ENERGY OF A KNOT: VARIATIONAL PRINCIPLES; MM-ENERGY.

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CONTENTS

1. INTRODUCTION

Let $S^1 = \mathbb{R}/(2\pi\mathbb{Z})$ be the circle and $\tau : S^1 \longrightarrow \mathbb{R}^3$ be a smooth knot. We will assume that $\tau(t)$ is the arc length parametrization. Denote by $D(t_1, t_2)$ the length of the minimal subarc between t_1 and t_2 on the circle. Let $|*|$ denote the absolute value of vectors in $\mathbb{R}^3$.

Following [1], we denote by

$$E(\tau) = E_f(\tau) = \iint\limits_{S^1 \times S^1} f(|\tau(t_1) - \tau(t_2)|, D(t_1, t_2))dt_1 dt_2$$

the energy of the knot τ, where $f(\rho, \alpha)$ satisfies the following conditions:

1) $f(\rho, \alpha) \in C^{1,1}(U)$, where $U = \{(\rho, \alpha)|0 < \rho \le \alpha, \alpha \le \pi\}$;

2) there exist the following limits:

$$\lim_{\substack{(\rho,\alpha)\in U \\ \rho\to 0, \rho/\alpha\to 1}} f(\rho, \alpha), \qquad \lim_{\substack{(\rho,\alpha)\in U \\ \rho\to 0, \rho/\alpha\to 1}} \frac{\partial f(\rho, \alpha)}{\partial \rho}, \qquad \lim_{\substack{(\rho,\alpha)\in U \\ \rho\to 0, \rho/\alpha\to 1}} \frac{\partial f(\rho, \alpha)}{\partial \rho}.$$

Almost all energies are not homothety invariant, so we will consider only knots of length 2π.

The energy of a knot is not an invariant of the topological class of this knot. If we make a smooth perturbation of a knot, its energy smoothly changes. We will consider energies with the following important properties. The energy is always positive. When a knot crossing tends to a double point, the energy tends to infinity. So every topological class of knots has a representative with the minimal value of energy. This knot is called a

normal form of the class. It is unknown whether each class has a unique normal form or not, i.e., whether the normal form for some energy is an invariant of the topological class or not. The normal forms satisfy the variational equations considered below.

Some energies have a physical meaning. For example $f = 1/(|\tau(t_1) - \tau(t_2)|)$ is the energy of a charged knot. Unfortunately, this energy is always infinite. As long as the charged knot does not break there must be some other forces which save the knot. Let us consider a model of such a restriction:

$$f = \frac{(D^2(t_1, t_2))}{(\tau(t_1) - |\tau(t_2)|)}.$$

For this energy we will develop our variational principles.

The study of knot energies began with the work of Moffatt (1969) [7], and was developed by him in [8] following Arnold's work [2]. The first steps in studying properties of the energies of knots were made by O'Hara [9, 10, 11] and the first variational principles for polygons in space were studied by Fukuhara [4].

The aim of this article is to prove that any extremal knot τ satisfies certain variational equations. The paper is organized as follows. We start in Section 2 with the definitions and formulations of the main theorem. In Section 3 we prove this theorem. In Section 4 we prove that the circle unknot always satisfies our extremal conditions. Unfortunately the integrals in the equations do not converge for all possible energies. For example, they do not converge in the case of the most famous energy: Möbius energy. We discuss this also in Section 4. Section 5 seems to be independent from the previous sections. In Section 5 we represent Mm-energy. The definition of this energy differs with one regarded above. Nevertheless besides its own properties Mm-energy has some similar with Möbius energy properties.

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2. Notation and definitions

Mostly we will work with knots of fixed length 2π. So let $S^1 = \mathbb{R}/(2\pi\mathbb{Z})$ be the circle and let $\tau : S^1 \longrightarrow \mathbb{R}^3$ denote some smooth knot of length 2π. Let $\tau(t)$ be the arc length parametrization.

By $\kappa(t)$ we denote the curvature at t and $R(t) = 1/\kappa(t)$, the radius of curvature at t.

Definition 2.1. Given a smooth knot $\tau : S^1 \longrightarrow \mathbb{R}^3$ and a point $t_0 \in S^1$, a *locally perturbed knot* is a knot (denoted by $\tau_{t_0,\varepsilon}$) such that
  a) $|\tau(t) - \tau_{t_0,\varepsilon}(t)| < \varepsilon^2$ if $D(t_0, t) \leq \varepsilon$ and $\tau(t) = \tau_{t_0,\varepsilon}(t)$ if $D(t_0, t) > \varepsilon$;
  b) $|\kappa(t) - \kappa_{t_0,\varepsilon}(t)| < \varepsilon$ for $D(t_0, t) < \varepsilon$;
  c) $\tau_{t_0,\varepsilon}(t_0 + \lambda) = \tau_{t_0,\varepsilon}(t_0) + \lambda\dot\tau_{t_0,\varepsilon}(t_0) + (\lambda^2/2)\ddot\tau_{t_0,\varepsilon}(t_0) + o(\varepsilon^2)$ if $D(t_0, t_0 + \lambda) \leq \varepsilon$.

Note that at the points $t_0 - \varepsilon$ and $t_0 + \varepsilon$ the curvature is not restricted.

The length of the knot $\tau_{t_0,\varepsilon}$ can change, but we regard knots of length 2π only. One of the ways to solve this problem is to consider the restriction of the set of locally perturbed

knots to the set of knots of constant length 2π, but this definition is unsatisfactory. Indeed, let a knot τ in some neighborhood of the point t_0 be a piece of a straight line. Then the set of locally perturbed knots at the point t_0 of length 2π consists of the knot τ only.

We will extend this set in the following way.

Definition 2.2. Let the length of $\tau_{t_0,\varepsilon}$ be $(1+\delta)2\pi$. The *locally perturbed length 2π knot* $\tilde{\tau}_{t_0,\varepsilon}$ is the knot obtained from $\tau_{t_0,\varepsilon}$ by homothety with coefficient $1/(1+\delta)$ and center at the origin. We also say that the knot $\tilde{\tau}$ is *associated* with the knot τ.

Consider any $\tau_{t_0,\varepsilon}$. We will show later that $\delta = c_1\varepsilon^3 + \circ(e^3)$. Thus by Definition 2.1 we have

$$|\tau_{t_0,\varepsilon}(t_1) - \tau_{t_0,\varepsilon}(t_2)| = |\tau(t_1) - \tau(t_2)| + c_2(t_1,t_2)\varepsilon^2 + o(e^2)$$

if $D(t_0,t_1) < \varepsilon$ or $D(t_0,t_2) < \varepsilon$. Then we may conclude that

$$E(\tau_{t_0,\varepsilon}) = E(\tau) + c_3\varepsilon^3 + o(\varepsilon^3) \quad \text{and} \quad E(\tilde{\tau}_{t_0,\varepsilon}) = E(\tau) + c_4\varepsilon^3 + o(\varepsilon^3).$$

The coefficients c_3 and c_4 of the term ε^3 will be called the *variation* and denoted by $Var(\tau_{t_0,\varepsilon})$ and $Var(\tilde{\tau}_{t_0,\varepsilon})$ respectively.

