\ -2\tilde{\lambda}_1 = \gamma_{17} - 2\tilde{\theta}\,, \tag{4.47}$$

which then give

$$2\gamma_7 + 2\gamma_{10} - \gamma_{12} = 0\,, \quad \gamma_9 - \gamma_{11} = -8\,, \quad \gamma_{13} = \gamma_{14}\,. \tag{4.48}$$

As a consequence of Eq. (4.20) the freedom $\delta\tilde{\theta} = -2\mu$ requires also $\delta\tilde{\lambda}_1 = -2\mu$. We may combine Eq. (4.44) with Eq. (4.47) to give $\gamma_{24} = -2\tilde{\lambda}_2$.

Altogether Eqs. (4.41), (4.43), (4.46), (4.48) give eight conditions which are all satisfied by the coefficients as calculated [21, 22]:

$$
\begin{gathered}
\gamma_1 = -1, \quad \gamma_2 = -\tfrac{1}{2}, \quad \gamma_3 = 2, \quad \gamma_4 = \tfrac{1}{4}\kappa, \quad \gamma_5 = -2\kappa + 4, \\
\gamma_6 = -\kappa, \quad \gamma_7 = \kappa, \quad \gamma_8 = \kappa + 4, \quad \gamma_9 = -5\kappa, \quad \gamma_{10} = \kappa, \\
\gamma_{11} = -5\kappa + 8, \quad \gamma_{12} = 4\kappa, \quad \gamma_{13} = -\kappa, \quad \gamma_{14} = -\kappa, \quad \gamma_{15} = -2, \\
\gamma_{16} = -4, \quad \gamma_{17} = -12, \quad \gamma_{18} = -10\kappa, \quad \gamma_{19} = -2\kappa, \quad \gamma_{20} = -8, \\
\gamma_{21} = 2, \quad \gamma_{22} = 4\kappa + 12, \quad \gamma_{23} = 12\kappa, \quad \gamma_{24} = -4\kappa,
\end{gathered} \tag{4.49}
$$

where $\kappa = 6\zeta(3)$. As mentioned earlier, the NSVZ form of the gauge β-function β_g is valid only in a specific renormalisation scheme (which differs from DRED at three loops). We are therefore obliged for consistency to use the result for the anomalous dimension corresponding to this NSVZ scheme. The required transformation was presented in Ref. [39] and its effect on $\gamma^{(3)}$ given in Ref. [40]. In fact it is only γ_{17} and γ_{22} which are affected.

Once the conditions on the γ_α in Eqs. (4.41), (4.43), (4.46), (4.48) are satisfied, the Λ_α, Θ_α etc maybe be assigned in accord with Eq. (4.47), with considerable arbitrariness; there is little to be gained from stating the residual relations amongst them.

In the Wess-Zumino case considered in Ref. [15] the existence of an a-function satisfying Eq. (1.2) implied that $\gamma_1 - 2\gamma_2 - \gamma_3$ was an invariant (in a sense described in Ref. [15]) but did not impose a specific value; thus showing that Eq. (4.17) is sufficient but not necessary. We might expect similar remarks to apply to the other conditions in Eq. (1.2). It is all the more striking that these conditions are in fact satisfied by the anomalous dimension as computed.

We may count the independent parameters in the anomalous dimension as we did in Section 2 for the Yukawa β-function. The essential Eqs. (4.13) and (4.17) are invariant under redefinitions of g as in Eq. (4.8) and taking $\delta Y = Y * h$ where

$$
\delta\beta_Y = Y * \delta\gamma \,, \qquad \delta\beta_{\bar{Y}} = \delta\gamma * \bar{Y} \,, \tag{4.50}
$$

and also assuming $\delta\tilde{\beta}_g$ is given in terms of $\delta\gamma$ in accord Eq. (4.5), for

$$
\delta\gamma = -\big(\delta g \partial_g + (Y * h) \circ \partial_Y\big)\gamma + \beta_{\bar{Y}} \circ \partial_{\bar{Y}} h \,. \tag{4.51}
$$

