POSITIVE MODEL STRUCTURES FOR ABSTRACT SYMMETRIC SPECTRA

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ABSTRACT. We give a general method of constructing positive stable model structures for symmetric spectra over an abstract simplicial symmetric monoidal model category. The method is based on systematic localization, in Hirschhorn's sense, of a certain positive projective model structure on spectra, where positivity basically means the truncation of the zero level. The localization above is by the set of stabilizing morphisms, or their truncated version.

1. INTRODUCTION

The aim of this paper is to give a systematic account of the method of constructing positive model structures for abstract symmetric spectra, used to prove one of the key theorems in [4]. Let first $\mathscr{S}$ be the category of topological symmetric spectra in the sense of [9], and let $\mathscr{T}$ be the homotopy category of $\mathscr{S}$ with respect to the stable model structure in it. Then $\mathscr{T}$ is equivalent to the standard topological stable homotopy category, whose Hom-groups encode the stable homotopy groups of CW-complexes. As it was shown in [12] (see also [2]), the category $\mathscr{S}$ admits another one, so-called positive, model structure whose homotopy category is the same as $\mathscr{T}$, but the positivity of this new structure gives rise to many good properties missing in the standard stable model structure. For example, if X is a topological symmetric spectrum, which is cofibrant in the positive model structure, then the natural morphism from the n-th homotopy symmetric power of X onto the honest n-th symmetric power of X is a stable weak equivalence, loc.cit. The latter result is important for our understanding of the stable homotopy groups through the Barratt-Priddy-Quillen theorem, see the modern approach in [14]. Another essential application of positive model structures in topology is that it yields a convenient model structure for commutative ring spectra, see [15].

On the other hand, following [8], one can get a general method for constructing stable homotopy categories, equally appropriate in topology and in $\mathbb{A}^1$-homotopy theory, where the initial category $\mathscr{C}$ is nothing but the

category of simplicial Nisnevich sheaves on smooth schemes over a base, see [10]. We start with a closed symmetric monoidal model category $\mathscr{C}$, which is, in addition, left proper and cellular, then take a cofibrant object T in $\mathscr{C}$ and look at the category of symmetric T-spectra $\mathscr{S}$ over $\mathscr{C}$. This category $\mathscr{S}$ possesses a stable model structure, and the corresponding homotopy category $\mathscr{T} = Ho(\mathscr{S})$ generalizes the topological stable homotopy category and the motivic one, loc.cit. A natural question is then how to extend the method of constructing positive model structures developed in topology to the level of generality, high enough to be applicable in motivic algebraic geometry, and in other reasonable settings.

In this paper, we give an affirmative answer to this question and show a universal method of constructing many positive structures, adjustable to particular needs. Basically, we follow the method from §14 in [12], keeping the level of generality as high as possible. A new thing, however, is that we systematically exploit the technique of localization of model categories from [5], which allows us to make the approach more conceptual and put an order on various model structures naturally arising in our considerations. In nutshell, we first take a projective model structure, truncate it in its 0-level, or any finite number of levels starting from the zero one, and then localize the truncated model structure by the stabilizing Hovey's ζ-morphisms between appropriately shifted T-spectra.

The application of positive model structures in [4] goes as follows. Let again $\mathscr{C}$ be a closed symmetric monoidal model category $\mathscr{C}$, left proper and cellular, T a cofibrant object in $\mathscr{C}$, and let $\mathscr{S}$ be the category of symmetric T-spectra over $\mathscr{C}$. Let X be a spectrum cofibrant with respect to the positive projective model structure in the category $\mathscr{S}$. Using the positive model structure at this high level of generality, we proved in [4] (see Theorem 55 there) that the natural morphism from the n-th homotopy symmetric power of X to its honest n-th symmetric power is a stable weak equivalence of symmetric spectra. As a consequence, symmetric powers preserve stable weak equivalences between positively cofibrant objects in $\mathscr{S}$. This result generalizes Lemma 15.5 in [12], and allowed us to derive symmetric powers in the abstract stable homotopy category $\mathscr{T}$, see Corollary 57 in [4].

It should be pointed out that positive model structures were utilized in [6] in the context of $\mathbb{A}^1$-homotopy theory of schemes, and they were also used to compare the geometric symmetric powers of motivic spectra with their left derived symmetric powers in [11]. The extreme level of generality of our construction makes it possible to apply positive model structures not only in the Morel-Voevodsky stable category of motivic symmetric spectra over a field, but much more beyond. For example, in [1] positive model structures were used for the study of commutative monoids in an abstract symmetric monoidal model category, and in [13] the methods and results of the present work are extended to algebra spectra over symmetric operads. Broadly speaking, the abstract positive model structures are required in

order to have a transferred model structure on commutative ring spectra, and on algebras over operads, which is clear from [15] and [13]. We also expect that our results are applicable in the context of [16].

The paper is organized as follows. In Section 2 we set up what exactly we want to construct, and fix notation and terminology. We recall some basic definitions on abstract symmetric spectra in Section 2, but the reader is advised to use Hovey's article [8] to repeat the details. In Section 3 we present our concept of positive stable model structures as systematic localizations of positive projective model structures on symmetric spectra. We have chosen to start with projective model structure, but injective model structures are good for our purposes too. Section 4 is devoted to deducing the needed results on loop-spectra in the abstract setting. Finally, in Section 5 we prove the main result (Theorem 10) saying that weak equivalences in the stable model structure are the same as weak equivalences in the positive model structure. This implies that the resulting stable homotopy category is the same.

2. Positive model structures: what to construct?

First we need to explain what do we mean by an abstract stable homotopy category. Our viewpoint is that it should be understood as the homotopy category of the category of symmetric spectra over a given simplicial model monoidal category $\mathscr{C}$, stabilizing smashing with T, where T is a cofibrant object T in $\mathscr{C}$. Such a general gadget generalizes both the topological stable homotopy category and the motivic one due to Morel and Voevodsky. Nowadays, in both cases, we should work with symmetric spectra as they provide a set of powerful monoidal properties of spectra, useful in applications. In our considerations we depart from the paper [8], which is basic to us.

