Infinite-Duration Poorman-Bidding Games*

Guy Avni† Thomas A. Henzinger‡ Rasmus Ibsen-Jensen§
IST Austria

Abstract

In two-player games on graphs, the players move a token through a graph to produce an infinite path, which determines the winner or payoff of the game. Such games are central in formal verification since they model the interaction between a non-terminating system and its environment. We study *bidding games* in which the players bid for the right to move the token. Two bidding rules have been defined. In *Richman* bidding, in each round, the players simultaneously submit bids, and the higher bidder moves the token and pays the other player. *Poorman* bidding is similar except that the winner of the bidding pays the "bank" rather than the other player. While poorman reachability games have been studied before, we present, for the first time, results on *infinite-duration* poorman games. A central quantity in these games is the *ratio* between the two players' initial budgets. The questions we study concern a necessary and sufficient ratio with which a player can achieve a goal. For reachability objectives, such *threshold ratios* are known to exist for both bidding rules. We show that the properties of poorman reachability games extend to complex qualitative objectives such as parity, similarly to the Richman case. Our most interesting results concern quantitative poorman games, namely poorman mean-payoff games, where we construct optimal strategies depending on the initial ratio, by showing a connection with *random-turn based games*. The connection in itself is interesting, because it does not hold for reachability poorman games. We also solve the complexity problems that arise in poorman bidding games.

1 Introduction

Two-player infinite-duration games on graphs are a central class of games in formal verification [3] and have deep connections to foundations of logic [34]. They are used to model the interaction between a system and its environment, and the problem of synthesizing a correct system then reduces to finding a winning strategy in a graph game [32]. Theoretically, they have been widely studied. For example, the problem of deciding the winner in a parity game is a rare problem that is in NP and coNP [21], not known to be in P, and for which a quasi-polynomial algorithm was only recently discovered [10].

A graph game proceeds by placing a token on a vertex in the graph, which the players move throughout the graph to produce an infinite path ("play") π. The game is zero-sum and π determines the winner or payoff. Two ways to classify graph games are according to the type of *objectives* of the players, and according to the *mode of moving* the token. For example, in *reachability games*, the objective of Player 1 is to reach a designated vertex t, and the objective of Player 2 is to avoid t. An infinite play π is winning for Player 1 iff it visits t. The simplest mode of moving is *turn based*: the vertices are partitioned between the two players and whenever the token reaches a vertex that is controlled by a player, he decides how to move the token.

We study a new mode of moving in infinite-duration games, which is called *bidding*, and in which the players *bid* for the right to move the token. The bidding mode of moving was introduced in [26, 27]

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†guy.avni@ist.ac.at

‡tah@ist.ac.at

§ribsen@ist.ac.at

for reachability games, where two bidding rules were defined. The first bidding rule, called *Richman* rule (named after David Richman), is as follows: Each player has a budget, and before each move, the players submit bids simultaneously, where a bid is legal if it does not exceed the available budget. The player who bids higher wins the bidding, pays the bid to other player, and moves the token. The second bidding rule, which we focus on in this paper and which is called *poorman* bidding in [26], is similar except that the winner of the bidding pays the "bank" rather than the other player. Thus, the bid is deducted from his budget and the money is lost. Note that while the sum of budgets is constant in Richman games, in poorman games, the sum of budgets shrinks as the game proceeds.

Bidding for moving is a general concept that is relevant in any setting in which a scheduler needs to decide the order in which selfish agents perform actions. For example, in a multi-process system, a scheduler decides the order in which the processes execute. Allowing the processes to bid for moving is one method to resolve this conflict, and it ensures that processes never starve, a property that is called *fairness*. Systems that use internal currency to prevent free-riding are called "scrip systems" [23], and are popular in databases for example. Other examples in which bidding for moving can be used to determine agent ordering include multi-rounded negotiations [35], sequential auctions [28], and local search for Nash equilibria [18].

Poorman bidding is appropriate in modeling settings in which the agents pay the scheduler to gain priority. In order to accept payment, a scheduler needs to be a selfish entity. An example of such a scheduler appears in *Blockchain* technology like the one used in Bitcoin or Etherium. Simplifying the technology, a blockchain is a log of transactions issued by clients and maintained by *miners*. In order to write to the log, clients send their transactions and an offer for a transaction fee to a miner, who has freedom to decide transaction priority. A vulnerability of Etherium was recognized in which a client was able to extract money by colluding with the miner and gaining priority for his transactions.[1] As another example, when the agents are buyers in a sequential auction, the scheduler models the auctioneer, and the agents' winning bids are its revenue [20].

An advantage of the poorman rule over the Richman rule is that they generalize easily to other important domains such as multi-player games, where the restriction of fixed sum of budgets in Richman bidding, is an obstacle. For example, it is easier to add to poorman games a re-charge of the budget, say at special vertices, which is natural in many applications. Also, our games are full-information games; a player knows the budget of the other player and can use this information in his biddings. Extending poorman rules to incorporate partial information is more natural. Historically, however, poorman bidding was less studied than Richman bidding since they are technically more difficult as we elaborate later.

A central quantity in bidding games is the *ratio* of the players' initial budgets. Formally, let $B_i \in \mathbb{R}_{\geq 0}$, for $i \in \{1, 2\}$, be Player i's initial budget. The *total initial budget* is $B = B_1 + B_2$ and Player i's *initial ratio* is B_i/B. The first question that arises in the context of bidding games is a necessary and sufficient initial ratio for a player to guarantee winning. For reachability games, it was shown in [26, 27] that such *threshold ratios* exist in every Richman and poorman reachability game: for every vertex v there is a ratio $\mathtt{Th}(v) \in [0, 1]$ such that (1) if Player 1's initial ratio exceeds $\mathtt{Th}(v)$, he can guarantee winning, and (2) if his initial ratio is less than $\mathtt{Th}(v)$, Player 2 can guarantee winning. This is a central property of the game, which is a form of *determinacy*, and shows that no ties can occur.[2]

An interesting probabilistic connection was observed in [26, 27] for reachability Richman games. For $r \in [0, 1]$, the *random turn-based game* that corresponds to a game $\mathcal{G}$ w.r.t. r, denoted $\mathtt{RTB}^r(\mathcal{G})$, is a special case of stochastic game [16] in which, rather than bidding for moving, in each round, independently, Player 1 is chosen to move with probability r and Player 2 moves with the remaining probability of $1 - r$. The probabilistic connection is the following: the probability with which Player 1 can guarantee reaching

[1] For further details see: `http://bit.ly/2obzyE7`.

[2] When the initial budget of Player 1 is exactly $\mathtt{Th}(v)$, the winner of the game depends on how we resolve draws in biddings, and our results hold for any tie-breaking mechanism.

his target in the uniform game $\texttt{RTB}^{0.5}(\mathcal{G})$ from a vertex v equals $1 - \texttt{Th}(v)$ in $\mathcal{G}$. For poorman reachability games, no such probabilistic connection is known. Moreover, such a connection is unlikely to exist since unlike in the Richman case, there are finite poorman games with irrational threshold ratios. The lack of a probabilistic connection makes poorman games technically more complicated.

More interesting, from the synthesis and logic perspective, are infinite winning conditions, but they have only been studied in the Richman setting previously [5]. We show, for the first time, existence of threshold ratios in qualitative poorman games with infinite winning conditions such as parity. The proof technique is similar to the one for Richman bidding: we show a linear reduction from poorman games with qualitative objectives to poorman reachability games.

Things get more interesting in poorman *mean-payoff games*, which are quantitative games; an infinite play π of the game is associated with a *payoff* $c \in \mathbb{R}_{\geq 0}$, which is Player 1's reward and Player 2's cost. Accordingly, we refer to the players in a mean-payoff game as Max and Min. The payoff of π is determined according to the weights it traverses and, as in the previous games, the bids are only used to determine whose turn it is to move. The central question in these games is: Given a value $c \in \mathbb{Q}$, what is the initial ratio that is necessary and sufficient for Max to guarantee a payoff of c? More formally, we say that c is the *value* with respect to a ratio $r \in [0, 1]$ if for every $\epsilon > 0$, we have (1) when Max's initial ratio is $r + \epsilon$, he can guarantee a payoff of at least c, and (2) intuitively, Max cannot hope for more: if Max's initial ratio is $r - \epsilon$, then Min can guarantee a payoff of at most c.

Our most technically-involved contribution is a construction of optimal strategies in poorman mean-payoff games, which depend on the initial ratio $r \in [0, 1]$. The crux of the solution is reasoning about strongly-connected games: we first reason on the bottom strongly-connected components of a game graph and extend the solution by, intuitively, playing a reachability game in the rest of the graph. Before describing our solution, let us highlight an interesting difference between Richman and poorman rules. With Richman bidding, it is shown in [5] that a strongly-connected Richman mean-payoff game has a value that does not depend on the initial ratio and only on the structure of the game. It thus seems reasonable to guess that the initial ratio would not matter with poorman bidding as well. We show, however, that this is not the case; the higher Max's initial ratio is, the higher the payoff he can guarantee. We demonstrate this phenomenon with the following simple game. Technically, each vertex in a mean-payoff game is labeled by a weight. Consider an infinite play π. The *energy* of a prefix π^n of length n of π, denoted $E(\pi^n)$, is the sum of the weights it traverses. The payoff of π is $\liminf_{n\to\infty} E(\pi^n)/n$.