Now all is prepared for the definition of a locally extremal point of a knot.

Definition 2.3. Any $t_0 \in S^1$ is called *locally extremal point* of τ if $Var(\tilde{\tau}_{t_0,\varepsilon}) = 0$ for each locally perturbed knot $\tilde{\tau}_{t_0,\varepsilon}$ of length 2π.

Definition 2.4. The knot τ is said to be *locally extremal* if all its points are locally extremal.

Let us find necessary and sufficient conditions for the point t_0 be locally extremal. We denote the vector product of two vectors a and b by $[a,b]$. By (a,b,c) we denote the mixed product (oriented volume) of the vectors a, b and c. Let $\dot{\tau}(t)$ be the velocity vector and $\ddot{\tau}(t)$ be the acceleration vector. Now we define the functions $\Psi(t_0,t)$ and $\Phi(t_0,t)$.

$$\Psi(t_0,t) = \begin{cases} \left(\frac{\dot{\tau}(t_0)}{|\dot{\tau}(t_0)|}, \frac{\ddot{\tau}(t_0)}{|\ddot{\tau}(t_0)|}, \frac{\tau(t)-\tau(t_0)}{|\tau(t)-\tau(t_0)|}\right) & \text{, if } \ddot{\tau}(t_0) \neq 0; \\ \left(\frac{\tau(t)-\tau(t_0)}{|\tau(t)-\tau(t_0)|}, \frac{\dot{\tau}(t_0)}{|\dot{\tau}(t_0)|}\right) & \text{, if } \ddot{\tau}(t_0) = 0. \end{cases}$$

$$\Phi(t_0,t) = \begin{cases} \left(\frac{\dot{\tau}(t_0)}{|\dot{\tau}(t_0)|}, \frac{\tau(t)-\tau(t_0)}{|\tau(t)-\tau(t_0)|}, \left[\frac{\dot{\tau}(t_0)}{|\dot{\tau}(t_0)|}, \frac{\ddot{\tau}(t_0)}{|\ddot{\tau}(t_0)|}\right]\right) & \text{, if } \ddot{\tau}(t_0) \neq 0; \\ 0 & \text{, if } \ddot{\tau}(t_0) = 0. \end{cases}$$

Note that $|\dot{\tau}(t_0)| = 1$ and $|\tau(t) - \tau(t_0)| \neq 0$ if $t \neq t_0$. Thus Ψ and Φ are well defined.

We also remark that $\Psi(t_0,t) = \sin\psi(t_0,t)$, where $\psi(t_0,t)$ is the angle between the vector $\tau(t) - \tau(t_0)$ and the oriented plane spanning of $\dot{\tau}(t_0)$ and $\ddot{\tau}(t_0)$. The function Φ has a similar representation: $\Phi(t_0,t) = \sin\phi(t_0,t)$, where $\phi(t_0,t)$ is the angle between the vector $\tau(t) - \tau(t_0)$ and the oriented plane spanning of $\dot{\tau}(t_0)$ and $[\dot{\tau}(t_0), \ddot{\tau}(t_0)]$. (See Fig. 1). These angles can be either positive or negative.
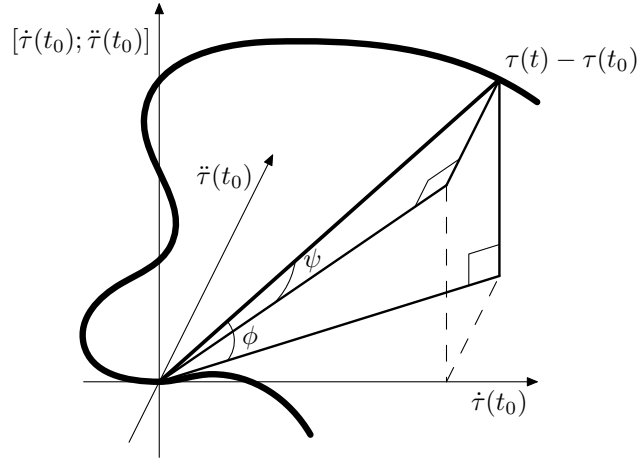
FIGURE 1. The geometric interpretation of $\psi(t_0, t)$ and $\phi(t_0, t)$.

Theorem 2.1. *Let τ be a smooth knot. The point t_0 is a locally extremal point of τ if and only if the following conditions hold:*

$$V_1(t_0) := \frac{2}{3R(t_0)}\left(4\int_{S^1}\left(f + R(t_0)\Phi(t_0, t)\frac{\partial f}{\partial \rho}\right)dt - \frac{1}{\pi}\iint_{S^1\times S^1}\left(2f + D(t_1, t_2)\frac{\partial f}{\partial \rho}+\right.\right.$$

$$\left.\left.|\tau(t_1) - \tau(t_2)|\frac{\partial f}{\partial \alpha}\right)dt_1 dt_2 + 2\iint_A \frac{\partial f}{\partial \alpha}dt_1 dt_2\right) = 0;$$

$$V_2(t_0) := \frac{4}{3R(t_0)}\int_S \frac{\partial f}{\partial \rho}\Psi(t_0, t)dt = 0.$$

Here $A \subset S^1 \times S^1$ is the set of points (t_1, t_2) such that $D(t_1, t_2) = D(t_1, t_0) + D(t_0, t_2)$.

Corollary 2.1. *A knot τ is locally extremal if and only if almost all of its points are locally extremal, i.e.,*