Taking

$$
\begin{aligned}
h &= \tilde{\alpha}\, S_1 + \tilde{\beta}\, g^4 S_2 + \tilde{\gamma}\, g^2 P C_R + \tilde{\delta}\, g^4 C_R^2 + (\tilde{\zeta}\, Q + \tilde{\xi}\, C_G) g^4 C_R \,, \\
\delta g &= g^5 (\tilde{\mu}\, Q^2 + \tilde{\nu}\, Q C_G + \tilde{\rho}\, C_G^2 + \tilde{\sigma}\, \mathrm{tr}[C_R^2]/n_V) \,,
\end{aligned} \tag{4.52}
$$

G_1^{NP}	G_2^{NP}

Table 7: Non planar Feynman diagrams used to define $\gamma'^{(3)}$

results in

$$
\begin{aligned}
\delta\gamma_1 = 2\tilde{\alpha}, \quad \delta\gamma_2 = \tilde{\alpha}, \quad \delta\gamma_5 = 2\tilde{\alpha} + \tilde{\beta} - \tilde{\gamma}, \quad \delta\gamma_6 = \tilde{\beta}, \\
\delta\gamma_7 = -\tilde{\beta}, \quad \delta\gamma_8 = \tilde{\gamma} - 2\tilde{\alpha}, \quad \delta\gamma_9 = -\tilde{\delta}, \quad \delta\gamma_{11} = -\tilde{\delta}, \\
\delta\gamma_{12} = -2\tilde{\beta}, \quad \delta\gamma_{13} = -\tilde{\xi}, \quad \delta\gamma_{14} = -\tilde{\xi}, \quad \delta\gamma_{15} = -\tilde{\zeta}, \\
\delta\gamma_{16} = -\tilde{\zeta}, \quad \delta\gamma_{18} = -2\tilde{\delta}, \quad \delta\gamma_{19} = -2\tilde{\xi}, \quad \delta\gamma_{20} = -2\tilde{\zeta}, \\
\delta\gamma_{21} = 4\tilde{\mu}, \quad \delta\gamma_{22} = 4\tilde{\nu}, \quad \delta\gamma_{23} = 4\tilde{\rho}, \quad \delta\gamma_{24} = 4\tilde{\sigma},
\end{aligned}
\tag{4.53}
$$

which leave Eqs. (4.41), (4.43), (4.46), (4.48) invariant. Allowing for such variations the number of independent parameters is therefore $24 - 10 = 14$ but the eight constraints in Eqs. (4.41), (4.43), (4.46), (4.48) reduce the number of free parameters in $\gamma^{(3)}$ to be reduced to six.

In the $g = 0$ case, the only coefficient in $\gamma^{(3)}$ with a κ-dependence, γ_4, corresponds to a non-planar graph. In the general case there is no such obvious association between non-planar Feynman graphs and coefficients in $\gamma^{(3)}$ with κ-dependence (evaluated using DRED). However, an intriguing observation is that a redefinition given by choosing

$$
\begin{aligned}
\tilde{\beta} = \kappa, \quad \tilde{\gamma} = -\,\kappa, \quad \tilde{\delta} = -2\kappa, \\
\tilde{\nu} = -\kappa, \quad \tilde{\rho} = -\,3\kappa, \quad \tilde{\sigma} = \kappa\,,
\end{aligned}
\tag{4.54}
$$