Example 1. Consider the mean-payoff bidding game that is depicted in Figure 1, where for convenience the weights are placed on the edges rather than the vertices. We take the viewpoint of Min in this example. We consider the case of $r = \frac{1}{2}$, and claim that the value with respect to $r = \frac{1}{2}$ is 0. Note that the players' choices upon winning a bid in the game are obvious, and the difficulty in devising a strategy is finding the right bids. Intuitively, Min copies Max's strategy. Suppose, for example, that Min starts with a budget of $1+\epsilon$ and Max starts with 1, for some $\epsilon > 0$. A strategy for Min that ensures a payoff of 0 is based on a stack of numbers as follows: In round i, if the stack is empty Min bids $\epsilon \cdot 2^{-i}$, and otherwise the first number of the stack. If Min wins, he removes the first number on the stack (if non-empty). If Max wins, Min pushes Max's winning bid on the stack. For example, suppose Max's bid in the first bidding is 0.2, he wins and Min pushes 0.2 on the empty stack. Suppose Max bids 0.3 in the second bidding, then he wins again since Min's bid is 0.2. Suppose Max bids 0.1 in the third bidding, then Min wins with his bid of 0.3 and pops it from the stack. In the next bidding his bid is 0.2. It is not hard to show that Min never bids higher than the available budget. Also, we can show that every Max win is eventually matched, thus Min's queue empties infinitely often and the energy hits 0 infinitely often. Since we use $\liminf$ in the definition of the payoff, Min guarantees a non-positive payoff. Showing that Max can guarantee a non-negative payoff with an initial ratio of $\frac{1}{2} + \epsilon$ is harder, and a proof for the general case can be found in Section 4.

We show that the value c decreases with Max's initial ratio r. We set $r = \frac{1}{3}$. Suppose, for example,

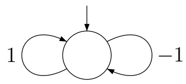


Figure 1: A mean-payoff game.

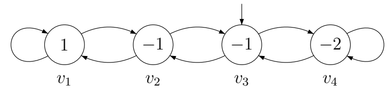


Figure 2: A second mean-payoff game.

that Min's initial budget is $2 + \epsilon$ and Max's initial budget is 1. We claim that Min can guarantee a payoff of $-1/3$. His strategy is similar to the one above, only that whenever Max wins with b, Min pushes b to the stack twice. Now, every Max win is matched by two Min wins, and the claim follows. $\square$

In order to solve strongly-connected poorman mean-payoff games, we identify a probabilistic connection for these games. Consider such a game $\mathcal{G}$ and a ratio $r \in [0, 1]$. Recall that $\texttt{RTB}^r(\mathcal{G})$ is a random turn-based game in which Max chooses the next move with probability r and Min with probability $1 - r$. The game $\texttt{RTB}^r(\mathcal{G})$ is a stochastic mean-payoff game, and its value, denoted $\texttt{MP}(\texttt{RTB}^r(\mathcal{G}))$, is the optimal expected payoff that the players can guarantee. The probabilistic connection we show is that the value in $\mathcal{G}$ with respect to r equals the value $\texttt{MP}(\texttt{RTB}^r(\mathcal{G}))$.

To the best of our knowledge, there was no natural known connection between bidding games and biased random turn-based games, i.e., $r \neq 0.5$. Uniform random turn-based games, i.e., with $r = 0.5$, have been extensively studied in the mathematics community. In the seminal work [31], a class of such games called *tug of war* games have been used to show the existence of the *infinity Laplacian* on a large class of sets. The infinity-laplacian problem is in turn connected to a discrete version of Dirichlet's problem. In Dirichlet's problem one gets a set S and the value of a function f of a subset of S and the goal is to find a continuous extension of f to S. The infinity Laplacian is similar only that a different local rule on the function is used rather than continuity.

Reachability games tend to be simpler than mean-payoff games, thus we find the existence of a probabilistic connection in poorman mean-payoff games surprising given the inexistence of such a connection for poorman reachability games. A corollary of the result is the following connection between poorman and Richman bidding. It is shown in [5] that the value of a Richman mean-payoff strongly-connected game $\mathcal{G}$ equals the value of the uniform game $\texttt{RTB}^{0.5}(\mathcal{G})$. Thus, the value of $\mathcal{G}$ when viewed as a Richman game equals the value of $\mathcal{G}$ when viewed as a poorman game with respect to the initial ratio 0.5. We are not aware of previous such connections between the two bidding rules.

Finally, we address, for the first time, complexity issues in poorman games; namely, we study the problem of finding threshold ratios in poorman games. We show that for qualitative games, the corresponding decision problem is in PSPACE using the existential theory of the reals [11]. For mean-payoff games, the problem of finding the value of the game with respect to a given ratio is also in PSPACE for general games, and for strongly-connected games, we show the value can be found in NP and coNP, and even in P for strongly-connected games with out-degree 2.

Further related work Beyond the works that are directly relevant to us, which we have compared to above, we list previous work on Richman games. To the best of our knowledge, since their introduction, poorman games have not been studied. Motivated by recreational games, e.g., bidding chess [8, 25], *discrete bidding games* are studied in [17], where the money is divided into chips, so a bid cannot be arbitrarily small unlike the bidding games we study. Non-zero-sum two-player Richman games were recently studied in [22].

2 Preliminaries

A graph game is played on a directed graph $G = \langle V, E \rangle$, where V is a finite set of vertices and $E \subseteq V \times V$ is a set of edges. The *neighbors* of a vertex $v \in V$, denoted $N(v)$, is the set of vertices $\{u \in V : \langle v, u \rangle \in E\}$, and we say that G has out-degree 2 if for every $v \in V$, we have $|N(v)| = 2$. A *path* in G is a finite or infinite sequence of vertices $v_1, v_2, \ldots$ such that for every $i \geq 1$, we have $\langle v_i, v_{i+1} \rangle \in E$.

Objectives An objective O is a set of infinite paths. In reachability games, Player 1 has a target vertex v_R and an infinite path is winning for him if it visits v_R. In *parity* games each vertex has a parity index in $\{1, \ldots, d\}$, and an infinite path is winning for Player 1 iff the maximal parity index that is visited infinitely often is odd. We also consider games that are played on a weighted graph $\langle V, E, w \rangle$, where $w : V \to \mathbb{Q}$. Consider an infinite path $\pi = v_1, v_2, \ldots$. For $n \in \mathbb{N}$, we use π^n to denote the prefix of length n of π. We call the sum of weights that π^n traverses the *energy* of the game, denoted $E(\pi^n)$. Thus, $E(\pi^n) = \sum_{1 \leq j < n} w(v_j)$. In *energy games*, the goal of Player 1 is to keep the energy level positive, thus he wins an infinite path iff for every $n \in \mathbb{N}$, we have $E(\pi^n) > 0$. Unlike the previous objectives, a path in a *mean-payoff* game is associated with a payoff, which is Player 1's reward and Player 2's cost. Accordingly, in mean-payoff games, we refer to Player 1 as Min and Player 2 as Max. We define the payoff of π to be $\liminf_{n \to \infty} \frac{1}{n} E(\pi^n)$. We say that Max wins an infinite path of a mean-payoff game if the payoff is non-negative.

Strategies and plays A *strategy* prescribes to a player which *action* to take in a game, given a finite *history* of the game, where we define these two notions below. For example, in turn-based games, a strategy takes as input, the sequence of vertices that were visited so far, and it outputs the next vertex to move to. In bidding games, histories and strategies are more complicated as they maintain the information about the bids and winners of the bids. Formally, a history is a sequence $\tau = v_0, \langle v_1, b_1, \ell_1 \rangle, \langle v_2, b_2, \ell_2 \rangle, \ldots, \langle v_k, b_k, \ell_k \rangle \in V \cdot (V \times \mathbb{R} \times \{1, 2\})^*$, where, for $j \geq 1$, in the j-th round, the token is placed on vertex v_{j-1}, the winning bid is b_j, and the winner is Player ℓ_j, and Player ℓ_j moves the token to vertex v_j. A strategy prescribes an action $\langle b, v \rangle$, where b is a bid that does not exceed the available budget and v is a vertex to move to upon winning. The winner of the bidding is the player who bids higher, where we assume there is some mechanism to resolve draws, and our results are not affected by what the mechanism is. More formally, for $i \in \{1, 2\}$, let B_i be the initial budgets of Player i, and, for a finite history π, let $W_i(\pi)$ be the sum of Player i winning bids throughout π. In Richman bidding, the winner of a bidding pays the loser, thus Player 1's budget following π is $B_1 - W_1 + W_2$. In poorman bidding, the winner pays the "bank", thus Player 1's budget following π is $B_1 - W_1$. Note that in poorman bidding, the loser's budget does not change following a bidding. An initial vertex together with two strategies for the players determine a unique infinite *play* π for the game. The vertices that π visits form an infinite path $path(\pi)$. Player 1 wins π according to an objective O iff $path(\pi) \in O$. We call a strategy f *winning* for Player 1 if for every strategy g of Player 2 the play they determine satisfies O. Winning strategies for Player 2 are defined dually.

Definition 2. (**Initial ratio**) Suppose the initial budget of Player i is B_i, for $i \in \{1, 2\}$, then the *total initial budget* is $B = B_1 + B_2$ and Player i's *initial ratio* is B_i/B. We assume $B > 0$.

The first question that arrises in the context of bidding games asks what is the necessary and sufficient initial ratio to guarantee an objective. We generalize the definition in [26, 27]:

Definition 3. (**Threshold ratios**) Consider a poorman or Richman game $\mathcal{G}$, a vertex v, and an initial ratio r and objective O for Player 1. The threshold ratio in v, denoted $\texttt{Th}(v)$, is a ratio in $[0, 1]$ such that

- if $r > \texttt{Th}(v)$, then Player 1 has a winning strategy that guarantees O is satisfied, and
- if $r < \texttt{Th}(v)$, then Player 2 has a winning strategy that violates O.