$$\int_{S^1}\left(V_1^2(t) + V_2^2(t)\right)dt = 0.$$

3. PROOFS

Let t_0 be any point of S^1. We choose orthonormal coordinates in $\mathbb{R}^3$ such that $\tau(t_0)$ is on the (X, Y)-plane, $\tau(t_0 - \varepsilon)$ and $\tau(t_0 + \varepsilon)$ lie symmetrically on the X-axis. If $\tau(t_0 - \varepsilon)$, $\tau(t_0)$ and $\tau(t_0 + \varepsilon)$ are on the same line, then we make any possible choice of the Y-axis. Finally, we choose the Z-axis such that the orientation of the (X, Y, Z)-space is positive (see Fig. 2a)).
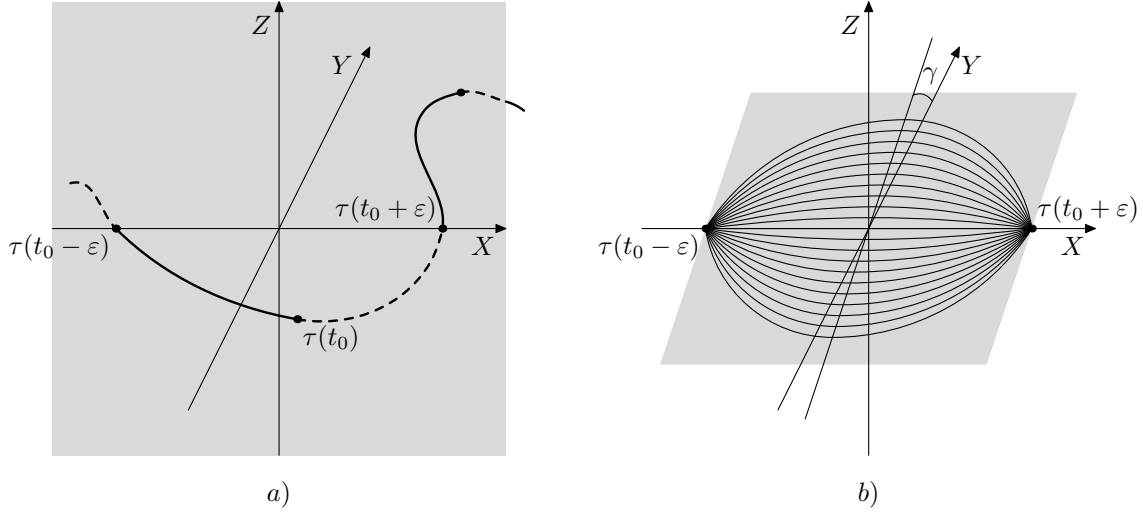
FIGURE 2. a)The choice of X, Y and Z-axes. b)The parabolic arcs (α, γ) where γ is fixed.

Let P_ε be the class of parabolic arcs and one segment such that all the parabolas have their vertex in the (Y, Z)-plane, $\tau(t_0 - \varepsilon)$ and $\tau(t_0 + \varepsilon)$ are the endpoints of the arcs, and the endpoints of the segment are $\tau(t_0 - \varepsilon)$ and $\tau(t_0 + \varepsilon)$. Each parabola can be specified by two parameters (λ, γ), where 2λ is the "acceleration" and γ is the angle between the (X, Y)-plane and the plane containing the parabola (see Fig. 2b)). Notice also that $(0, \gamma)$ is some segment.

Denote by $M_{P,t_0,\varepsilon}$ the 2-dimensional set of knots $\tau_{t_0,\varepsilon,\lambda,\gamma}$, where the curve connecting $\tau(t_0 - \varepsilon)$ and $\tau(t_0 + \varepsilon)$ belongs to the class P_ε with the following property: the knot $(\tau_{t_0,\varepsilon,\lambda,\gamma} + \tau)/2$ is a locally perturbed knot. Denote by $\tilde{M}_{P,t_0,\varepsilon}$ the set of knots associated with the knots in the class P_ε.

Theorem 3.1. *Let τ be a smooth knot. The point t_0 is a locally extremal point if and only if $Var(\tilde{\tau}_{t_0,\varepsilon}) = 0$ for each locally perturbed (at t_0) knot $\tilde{\tau}_{t_0,\varepsilon} \in \tilde{M}_{P,t_0,\varepsilon}$.*

Proof of Theorem 3.1.
We begin the proof with the following lemma.

Lemma 3.1. *Let $C = \{(x, y, z) \in \mathbb{R}^3 | \sqrt{y^2 + z^2} < r, |x| < s\}$ be a cylinder. Suppose a point moves inside C with velocity of constant modulus 1 and so that the absolute value of its acceleration is bounded by K (see Fig.3). Let $x(0) = -s$, $x(T) = s$, $s \gg r$ and $K < 1/(4r)$. Then the length of the trajectory of a point (i.e. T) is bounded:*

$$T < \frac{2s}{\sqrt{1 - 4Kr}}.$$

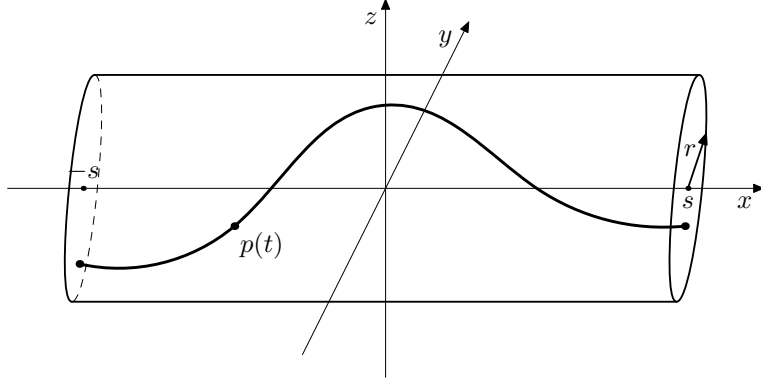FIGURE 3. The trajectory of the point $p(t)$ inside the cylinder C.

First let us prove that $\dot{y}^2(t_0) < 2Kr$. We first consider the case for which $x(t_0) < 0$, $\dot{x}(t_0) > 0$ and $y(t_0) > 0$; then

$$y(t) = y(t_0) + \int_{t_0}^{t} \dot{y}(\xi)d\xi < r.$$

By the assumption, we have

$$\dot{y}(\xi) = \dot{y}(t_0) + \int_{t_0}^{\xi} \ddot{y}(\zeta)d\zeta > \dot{y}(t_0) - \int_{t_0}^{\xi} Kd\zeta = \dot{y}(t_0) - (\xi - t_0)K.$$

It follows that

$$y(t) > y(t_0) + \int_{t_0}^{t} \dot{y}(x_0) - (\xi - t_0)Kd\xi = y(t_0) + (t - t_0)\dot{y}(t_0) - \frac{(t - t_0)^2}{2}K.$$

But $y(t_0) > -r$ and $y(t) < r$, so

$$(t - t_0)\dot{y}(t_0) - \frac{(t - t_0)^2}{2}K - 2r < 0.$$

By assumption $x < 0$ and $s \gg r$, so the vertex of the parabola is at the point $t - t_0 = \dot{y}(t_0)/K < s$. This yields the inequality $\dot{y}^2(t_0) < 2Kr$.