(and the remaining coefficients in Eq. (4.52) set to zero) gives a redefined $\gamma^{(3)}$

$$
\gamma'^{(3)} = \gamma^{(3)}|_{\kappa=0} + \gamma_4 G_4^{\gamma} + \kappa G_1^{\rm NP} + 2\kappa G_2^{\rm NP}\,, \tag{4.55}
$$

where

$$
\begin{aligned}
G_1^{\rm NP} &= G_9^{\gamma} + G_{10}^{\gamma} + g^4(-2C_R S_2 + P C_R^2 + C_G S_2 - C_G P C_R) + 2g^6(C_R^3 - C_G C_R^2), \\
G_2^{\rm NP} &= G_9^{\gamma} - G_{10}^{\gamma} + g^4(P C_R^2 + C_G P C_R) + 2g^6(C_R^3 + C_G C_R^2)\,,
\end{aligned}
\tag{4.56}
$$

are the contributions corresponding to the Feynman diagrams shown in Table (7). The implication is that there is a scheme in which the κ-dependent terms in $\gamma^{(3)}$ are generated solely by non-planar diagrams.

5 Reduction of non-supersymmetric results to supersymmetric case

In this section we shall check that the a-function obtained using the methods of Section 2 for a general theory is compatible, upon specialisation to the supersymmetric case, with the a-function presented in Section 4 (at least up to two loops). The reduction of the non-supersymmetric theory presented in Section 2 to the supersymmetric case (with $n_\psi = n_V + n_C$, $n_\varphi = 2n_C$) may be accomplished by writing

$$\varphi_a \to \begin{pmatrix} \phi_i \\ \bar{\phi}^i \end{pmatrix}, \quad \bar{\phi}^i = (\phi_i)^*, \quad \psi_i \to \begin{pmatrix} \psi_i \\ \lambda_A \end{pmatrix}, \quad i = 1 \dots n_C, \tag{5.1}$$

and with $y_a\varphi_a = y^i\phi_i + \bar{y}_i\bar{\phi}^i$,

$$y^i \to \begin{pmatrix} Y^{ijk} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2}g(R_B)_j{}^i \\ 0 & 0 & \sqrt{2}g(R_A^T)^i{}_k & 0 \end{pmatrix},$$
$$\bar{y}_i \to \begin{pmatrix} 0 & \sqrt{2}g(R_B^T)^j{}_i & 0 & 0 \\ \sqrt{2}g(R_A)_i{}^k & 0 & 0 & 0 \\ 0 & 0 & \bar{Y}_{ijk} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{5.2}$$

where λ is the gaugino field. $\hat{\bar{y}}_i$ and $\hat{y}^i$ may be obtained from $\bar{y}_i$ and y^i by interchanging the upper left and lower right 2×2 blocks of the 4×4 matrices. We also have

$$t_A^\varphi \to \begin{pmatrix} R_A & 0 \\ 0 & -R_A^T \end{pmatrix}, \quad t_A^\psi \to \begin{pmatrix} R_A & 0 \\ 0 & R_A^{\rm ad} \end{pmatrix}, \quad (R_A^{\rm ad})_{BC} = -if_{ABC}, \tag{5.3}$$

and consequently, from Eq. (2.6),

$$R^\varphi \to 2T_R, \quad R^\psi \to T_R + C_G. \tag{5.4}$$

The scalar potential is now given by

$$V = \tfrac{1}{4}\bar{Y}_{ijm}Y^{klm}\bar{\phi}^i\bar{\phi}^j\phi_k\phi_l - g^2\tfrac{1}{2}(\bar{\phi}R_A\phi)(\bar{\phi}R_A\phi). \tag{5.5}$$

In making the reduction from the general theory to the supersymmetric case, we must start from two-loop β-functions corresponding to DRED, since the RG functions used in Section 3 were evaluated using this scheme; as we mentioned earlier, the DRED and NSVZ schemes coincide up to the two-loop order we are considering in this Section. We use the results given in Ref. [42], which may be obtained from the DREG results by a coupling redefinition as in Eq. (2.17) given by

$$\mu_4 = -\tfrac{1}{2}, \quad \mu_5 = 1, \quad \nu_1 = \tfrac{1}{6}, \tag{5.6}$$