Recall that we say that Max wins a mean-payoff game $\mathcal{G} = \langle V, E, w \rangle$ if the mean-payoff value is non-negative. Finding $\texttt{Th}(v)$ for a vertex v in $\mathcal{G}$ thus answers the question of what is the minimal ratio of the initial budget that guarantees winning. A more refined question asks what is the optimal payoff Max can guarantee with an initial ratio r. Formally, for a constant $c \in \mathbb{Q}$, let $\mathcal{G}^c$ be the mean-payoff game that is obtained from $\mathcal{G}$ by decreasing all weights by c.

Definition 4. (**Mean-payoff values**) Consider a mean-payoff game $\mathcal{G} = \langle V, E, w \rangle$ and a ratio $r \in [0, 1]$. The value of $\mathcal{G}$ with respect to c, denoted $\mathtt{MP}^r(\mathcal{G}, v)$, is such that $\mathtt{Th}(v) = r$ in $\mathcal{G}^c$.

Random turn-based games In a *stochastic game* the vertices of the graph are partitioned between two players and a *nature* player. As in turn-based games, whenever the game reaches a vertex of Player i, for $i = 1, 2$, he choses how the game proceeds, and whenever the game reaches a vertex v that is controlled by nature, the next vertex is chosen according to a probability distribution that depends only on v.

Consider a game $\mathcal{G} = \langle V, E \rangle$. The *random-turn based game* with ratio $r \in [0, 1]$ that is associated with $\mathcal{G}$ is a stochastic game that intuitively simulates the fact that Player 1 chooses the next move with probability r and Player 2 chooses with probability $1 - r$. Formally, we define $\mathtt{RTB}^r(\mathcal{G}) = \langle V_1, V_2, V_N, E, \Pr, w \rangle$, where each vertex in V is split into three vertices, each controlled by a different player, thus for $\alpha \in \{1, 2, N\}$, we have $V_\alpha = \{v_\alpha : v \in V\}$, nature vertices simulate the fact that Player 1 chooses the next move with probability r, thus $\Pr[v_N, v_1] = r = 1 - \Pr[v_N, v_2]$, and reaching a vertex that is controlled by one of the two players means that he chooses the next move, thus $E = \{\langle v_\alpha, u_N \rangle : \langle v, u \rangle \in E \text{ and } \alpha \in \{1, 2\}\}$. When $\mathcal{G}$ is weighted, then the weights of v_1, v_2, and v_N equal that of v.

Fixing two strategies f and g for the two players in a stochastic game results in a Markov chain, which in turn gives rise to a probability distribution $D(f, g)$ over infinite sequences of vertices. A strategy f is *optimal* w.r.t. an objective O if it maximizes $\sup_f \inf_g \Pr_{\pi \sim D(f,g)}[\pi \in O]$. For the objectives we consider, it is well-known that optimal strategies exist, which are, in fact, *positional*; namely, strategies that only depend on the current position of the game and not on its history.

Definition 5. (**Values**) Let $r \in [0, 1]$. For a qualitative game $\mathcal{G}$, the *value* of $\mathtt{RTB}^r(\mathcal{G})$, denoted $val(\mathtt{RTB}^r(\mathcal{G}))$, is the probability that Player 1 wins when he plays optimally. For a mean-payoff game $\mathcal{G}$, the *mean-payoff value* of $\mathtt{RTB}^r(\mathcal{G})$, denoted $\mathtt{MP}(\mathtt{RTB}^r(\mathcal{G}))$, is the maximal expected payoff Max obtains when he plays optimally.

3 Poorman Parity Games

For qualitative objectives, poorman games have mostly similar properties to the corresponding Richman games. We start with reachability objectives, which were studied in [27, 26]. The objective they study is slightly different than ours. We call their objective *double-reachability*: both players have targets and the game ends once one of the targets is reached. As we show below, for our purposes, the variants are equivalent since there are no draws in finite-state poorman and Richman double-reachability games.

Consider a double-reachability game $\mathcal{G} = \langle V, E, u_1, u_2 \rangle$, where, for $i = 1, 2$, the target of Player i is u_i. In both Richman and poorman bidding, trivially Player 1 wins in u_1 with any initial budget and Player 2 wins in u_2 with any initial budget, thus $\mathtt{Th}(u_1) = 0$ and $\mathtt{Th}(u_2) = 1$. For $v \in V$, let $v^+, v^- \in N(v)$ be such that, for every $v' \in N(v)$, we have $\mathtt{Th}(v^-) \leq \mathtt{Th}(v') \leq \mathtt{Th}(v^+)$.

Theorem 6. *[27, 26] Threshold ratios exist in Richman and poorman reachability games. Moreover, consider a double-reachability game $\mathcal{G} = \langle V, E, u_1, u_2 \rangle$.*

- *In Richman bidding, for $v \in V \setminus \{u_1, u_2\}$, we have $\mathtt{Th}(v) = \frac{1}{2}\big(\mathtt{Th}(v^+) + \mathtt{Th}(v^-)\big)$, and it follows that $\mathtt{Th}(v) = val(\mathtt{RTB}^{0.5}(\mathcal{G}, v))$ and that $\mathtt{Th}(v)$ is a rational number.*
- *In poorman bidding, for $v \in V \setminus \{u_1, u_2\}$, we have $\mathtt{Th}(v) = \mathtt{Th}(v^+)/\big(1 - \mathtt{Th}(v^-) + \mathtt{Th}(v^+)\big)$. There is a game $\mathcal{G}$ and a vertex v with an irrational $\mathtt{Th}(v)$.*

Proof. The proof here is similar to [26] and is included for completeness, with a slight difference: unlike [26], we consider games in which one of the targets is not reachable. The Richman case is irrelevant for us and we leave it out.

Consider a poorman double-reachability game $\mathcal{G} = \langle V, E, u_1, u_2 \rangle$. It is shown in [26] that there exists a unique function $f : V \to [0,1]$ that satisfies the following conditions: we have $f(u_1) = 0$ and $f(u_2) = 1$, and for every $v \in V$, we have $f(v) = \frac{f(v^+)}{1+f(v^+)-f(v^-)}$, where $v^+, v^- \in N(v)$ are the neighbors of v that respectively maximize and minimize f, i.e., for every $v' \in N(v)$, we have $f(v^-) \leq f(v') \leq f(v^+)$.

We claim that for every $v \in V$, we have $\texttt{Th}(v) = f(v)$. Our argument will be for Player 1 and duality gives an argument for Player 2. Suppose Player 1's budget is $f(v) + \epsilon$ and Player 2's budget is $1 - f(v)$, for some $\epsilon > 0$. Note that we implicitly assume that $f(v) < 1$. In case $f(v) = 1$ we do not show anything, but still, our dual strategy for Player 2 ensures that u_2 is visited, when the initial budget for Player 2 is positive. We describe a Player 1 strategy that forces the game to u_1.

Similar to [26], we divide Player 1's budget ratio into his *real budget* and a *slush fund*. We will ensure the following invariants:

1. Whenever we are in state v, if x is Player 1's real budget and y is Player 2's budget, then $f(v) = x/(x+y)$.

2. Every time Player 2 wins a bidding the slush fund increases by a constant factor. Formally, there exists a constant $c > 1$, such that when ϵ_0 is the initial slush fund and ϵ_i is the slush fund after Player 2 wins for the i-th time, we have that $\epsilon_i > c \cdot \epsilon_{i-1}$, for all $i \geq 1$.

Note that these invariants are satisfied initially.

We describe a Player 1 strategy. Consider a round in vertex v in which Player 1's real budget is x', Player 2's budget is y' and the last time Player 2 won (or initially, in case Player 2 has not won yet) his slush fund was ϵ'. Player 1's bid is $\Delta(v) \cdot x' + \delta_v \cdot \epsilon'$, where we define $\Delta(v)$ and δ_v below. Upon winning, Player 1 moves to v^-, i.e., to the neighbor that minimizes $f(v)$, or, when $f(v) = 0$, he moves to a vertex closer to u_1. Upon winning, Player 1 pays $\Delta(v) \cdot x'$ from his real budget and $\delta_v \cdot \epsilon'$ from his slush fund.

For $v \in V \setminus \{u_1, u_2\}$, if $f(v) > 0$ and $f(v^-) < 1$, let $\Delta(v) = \frac{f(v)-f(v^-)}{f(v)(1-f(v^-))}$ and otherwise, let $\Delta(v) = 0$. In App. A we choose δ_v, for $v \in V$, and show that our choice implies that Player 1's strategy maintains the invariant above. Note that the second invariant indicates that Player 2 cannot win more than a finite number of times, since whenever he wins, the slush fund increases by a constant and the slush fund cannot exceed 1, because then it would be bigger than the total budget. This in turn shows that eventually Player 1 wins n times in a row, which ensures that the play reaches u_1.

We show existence of threshold budgets in a poorman reachability game $\mathcal{G} = \langle V, E, u_1 \rangle$. Let $S \subseteq V$ be the set of vertices that have no path to u_1. Clearly, for $v \in S$, we have $\texttt{Th}(v) = 1$. Let $\mathcal{G}' = \langle V', E', u_1, u_2 \rangle$ be the double-reachability game that is obtained from $\mathcal{G}$ by setting $V' = V \setminus S$ and Player 2's target u_2 to be a vertex in S. For $v \in V'$, we claim that $\texttt{Th}(v)$ in $\mathcal{G}'$ equals $\texttt{Th}(v)$ in $\mathcal{G}$. Indeed, if Player 1's ratio exceeds $\texttt{Th}(v)$ he can draw the game to u_1 and if Player 2's ratio exceeds $1 - \texttt{Th}(v)$ he can draw the game to S.