Let $\mathscr{C}$ be a closed symmetric monoidal model category with the monoidal product $\wedge$. This notation is the tradition coming from the pointed setting needed to make the homotopy category of spectra additive. Respectively, the coproduct will be denoted by $\vee$.

Next, we assume that the model structure in $\mathscr{C}$ is left proper and cellular. Left properness means that the push-out of a weak equivalence along a cofibration is again a weak equivalence, and cellularity means that $\mathscr{C}$ is cofibrantly generated by a set of generating cofibrations I and a set of trivial generating cofibrations J, the domains and codomains of morphisms in I are compact relative to I, the domains of morphisms in J are small relative to the cofibrations, and cofibrations are effective monomorphisms. To avoid any misunderstanding in using this complicated terminology we would recommend the reader to consult with [7], [8] and [5]. Suppose, moreover, that the domains of the generating cofibrations I in $\mathscr{C}$ are cofibrant, which is needed to satisfy the assumptions of Theorem 8.11 in [8].

For simplicity, we shall also assume that $\mathscr{C}$ is simplicial, and that the simplicial structure is compatible with the structure of a closed symmetric monoidal model category. This will be used in the proofs of Proposition 4 and Corollary 8 merely in order to avoid the bulky work with functional complexes. However, Proposition 4 and Corollary 8, as well as the main Theorem 10, are true without this assumption.

Let Σ be a disjoint union of symmetric groups Σ_n for all $n \geq 0$, where $\Sigma_0 = \emptyset$ and all groups are considered as one object categories. Let $\mathscr{C}^\Sigma$ be the category of symmetric sequences over $\mathscr{C}$, i.e. functors from Σ to $\mathscr{C}$. Since $\mathscr{C}$ is closed symmetric monoidal, so is the category $\mathscr{C}^\Sigma$. The monoidal product in $\mathscr{C}^\Sigma$ is given by the formula

$$(X \wedge Y)_n = \vee_{i+j=n} \Sigma_n \times_{\Sigma_i \times \Sigma_j} (X_i \wedge Y_j) .$$

Here $\Sigma_i \times \Sigma_j$ is embedded into Σ_n in a way, such that Σ_i permutes the first i elements and Σ_j permutes the last j elements in $\{1, \ldots n\}$. The object $\Sigma_n \times_{\Sigma_i \times \Sigma_j} (X_i \wedge Y_j)$ is nothing but the quotient of $\Sigma_n \times (X_i \wedge Y_j)$ by $\Sigma_i \times \Sigma_j$, where the action of $\Sigma_i \times \Sigma_j$ is natural on $X_i \wedge Y_j$, and by right translations on Σ_n. The action of Σ_n is given by the left translation on the left multiple in $\Sigma_n \times_{\Sigma_i \times \Sigma_j} (X_i \wedge Y_j)$. Notice that this construction is denoted by $\mathrm{cor}_{\Sigma_i \times \Sigma_j}^{\Sigma_n} (X_i \wedge Y_j)$ in [4].

Let T be a cofibrant object in $\mathscr{C}$, and let $S(T)$ be the free commutative monoid on the symmetric sequence $(\emptyset, T, \emptyset, \emptyset, \ldots)$, i.e. the symmetric sequence

$$S(T) = (T^0, T^1, T^2, T^3, \ldots) ,$$

where $T^0 = \mathbb{1}$ is the unit, $T^1 = T$ and Σ_n acts on T^n by permutation of factors. Then a symmetric T-spectrum is nothing but a module over $S(T)$ in $\mathscr{C}^\Sigma$. Explicitly, a symmetric spectrum X is a sequence of objects

$$X_0 , \quad X_1 , \quad X_2 , \quad X_3 , \quad \ldots$$

in $\mathscr{C}$ together with Σ_n-equivariant morphisms

$$X_n \wedge T \longrightarrow X_{n+1} ,$$

such that for all $n, i \geq 0$ the composite

$$X_n \wedge T^i \longrightarrow X_{n+1} \wedge T^{i-1} \rightarrow \cdots \rightarrow X_{n+i}$$

is $\Sigma_n \times \Sigma_i$-equivariant.

Let

$$\mathscr{S} = \mathrm{Spt}^\Sigma(\mathscr{C}, T)$$

be the category of symmetric T-spectra over $\mathscr{C}$. There is a natural closed symmetric monoidal structure on $\mathscr{S}$ given by the product of modules over the commutative monoid $S(T)$.

A model structure on $\mathscr{S}$ can be constructed as a localization of the so-called projective model structure coming from the model structure on $\mathscr{C}$, using the main result of [5] (Theorem 4.1.1).

Namely, for any non-negative n we consider the evaluation functor

$$\mathrm{Ev}_n : \mathscr{S} \longrightarrow \mathscr{C}$$

sending any symmetric spectrum X to its n-th level. Each Ev_n has a left adjoint

$$F_n : \mathscr{C} \longrightarrow \mathscr{S} ,$$

which can be constructed as follows. Let $\tilde{F}_n$ be a functor sending any object X in $\mathscr{C}$ to the symmetric sequence

$$(\emptyset, \dots, \emptyset, \Sigma_n \times X, \emptyset, \emptyset, \dots) ,$$

where $\emptyset$ is the initial object in $\mathscr{C}$. Then

$$F_n X = \tilde{F}_n X \wedge S(T) ,$$

see Definition 7.3 in [8].

Let now

$$I_T = \cup_{n \geq 0} F_n I \qquad \text{and} \qquad J_T = \cup_{n \geq 0} F_n J ,$$

where $F_n I$ is the set of all the morphisms of type $F_n f$, $f \in I$, and the same for $F_n J$. Let also

$$W_T$$

be the class of projective weak equivalences, i.e. level weak equivalences, which means that for any morphism $f : X \rightarrow Y$ in W_T the morphism $f_n : X_n \rightarrow Y_n$ is a weak equivalence in $\mathscr{C}$ for all $n \geq 0$.