G_1^S G_2^S G_3^S G_4^S

G_5^S G_6^S G_7^S G_8^S

Table 8: Contributions to $A^{(4)}$ in the supersymmetric case

with all other coefficients set to zero.[4] We list here the values of the coefficients in Eq. (2.16) which change under this redefinition (as may be easily checked using Eqs. (2.19), (2.16)):

$$\begin{aligned} c_6^{\rm DR} = 2 \qquad c_7^{\rm DR} = -1, \qquad c_{13}^{\rm DR} = -\tfrac{5}{4}, \qquad c_{14}^{\rm DR} = \tfrac{1}{4}, \qquad c_{15}^{\rm DR} = \tfrac{7}{2}, \\ c_{19}^{\rm DR} = 7, \quad c_{20}^{\rm DR} = -\tfrac{1}{12}\left(138C_G - 12R^{\psi} - 9R^{\varphi}\right), \\ c_{26}^{\rm DR} = \tfrac{7}{2}, \quad c_{28}^{\rm DR} = \tfrac{1}{12}(59C_G + 4R^{\psi} + R^{\varphi}), \end{aligned} \tag{5.7}$$

The DRED β-function leads to the following alterations in the coefficients in Eq. (2.26); the others are the same as in Eq. (2.31).

$$\begin{aligned} A_{13}^{\rm DR} = -\tfrac{5}{24}, \quad A_{14}^{\rm DR} = -\tfrac{1}{12}, \quad A_{15}^{\rm DR} = \tfrac{7}{12}, \quad A_{16}^{\rm DR} = -\tfrac{1}{3}, \quad A_{17}^{\rm DR} = \tfrac{1}{6}, \\ A_{19}^{\rm DR} = \tfrac{1}{6}, \quad A_{20}^{\rm DR} = \tfrac{5}{48}, \quad A_{22}^{\rm DR} = \tfrac{1}{4}, \quad A_{23}^{\rm DR} = \tfrac{1}{2}, \quad A_{25}^{\rm DR} = -\tfrac{5}{2}, \\ A_{26}^{\rm DR} = -\tfrac{1}{72}[138C_G - 12R^{\psi} - 9R^{\varphi}] - t_1\beta_0, \\ A_{27}^{\rm DR} = \tfrac{1}{144}[59C_G + 4R^{\psi} + R^{\varphi}] - t_2\beta_0, \end{aligned} \tag{5.8}$$

These changes are a consequence of making the transformations Eq. (5.6) and also

$$\delta g = -\frac{g^3}{72 n_V}\,\mathrm{tr}[y_a \hat{y}_a \hat{C}_R^{\psi}]\,, \tag{5.9}$$

upon $A^{(3)}$ and $A^{(2)}$ respectively in Eq. (2.12). Presumably the transformation in Eq. (5.9) represents a part of the two-loop transformation from DREG to DRED (namely the Yukawa dependent contribution to the transformation of g). To the best of our knowledge this has

[4] In general the DRED β-functions obtained using (5.6) do not correspond to a diagrammatic calculational scheme. There is an alternative implementation of DRED based on a calculational scheme, but involving the use of additional "evanescent" couplings. It is this scheme we referred to specifically as DRED in Ref. [42]. The two versions of DRED agree in the case of supersymmetry, which is our focus of interest here; but see Refs. [44, 45] for further discussion of the general case.

not been computed in full, though results have been given for the pure gauge case in Ref. [43].

Inserting Eqs. (5.2), (5.3), (5.5) into the expressions for the various contributions to $\tilde{A}^{(4)}$, depicted in Table (8), we find