Finally, we describe a poorman double-reachability game $\mathcal{G}$ with irrational threshold budgets. The vertices are u_1, v_1, v_2, and u_2, and the edges are $u_1 \leftarrow v_1 \leftrightarrow v_2 \rightarrow u_2$. Solving the equation above we get $\texttt{Th}(v_1) = (\sqrt{5} - 1)/2$ and $\texttt{Th}(v_2) = (3 - \sqrt{5})/2$. □

We continue to study poorman games with richer objectives.

Theorem 7. *Poorman parity games are linearly reducible to poorman reachability games. Specifically, threshold ratios exist in poorman parity games.*

Proof. The crux of the proof is to show that in a bottom strongly-connected component (BSCC, for short) of $\mathcal{G}$, one of the players wins with every initial budget. Thus, the threshold ratios for vertices in BSCCs are either 0 or 1. For the rest of the vertices, we construct a reachability game in which a player's goal is to reach a BSCC that is "winning" for him.

Formally, consider a strongly-connected poorman parity game $\mathcal{G} = \langle V, E, p \rangle$. We claim that there is $\alpha \in \{0, 1\}$ such that for every $v \in V$, we have $\texttt{Th}(v) = \alpha$, i.e., when $\alpha = 0$, Player 1 wins with any positive initial budget, and similarly for $\alpha = 1$. Moreover, deciding which is the case is easy: let $v_{Max} \in V$ be the vertex with maximal parity index, then $\alpha = 0$ iff $p(v_{Max})$ is odd.

Suppose $p(v_{Max})$ is odd and the proof for an even $p(v_{Max})$ is dual. We prove in two steps. First, following the proof of Theorem 6, we have that when Player 1's initial budget is $\epsilon > 0$, he can draw the game to v_{Max} once. Second, we show that Player 1 can reach v_{Max} infinitely often when his initial budget is $\epsilon > 0$. Player 1 splits his budget into parts $\epsilon_1, \epsilon_2, \ldots$, where $\epsilon_i = \epsilon \cdot 2^{-i}$, for $i \geq 1$, thus $\sum_{i \geq 1} \epsilon_i = \epsilon$. Then, for $i \geq 0$, following the i-th visit to v_{Max}, he plays the strategy necessary to draw the game to v_{Max} with initial budget ϵ_{i+1}.

We turn to show the reduction from poorman parity games to poorman double-reachability games. Consider a poorman parity game $\mathcal{G} = \langle V, E, p \rangle$. Let $S \subseteq V$ be a BSCC in $\mathcal{G}$. We call S *winning* for Player 1 if the vertex v_{Max} with highest parity index in S has odd $p(v_{Max})$. Dually, we call S winning for Player 2 if $p(v_{Max})$ is even. Indeed, the claim above implies that for every S that is winning for Player 1 and $v \in S$, we have $\texttt{Th}(v) = 0$, and dually for Player 2. Let $\mathcal{G}'$ be a poorman double-reachability game that is obtained from $\mathcal{G}$ by setting the BSCCs that are winning for Player 1 in $\mathcal{G}$ to be his target in $\mathcal{G}'$ and the BSCCs that are winning for Player 2 in $\mathcal{G}$ to be his target in $\mathcal{G}'$. Similar to the proof of Theorem 6, we have that $\texttt{Th}(v)$ in $\mathcal{G}$ equals $\texttt{Th}(v)$ in $\mathcal{G}'$, and we are done. □

4 Poorman Mean-Payoff Games

This section consists of our most technically challenging contribution. We construct optimal strategies for the players in poorman mean-payoff games. The crux of the solution regards strongly-connected mean-payoff games, which we develop in the first three sub-sections.

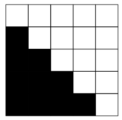
Consider a strongly-connected game $\mathcal{G}$ and an initial ratio $r \in [0, 1]$. We claim that the value in $\mathcal{G}$ w.r.t. r does not depend on the initial vertex. For a vertex v in $\mathcal{G}$, recall that $\texttt{MP}^r(\mathcal{G}, v)$ is the maximal payoff Max can guarantee when his initial ratio in v is $r + \epsilon$, for every $\epsilon > 0$. We claim that for every vertex $u \neq v$ in $\mathcal{G}$, we have $\texttt{MP}^r(\mathcal{G}, u) = \texttt{MP}^r(\mathcal{G}, v)$. Indeed, as in Theorem 7, Max can play as if his initial ratio is $\epsilon/2$ and draw the game from u to v, and from there play using an initial ratio of $r + \epsilon/2$. Since the energy that is accumulated until reaching v is constant, it does not affect the payoff of the infinite play starting from v.

We write $\texttt{MP}^r(\mathcal{G})$ to denote the value of $\mathcal{G}$ w.r.t. r. We show the following probabilistic connection: the value $\texttt{MP}^r(\mathcal{G})$ equals the value $\texttt{MP}(\texttt{RTB}^r(\mathcal{G}))$ of the random turn-based mean-payoff game $\texttt{RTB}^r(\mathcal{G})$ in which Max chooses the next move with probability r and Min with probability $1 - r$.

4.1 Warm up: solving a simple game

In this section we solve a simple game through which we demonstrate the ideas of the general case. Recall that in an energy game, Min wins a finite play if the sum of weights it traverses, a.k.a. the energy, is 0 and Max wins an infinite play in which the energy stays positive throughout the play. Consider the game depicted in Fig. 1 and view the game as an energy game. It is shown in [26] that if the initial energy is $k \in \mathbb{N}$, then Max wins iff his initial ratio exceeds $\frac{k+2}{2k+2}$. We describe an alternative proof for the first implication.

We need several definitions. For $k \in \mathbb{N}$, let S_k be the square of area k^2. In Fig. 3, we depict S_5. We split S_k into unit-area boxes such that each of its sides contains k boxes. A diagonal in S_k splits it into a smaller black triangle and a larger white one. For $k \in \mathbb{N}$, we respectively denote by t_k and T_k the areas of the smaller black triangle and the larger white triangle of S_k. For example, we have $t_5 = 10$ and $T_5 = 15$, and in general $t_k = \frac{k(k-1)}{2}$ and $T_k = \frac{k(k+1)}{2}$.



	2	3	4	5	6	
t_k	1	3	6	10	15	$\dots$
T_k	3	6	10	15	21	

Figure 3: The square S_5 with area 25 and the sizes of some triangles.

Suppose the game starts with energy $\kappa \in \mathbb{N}$. We show that Max wins when his ratio exceeds $\frac{\kappa+2}{2\kappa+2}$, which equals $\frac{T_{\kappa+1}}{(\kappa+1)^2}$. For ease of presentation, it is convenient to assume that the players' ratios add up to $1+\epsilon_0$, Max's initial ratio is $\frac{T_{\kappa+1}}{(\kappa+1)^2}+\epsilon_0$, and Min's initial ratio is $\frac{t_{\kappa+1}}{(\kappa+1)^2}$. For $j \geq 0$, we think of ϵ_j as Max's slush fund in the j-th round of the game, though its role here is somewhat less significant than in Theorem 6. Consider a play π. We think of changes in energy throughout π and changes in budget ratio as representing two walks on two sequences. The *energy sequence* is $\mathbb{N}$ and the *budget sequence* is $\{t_k/S_k : k \in \mathbb{N}\}$, with the natural order in the two sets. We show a strategy for Max that maintains the invariant that whenever the energy is $k \in \mathbb{N}$, then Max's ratio is greater than $T_{k+1}/(k+1)^2$. That is, whenever Max wins a bidding, both sequences take a "step up" and when he loses, both sequences take a "step down".

We describe Max's strategy. Upon winning a bidding, Max chooses the $+1$ edge and we assume Min chooses the -1 edge. The challenge is to find the right bids. Suppose the energy level is k at the j-th round. Thus, Max and Min's ratio are respectively $T_{k+1}/(k+1)^2+\epsilon_j$ and $t_{k+1}/(k+1)^2$. In other words, Min owns t_{k+1} boxes and Max owns a bit more than T_{k+1} boxes. Max's bid consists of two parts. Max bids $1/(k+1)^2+\epsilon_j/2$, or in other words, a single box and half of his slush fund. We first show how the strategy maintains the invariant and then how it guarantees that an energy of 0 is never reached. Suppose first that Max wins the bidding. The total number of boxes decreases by one to $(k+1)^2-1$, his slush fund is cut by half, and Min's budget is unchanged. Thus, Max's ratio of the budget is more than $(T_{k+1}-1)/\big((k+1)^2-1\big)$, which equals $T_{k+2}/(k+2)^2$. In other words, after normalization, Max owns more than T_{k+2} boxes and Min owns t_{k+2} boxes; the budget sequence takes a step up, matching the increase of 1 in the energy. The other case is when Min wins the bidding, the energy decreases by 1, and we show that the budget sequences takes a step down. Since Max bids more than one box, Min overbids, and in the worst case, he bids 1 box. Max's new ratio is more than $T_{k+1}/((k+1)^2-1) = T_k/k^2$. For example, let $k=4$. Following a Max win, Max's ratio is $\frac{T_5-1}{t_5+T_5-1} = \frac{15-1}{25-1} = \frac{21}{36} = \frac{T_6}{t_6+T_6}$ and upon losing, Max's ratio is $\frac{15}{25-1} = \frac{10}{16} = \frac{T_4}{t_4+T_4}$.