For technical reasons, we prefer to use different symbols to denote a category and a model structure in it. The projective model structure

$$\mathscr{M} = (I_T, J_T, W_T)$$

is generated by the set of generating cofibrations I_T and the set of trivial generating cofibrations J_T. As the model structure in $\mathscr{C}$ is left proper and cellular, the projective model structure in $\mathscr{S}$ is left proper and cellular too, see Theorem 8.2 in [8]. In particular, the class of cofibrations in $\mathscr{M}$ is equal to the class I_T-cof. Recall that I_T-cof refers to the class of maps having the left lifting property with respect to I_T-inj, and the latter is the class of maps that have the right lifting property with respect to I_T.

For any two non-negative integers m and n, $m \geq n$, consider the embedding of the group Σ_{m-n} into the group Σ_m, such that Σ_{m-n} permutes the last $m - n$ elements in the m elements set. For any object X in $\mathscr{C}$ let Σ_{m-n} act on $X \wedge T^{m-n}$ by permuting factors in T^{m-n}. Then $F_n X$ can be computed by the formula

$$(F_n X)_m = \mathrm{cor}^{\Sigma_m}_{\Sigma_{m-n}}(X \wedge T^{m-n}) \, ,$$

see §7 in [8]. In particular,

$$\mathrm{Ev}_{n+1} F_n X = \mathrm{cor}^{\Sigma_{n+1}}_{\Sigma_1}(X \wedge T) = \Sigma_{n+1} \times (X \wedge T) \, .$$

Let now

$$\zeta_n^X : F_{n+1}(X \wedge T) \longrightarrow F_n(X)$$

be the adjoint to the morphism

$$X \wedge T \longrightarrow \mathrm{Ev}_{n+1} F_n X = \Sigma_{n+1} \times (X \wedge T)$$

induced by the canonical embedding of the trivial group Σ_1 into Σ_{n+1}.

For any set of morphisms U let $\mathrm{dom}(U)$ and $\mathrm{codom}(U)$ be the set of domains and codomains of morphisms from U, respectively. Let then

$$S = \{\zeta_n^X \mid X \in \mathrm{dom}(I) \cup \mathrm{codom}(I) \, , \ n \geq 0\}$$

be the set of stabilizing morphisms. Then a *stable model structure*

$$\mathscr{M}_S = (I_T, J_{T,S}, W_{T,S})$$

in $\mathscr{S}$ is defined to be the Bousfield localization of the projective model structure with respect to the set S in the sense of Definition 3.1.1 in [5]. It is generated by the same set of generating cofibrations I_T, and by a new set of trivial generating cofibrations $J_{T,S}$. Here $W_{T,S}$ is the class of stable weak equivalences, i.e. new weak equivalences obtained as a result of the localization.

Let

$$\mathscr{T} = \mathscr{S}[W_{T,S}^{-1}]$$

be the localization of $\mathscr{S}$ with respect to the class $W_{T,S}$, i.e. the homotopy category of $\mathscr{S}$ with respect to weak equivalences in $W_{T,S}$. Then we call $\mathscr{T}$ an abstract stable homotopy category of symmetric spectra over $\mathscr{C}$ which stabilizes smashing by T. As the functor $(- \wedge T)$ is a Quillen autoequivalence of $\mathscr{S}$, with respect to the model structure $\mathscr{M}_S$, Theorem 8.10 in [8], it induces an autoequivalence on the homotopy category $\mathscr{T}$, as required.

By Hovey's result, see §10 in [8], the homotopy category $\mathscr{T}$ is equivalent to the homotopy category of ordinary T-spectra provided the cyclic permutation on $T \wedge T \wedge T$ is left homotopic to the identity morphism.

Let now

$$S^+ = \{\zeta_n^X \mid X \in \mathrm{dom}(I) \cup \mathrm{codom}(I) \, , \ n > 0\}$$

be the positive stabilizing set. Our aim is actually to find a new model structure $\mathscr{M}^+$, generated by a new set I_T^+ of generating cofibrations, and a new set of generating trivial cofibrations J_T^+, having a new class of weak equivalences W_T^+

$$\mathscr{M}^+ = (I_T^+, J_T^+, W_T^+) \, ,$$

such that weak equivalences in $\mathscr{M}^+$ would be those morphisms $f : X \to Y$ in which $f_n : X_n \to Y_n$ is a weak equivalence in $\mathscr{C}$ for all $n > 0$, and if

$$\mathscr{M}_{S^+}^+ = (I_T^+, J_{T,S^+}^+, W_{T,S^+}^+)$$

is a localization of $\mathscr{M}^+$ with respect to the above set S^+ then

$$W_{T,S} = W_{T,S^+}^+ \, .$$

Since now the desired model structure $\mathscr{M}_{S^+}^+$ will be called a *positive stable model structure* whose fibrations, cofibrations and weak equivalences will be called *positive fibrations, cofibrations and stable weak equivalences.*

3. Positive projective model structures

Let

$$I_T^+ = \cup_{n>0} F_n(I) \, ,$$
$$J_T^+ = \cup_{n>0} F_n(J)$$

and let

$$W_T^+$$

be the set of morphisms $f : X \to Y$, such that $f_n : X_n \to Y_n$ is a weak equivalence in $\mathscr{C}$ for all $n > 0$. First we will prove a proposition saying that I_T^+, J_T^+ and W_T^+ generate a model structure in $\mathscr{S}$. We will systematically use the terminology from §2.1 of the book [7].

Proposition 1. *The above sets I_T^+, J_T^+ and W_T^+ do satisfy the conditions of Theorem 2.1.19 in [7], so that they generate a model structure, denoted by $\mathscr{M}^+$ with the set of generating cofibrations I_T^+, the set of trivial generating cofibrations J_T^+, and whose weak equivalences are W_T^+. In particular, the set of cofibrations in $\mathscr{M}^+$ is the set I_T^+-cof, the set of trivial cofibrations is J_T^+-cof, and weak equivalences in $\mathscr{M}^+$ are W_T^+.*

Proof. We will use the fact that $\mathscr{M} = (I_T, J_T, W_T)$, and so the sets I_T, J_T and W_T satisfy the conditions of Theorem 2.1.19 in [7].