The proof for the cases in which $\dot{x}(t_0) > 0$ and $y(t_0) < 0$; $\dot{x}(t_0) < 0$ and $y(t_0) > 0$; $\dot{x}(t_0) < 0$ and $y(t_0) < 0$ is similar.
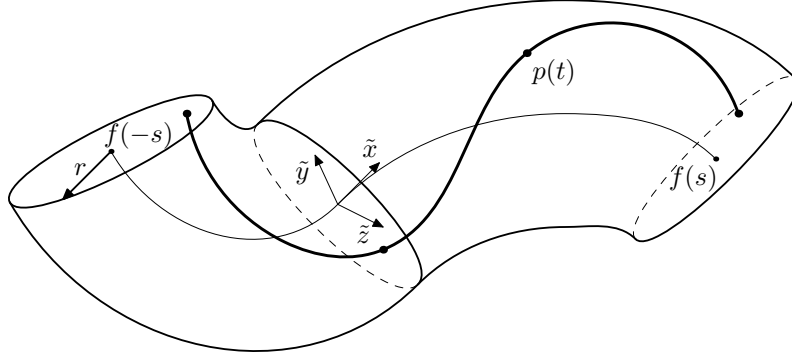
Secondly, we claim that $\dot{z}^2(t_0) < 2Kr$. The proof is similar to the inequality for $\dot{y}^2(t_0)$.

By the previous statements, it follows that

$$\dot{x}^2(t_0) = 1 - \dot{y}^2(t_0) - \dot{z}^2(t_0) > 1 - 4Kr > 0$$

for every $t_0 \in [0, T]$. So we have $T < 2s/(1 - 4Kr)$

This completes the proof of Lemma 3.1

FIGURE 4. The trajectory of the point p(t) inside the cylinder C_f.

We continue the proof with a generalization of the previous lemma.

Lemma 3.2. *Let $f : [-s, s] \longmapsto \mathbb{R}^3$ be a unit-length smooth map, let the curvature of f be bounded ($|\ddot{f}(t)| < K_1$) and $sK_1 < 1$. Let $D^2(t) \in \mathbb{R}^3$, where $t \in [-s, s]$ is the disk of radius r centered at $f(t)$ with the plane of the disc orthogonal to $\dot{f}$. Let also $rK_1 < 1$. Denote by $C_f = \bigcup_{[-s,s]} D^2(t)$ the tubular neighborhood of the curve f. Suppose a point $p(t) = (x(t), y(t), z(t))$ moves inside C_f (see Fig. 4) with velocity of constant absolute value 1 and let the absolute value of its acceleration be bounded by K_2. Let $p(0) \in D^2(-s)$, $p(T) \in D^2(s)$. Let $s \gg r$ and*

$$K_2 + \frac{1}{1 - rK_1}K_1 < \frac{1}{4r}.$$

Then the length of the trajectory of the point (i.e., T) is bounded and

$$2s(1 - rK_1) < T < \frac{2s(1 + rK_1)}{\sqrt{1 - 4K_2 r}}.$$

Let us define $\tilde{x} = t$.

Now we describe some map π from C_f to the standard cylinder C (see Fig. 3). Let

$$\pi(D^2(\tilde{x})) = \{(\tilde{x}, y, z) \in \mathbb{R} | \sqrt{y^2 + z^2}\}$$

be isometric images of the disk $D^2(\tilde{x})$ for each $\tilde{x} \in [-s, s]$. If we fix a preimage $\tilde{y}$-axis of the y-axis and a preimage $\tilde{z}$-axis of the z-axis in the disc $D^2(\tilde{x})$ for each $\tilde{x} \in [-s, s]$, then the map will be completely described. As long as $sK_1 < 1$ and $rK_1 < 1$, this map is well defined and the manifold $N_f = \bigcup_{[-s,s]} \partial D^2(t)$ with boundary $\partial D^2(-s) \cup \partial D^2(s)$ is smooth.

Let $\pi(\tilde{y}_{-s}) = (-s, r, 0)$ for some $\tilde{y}_{-s} \in \partial D^2(-s)$. Consider the vector field on N_f with the following property: if the point q lies on the circle $\partial D^2(\tilde{x})$, then the vector v_q equals $\dot{f}(\tilde{x})$; this means that v_q is the unit-length vector orthogonal to the disc $D^2(\tilde{x})$ with the corresponding direction. Denote the integral trajectory of this field passing through the point $\tilde{y}_{-s}$ by $\tilde{y} = \{\tilde{y}(\tilde{x}) | \tilde{x} \in [-s, s]\}$. This trajectory defines the $\tilde{y}$ coordinate in each disc

$D^2(\tilde{x})$. Finally we define the unit-length $\tilde{z}$-vector as the vector product of the unit-length $\tilde{x}$-vector and unit-length $\tilde{y}$-vector (in each $D^2(\tilde{x})$).

The image $\pi(p)$ of the point p moves inside C. We denote $\pi(p)$ by $\hat{p}$. Notice that

$$\frac{|\dot{p}(t)|}{|\dot{\hat{p}}(t)|} = \frac{1}{|\dot{\hat{p}}(t)|} \in [1 - rK_1, 1 + rK_1].$$

Note also that if the curvature of the trajectory is K at some point $p(t)$, then the curvature of the image of this trajectory will be

$$\hat{K} < K + \frac{1}{\frac{1}{K_1} - r}$$

at the point $\hat{p}(t)$, as can be easily shown.

Now Lemma 3.2 follows from Lemma 3.1.

We continue the proof of Theorem 3.1. Let $\tau_{t_0,\varepsilon}$ be any locally perturbed knot at the point t_0 and let $t \in S^1$ such that $D(t_0, t) < \varepsilon$. Consider

$$P_{\tau_{t_0,\varepsilon}}(t) = \tau_{t_0,\varepsilon}(t_0) + (t - t_0)\big(\dot{\tau}_{t_0,\varepsilon}(t_0) + c_1\big) + \frac{(t - t_0)^2}{2}\big(\ddot{\tau}(t_0) + c_2\big)$$

We choose the constants $c_1 = o(\varepsilon^2)$ and $c_2 = o(\varepsilon^2)$ so that

$$P_{\tau_{t_0,\varepsilon}}(t + \varepsilon) = \tau_{t_0,\varepsilon}(t + \varepsilon), \quad P_{\tau_{t_0,\varepsilon}}(t - \varepsilon) = \tau_{t_0,\varepsilon}(t - \varepsilon).$$

Here we take the unit-length parametrization and denote the length of curves by $l(*)$. Then $P_{\tau_{t_0,\varepsilon}}(t)$ is a parabolic arc in the ε-neighborhood of the point t_0. From Lemma 3.2 it follows that $|P_{\tau_{t_0,\varepsilon}}(t) - \tau_{t_0,\varepsilon}(t)| = o(\varepsilon^2)$ and also $l(P_{\tau_{t_0,\varepsilon}}(t)) = l(\tau_{t_0,\varepsilon}) + o(\varepsilon^3)$. So $E_{\tau_{t_0,\varepsilon}} - E_{P_{\tau_{t_0,\varepsilon}}(t)} = o(\varepsilon^3)$.

Consider the perturbed curve $\tau_{t_0,\varepsilon,\lambda,\gamma}$ passing through the point $\tau_{t_0,\varepsilon}(t_0)$. We have

$$|\tau_{t_0,\varepsilon,\lambda,\gamma}(t) - \tau_{t_0,\varepsilon}(t)| < \varepsilon.$$

We also have $E_{\tau_{t_0,\varepsilon,\lambda,\gamma}} - P_{E_{\tau_{t_0,\varepsilon}}} = o(\varepsilon^3)$.