We conclude by showing that the energy never reaches 0 by showing that the walk on the budget sequence never reaches the first element. Suppose the energy is $k=1$ in the j-th round, thus according to the invariant, Max's ratio is $\frac{3}{4}+\epsilon_j$ and Min's ratio is $\frac{1}{4}$. Recall that Max bids $\frac{1}{(k+1)^2}+\epsilon_j/2$ at energy k, thus he bids $\frac{1}{4}+\epsilon_j/2$ at energy 1, and necessarily wins the bidding, implying that the energy increases.

4.2 Defining a richer budget sequence

The game studied in the previous section is very simple. In this section we generalize the budget sequence that is used there so that we can handle arbitrary strongly-connected graphs. We proceed in two steps. Note that the budget sequence that is used in the previous section tends to 0.5 as the initial energy increases. It can thus be used for an initial ratio $r=0.5$. The first generalization allows us to deal with $r \neq 0.5$. Recall the geometric intuition in the previous section. For some $k \in \mathbb{N}$, Min owns the smaller black triangle t_k and Max's owns the larger white triangle T_k. The total area of the square is t_k+T_k. Let $\mu, \nu \in \mathbb{Q}_{>0}$. We generalize the sequence by setting Min's budget to be μ black triangles and Max's budget to be ν white triangles. The total budget, or area, is thus $\mu \cdot t_k + \nu \cdot T_k$ and Max's initial ratio is $r = \frac{\nu \cdot T_k}{\nu \cdot T_k + \mu \cdot t_k}$. For example, set $k=5, \mu=2$, and $\nu=3$. Then, Min has $2 \cdot t_5 = 2 \cdot 10$ boxes and Max has $3 \cdot T_5 = 3 \cdot 15$ boxes. It is nice to note the following property, which can of course be generalized: a Min win with a bid of 2 results in a step down, indeed $\frac{3T_5}{2t_5+3T_5-2} = \frac{3 \cdot 15}{2 \cdot 10 + 3 \cdot 15 - 2} = \frac{3 \cdot T_4}{2 \cdot t_4 + 3 \cdot T_4}$, and a Max win with a bid of 3 results in a step up, indeed $\frac{3\cdot T_5-3}{2\cdot t_5+3\cdot T_5-3} = \frac{3\cdot T_6}{2\cdot t_6+3\cdot T_6}$.

We make a second generalization. Rather than restricting to a discrete domain in which k gets values in $\mathbb{N}$, we replace k with a variable x whose domain is the real numbers. We define a function $\mathcal{R}^r : \mathbb{R} \to \mathbb{R}$ by $\mathcal{R}^r(x) = \frac{\nu\cdot T_x}{\mu\cdot t_x+\nu\cdot T_x}$. Note that $\lim_{x\to\infty} \mathcal{R}^r(x) = \frac{\nu}{\mu+\nu}$, and that the limit is reached from above.

We describe the intuition of how the following lemma is used. A play is going to induce a walk on a budget sequence $B \subseteq \mathbb{R}$. Max's strategy will ensure that whenever the walk reaches $x \in B$, Max's ratio is greater than $\mathcal{R}^r(x)$. In the first part of the lemma Min bids $\mu \cdot y$, wins the bidding, and the walk proceeds down y steps. In the second part, Max bids $\nu \cdot y$, wins the bidding, and the walk proceeds up y steps. The proof can be found in App. B.

Lemma 8. *Consider $\mu, \nu \in \mathbb{Q}_{>0}$ and $0 < y \leq 1$ such that $\mu > \nu \cdot y$ when $\nu > \mu$ and $\nu > \mu \cdot y$ when $\mu > \nu$. Then, for every $x \geq 1$ both of the following hold*

$$\frac{\nu \cdot T_x}{\mu \cdot t_x + \nu \cdot T_x - \mu \cdot y} \leq \frac{\nu \cdot T_{x-y}}{\mu \cdot t_{x-y} + \nu \cdot T_{x-y}} \text{ and } \frac{\nu \cdot T_x - \nu \cdot y}{\mu \cdot t_x + \nu \cdot T_x - \nu \cdot y} \leq \frac{\nu \cdot T_{x+y}}{\mu \cdot t_{x+y} + \nu \cdot T_{x+y}}$$

4.3 The potential and strength of vertices

In an arbitrary strongly-connected game the bids in the different vertices cannot be the same. In this section we develop a technique to determine the "importance" of a node v, which we call its *strength* and measures how high the bid should be in v compared with the other nodes.

Consider a strongly-connected game $\mathcal{G} = \langle V, E, w\rangle$ and $r \in [0, 1]$. Recall that $\texttt{RTB}^r(\mathcal{G})$ is a random-turn based game in which Max chooses the next move with probability r and Min with probability $1 - r$. A *positional strategy* is a strategy that always chooses the same action (edge) in a vertex. It is well known that there exist optimal positional strategies for both players in stochastic mean-payoff games.

Consider two optimal positional strategies f and g in $\texttt{RTB}^r(\mathcal{G})$, for Min and Max, respectively. For a vertex $v \in V$, let $v^-, v^+ \in V$ be such that $f(v_{Min}) = v^-$ and $g(v_{Max}) = v^+$. The *potential* of v, denoted $\text{Pot}^r(v)$, is a known concept in probabilistic models and its existence is guaranteed [33]. We use the potential to define the *strength* of v, denoted $\text{St}^r(v)$, which intuitively measures how much the potentials of the neighbors of v differ. We assume w.l.o.g. that $\texttt{MP}(\texttt{RTB}^r(\mathcal{G})) = 0$ as otherwise we can decrease all weights by this value. Let $r = \frac{\nu}{\nu+\mu}$. The potential and strengths of v are functions that satisfy the following:

$$\text{Pot}^r(v) = \frac{\nu \cdot \text{Pot}^r(v^+) + \mu \cdot \text{Pot}^r(v^-)}{\mu + \nu} + w(v) \text{ and } \text{St}^r(v) = \nu\mu \cdot \frac{\text{Pot}^r(v^+) - \text{Pot}^r(v^-)}{\mu + \nu}$$

There are optimal strategies for which $\text{Pot}^r(v^-) \leq \text{Pot}^r(v') \leq \text{Pot}^r(v^+)$, for every $v' \in N(v)$, which can be found for example using the strategy iteration algorithm.

Consider a finite path $\pi = v_1, \ldots, v_n$ in $\mathcal{G}$. We intuitively think of π as a play, where for every $1 \leq i < n$, the bid of Max in v_i is $\text{St}(v_i)$ and he moves to v_i^+ upon winning. Thus, if $v_{i+1} = v_i^+$, we say that Max won in v_i, and if $v_{i+1} \neq v_i^+$, we say that Max lost in v_i. Let $W(\pi)$ and $L(\pi)$ respectively be the indices in which Max wins and loses in π. We call Max wins *investments* and Max loses *gains*, where intuitively he *invests* in increasing the energy and *gains* a higher ratio of the budget whenever the energy decreases. Let $G(\pi)$ and $I(\pi)$ be the sum of gains and investments in π, respectively, thus $G(\pi) = \sum_{i\in L(\pi)} \text{St}(v_i)$ and $I(\pi) = \sum_{i\in W(\pi)} \text{St}(v_i)$. Recall that the energy of π is $E(\pi) = \sum_{1\leq i<n} w(v_i)$. The following lemma connects the strength, potential, and energy.

Lemma 9. *Consider a strongly-connected game $\mathcal{G}$, a ratio $r = \frac{\nu}{\mu+\nu} \in (0, 1)$ such that $\texttt{MP}(\texttt{RTB}^r(\mathcal{G})) = 0$, and a finite path π in $\mathcal{G}$ from v to u. Then, $\text{Pot}^r(v) - \text{Pot}^r(u) \leq E(\pi) + G(\pi)/\mu - I(\pi)/\nu$.*

Proof. We prove by induction on the length of π. For $n = 1$, the claim is trivial since both sides of the equation are 0. Suppose the claim is true for paths of length n and we prove for paths of length $n+1$. We distinguish between two cases. In the first case, Max wins in v, thus the second vertex in π is v^+. The case where Min wins is proven similarly. Let π' be the prefix of π starting from v^+. Note that since Max wins the first bidding, we have $G(\pi) = G(\pi')$ and $I(\pi) = \text{St}(v) + I(\pi')$. Also, we have $E(\pi) = E(\pi') + w(v)$. Combining these, we have $E(\pi) + \frac{G(\pi)}{\mu} - \frac{I(\pi)}{\nu} = E'(\pi) + w(v) + \frac{G(\pi')}{\mu} - \frac{I(\pi')}{\nu} - \frac{\text{St}(v)}{\nu}$. By the induction hypothesis, we have $\text{Pot}^r(v^+) - \text{Pot}^r(u) \leq E(\pi') + G(\pi')/\mu - I(\pi')/\nu$. Combining these with the definition of $\text{St}(v)$, we have the following.

$$E(\pi) + \frac{G(\pi)}{\mu} - \frac{I(\pi)}{\nu} \geq -\frac{\text{St}(v)}{\nu} + \text{Pot}^r(v^+) + w(v) - \text{Pot}^r(u) =$$

$$= \mu \cdot \frac{-\text{Pot}^r(v^+) + \text{Pot}^r(v^-)}{\mu + \nu} + \text{Pot}^r(v^+) + w(v) - \text{Pot}^r(u) =$$

$$= \text{Pot}^r(v) - \text{Pot}^r(u)$$

□

Example 10. Consider the game depicted in Fig. 2. Max always proceeds left and Min always proceeds right, so, for example, we have $v_2^+ = v_1$ and $v_2^- = v_3$ We have $P^{\frac{2}{3}}(v_1) = 6$, $P^{\frac{2}{3}}(v_2) = 3$, $P^{\frac{2}{3}}(v_3) = 0$, and $P^{\frac{2}{3}}(v_4) = -3$. Thus, the strengths are $\text{St}(v_1) = 2$, $\text{St}(v_2) = 4$, $\text{St}(v_3) = 4$, and $\text{St}(v_4) = 2$. Consider the path $\pi = v_0, v_1, v_2, v_2, v_1, v_0$ in which Max wins the first three bids and loses the last two, thus $G(\pi) = 2+4$ and $I(\pi) = 4 + 4 + 2 = 10$. We have $E(\pi) = -1$ since the last vertex does not contribute to the energy. The left-hand side of the expression in Lemma 9 is 0, and the right-hand side is $-1 + 6/1 - 10/2 = 0$. □

4.4 Putting it all together

In this section we combine the ingredients developed in the previous sections to solve arbitrary strongly-connected mean-payoff games.