First condition

The first condition from Theorem 2.1.19 in [7] is satisfied automatically.

Second condition

Since

$$I_T^+ \subset I_T \, ,$$

we get

$$\operatorname{dom}(I_T^+) \subset \operatorname{dom}(I_T) \ ,$$

and

$$I_T^+\text{-cell} \subset I_T\text{-cell} \ .$$

By the property 2 from Hovey's theorem, applied to $\mathscr{M}$, we have that $\operatorname{dom}(I_T)$ are small relative to I_T-cell. Since $\operatorname{dom}(I_T^+) \subset \operatorname{dom}(I_T)$, even more so the set $\operatorname{dom}(I_T^+)$ is small relative to I_T-cell. As I_T^+-cell is a subset in I_T-cell, even more so the set $\operatorname{dom}(I_T^+)$ is small relative to the smaller class I_T^+-cell.

Third condition

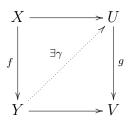Everything is the same as in the case of the second condition, but we need to replace I by J.

Fourth condition

First we look at the chain of the obvious inclusions

$$J_T^+\text{-cell} \subset J_T\text{-cell} \subset W_T \subset W_T^+ \ .$$

Now we need to show that J_T^+-cell $\subset I_T^+$-cof. Notice that the class J_T^+-cell consists of transfinite compositions of push-outs of morphisms from J_T^+ and the class I_T^+-cof is closed under transfinite compositions and push-outs, see the proof of Lemma 2.1.10 on page 31 in [7]. This is why, in order to show that J_T^+-cell $\subset I_T^+$-cof , it is enough to prove that $J_T^+ \subset I_T^+$-cof .

We need some more terminology. Let $\mathscr{X}$ be a category, and let A and B be two classes of morphisms in it. We will say that the the pair $\{A, B\}$ has the lifting property (LP, for short) if for any morphism $f : X \to Y$ from A, and any morphism $g : U \to V$ from B, and any commutative square

there exists a morphism γ keeping the diagram commutative.

Let now $\mathscr{X}$ and $\mathscr{Y}$ be two categories, and let $F : \mathscr{X} \rightleftarrows \mathscr{Y} : G$ be two adjoint functors, F from the left, and G from the right. Let A be a class of morphisms in $\mathscr{X}$, and let B be a class of morphisms in $\mathscr{Y}$. Then $\{A, G(B)\}$ has the LP if and only if $\{F(A), B\}$ has the LP.

Using this, and also taking into account that the class of fibrations in a cofibrantly generated model category coincides with the class J-inj, see Definition 2.1.17 (3) in [7], we get that

$$J_T^+\text{-inj} = \{f : X \to Y \text{ in } \mathscr{S} \mid \forall n > 0 \ \operatorname{Ev}_n(f) \text{ is a fibration in } \mathscr{C}\} \ ,$$

i.e. the class J_T^+-inj is the class of positive level fibrations in $\mathscr{S}$.

Similarly,

$$I_T^+\text{-inj} = \{f : X \to Y \text{ in } \mathscr{S} \mid \forall n > 0 \ \mathrm{Ev}_n(f) \text{ is a trivial fibration in } \mathscr{C}\} \ .$$

It follows that

$$I_T^+\text{-inj} \subset J_T^+\text{-inj} \ .$$

By definition, it means that all morphisms in I_T^+-inj have the right lifting property with respect to all morphisms from J_T^+. Then it means that

$$J_T^+ \subset I_T^+\text{-cof} \ ,$$

as required.

As a result,

$$J_T^+\text{-cell} \subset W_T^+ \cap I_T^+\text{-cof} \ ,$$

and the fourth condition is done.

Fifth and sixth condition

The above descriptions of the classes J_T^+-inj and I_T^+-inj give that

$$J_T^+\text{-inj} \cap W_T^+ = I_T^+\text{-inj} \ .$$

This gives the conditions five and six in Theorem 2.1.19 in Hovey's book.

Thus, the sets I_T^+, J_T^+ and W_T^+ generate a model structure in $\mathscr{S}$, denoted by $\mathscr{M}^+$, such that weak equivalences in it are those morphisms $f : X \to Y$ in which $f_n : X_n \to Y_n$ is a weak equivalence in $\mathscr{C}$ for all $n > 0$.

$\square$

Corollary 2. *A morphism $f : X \to Y$ in $\mathscr{S}$ is a fibration in $\mathscr{M}^+$ if and only if $f_n : X_n \to Y_n$ is a fibration in $\mathscr{C}$ for any $n > 0$. A morphism $f : X \to Y$ in $\mathscr{S}$ is a cofibration in $\mathscr{M}^+$ if and only if f is a cofibration in $\mathscr{M}$ and $f_0 : X_0 \to Y_0$ is an isomorphism. In particular, an object X in $\mathscr{S}$ is cofibrant in $\mathscr{M}^+$ if and only if X is cofibrant in $\mathscr{M}$ and $X_0 = *$.*

Proof. The corollary can be proved using the definition of I_T^+, J_T^+, left lifting and the adjunction between F_n and Ev_n. $\square$

4. Loop spectra

Let $\mathscr{D}$ be a simplicial closed symmetric monoidal model category. In particular, for any object X in $\mathscr{D}$ the functor $- \wedge X$ has right adjoint functor $\underline{\mathrm{Hom}}(X, -)$. This is nothing but the function object whose value $\underline{\mathrm{Hom}}(X, Y)$, for any object Y in $\mathscr{D}$, can be viewed as "functions" from X to Y. Certainly, $\underline{\mathrm{Hom}}(-, -)$ is a bifunctor from $\mathscr{D}^{\mathrm{op}} \times \mathscr{D}$ to $\mathscr{D}$.