$$\begin{aligned}
&G_1^A \to 2(G_1^S - 9G_3^S + 6G_4^S + 6G_5^S - 12G_6^S - 3(C_G - 2T_R)G_7^S + 4G_8^S \ldots)\\
&G_2^A \to 6(G_2^S + 2G_3^S - 4G_4^S - 6G_5^S + (C_G - 2T_R)G_7^S + \ldots)\\
&G_3^A \to -5G_3^S + 16G_5^S - 16G_6^S + 4C_G G_7^S + 2G_8^S + \ldots\\
&G_4^A \to 2(9G_5^S - 6G_6^S + 3C_G G_7^S + G_8^S + \ldots), \quad G_5^A \to 2(G_1^S - 6G_3^S + 12G_5^S + \ldots)\\
&G_6^A \to 2(G_2^S - 4G_4^S + 4G_6^S - 8T_R G_7^S + \ldots),\\
&G_7^A \to 2S_2 - 4G_3^S - 12G_4^S + 24G_5^S + 16G_6^S - 8T_R G_7^S + \ldots,\\
&G_8^A \to 2G_1^S - 24G_3^S + 96G_5^S + \ldots\\
&G_9^A \to 2(G_3^S + 2G_4^S - 2G_5^S - 4G_6^S + 2T_R G_7^S + \ldots \quad G_{10}^A \to 2(-G_3^S + 4G_5^S + \ldots)\\
&G_{11}^A \to 4(2G_3^S - 8G_5^S + \ldots), \quad G_{12}^A \to 4(G_4^S + 2G_5^S + \ldots)\\
&G_{13}^A \to 2(G_3^S - 4G_5^S + \ldots), \quad G_{14}^A \to 2(G_4^S - 2G_6^S + 2C_G G_7^S + \ldots)\\
&G_{15}^A \to 2(G_4^S - 2G_5^S - 2G_6^S + \ldots) \quad G_{16}^A \to 2(G_5^S + 2G_6^S - C_G G_7^S + \ldots), \quad G_{17}^A \to 8G_5^S + \ldots\\
&G_{18}^A \to \tfrac{1}{2}G_3^S - 4G_5^S - 2C_G G_7^S + \ldots, \quad G_{19}^A \to 2G_4^S - 4G_5^S - 8G_6^S + 4C_G G_7^S + \ldots,\\
&G_{20}^A \to 2G_3^S - 16G_5^S \ldots, \quad G_{21}^A \to 3G_5^S - 2G_6^S + \tfrac{1}{2}C_G G_7^S + \ldots, \quad G_{22}^A \to 2G_5^S + \ldots\\
&G_{23}^A \to 2G_6^S + \ldots, \quad G_{24}^A \to 4G_5^S + \ldots, \quad G_{25}^A \to 4G_6^S + \ldots\\
&G_{26}^A \to 2G_7^S + \ldots, \quad G_{27}^A \to 2G_7^S + \ldots,\\
&G_{28}^A \to \tfrac{3}{2}G_1^S + 3G_2^S + 12G_3^S + 24G_4^S + 24G_5^S + 48G_6^S + \ldots\,, \qquad (5.10)
\end{aligned}$$

where again we do not display the purely gauge-coupling dependent terms.

It is then straightforward to show, using the DRED values of the coefficients from Eqs. (2.31), (5.8) that $\tilde{A}^{(4)}$ reduces as

$$\tilde{A}^{(4)} \to \tfrac{1}{3}\mathrm{tr}[(\gamma^{(1)})^3] + \left(\alpha - \tfrac{1}{36}\right)\beta_Y^{(1)} \circ \beta_{\bar{Y}}^{(1)} + \left[-\tfrac{1}{2} + 8(t_1 + t_2)\right]\beta_g^{(1)} g\,\mathrm{tr}[PC_R]\,. \qquad (5.11)$$