Theorem 11. *Consider a strongly-connected poorman mean-payoff game $\mathcal{G}$ and a ratio $r \in [0, 1]$. The value of $\mathcal{G}$ with respect to r equals the value of the random-turn based mean-payoff game $\texttt{RTB}^r(\mathcal{G})$ in which Max chooses the next move with probability r, thus $\texttt{MP}^r(\mathcal{G}) = \texttt{MP}(\texttt{RTB}^r(\mathcal{G}))$.*

Proof. We assume w.l.o.g. that $\texttt{MP}(\texttt{RTB}^r(\mathcal{G})) = 0$ since otherwise we decrease this value from all weights. Also, the case where $r \in \{0, 1\}$ is easy since $\texttt{RTB}^r(\mathcal{G})$ is a graph and in $\mathcal{G}$, one of the players can win all biddings. Thus, we assume $r \in (0, 1)$. Recall that $\texttt{MP}(\pi) = \liminf_{n\to\infty} \frac{E(\pi^n)}{n}$. We show a Max strategy that, when the game starts from a vertex $v \in V$ and with an initial ratio of $r + \epsilon$, guarantees that the energy is bounded below by a constant, which implies $\texttt{MP}(\pi) \geq 0$.

Note that showing such a strategy for Max suffices to prove $\texttt{MP}^r(\mathcal{G}) = 0$ since our definition for a payoff favors Min. Consider the game $\mathcal{G}'$ that is obtained from $\mathcal{G}$ by multiplying all weights by -1. We have $\texttt{MP}(\texttt{RTB}^r(\mathcal{G}')) = -\texttt{MP}(\texttt{RTB}^r(\mathcal{G})) = 0$. Associate Min in $\mathcal{G}$ with Max in $\mathcal{G}'$, thus an initial ratio of $r - \epsilon$ for Min in $\mathcal{G}$ is associated with an initial ratio of $r + \epsilon$ of Max in $\mathcal{G}'$. Let f be a Max strategy in $\mathcal{G}'$ that guarantees a non-negative payoff. Suppose Min plays in $\mathcal{G}$ according to f and let π be a play when Max plays some strategy. Since f guarantees a non-negative payoff in $\mathcal{G}'$, we have $\limsup_{n\to\infty} E(\pi^n)/n \leq 0$ in $\mathcal{G}$, and in particular $\texttt{MP}(\pi) = \liminf_{n\to\infty} E(\pi^n)/n \leq 0$.

Before we describe Max's strategy, we need several definitions. Let $S = \max_{v\in V} |\text{St}(v)|$ and $r = \frac{\nu}{\nu+\mu}$. We choose $0 < \beta \leq 1$ such that $\beta \cdot \nu \cdot S < 1$ and $\beta \cdot \mu \cdot \nu \cdot S < \frac{\mu}{\nu}$. Let $B = \{\beta \cdot i : i \in \mathbb{N}\}$. We choose $x_0 \in B$ such that Max's ratio is greater than $\mathcal{R}^r(x_0)$, which is possible since $\mathcal{R}^r$ tends to $1 - r$ from above.

Suppose Max is playing according to the strategy we describe below and Min is playing according to some strategy. The play induces a walk on B, which we refer to as the *budget walk*. Max's strategy guarantees the following:

Invariant: Whenever the budget walk reaches an $x \in B$, then Max's ratio is greater than $\mathcal{R}^r(x)$.

The walk starts in x_0 and the invariant holds initially due to our choice of x_0. Suppose the token is placed on the vertex $v \in V$ and the walk reaches x. Max bids $\text{St}(v) \cdot \beta \cdot \mu \cdot \nu \cdot (D^r(x))^{-1}$, where $D^r(x)$ is the denominator of $\mathcal{R}^r(x)$, and he moves to v^+ upon winning. If Max loses, the walk proceeds down to $x - \nu \cdot \text{St}(v) \cdot \beta$, and by Lemma 8, the invariant is maintained. If Max wins, the walk proceeds up to $x + \mu \cdot \text{St}(v) \cdot \beta$, and by the other part of Lemma 8, and the invariant is maintained.

Claim: For every Min strategy, the budget walk never reaches $x = 1$.

Recall that S is the maximal strength of a vertex. The largest step down on the budget sequence following a Min win is at most $\beta \cdot S \cdot \nu$. Thus, before crossing 1, the walk must visit $1 + k \cdot \beta$, for some $1 \leq k \leq S \cdot \nu + 1$. Suppose the walk visits $x = 1 + k \cdot \beta$ at vertex $v \in V$ with $\text{St}(v) \cdot \nu > k$. Thus, if Min wins the bidding, the walk crosses 1. We claim that Max wins the bidding. Let $z = \text{St}(v) \cdot \nu \cdot \beta$. Recall that Max's bid at v is $\frac{\mu \cdot z}{D(x)}$ and that Max's ratio is greater than $\mathcal{R}^r(x)$, thus Min's ratio is less than $1 - \mathcal{R}^r(x) \leq \frac{\mu \cdot z(z-1)/2}{D(x)}$. For every $0 < z < 1$, we have $z > \frac{z(z-1)}{2}$, and our choice of β implies that indeed $0 < z < 1$. Thus, Max's bid exceeds Min's budget, so he wins the bidding, and we are done.

Claim: The energy throughout a play is bounded from below. Formally, there exists a constant $c \in \mathbb{R}$ such that for every Min strategy and a finite play π, we have $E(\pi) \geq c$.

Consider a finite play π. We view π as a sequence of vertices in $\mathcal{G}$. Recall that the budget walk starts at x_0, and that $G(\pi)$ and $I(\pi)$ represent sums of strength of vertices. Suppose the budget walk reaches x following the play π, then $x = x_0 - G(\pi) \cdot \nu \cdot \beta + I(\pi) \cdot \mu \cdot \beta$. Recall that for every $v \in V$, we have $\text{St}(v) \geq -S$. Rephrasing Lemma 9, we have $\frac{-G(\pi) \cdot \nu + I(\pi) \cdot \mu}{\nu \cdot \mu} \leq 2S + E(\pi)$. Thus, $\frac{x - x_0}{\beta \mu \nu} \leq 2S + E(\pi)$. By the claim above $x \geq 1$. It follows that $\frac{1 - x_0}{\beta \mu \nu} - 2S \leq E(\pi)$, and we are done. ☐

Remark 12. An interesting connection between poorman and Richman biddings arrises from Theorem 11. Consider a strongly-connected mean-payoff game $\mathcal{G}$. For an initial ratio $r \in [0, 1]$, let $\texttt{MP}^r_{\mathcal{P}}(\mathcal{G})$ denote the value of $\mathcal{G}$ with respect to r with poorman bidding. In [5], the authors show that the value with Richman bidding does not depend on r, thus we denote it by $\texttt{MP}_{\mathcal{R}}(\mathcal{G})$. Moreover, they show a probabilistic connection: $\texttt{MP}_{\mathcal{R}}(\mathcal{G})$ equals the value in the RTB in which the players are selected uniformly, thus $\texttt{MP}_{\mathcal{R}}(\mathcal{G}) = \texttt{MP}(\texttt{RTB}^{0.5}(\mathcal{G}))$. Our results show that poorman games with initial ratio 0.5 coincide with Richman games. Indeed, we have $\texttt{MP}_{\mathcal{R}}(\mathcal{G}) = \texttt{MP}^{0.5}_{\mathcal{P}}(\mathcal{G})$. To the best of our knowledge such a connection between the two bidding rules has not been identified before.

Remark 13. The proof technique in Theorem 11 extends to poorman energy games. Consider a strongly-connected mean-payoff game $\mathcal{G}$, and let $r \in [0, 1]$ such that $\texttt{MP}^r(\mathcal{G}) = 0$. Now, view $\mathcal{G}$ as a poorman energy game. The proof of Theorem 11 shows that when Max's initial ratio is $r + \epsilon$, there exists an initial energy level from which he can win the game. On the other hand, when Max's initial ratio is $r - \epsilon$, Min can win the energy game from every initial energy. Indeed, consider the game $\mathcal{G}'$ that is obtained from $\mathcal{G}$ by multiplying all weights by -1. Again, using Theorem 11 and associating Min with Max, Min can keep the energy level bounded from above, which allows him, similar to the qualitative case, to play a strategy in which he either wins or increases his ratio by a constant. Eventually, his ratio is high enough to win arbitrarily many times in a row and drop the energy as low as required.