Being a simplicial category, $\mathscr{D}$ also has a bifunctor $\mathrm{Map}(-, -)$ from $\mathscr{D}^{\mathrm{op}} \times \mathscr{D}$ to the category of simplicial sets $\triangle^{op}\mathscr{S}ets$ with all nice adjunctions, see [7] and [3]. Since the setting is symmetric and simplicial, we will systematically

ignore the difference between the left and right versions of <u>Hom</u> and Map, see a remark on page 131 in [7].

For any simplicial set U we have that its n-th level U_n is canonically isomorphic to the Hom-set $\mathrm{Hom}_{\triangle^{op}\mathscr{S}ets}(\Delta^n, U)$. Using the adjunction between $\mathrm{Map}(X,-)$ and $X \wedge -$, see [3], we obtain that $\mathrm{Hom}_{\triangle^{op}\mathscr{S}ets}(\Delta^n, \mathrm{Map}(X,Y))$ is isomorphic to $\mathrm{Hom}_{\mathscr{D}}(X \wedge \Delta^n, Y)$. Then,

$$\mathrm{Map}(X,Y)_n \simeq \mathrm{Hom}_{\mathscr{D}}(X \wedge \Delta^n, Y) \ .$$

Objects $\mathrm{Map}(X,Y)$ come from the simplicial structure of the category $\mathscr{D}$. To provide them with a homotopical meaning we need to replace X and Y by their cofibrant and fibrant replacements QX and RY respectively. Then let

$$\mathrm{map}(X,Y) = \mathrm{Map}(QX, RY) \ ,$$

so that we obtain yet another bifunctor $\mathrm{map}(-,-)$ from the category $\mathscr{D}^{\mathrm{op}} \times \mathscr{D}$ to $\triangle^{op}\mathscr{S}ets$, see [8], Section 2.

Now let $\mathscr{D}$ be the category of symmetric spectra $\mathscr{S}$. Let Q and R be cofibrant and fibrant replacement functors with respect to the model structure $\mathscr{M}$, and let Q^+ and R^+ be cofibrant and fibrant replacement functors with respect to the model structure $\mathscr{M}^+$. Cofibrations do not change when passing to localizations, so that Q remains the same in the localizations of the model structure $\mathscr{M}$ by S or S^+, and Q^+ remains the same in the localizations of the model structure $\mathscr{M}^+$ by S or S^+. Respectively, we define two bifunctors

$$\mathrm{map}(X,Y) = \mathrm{Map}(QX, RX)$$

and

$$\mathrm{map}^+(X,Y) = \mathrm{Map}(Q^+X, R^+X)$$

from $\mathscr{S}^{\mathrm{op}} \times \mathscr{S}$ to $\triangle^{op}\mathscr{S}ets$.

Next, following [9] (and [12]), for any spectrum X in $\mathscr{S}$ let

$$\Theta X := \underline{\mathrm{Hom}}(F_1(T), X) \ ,$$

and let

$$\theta : X \longrightarrow \Theta X$$

be the morphism induced by the morphism

$$\zeta_0^{\mathbb{1}} : F_1(T) \to F_0(\mathbb{1}) \ .$$

It is useful to interpret the functor Θ as a loop spectrum. Indeed, if

$$s_- : \mathscr{S} \longrightarrow \mathscr{S}$$

is a shift functor

$$s_- = \underline{\mathrm{Hom}}(F_1(\mathbb{1}), -) \ ,$$

see Definition 8.9 in [8], then Θ is isomorphic to the composition of s_- and a loop-spectrum functor

$$(-)^T = \underline{\mathrm{Hom}}(F_0(T), -) : \mathscr{S} \longrightarrow \mathscr{S} \ ,$$

loc.cit.

We also have iterations
$$\Theta^0 X = X \ ,$$
$$\Theta^n X := \Theta(\Theta^{n-1}X) \ ,$$

and
$$\theta^n : X \longrightarrow \Theta^n X \ ,$$

being a composition of morphisms $\Theta^i(\theta) : \Theta^i X \to \Theta^{i+1}X$ for all $i = 0, \ldots, n-1$.

We can also take the colimit
$$\Theta^\infty X = \operatorname{colim}_n \Theta^n X$$

with respect to the morphisms $\Theta^i(\theta)$, and consider the corresponding morphism
$$\theta^\infty : X \longrightarrow \Theta^\infty X \ .$$

The meaning of the above constructions comes from topology. Indeed, if $\mathscr{C}$ is the category of pointed simplicial sets $\triangle^{op}\mathscr{S}ets_*$, then $\mathbb{1}$ is the 0-dimensional sphere
$$S^0 = \partial\Delta[1] \ ,$$

T is the simplicial circle
$$S^1 = \Delta[1]/\partial\Delta[1] \ ,$$

and $\mathscr{S}$ is the category of topological symmetric spectra from [9]. For any pointed simplicial set Y let
$$X = F_0(Y) = \Sigma^\infty Y$$

be the symmetric S^1-suspension spectrum of Y. Then, by Proposition 2.2.6 (3) in [9], we have the following isomorphisms of simplicial sets,
$$\operatorname{Map}(F_1(S^1), X) \simeq \operatorname{Map}(S^1, \operatorname{Ev}_1 X) =$$
$$= \operatorname{Map}(S^1, S^1 \wedge Y) = \Omega\Sigma Y \ .$$

The latter is the simplicial set of loops in the suspension ΣY of the pointed simplicial set Y. Similarly, by adjunction between F_0 and Ev_0 we have that
$$Y \simeq \operatorname{Map}(S^0, Y) \simeq \operatorname{Map}(F_0(S^0), F_0(Y)) = \operatorname{Map}(F_0(S^0), X) \ .$$

As the suspension Σ is left adjoint to the loop-functor Ω, the identity morphism $\operatorname{id} : \Sigma Y \to \Sigma Y$ gives a morphism $\theta' : Y \to \Omega\Sigma Y$. In view of the above isomorphisms, θ' is nothing but the morphism $\operatorname{Map}(\zeta_0^{S^0}, X)$, induced by the morphism $\zeta_0^{S^0} : F_1(S^1) \to F_0(S^0)$. In other words, θ is a "spectralized" morphism θ' obtained by replacing Homs by internal Homs in $\mathscr{S}$.