This expression can readily be shown, with the aid of Eqs. (4.10), (4.5), to be equivalent at this order to Eq. (4.13). The explicit form at this order was already given in Ref. [20]; with our notation and conventions, this corresponds to $\Lambda^{(1)} = Y$ as in Eq. (4.22) and with

$$\beta_Y \circ H \circ \beta_{\bar{Y}} = \alpha\, \beta_Y \circ \beta_{\bar{Y}}\,, \qquad (5.12)$$

and

$$\lambda = g^2 + \tilde{\lambda} g^4 Q + \frac{\tilde{\lambda}_1}{n_V} g^4 \mathrm{tr}[PC_R] + \ldots\,, \qquad (5.13)$$

where we have picked out the terms which can contribute to $Q\mathrm{tr}[PC_R]$. We find using Eqs. (4.5), (4.31), (4.47), (4.49), (5.11) that we require

$$t_1 + t_2 = \frac{29}{16}\,. \qquad (5.14)$$

These coefficients correspond to a three-loop calculation (see Eq. (2.23)) and, in view of Eq. (4.47), depend on the value of γ_{17}, which has a different value for the NSVZ scheme than for DRED. It is beyond the scope of this article to consider how Eq. (5.14) would be modified within DRED or indeed within DREG. Since our whole approach is predicated on the NSVZ scheme, it would probably be naive to assume that the DRED form of Eq. (5.14) would be obtained simply by using the DRED result for γ_{17}.

Eq. (5.11) extends the result of Eq. (7.30) in Ref. [15] (with $a = 3\alpha - \frac{1}{12}$) to the gauge case–once again, modulo pure gauge terms which are not captured by the methods used in Section 2. We see again the ambiguity in the form of $\tilde{A}$ expressed in general by Eq. (2.27).

Of course this check is guaranteed to work but nevertheless given the indirect manner in which we have obtained $\tilde{A}$ and the possibility of subtleties regarding scheme dependence, it is satisfying to "close the loop" in this fashion.

Finally, we remark that although the form for $\tilde{A}$ presented in Eq. (5.11) is appealingly simple (arguably even more so than Eq. (4.13)), the obvious extension to higher loops does not appear to be viable.

6 Conclusions

In this article we have extended the results of Ref. [15] to the case of general gauge theories. In the non-supersymmetric case we have constructed the terms in the four-loop a-function containing Yukawa or scalar contributions, using the two-loop Yukawa β-function and one-loop scalar β-function. Our main result here is Eq. (2.26) with Eq. (2.31). This enabled a comparison with similar terms in the three-loop gauge β-function. In general, as a consequence of the properties of the coupling-constant metric, one can obtain information on the $(n+1)$-loop gauge β-function from the n- (and lower-) loop Yukawa β-function and the $(n-1)$ (and lower) loop scalar β-function. This is reminiscent of the way in which the $(n+1)$-loop gauge β-function is determined by the lower order anomalous dimensions in a supersymmetric theory, via the NSVZ formula.

In the supersymmetric case we have given a general sufficient condition for the exact a-function of Refs. [17–19], given in Eq. (4.13), to be valid, and shown that it is satisfied by the three-loop anomalous dimension. This condition is displayed in Eq. (4.17) and is our main result for the supersymmetric case.

One feature of interest is that Eq. (4.17) imposes extra conditions on the anomalous dimension beyond the mere requirements of integrability from Eq. (1.2); but which are nevertheless satisfied by the explicit results as computed. Indeed we remark here (without giving further details since it is beyond our remit in this article on the gauged case) that we have observed similar features in the Wess-Zumino model at four loops, using the results of Ref. [41].

These properties certainly hint that there might be some underlying reason why Eq. (4.17) *must* be satisfied; it would be interesting to explore this further. If this were indeed the case, one could imagine exploiting Eq. (4.17) to expedite higher-order calculations of the anomalous dimension such as the full gauged case at four loops; possibly combined with additional information such as the necessary vanishing of γ in the $N = 2$ case. Unfortunately, a preliminary check indicates that these constraints are far from sufficient to determine γ completely, even at three loops; and therefore a considerable quantity of perturbative calculation would still be unavoidable.

Finally, in Ref. [15] we explored in some detail the freedoms to redefine the various quantities we have considered, and it would be interesting to extend these discussions to the current gauged case. In particular it would be useful to extend Eq. (4.17), which in its current form is predicated upon the NSVZ renormalisation scheme, to a form valid for *any* scheme.

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