4.5 Extention to general mean-payoff games

We extend the solution in the previous sections to general graphs in a similar manner to the qualitative case; we first reason about the BSCCs of the graph and then construct an appropriate reachability game on the rest of the vertices. Recall that, for a vertex v in a mean-payoff game, the ratio $\texttt{Th}(v)$ is a necessary and sufficient initial ratio to guarantee a payoff of 0.

Consider a poorman mean-payoff game $\mathcal{G} = \langle V, E, w \rangle$. Recall that, for $v \in V$, $\texttt{Th}(v)$ is the necessary and sufficient initial ratio for Max to guarantee a non-positive payoff. Let $S_1, \ldots, S_k \subseteq V$ be the BSCCs of $\mathcal{G}$ and $S = \bigcup_{1 \leq i \leq k} S_i$. For $1 \leq i \leq k$, the poorman mean-payoff game $\mathcal{G}_i = \langle S_i, E_{|S_i}, w_{|S_i} \rangle$ is a strongly-connected game. We define $r_i \in [0, 1]$ as follows. If there is an $r \in [0, 1]$ such that $\texttt{MP}^r(\mathcal{G}_i) = 0$, then $r_i = r$. Otherwise, if for every r, we have $\texttt{MP}^r(\mathcal{G}_i) > 0$, then $r_i = 0$, and if for every r, we have $\texttt{MP}^r(\mathcal{G}_i) < 0$, then $r_i = 1$. By Theorem 11, for every $v \in S_i$, we have $\texttt{Th}(v) = r_i$. We construct a *generalized reachability game* $\mathcal{G}'$ that corresponds to $\mathcal{G}$ by replacing every S_i in $\mathcal{G}$ with a vertex u_i. Player 1 wins a path in $\mathcal{G}$ iff it visits some u_i and when it visits u_i, Player 1's ratio is at least r_i. It is not hard to generalize the proof of Theorem 6 to generalized reachability poorman games and obtain the following.

Theorem 14. *The threshold ratios in a poorman mean-payoff game $\mathcal{G}$ coincide with the threshold ratios in the generalized reachability game that corresponds to $\mathcal{G}$.*

5 Computational Complexity

We study the complexity of finding the threshold ratios in poorman games. We formalize this search problem as the following decision problem. Recall that threshold ratios in poorman reachability games may be irrational (see Theorem 6).

THRESH-BUD Given a bidding game $\mathcal{G}$, a vertex v, and a ratio $r \in [0, 1] \cap \mathbb{Q}$, decide whether $\texttt{Th}(v) \geq r$.

Theorem 15. *For poorman parity games, THRESH-BUD is in PSPACE.*

Proof. To show membership in PSPACE, we guess the optimal moves for the two players. To verify the guess, we construct a program of the *existential theory of the reals* that uses the relation between the threshold ratios that is described in Theorem 6. Deciding whether such a program has a solution is known to be in PSPACE [11]. Formally, given a poorman parity game $\mathcal{G} = \langle V, E, p \rangle$ and a vertex $v \in V$, we guess, for each vertex $u \in V$, two neighbors $u^+, u^- \in N(u)$. We construct the following program. For every vertex $u \in V$, we introduce a variable x_u, and we add constraints so that a satisfying assignment to x_u coincides with the threshold ratio in u. Consider a BSCC S of $\mathcal{G}$. Recall that the threshold ratios in S are all either 0 or 1, and verifying which is the case can be done in linear time. Suppose the threshold ratios are $\alpha \in \{0, 1\}$. We add constraints $x_u = \alpha$, for every $u \in S$. For every vertex $u \in V$ that is not in a BSCC, we have constraints $x_u = \frac{x_{u^+}}{1 - x_{u^-} + x_{u^+}}$ and $x_{u^-} \leq x_{u'} \leq x_{u^+}$, for every $u' \in N(u)$. By Theorems 6 and 7, a satisfying assignment assigns to x_u the ratio $\texttt{Th}(u)$. We conclude by adding a final constraint $x_v \geq r$. Clearly, the program has a satisfying assignment iff $\texttt{Th}(v) \geq r$, and we are done. □

We continue to study mean-payoff games.

Theorem 16. *For poorman mean-payoff games, THRESH-BUD is in PSPACE. For strongly-connected games, it is in NP and coNP. For strongly-connected games with out-degree 2, THRESH-BUD is in P.*

Proof. To show membership in PSPACE, we proceed similarly to the qualitative case, and show a nondeterministic polynomial-space that uses the existential theory of the reals to verify its guess. Given a game $\mathcal{G}$, we construct a program that finds, for each BSCC S of $\mathcal{G}$, the threshold ratio for all the vertices in V. We then extend the program to propagate the threshold ratios to the rest of the vertices, similar to Theorem 14. Given a strongly-connected game $\mathcal{G}$ and a ratio $r \in [0, 1]$, we construct $\texttt{RTB}^r(\mathcal{G})$ in linear time. Then, deciding whether $\texttt{MP}(\texttt{RTB}^r(\mathcal{G})) \geq 0$, is known to be in NP and coNP.

The more challenging case is the solution for strongly-connected games with out-degree 2. Consider such a game $\mathcal{G} = \langle V, E, w \rangle$ and $r \in [0, 1]$. We construct an MDP $\mathcal{D}$ on the structure of $\mathcal{G}$ such that $\texttt{MP}(\mathcal{D}) = \texttt{MP}^r(\mathcal{G})$. Since finding $\texttt{MP}(\mathcal{D})$ is known to be in P, the claim follows. When $r \geq \frac{1}{2}$, then $\mathcal{D}$ is a max-MDP, and when $r < \frac{1}{2}$, it is a min-MDP. Assume the first case, and the second case is similar. We split every vertex $v \in V$ in three, where $v \in V_{Max}$ and $v_1, v_2 \in V_N$. Suppose $\{u_1, u_2\} = N(v)$. Intuitively, moving to v_1 means that Max prefers moving to u_1 over u_2. Thus, we have $\Pr[v_1, u_1] = r = 1 - \Pr[v_1, u_2]$ and $\Pr[v_2, u_1] = 1 - r = 1 - \Pr[v_2, u_2]$. It is not hard to see that $\texttt{MP}(\mathcal{D}) = \texttt{MP}^r(\mathcal{G})$. □

6 Discussion

We studied for the first time infinite-duration poorman bidding games. We show the existence of threshold ratios for poorman games with qualitative objectives and give, to the best of our knowledge, the first complexity upper bounds on finding threshold ratios. For poorman mean-payoff games, we construct optimal strategies with respect to the initial ratio of the budgets and show an interesting probabilistic connection for these games.

Historically, poorman bidding has been studied less than Richman bidding, but the reason was technical difficulty, not lack of motivation. On the contrary, we believe that poorman bidding is as motivated as Richman bidding, if not more so, particularly since they are easier to generalize. Poorman bidding has been less approachable since, e.g., poorman reachability games do not necessarily have rational threshold ratios. We expect that the structure we find here, namely the probabilistic connection for poorman bidding, will make these game more approachable and assist in introducing concepts like multiple-players, recharging stations, and partial information to bidding games, which are hard to add to Richman bidding.

This work belongs to a line of works that transfer concepts and ideas between the areas of formal verification and algorithmic game theory [30], two fields with a different take on game theory and with complementary needs. For example, formal reasoning about multi-agent safety critical systems, e.g., components of an autonomous car, requires insights on rationality. On the other side, formally verifying the correctness of auctions or reasoning about ongoing auctions, are both challenges that can benefit from the experience of the formal methods community. Examples of works in the intersection of the two fields include logics for specifying multi-agent systems [2, 14, 29], studies of equilibria in games related to synthesis and repair problems [13, 12, 19, 1], non-zero-sum games in formal verification [15, 9], and applying concepts from formal methods to *resource allocation games* such as rich specifications [7], efficient reasoning about very large games [4, 24], and a dynamic selection of resources [6].

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A Proof of Theorem 6

Let $\Delta_{\min}$ be the smallest positive number such that $f(v) = \Delta_{\min}$ for some v, and $\Delta_{\min} = 1$ if $f(v) = 0$ for all $v \in V$. Let δ_1 be 1 and δ_i be such that

$$\sum_{j=1}^{i-1} \delta_j < \Delta_{\min}/2\delta_i \ \ ,$$

for all $i \in \{2, \ldots, |V|\}$. Also, let γ be such that $\sum_{j=1}^{|V|} \delta_j < 1/\gamma$. For each state v (such that $f(v) > 0$), consider that Player 1 wins all bids and let dist(v) be the number of bids before the play ends up in u_1 starting from v. When $f(v) = 0$, let dist(v) be the length of the shortest path from v to u_1. Then, $\delta_v = \gamma\delta_i$, for $i = |V| - \text{dist}(v)$.

In case Player 1 wins, his real budget becomes $x' - \Delta(v)x'$, and Player 2's budget stays y'. In that case, Player 1's new real budget ratio becomes $\frac{(1-\Delta(v))x'}{(1-\Delta(v))x'+y'} = f(v^-)$, and the invariants are thus satisfied. (His slush fund also decreases by $\delta_v\epsilon'$. We will not proof anything about the slush fund in this case, except noting that it stays positive).