Iterating the process we would see that the morphisms $\theta^n : X \to \Theta^n X$ come from the morphisms $Y \to \Omega^n\Sigma^n Y$, and the morphism $\theta^\infty : X \to \Theta^\infty X$ comes from the morphism $Y \to \Omega^\infty\Sigma^\infty Y$. The simplicial set $\Omega^\infty\Sigma^\infty Y$ is sometimes denoted by QY.
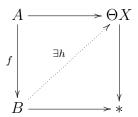
Now we come back to the category $\mathscr{S}$ of abstract symmetric T-spectra over the general category $\mathscr{C}$. Recall that $\mathscr{C}$ is a closed symmetric monoidal model category, left proper and cellular, and T is a cofibrant object in $\mathscr{C}$.

Remark 3. By Theorem 8.8 in [8], for any cofibrant object A in $\mathscr{C}$ and any $n \geq 0$, the morphism ζ_n^A is a stable weak equivalence. The same argument as in loc.cit. shows that for any $n > 0$, the morphism ζ_n^A is a positive stable weak equivalence, i.e. it is an S^+-local weak equivalence with respect to the positive model structure $\mathscr{M}^+$.
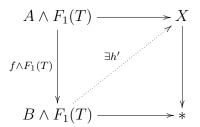
Proposition 4. *Let X be an S^+-local object in $\mathscr{S}$ with respect to the positive projective model structure $\mathscr{M}^+$. Then:*

> *(i) ΘX is an S-local object with respect to the projective model structure $\mathscr{M}$, and*
> *(ii) the morphism $\theta : X \to \Theta X$ is a weak equivalence in the model structure $\mathscr{M}^+$.*

Proof. First of all we need to show that ΘX is fibrant in $\mathscr{M}$. Let $f : A \to B$ be a trivial cofibration in $\mathscr{M}$, and consider the following commutative square

$$
\begin{array}{ccc}
A & \longrightarrow & \Theta X \\
\downarrow{\scriptstyle f} & \overset{\exists h}{\nearrow} & \downarrow \\
B & \longrightarrow & *
\end{array}
$$

We need to find a morphism $h : B \to \Theta X$ completing the diagram to a commutative one. By adjunction between $- \wedge F_1(T)$ and $\underline{\mathrm{Hom}}(F_1(T), -)$ the lifting h exists if and only if there exists a lifting h' making the diagram

$$
\begin{array}{ccc}
A \wedge F_1(T) & \longrightarrow & X \\
\downarrow{\scriptstyle f \wedge F_1(T)} & \overset{\exists h'}{\nearrow} & \downarrow \\
B \wedge F_1(T) & \longrightarrow & *
\end{array}
$$

commutative. The object $F_1(T)$ is cofibrant in $\mathscr{M}$ because T is cofibrant and the functor F_1 is left Quillen with respect to the model structure $\mathscr{M}$. Since $\mathscr{M}$ is a monoidal model category by Theorem 8.3 in [8], we obtain that $f \wedge F_1(T)$ is a trivial cofibration in $\mathscr{M}$. The specificity of the spectrum $F_1(T)$ yields that $(A \wedge F_1(T))_0 = *$ and $(B \wedge F_1(T))_0 = *$, so that $(f \wedge F_1(T))_0$ is an isomorphism. Therefore, $f \wedge F_1(T)$ is a trivial cofibration not only in $\mathscr{M}$ but also in $\mathscr{M}^+$. Since X is fibrant in $\mathscr{M}^+$, because it is S^+-local with respect to $\mathscr{M}^+$ by assumption, the required h' exists.

Thus, ΘX is fibrant in $\mathscr{M}$, and we can start to prove the first part of the proposition. In order to show that ΘX is S-local, with respect to $\mathscr{M}$, we need to show that for any $U \in \mathrm{dom}(I) \cup \mathrm{codom}(I)$, and for any non-negative integer n the morphism

$$(\zeta_n^U)^* : \mathrm{map}(F_n(U), \Theta X) \longrightarrow \mathrm{map}(F_{n+1}(T \wedge U), \Theta X)$$

is a weak equivalence of simplicial sets. As U is cofibrant, the spectrum $F_n(U)$ is cofibrant in $\mathscr{M}$ too. The spectrum ΘX is fibrant in $\mathscr{M}$. Therefore, the simplicial set

$$\mathrm{map}(F_n(U), \Theta X)$$

is weakly equivalent to the simplicial set

$$\mathrm{Map}(F_n(U), \Theta X) \simeq \mathrm{Hom}_{\mathscr{S}}(F_n(U) \wedge \Delta^\bullet, \Theta X)$$

By the adjunction between $- \wedge F_1(T)$ and $\underline{\mathrm{Hom}}(F_1(T), -)$ we get an isomorphism

$$\mathrm{Hom}_{\mathscr{S}}(F_n(U) \wedge \Delta^\bullet, \Theta X) \simeq \mathrm{Hom}_{\mathscr{S}}(F_n(U) \wedge F_1(T) \wedge \Delta^\bullet, X) \ .$$

But

$$\mathrm{Hom}_{\mathscr{S}}(F_n(U) \wedge F_1(T) \wedge \Delta^\bullet, X) \simeq \mathrm{Map}(F_n(U) \wedge F_1(T), X) \ .$$

Besides, the spectrum $F_n(U) \wedge F_1(T)$ is cofibrant in $\mathscr{M}^+$, and X is fibrant in $\mathscr{M}^+$ by assumption, so that

$$\mathrm{Map}(F_n(U) \wedge F_1(T), X) \sim \mathrm{map}^+(F_n(U) \wedge F_1(T), X) \ ,$$

where $\sim$ stays for weak equivalences of simplicial sets. As a result, we obtain that

$$\mathrm{map}(F_n(U), \Theta X) \sim \mathrm{map}^+(F_n(U) \wedge F_1(T), X) \ .$$