Finally we conclude that $E_{\tau_{t_0,\varepsilon,\lambda,\gamma}} - E_{\tau_{t_0,\varepsilon}} = o(\varepsilon^3)$.

One can see that the knot $\tau_{t_0,\varepsilon,\lambda,\gamma}$ belongs $M_{P,t_0,\varepsilon}$. We note again that $l(P_{\tau_{t_0,\varepsilon}}(t)) = l(\tau_{t_0,\varepsilon}) + o(\varepsilon^3)$. Hence

$$E_{\tilde{\tau}_{t_0,\varepsilon,\lambda,\gamma}} - E_{\tilde{\tau}_{t_0,\varepsilon}} = o(\varepsilon^3).$$

By definition, the knot $\tilde{\tau}_{t_0,\varepsilon,\lambda,\gamma}$ belongs $\tilde{M}_{P,t_0,\varepsilon}$. This completes the proof of Theorem 3.1.

Proof of Theorem 2.1 Without loss of generality, we put

$$t_0 = 0, \qquad \gamma = o(1), \quad \text{and} \quad \lambda = 1/(2R(0)) + o(1),$$

where $R(0)$ is the radius of curvature at the point 0. According to Theorem 3.1, we can consider only the class $\tilde{M}_P$ of knots. Let $\tilde{\tau}_{0,\varepsilon,\lambda,\gamma}$ be a knot in $\tilde{K}_P$. Denote

$$\Delta := \left[\frac{\varepsilon}{1 + \delta}, \frac{\varepsilon}{1 + \delta}\right] \subset S^1.$$

Now note that for any τ we have

$$E(\tau) = \iint\limits_{S^1 \times S^1} f\,dxdy = 2 \iint\limits_{\Delta \times S^1} f\,dxdy - \iint\limits_{\Delta \times \Delta} f\,dxdy + \iint\limits_{A \backslash (\Delta \times S^1 \bigcup S^1 \times \Delta)} f\,dxdy +$$

$$\iint\limits_{S^1 \times S^1 \backslash A} f\,dxdy =: 2E_1(\tau) - E_2(\tau) + E_3(\tau) + E_4(\tau).$$

Here $f = f(\rho(\tau(x), \tau(y)), \alpha(\tau(x), \tau(y)))$. Further note that

$$Var(\tau) = 2Var_1(\tau) - Var_2(\tau) + Var_3(\tau) + Var_4(\tau),$$

where Var_i is the variation of E_i.

First we calculate Var_1. We recall that $\sin \phi = \Phi$ and $\sin \psi = \Psi$.

Lemma 3.3.

$$Var_1(\tilde{\tau}_{0,\varepsilon,\lambda,\gamma}) = \frac{4}{3}\Big(\int\limits_{S^1} \frac{f}{R(0)} + \sin\phi \frac{\partial f}{\partial \rho} dy\Big)\Big(\lambda - \frac{1}{2R(0)}\Big) + \frac{2}{3R(0)}\Big(\int\limits_{S^1} \sin\psi \frac{\partial f}{\partial \rho} dy\Big)\gamma.$$

The length of the arc of the parabola is $2\varepsilon + \frac{2}{3}\lambda^2\varepsilon^3 + o(e^3)$. So $\delta = \frac{2}{3}\lambda^2\varepsilon^3$. Note also that the coefficient of homothety is $o(e^2)$ and thus $Var_1(\tau_{0,\varepsilon,\lambda,\gamma}) = Var_1(\tilde{\tau}_{0,\varepsilon,\lambda,\gamma})$. Let

$$(a, b, c) = (a(t), b(t), c(t)) = \tau_{0,\varepsilon,\lambda,\gamma}(t), \quad \ell = \ell(t) = \sqrt{a(t)^2 + b(t)^2 + c(t)^2}, \quad f = f(\rho, \alpha).$$

Thus we have

$$E_1(\tau_{0,\varepsilon,\lambda}) = \int\limits_{S^1} \int\limits_{-\varepsilon}^{\varepsilon} \Big(\Big[1 + (2\lambda t_1)^2\Big]^{1/2}$$

$$f\Big(\big[((t_1 - a(t_2))^2 + ((\lambda t_1^2 - \lambda\varepsilon^2)\cos\gamma - b(t_1))^2 + ((\lambda t_1^2 - \lambda\varepsilon^2)\sin\gamma - c(t_1))^2\big]^{1/2},$$

$$D(t_1, t_2)\Big)\Big)dt_1 dt_2 + o(\varepsilon^3) =$$

$$\int\limits_{S^1} \int\limits_{-\varepsilon}^{+\varepsilon} \Big(\Big(1 + 2\lambda^2 t_1^2 + o(t_1^2)\Big)\Big(f + \Big(\frac{t_1^2 - at_1 + \lambda(\varepsilon^2 - t_1^2)(b\cos\gamma + c\sin\gamma)}{\ell} - \frac{a^2 t_1^2}{2\ell^2}\Big)\times$$

$$\frac{\partial f}{\partial \rho} + \frac{a^2 t_1^2 \frac{\partial^2 f}{\partial \rho^2}}{2\ell} + D(0, t_1)\frac{\partial f}{\partial \alpha}\Big)\Big)dt_1 dt_2 + o(\varepsilon^3) =$$

$$= \int\limits_{S^1} \int\limits_{-\varepsilon}^{+\varepsilon} \left(2\lambda^2 t_1{}^2 f + f + \left(\frac{t_1{}^2 - a t_1 + \lambda(\varepsilon^2 - t_1{}^2)(b\cos\gamma + c\sin\gamma)}{\ell} - \right.\right.$$

$$\left.\left. \frac{a^2 t_1{}^2}{2\ell^2} \right) \frac{\partial f}{\partial\rho} + \frac{a^2 t_1{}^2 \frac{\partial^2 f}{\partial\rho^2}}{2\ell} + D(t_1, 0)\frac{\partial f}{\partial\alpha} \right) dt_1 dt_2 + o(\varepsilon^3) =$$

$$= \int\limits_{S^1} \left(\frac{2\lambda^2\varepsilon^3}{3} f + \varepsilon f + \left(\frac{2\varepsilon^3 + 4\lambda\varepsilon^3(b\cos\gamma + c\sin\gamma)}{3\ell} - \frac{a^2\varepsilon^3}{2\ell^2} \right)\frac{\partial f}{\partial\rho} + \frac{a^2\varepsilon^3 \frac{\partial^2 f}{\partial\rho^2}}{2\ell} \right) dt_2 +$$

$$\int\limits_{S^1} \int\limits_{-\varepsilon}^{+\varepsilon} D(t_1, 0)\frac{\partial f}{\partial\alpha} dt_1 dt_2 + o(\varepsilon^3)$$