In case Player 2 wins, Player 1's real budget stays x' and Player 2's budget is at most $y' - \Delta(v)x' - \delta_v\epsilon'$. By construction, we have that if Player 2's budget became $y' - \Delta(v)x'$, then Player 1's budget ratio becomes $\frac{x'}{x'+y'-\Delta(v)x'} = f(v^+)$, so even if Player 2 moves to v^+, Player 2 has paid $\delta_v\epsilon'$ too much for Player 1's real budget ratio to be $f(v^+)$. Thus, the first invariant is satisfied. Note that this also indicates that $f(v^+) \neq 1$, in this case, since otherwise Player 1's budget ratio must be above 1, indicating that Player 2's budget is negative. When $f(v^+) > 0$, we can move $\delta_v\epsilon' f(v^+)/(1 - f(v^+)) \geq \delta_v\epsilon'\Delta_{\min}$ into the slush fund. When $f(v^+) = 0$, the new slush fund is $\delta_v\epsilon'$. Let j be such that $\delta_j = \delta_v$. By construction of δ_v, we have that since the last time Player 2 won a bidding (or since the start if Player 2 never won a bid before), we have subtracted at most $\epsilon' \sum_{i=j+1}^{|V|} \delta_i$ from the slush fond and now we have added $\delta_j\epsilon'\Delta_{\min}$. But δ_i was chosen such that $\sum_{i=j+1}^{|V|} \delta_i$ was below $\delta_v\Delta_{\min}/2$. Hence, we have added $\delta_v\epsilon'\Delta_{\min}$ to the previous content of ϵ'. Because δ_v and $\Delta_{\min}$ are constants, we have thus increased the slush fund by a constant factor. The invariants are thus satisfied in this case.

B Proof of Lemma 8

Recall that $t_\ell = \frac{\ell(\ell-1)}{2}$ and $T_\ell = \frac{\ell(\ell+1)}{2}$. We will use that they are strictly increasing in ℓ, $t_1 = 0$ and $T_1 = 1$. We will first argue that

$$\frac{\mu \cdot t_x - \mu \cdot y}{\mu \cdot t_x + \nu \cdot T_x - \mu \cdot y} \leq \frac{\mu \cdot t_{x-y}}{\mu \cdot t_{x-y} + \nu \cdot T_{x-y}} \ \ .$$

Before doing so we will show that the denominators in the inequality are positive, i.e. that $\mu \cdot t_x + \nu \cdot T_x - \mu \cdot y > 0$ and that $\mu \cdot t_{x-y} + \nu \cdot T_{x-y} > 0$.

We have that

$$\mu \cdot t_x + \nu \cdot T_x - \mu \cdot y \geq \mu \cdot t_1 + \nu \cdot T_1 - \mu \cdot y = \nu - \mu \cdot y > 0 \ \ ,$$

using that T_ℓ and t_ℓ are increasing, $x \geq 1$, $t_1 = 0$ and $T_1 = 1$ and $\nu > \mu \cdot y$, even if $\mu > \nu$.

Also,

$$\begin{aligned} &\mu \cdot t_{x-y} + \nu \cdot T_{x-y} \\ &= \mu \cdot \frac{(x-y)(x-y-1)}{2} + \nu \cdot \frac{(x-y)(x-y+1)}{2} \\ &= (\mu + \nu) \cdot \frac{(x-y)(x-y+1)}{2} - (x-y)\mu \end{aligned}$$

We consider two cases, $\mu = \nu$ and $\mu \neq \nu$. In the former case, if $\mu = \nu$, then

$$\begin{aligned} &2\mu \cdot \frac{(x-y)(x-y+1)}{2} - (x-y)\mu \\ &= \mu(x-y)(x-y+1) - (x-y)\mu > 0 \ \ , \end{aligned}$$

using that $x > y$ and $\mu > 0$. Otherwise, consider that $\mu \neq \nu$.

$$
\begin{aligned}
&(\mu+\nu) \cdot \frac{(x-y)(x-y+1)}{2} - (x-y)\mu \\
&\geq (\mu+\nu) \cdot \frac{(x-y)(2-y)}{2} - (x-y)\mu \\
&\geq \nu(x-y) - \frac{(x-y)(y\mu + y\nu)}{2} \\
&> \nu(x-y) - \frac{(x-y)(\nu + y\nu)}{2} \geq 0
\end{aligned}
$$

using that $x \geq 1$, $x > y$, $y \leq 1$, $\nu > 0$ and $\nu > \mu \cdot y$, even if $\mu > \nu$.

Using that, we consider

$$\frac{\mu \cdot t_x - \mu \cdot y}{\mu \cdot t_x + \nu \cdot T_x - \mu \cdot y} \leq \frac{\mu \cdot t_{x-y}}{\mu \cdot t_{x-y} + \nu \cdot T_{x-y}} \ .$$

and see that

$$
\begin{aligned}
\frac{\mu \cdot t_x - \mu \cdot y}{\mu \cdot t_x + \nu \cdot T_x - \mu \cdot y} &\leq \frac{\mu \cdot t_{x-y}}{\mu \cdot t_{x-y} + \nu \cdot T_{x-y}} \Leftrightarrow \\
(\mu \cdot t_x - \mu \cdot y)(\mu \cdot t_{x-y} + \nu \cdot T_{x-y}) &\leq \\
&(\mu \cdot t_{x-y})(\mu \cdot t_x + \nu \cdot T_x - \mu \cdot y) \Leftrightarrow \\
\mu \cdot \nu \cdot t_x \cdot T_{x-y} - \mu \cdot \nu \cdot y \cdot T_{x-y} &\leq \mu \cdot \nu \cdot t_{x-y} \cdot T_x \Leftrightarrow \\
t_x \cdot T_{x-y} - y \cdot T_{x-y} &\leq t_{x-y} \cdot T_x \Leftrightarrow \\
\frac{x(x-1)(x-y)(x-y+1)}{4} - y \cdot \frac{(x-y)(x-y+1)}{2} &\leq \\
\frac{(x-y)(x-y-1)x(x+1)}{4} &\Leftrightarrow \\
\frac{x^3 - x^2y + xy - x}{2} - y \cdot (x-y+1) &\leq \frac{x^3 - x^2y - xy - x}{2} \Leftrightarrow \\
xy &\leq y(x-y+1) \Leftrightarrow \\
0 &\leq y - y^2
\end{aligned}
$$

Note that $0 \leq y - y^2$ because $0 < y \leq 1$.

Next, we consider

$$\frac{\mu \cdot t_x}{\mu \cdot t_x + \nu \cdot T_x - \nu \cdot y} \leq \frac{\mu \cdot t_{x+y}}{\mu \cdot t_{x+y} + \nu \cdot T_{x+y}}$$

Like before we first consider the denominators of the inequality. We have that $\mu \cdot t_{x+y} + \nu \cdot T_{x+y} > 0$ because $x + y \geq 1$, t_ℓ and T_ℓ are increasing, $t_1 = 0$ and $T_1 = 1$ and $\mu, \nu > 0$. To show that

$$\mu \cdot t_x + \nu \cdot T_x - \nu \cdot y > 0 \ ,$$

we consider two cases, $y = 1$ and $y < 1$. If $y = 1$ then $x > 1$ (because $x > y$), implying that $T_x > 1$ and thus $\nu \cdot T_x - \nu \cdot y > 0$ (also $\mu \cdot t_x > 0$). If $y < 1$, then

$$\mu \cdot t_x + \nu \cdot T_x - \nu \cdot y \geq \nu - \nu \cdot y > 0 \ ,$$

using that T_ℓ and t_ℓ are increasing in ℓ, $x \geq 1$ and $t_1 = 0$ and $T_1 = 1$.

Using that, we consider

$$\frac{\mu \cdot t_x}{\mu \cdot t_x + \nu \cdot T_x - \nu \cdot y} \leq \frac{\mu \cdot t_{x+y}}{\mu \cdot t_{x+y} + \nu \cdot T_{x+y}}$$

and see that

$$
\begin{aligned}
&\frac{\mu \cdot t_x}{\mu \cdot t_x + \nu \cdot T_x - \nu \cdot y} \leq \\
&\qquad \frac{\mu \cdot t_{x+y}}{\mu \cdot t_{x+y} + \nu \cdot T_{x+y}} \Leftrightarrow \\
&(\mu \cdot t_x)(\mu \cdot t_{x+y} + \nu \cdot T_{x+y}) \leq \\
&\qquad (\mu \cdot t_{x+y})(\mu \cdot t_x + \nu \cdot T_x - \nu \cdot y) \Leftrightarrow \\
&\mu \cdot \nu \cdot t_x \cdot T_{x+y} \leq \\
&\qquad \mu \cdot \nu \cdot t_{x+y} \cdot T_x - \mu\nu \cdot y \cdot t_{x+y} \Leftrightarrow \\
&t_x \cdot T_{x+y} \leq t_{x+y} \cdot T_x - y \cdot t_{x+y} \Leftrightarrow \\
&\frac{x(x-1)(x+y)(x+y+1)}{4} \leq \\
&\frac{(x+y)(x+y-1)x(x+1)}{4} - y \cdot \frac{(x+y)(x+y-1)}{2} \Leftrightarrow \\
&\frac{x^3 + x^2y - xy - x}{2} \leq \\
&\frac{x^3 + x^2y + xy - x}{2} - y(x+y-1) \Leftrightarrow \\
&-xy \leq -y(x+y-1) \Leftrightarrow \\
&0 \leq y - y^2
\end{aligned}
$$

Again, note that $0 \leq y - y^2$ because $0 < y \leq 1$.