Similarly, we get a weak equivalence

$$\mathrm{map}(F_{n+1}(U \wedge T), \Theta X) \sim \mathrm{map}^+(F_n(U \wedge T) \wedge F_1(T), X) \ .$$

Consider now a commutative square

$$
\begin{array}{ccc}
\mathrm{map}(F_n(U), \Theta X) & \xrightarrow{\ (\zeta_n^U)^*\ } & \mathrm{map}(F_{n+1}(U \wedge T), \Theta X) \\
\Big\downarrow{\sim} & & \Big\downarrow{\sim} \\
\mathrm{map}^+(F_n(U) \wedge F_1(T), X) & \xrightarrow{\ (\zeta_{n+1}^{U \wedge T})^*\ } & \mathrm{map}^+(F_{n+1}(U \wedge T) \wedge F_1(T), X)
\end{array}
$$

As X is S^+-local with respect to $\mathscr{M}^+$ by assumption, and the morphism $\zeta_{n+1}^{U \wedge T}$ is an S^+-local weak equivalence by Remark 3, the bottom horizontal morphism in the above diagram is a weak equivalence of simplicial sets. And, as we have seen just now, the vertical morphisms are weak equivalences of simplicial sets. Then the top horizontal morphism $(\zeta_n^U)^*$ is a weak equivalence of simplicial sets as well, and (i) is done.

To prove (ii) all we need to show is that the morphism $\theta_n : X_n \to (\Theta X)_n$ is a weak equivalence in $\mathscr{C}$ for all $n \geq 1$. Since X is fibrant in $\mathscr{M}^+$ by assumption, X_n is fibrant in $\mathscr{C}$, provided $n \geq 1$. The object $(\Theta X)_n$ is fibrant in $\mathscr{C}$ because ΘX is fibrant in $\mathscr{M}$ by (i). Therefore, it is enough to show that for any cofibrant object B in $\mathscr{C}$ the corresponding morphism

$$(\theta_n)_* : \mathrm{map}(B, X_n) \longrightarrow \mathrm{map}(B, (\Theta X)_n)$$

is a weak equivalence of simplicial sets.

Recall that $\mathscr{C}$ is simplicial. As B is cofibrant and $\mathrm{Ev}_n(X)$ is fibrant in $\mathscr{C}$, we have that

$$\mathrm{map}(B, \mathrm{Ev}_n(X)) \sim \mathrm{Map}(B, \mathrm{Ev}_n(X)) = \mathrm{Hom}_{\mathscr{C}}(B \wedge \Delta^\bullet, \mathrm{Ev}_n(X)) \ .$$

By the adjunction between F_n and Ev_n we have that

$$\mathrm{Hom}_{\mathscr{C}}(B \wedge \Delta^\bullet, \mathrm{Ev}_n(X)) \simeq \mathrm{Hom}_{\mathscr{S}}(F_n(B \wedge \Delta^\bullet), X)$$

But

$$F_n(B \wedge \Delta^\bullet) \simeq F_n(B) \wedge \Delta^\bullet$$

by the definition of the action of simplicial sets on spectra. Therefore, we obtain

$$\mathrm{map}(B, \mathrm{Ev}_n(X)) \sim \mathrm{Hom}_{\mathscr{S}}(F_n(B \wedge \Delta^\bullet), X) \simeq \mathrm{Hom}_{\mathscr{S}}(F_n(B) \wedge \Delta^\bullet, X) =$$

$$= \mathrm{Map}(F_n(B), X) \ .$$

Similarly, we get a weak equivalence

$$\mathrm{map}(B, \mathrm{Ev}_n(\Theta X)) \sim \mathrm{Map}(F_n(B), \Theta X) \ .$$

Now we look at the commutative square

$$
\begin{array}{ccc}
\mathrm{map}(B, \mathrm{Ev}_n(X)) & \xrightarrow{\ (\theta_n)_*\ } & \mathrm{map}(B, \mathrm{Ev}_n(\Theta X)) \\
\Big\downarrow{\scriptstyle\sim} & & \Big\downarrow{\scriptstyle\sim} \\
\mathrm{Map}(F_n(B), X) & \xrightarrow{\ \theta_*\ } & \mathrm{Map}(F_n(B), \Theta X)
\end{array}
$$

Changing the bottom horizontal row by means of the adjunction

$$\mathrm{Map}(F_n(B), \Theta X) \simeq \mathrm{Map}(F_n(B) \wedge F_1(T), X) \simeq \mathrm{Map}(F_{n+1}(B \wedge T), X)$$

we get a new commutative square

$$
\begin{array}{ccc}
\mathrm{map}(B, X_n) & \xrightarrow{\ (\theta_n)_*\ } & \mathrm{map}(B, \Theta X_n) \\
\Big\downarrow{\scriptstyle\sim} & & \Big\downarrow{\scriptstyle\sim} \\
\mathrm{Map}(F_n(B), X) & \xrightarrow{\ (\zeta_n^B)^*\ } & \mathrm{Map}(F_{n+1}(B \wedge T), X)
\end{array}
$$

As $n \geq 1$, the morphism ζ_n^B is an S^+-local weak equivalence by Remark 3, and the objects $F_n(B)$ and $F_{n+1}(B \wedge T)$ are cofibrant in $\mathscr{M}^+$. Since X is S^+-local, the bottom horizontal morphism in the last commutative square is a weak equivalence of simplicial sets. Since the vertical morphisms are weak equivalences, we obtain that the top horizontal $(\theta_n)_*$ is a weak equivalence of simplicial sets. $\qquad\square$

If $\mathscr{A}$ and $\mathscr{A}'$ are two model structures on the same category $\mathscr{B}$ then we will use the symbols $Ho(\mathscr{A})$ and $Ho(\mathscr{A}')$ for the homotopy categories of the category $\mathscr{B}$ with respect to the model structures $\mathscr{A}$ and $\mathscr{A}'$ respectively. We also will be using the following lemma.