This yields

$$dE_1(\lambda, \gamma) = d\left(\int\limits_{S^1} \left(\frac{2\lambda^2\varepsilon^3}{3} f + \left(\frac{4\lambda\varepsilon^3(b\cos\gamma + c\sin\gamma)}{3\ell} \right)\frac{\partial f}{\partial\rho} \right) dt_2 + o(\varepsilon^3) \right) =$$

$$\left(\int\limits_{S^1} \left(\frac{4\lambda\varepsilon^3}{3} f + \left(\frac{4\varepsilon^3(b\cos\gamma + c\sin\gamma)}{3\ell} \right)\frac{\partial f}{\partial\rho} \right) dt_2 + o(\varepsilon^3) \right) d\lambda +$$

$$\left(\int\limits_{S^1} \left(\frac{4\varepsilon^3\lambda(-b\sin\gamma + c\cos\gamma)}{3\ell} \right)\frac{\partial f}{\partial\rho} dt_2 + o(\varepsilon^3) \right) d\gamma.$$

Finally we substitute

$$\frac{b}{\ell} = \sin\phi, \quad \frac{c}{\ell} = \sin\psi, \quad \gamma = o(1), \quad \lambda = \frac{1}{2R(0)} + o(1),$$

where $R(0)$ is the radius of curvature at the point 0, obtaining

$$Var_1(\tilde{\tau}_{0,\varepsilon,\lambda,\gamma}) = \frac{4}{3}\left(\int\limits_{S^1} \frac{f}{R(0)} + \sin\phi\frac{\partial f}{\partial\rho} dt_2 \right)\left(\lambda - \frac{1}{2R(0)} \right) + \frac{2}{3R(0)}\left(\int\limits_{S^1} \sin\psi\frac{\partial f}{\partial\rho} dt_2 \right)\gamma.$$

The proof of Lemma 3.3 is complete.

Lemma 3.4. $Var_2 = 0$.

Since $E_2(\tau) - E_2(\tilde{\tau}_{0,\varepsilon}) = o(\varepsilon^3)$, we immediately have $Var_2 = 0$.

Lemma 3.5.

$$Var_3 = \left(\frac{2}{3\pi R(0)} \iint\limits_{A\setminus(\Delta\times S^1 \bigcup S^1\times\Delta)} -2f - \ell\frac{\partial f}{\partial\rho} + (2\pi - D(t_1, t_2))f_\lambda dt_1 dt_2 \right)\left(\lambda - \frac{1}{2R(0)} \right).$$

The following calculations prove this lemma.

$$E_3(\tau_{0,\varepsilon,\lambda}) = \iint\limits_{A\backslash(\Delta\times S^1 \bigcup S^1\times\Delta)} f d\tilde{t}_1 d\tilde{t}_2 =$$

$$\iint\limits_{A\backslash(\Delta\times S^1 \bigcup S^1\times\Delta)} f\left(\ell\left(1 - \frac{2\lambda^2\varepsilon^3}{3\pi}\right), D\left(\left(|t_2 - t_1| + \frac{4\lambda^2\varepsilon^3}{3}\right)\left(1 - \frac{2\lambda^2\varepsilon^3}{3\pi}\right), 0\right)\right)$$

$$d\left(\left(1 - \frac{2\lambda^2\varepsilon^3}{3\pi}\right)t_1\right) d\left(\left(1 - \frac{2\lambda^2\varepsilon^3}{3\pi}\right)t_2\right) + o(\varepsilon^3) =$$

$$\iint\limits_{A\backslash(\Delta\times S^1 \bigcup S^1\times\Delta)} \left(f + \left(-\frac{2}{\pi}f - \frac{\ell}{\pi}\frac{\partial f}{\partial\rho} + 2f_\lambda - \frac{D(t_1,t_2)f_\lambda}{\pi}\right)\frac{2\lambda^2\varepsilon^3}{3}\right) dt_1 dt_2 + o(\varepsilon^3).$$

Let us remark that

$$\iint\limits_{\Delta\times S^1 \bigcup S^1\times\Delta} \left(-\frac{2f}{\pi} - \frac{\ell\frac{\partial f}{\partial\rho}}{\pi} + 2f_\lambda - \frac{D(t_1,t_2)f_\lambda}{\pi}\right)\frac{2\lambda^2\varepsilon^3}{3} dt_1 dt_2 = o(\varepsilon^3).$$

Therefore

$$Var_3 = \left(\frac{2}{3\pi R(0)}\iint\limits_{A\backslash(\Delta\times S^1 \bigcup S^1\times\Delta)} -2f - \ell\frac{\partial f}{\partial\rho} + \left(2\pi - D(t_1,t_2)\right)f_\lambda dt_1 dt_2\right)\left(\lambda - \frac{1}{2R(0)}\right).$$

Lemma 3.6.

$$Var_4 = \left(\frac{2}{3\pi R(0)}\iint\limits_{S^1\times S^1\backslash A} -2f - \ell\frac{\partial f}{\partial\rho} - D(t_1,t_2)\frac{\partial f}{\partial\alpha} dt_1 dt_2\right)\left(\lambda - \frac{1}{2R(0)}\right)$$

The proof of this lemma is similar to the previous one.
Lemmas 3.3-3.6 complete the proof of Theorem 2.1.

4. Corollaries

In [1] it is shown that the circle is not always the global maximum, or the global minimum for the energy considered. Let us show that circle is a locally extremal knot for any energy E satisfying the conditions 1), 2) of the Introduction.

Corollary 4.1. *The circle is always a locally extremal knot.*

If τ is a circle, then

$$\ell(t_1, t_2) = 2\sin\frac{t_2 - t_1}{2}, \quad R(t_1) = 1, \quad \psi(t_1, t_2) = 0, \quad \phi = \frac{t_2 - t_1}{2}.$$

So $V_2(t_1) = 0$ for any $t_1 \in S^1$. Further

$$V_1(t_1) = \frac{1}{3}\left(8 \int\limits_{S^1} f + \sin\left(\frac{|t_1|}{2}\right) f_\rho dt_1 - \frac{2}{\pi} \iint\limits_{S^1 \times S^1} 2f + 2\sin\left(\frac{|t_2 - t_1|}{2}\right)\frac{\partial f}{\partial \rho} \right.$$

$$\left. + D(t_1, t_2)\frac{\partial f}{\partial \alpha} dt_1 dt_2 + 4 \iint\limits_{A} \frac{\partial f}{\partial \alpha} dt_1 dt_2 \right) =$$

$$\frac{1}{3}\left(8 \int\limits_{S^1} f + \sin\left(\frac{|t_1|}{2}\right)\frac{\partial f}{\partial \rho} dt_1 - 4 \int\limits_{S^1} 2f + 2\sin\left(\frac{|t_1|}{2}\right)\frac{\partial f}{\partial \rho} + D(0, t_1)\frac{\partial f}{\partial \alpha} dt_1 + \right.$$

$$4 \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2D(t_1, t_2)\frac{\partial f}{\partial \alpha} dt_1 dt_2 \left.\right) \overset{f(\rho,\alpha)=f(\rho,2\pi-\alpha)}{=} -4 \int\limits_{S^1} D(0, t_1)\frac{\partial f}{\partial \alpha} dt_1 + 4 \int\limits_{S^1} D(0, t_1)\frac{\partial f}{\partial \alpha} dt_1 = 0.$$

Therefore any point of the circle is a locally extremal point. Hence the circle is locally extremal. The corollary is proved.

Now let us say a few worlds about Möbius energy which is (in the version from [3])

$$f_M = \frac{1}{|\tau(t_1) - \tau(t_2)|^2} - \frac{1}{D^2(t_1, t_2)}.$$

It has many remarkable properties (see [9] and [3]). Möbius energies of homothetic knots are equal. This energy is invariant for Möbius transformations (see also Section 5). The variational equations and the gradient flow equation of Möbius energy was studied in [3].

Unfortunately, for Möbius energy, the variation Var is always infinite, and this mean that we can not perturb the knot in the way considered above.

The main property of Möbius energy is as follows. When a knot crossing tends to a double point, the energy tends to infinity. The energy is always positive. So every topological type of knot has a representative with minimal value of energy, some normal form.

Notice that the main part of Möbius energy is $1/|\tau(t_1) - \tau(t_2)|^2$. The other part $1/D^2(t_1, t_2)$ is only a normalization that makes the integral convergent. So let us make another normalization of the "main part" of Möbius energy. In this case we often lose the invariance for Möbius transformations. Let us consider the following energy:

$$\tilde{f} = \frac{D^3(x, y)}{|\tau(x), \tau(y)|^2}.$$

It is easily seen that this energy on one hand has the above property and on the other we can use our variational principles. Note also that such an energy is the same for homothetic knots.

Corollary 4.2. *We present V_1 and V_2 for this energy:*

$$V_1(t_0) = \frac{2}{3R(t_0)}\left(4\int_{S^1}\left(\frac{|\tau(t)-\tau(t_0)|^3}{D(t,t_0)^2}\left(1-2\frac{R(t_0)}{D(t,t_0)}\Phi(t_0,t)\right)\right)dt-\right.$$

$$\left.\frac{3}{\pi}\iint_{S^1\times S^1}\frac{|\tau(t_2)-\tau(t_1)|^3}{D(t_2,t_1)^2}dt_1dt_2+6\iint_A\frac{|\tau(t_2)-\tau(t_1)|^2}{D(t_2,t_1)^2}dt_1dt_2\right);$$

$$V_2(t_0) = -\frac{8}{3R(t_0)}\int_S\frac{|\tau(t_1)-\tau(t_2)|^3}{D(t_0,t)^3}\Psi(t_0,t)dt.$$

5. Definition and some basic properties of Mm-energy

In this section we define the Mm-energy of a knot. The nature of this energy differs from the energies considered in the previous sections.

Let us fix some point t_0 on the circle and define the real number $f_{Mm}(t_0)$. Consider the map $\rho_{t_0} : S^1 \longrightarrow \mathbb{R}$ such that $\rho_{t_0}(t) = |\tau(t) - \tau(t_0)|$. Let us note that the map τ is smooth. Hence ρ_{t_0} is also smooth except for one point t_0. If the number of maximums and minimums is finite, then we define the function f_{Mm} as follows:

$$f_{Mm}(t_0) = \frac{1}{\rho_{t_0}(t_M)} + \sum_{t_{m_i}\in U_1}\frac{1}{\rho_{t_0}(t_{m_i})} - \sum_{t_{M_j}\in U_2}\frac{1}{\rho_{t_0}(t_{M_j})},$$

where t_M is one of the points where the function ρ_{t_0} achieves its global maximum; U_1 is the set of all points of the circle, except the point t_0, where the function ρ_{t_0} has local minimums; U_2 is the set of all points of the circle, except the point t_M, where the function ρ_{t_0} has local maximums (see Fig. 5). Here we suppose $t_0 < t_* < t_0 + 2\pi$. In the case of an infinite number of maximums and minimums we make a small smooth perturbation $\tilde{\rho}_{t_0}$ so that the number of minimums and maximums becomes finite. Now we can calculate the value of $\tilde{f}_{Mm}(t_0)$ for the function $\tilde{\rho}_{t_0}$ as it was made before. Finally we define the $f_{Mm}(t_0)$ as the limit of $\tilde{f}_{Mm}(t_0)$ in the C^∞-topology.

Now we define the Mm-energy.

Definition 5.1. We call *Mm-energy* of the given knot the following number:

$$E_{Mm}(\tau) = \int_{S^1} f_{Mm}(t)dt,$$

if the integral converges.

Remark 5.1. Consider some small smooth perturbation of a knot. Then for any point t_0 of the circle the function ρ_{t_0} is also perturbed in a smooth way. At a generic point four possible modifications in the sums of f_{Mm} can occur: small changes of the values of the
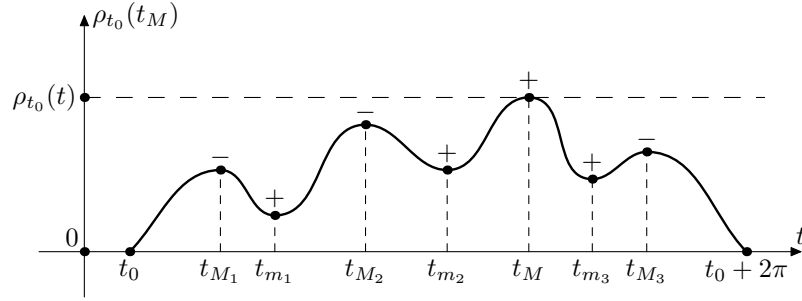
FIGURE 5. The function ρ_{t_0}.

maximums and minimums; the death of one maximum and of the neighboring minimum; conversely, the birth of one maximum and minimum at some point; a local maximum close to the global maximum can become the global maximum. In all these cases the variation of the resulting f_{Mm} is small. This is the reason why the Mm-energy depends on small perturbations of knots continuously.

Further we formulate the basic properties of Mm-energy.

Proposition 5.1. *The Mm-energy is greater than or equal to 2.*

Consider the sum

$$f_{Mm}(t_0) = \frac{1}{\rho_{t_0}(t_M)} + \sum_{t_{m_i} \in U_1} \frac{1}{\rho_{t_0}(t_{m_i})} - \sum_{t_{M_j} \in U_2} \frac{1}{\rho_{t_0}(t_{M_j})}$$

We can fix the ordering of the minimums and the maximums in the standard way:

$$t_0 < t_{M_1} < t_{m_1} < \ldots < t_{M_k} < t_{m_k} < t_M < t_{m_{k+1}} < t_{M_{k+1}} < \ldots < t_{m_n} < t_{M_n} < t_0 + 2\pi.$$

Then we have

$$f_{Mm}(t_0) = \sum_{i=0}^{k} \left(\frac{1}{\rho_{t_0}(t_{m_i})} - \frac{1}{\rho_{t_0}(t_{M_i})} \right) + \frac{1}{\rho_{t_0}(t_M)} + \sum_{i=k+1}^{n} \left(\frac{1}{\rho_{t_0}(t_{m_i})} - \frac{1}{\rho_{t_0}(t_{M_i})} \right) \geq$$
$$0 + \frac{1}{\rho_{t_0}(t_M)} + 0 = \frac{1}{\rho_{t_0}(t_M)}.$$

Finally, note that the length of the knot is 2π, hence the function $\rho_{t_0}(t_M)$ is smaller than or equal to π. Therefore

$$E_{Mm}(\tau) = \int_{S^1} f_{Mm}(t)dt \leq \int_{S^1} \frac{1}{\rho_t(t_{M_t})}dt \leq \int_{S^1} \frac{1}{\pi}dt = \frac{2\pi}{\pi} = 2.$$

This completes the proof of Proposition 5.1.

Proposition 5.2. *The Mm-energy is an invariant of homothety.*

Suppose τ is a knot of length 2π and $\tilde{\tau}$ is a homothetic knot of length $2l\pi$, where l is the coefficient of homothety. Then $d\tilde{t} = ldt$ and $\tilde{\rho}(\tilde{t}) = l\rho(t)$ for any t, and so $\tilde{f}_{Mm}(\tilde{t}) = f_{Mm}(t)/l$. Thus we obtain

$$E_{Mm}(\tilde{\tau}) = \int\limits_{S^1} \tilde{f}_{Mm}(\tilde{t})d\tilde{t} = \int\limits_{S^1} \frac{f_{Mm}(t)}{l}ldt = \int\limits_{S^1} f_{Mm}(t)dt = E_{Mm}(\tau).$$

Proposition 5.2 is proven.

So we can consider knots without any restriction on their lengths.

Proposition 5.3. *When two branches of the knot tends to a double crossing, the Mm-energy tends to infinity.*

Consider a smooth family $\{\tau_\lambda | \lambda \in [0,1]\}$ such that τ_0 is a smooth knot with double crossing and $\tau_\lambda, \lambda \neq 0$ is a smooth knot without any double crossing. For every ε we can choose a sufficiently small λ satisfying the following conditions: there exist two points t_1 and t_2 with $|t_1 - t_2| < \varepsilon^2$ such that the functions ρ_{t_1} and ρ_{t_2} have global minima at the points t_2 and t_1 correspondingly; and the ball $B_{\varepsilon,p}$ of radius ε with center at the midpoint p of the segment $[\tau_\lambda(t_1), \tau_\lambda(t_2)]$ has only two connected components of a knot τ_λ inside.

The family is smooth, hence the curvature of all knots is bounded by some N. If $\varepsilon < 1/N$, then every point t of the knot τ_λ inside the ball $B_{\varepsilon/2,p}$ has one extremum (i.e., the global minimum) of the function ρ_t inside the ball $B_{\varepsilon,p}$, and every point t of this knot inside the ball $B_{\varepsilon,p}$ has no more than one extremum (i.e., the global minimum) of ρ_t inside the ball $B_{\varepsilon,p}$. Let us estimate the energy inside the ball $B_{\varepsilon,p}$.

$$E_{Mm}(\tau_\lambda \cap B_{\varepsilon,p}) > 4\int\limits_{\frac{\varepsilon^2}{2}}^{\frac{\varepsilon}{2}} \frac{1}{t + \frac{\varepsilon^2}{2}}dt = 4\ln(t + \frac{\varepsilon^2}{2})\Big|_{\frac{\varepsilon^2}{2}}^{\frac{\varepsilon}{2}} = 4\ln\frac{\frac{e}{2} + \frac{e^2}{2}}{\varepsilon^2} > 4\ln\frac{2}{\varepsilon}.$$

The other terms (we ignore the global minimum of ρ_t) of the function f_{Mm} changes in a smooth way, hence the Mm-energy grows to infinity.

Therefore Mm-energy separates knots from different topological classes.

The following property is an essential property of Mm-energy.

Proposition 5.4. *The Mm-energy is well defined for piecewise smooth knots with obtuse angles.*

If some point t is "near" the angle then the function ρ_t is monotone function in some neighborhood of the vertex of an angle and hence there are no minima or maxima of ρ_t in this neighborhood.

In particular, the Mm-energy is well defined for piecewise linear knots with obtuse angles. So we can consider piecewise linear approximations of smooth knots and take the restriction to the set of piecewise linear knots. This property allows us to develop computer experiments in calculating normal forms for Mm-energies of topological classes of knots and the values of Mm-energies for this normal forms.
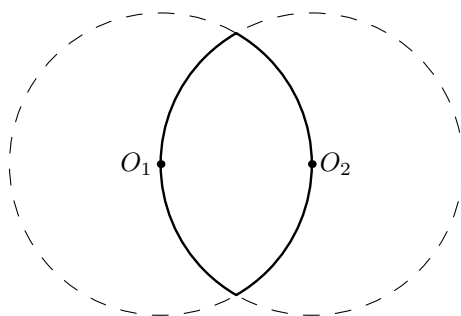
FIGURE 6. Mm-energy of this knot is $2\ln(\frac{7+4\sqrt{3}}{3})$.

Now we calculate Mm-energy for some knots. First we find the Mm-energy of the circle τ_0

$$E_{Mm}(\tau_0) = \int_{S^1} \frac{1}{2}dt = \pi.$$

Unfortunately the circle is not the normal form for the class of trivial knots. An example of the trivial knot with Mm-energy less than π is shown on Figure 6. This knot is a union of two arcs of the circle. Direct calculations shows that the Mm-energy of this knot is $2\ln(\frac{7+4\sqrt{3}}{3}) \approx 3.070607 < \pi$.

Computer experiments provide upper bounds for the Mm-energies of the normal forms for some topological classes (see the table behind).

CLASSES OF KNOTS	THE UPPER BOUNDS FOR THE ENERGIES OF NORMAL FORMS
the class of the circle	3.044012
the class of the trefoil	13.152759
the class of the figure-eight	19.450447
the class of 5_1	26.498108
the class of 5_2	27.168222
the class of 6_1	34.469191
the class of 6_2	35.466138
the class of 6_3	37.683129
the class of the connected sum of right and left trefoils	25.734616
the class of the connected sum of two right trefoils	26.748901

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