Lemma 5. *The pair of functors*

$$(- \wedge F_1(T), \Theta)$$

is a Quillen adjunction between $\mathscr{M}$ and $\mathscr{M}^+$. In particular, there exists right derived functor $R\Theta : Ho(\mathscr{M}^+) \to Ho(\mathscr{M})$.

Proof. Let f be a (trivial) cofibration in the model structure $\mathscr{M}$. As the model structure $\mathscr{M}$ is compatible with the monoidal structure in $\mathscr{S}$, the morphism $f \wedge F_1(T)$ is also a (trivial) cofibration in $\mathscr{M}$. Since $(F_1(T))_0 = *$ the morphism $(f \wedge F_1(T))_0$ is an isomorphism. Since $- \wedge F_1(T)$ has right adjoint Θ, we are done. $\qquad\square$

5. Positive weak equivalences are stable

In this section we will show that any weak equivalence in the positive model structure is a weak equivalence in the stable model structure. This is a consequence of the previous results and the following general effect.

Lemma 6. *Let $\mathscr{D}$ be a closed symmetric monoidal model category with a product $\wedge$ and unit $\mathbb{1}$. Suppose $\mathscr{D}$ is cofibrantly generated and that the domains of the generating cofibrations are cofibrant. Let U be cofibrant, X, Y fibrant objects, and let*

$$u : U \to \mathbb{1}, \quad f : X \to Y$$

be two morphisms, all in $\mathscr{D}$. Denote by (u) the set of morphisms

$$V \wedge u : V \wedge U \to V,$$

where V runs through domains and codomains of generating cofibrations in $\mathscr{D}$. Suppose furthermore that the morphism

$$f_* : \underline{\mathrm{Hom}}(U, X) \longrightarrow \underline{\mathrm{Hom}}(U, Y)$$

is a weak equivalence in $\mathscr{D}$. Then, f is a weak equivalence in the Bousfield localized category $\mathscr{D}_{(u)}$.

Proof. Let $q : Q\mathbb{1} \to \mathbb{1}$ be a cofibrant replacement of the unit in $\mathscr{D}$. In the commutative diagram

$$
\begin{array}{ccc}
Q\mathbb{1} \wedge U & \xrightarrow{\;Q\mathbb{1} \wedge u\;} & Q\mathbb{1} \\
\downarrow{\scriptstyle q \wedge U} & & \downarrow{\scriptstyle q} \\
U & \xrightarrow{\;\;\;u\;\;\;} & \mathbb{1}
\end{array}
$$

the morphism q is a weak equivalence in $\mathscr{D}$ by definition, and the morphism $q \wedge U$ is a weak equivalence in $\mathscr{D}$ by one of the axioms of the monoidal model structure. The category $\mathscr{D}_{(u)}$ is a closed monoidal model category by Proposition 36 in [4]. Therefore, for any cofibrant object V in $\mathscr{D}$ the morphism $V \wedge u$ is a weak equivalence in $\mathscr{D}_{(u)}$ by Lemma 35 in loc.cit[1]. In particular, $Q\mathbb{1} \wedge u$ is a weak equivalence in $\mathscr{D}_{(u)}$. Therefore, u is a weak equivalence in $\mathscr{D}_{(u)}$.

The morphism u defines a morphism

$$
u^* : X \simeq \underline{\mathrm{Hom}}(\mathbb{1}, X) \to \underline{\mathrm{Hom}}(U, X) \, .
$$

Let

$$
r : X \longrightarrow R_{(u)}X
$$

be a fibrant replacement in $\mathscr{D}_{(u)}$. As $R_{(u)}X$ is fibrant, U is cofibrant and u is a weak equivalence in the closed monoidal model category $\mathscr{D}_{(u)}$, the morphism

$$
u^* : R_{(u)}X \simeq \underline{\mathrm{Hom}}(\mathbb{1}, R_{(u)}X)) \longrightarrow \underline{\mathrm{Hom}}(U, R_{(u)}X)
$$

is a weak equivalence in $\mathscr{D}_{(u)}$ by Lemma 4.2.7 in [7]. The morphism r is a weak equivalence in $\mathscr{D}_{(u)}$ by definition. As the square

$$
\begin{array}{ccc}
X & \xrightarrow{\;\;u^*\;\;} & \underline{\mathrm{Hom}}(U, X) \\
\downarrow{\scriptstyle r} & & \downarrow{\scriptstyle r_*} \\
R_{(u)}X & \xrightarrow{\;\;u^*\;\;} & \underline{\mathrm{Hom}}(U, R_{(u)}X)
\end{array}
$$

is commutative, the composition

$$
X \xrightarrow{\;u^*\;} \underline{\mathrm{Hom}}(U, X) \xrightarrow{\;r_*\;} \underline{\mathrm{Hom}}(U, R_{(u)}X)
$$

is an isomorphism in the homotopy category $Ho(\mathscr{D}_{(u)})$, which means that X is functorially a retract of $\underline{\mathrm{Hom}}(U, X)$ in $Ho(\mathscr{D}_{(u)})$.

---

[1]Proposition 36 and Lemma 35 from [4] do not use in any way positive model structures on symmetric spectra

In particular, f is a retract of an isomorphism $f_* : \underline{\mathrm{Hom}}(U, X) \to \underline{\mathrm{Hom}}(U, Y)$ in $Ho(\mathscr{D}_{(u)})$. As a retract of an isomorphism is an isomorphism, f is an isomorphism in $Ho(\mathscr{D}_{(u)})$, and so it is a weak equivalence in $\mathscr{D}_{(u)}$. $\qquad\square$

Proposition 7. *Any positive weak equivalence is a stable weak equivalence.*

Proof. Let $f : X \to Y$ be a positive weak equivalence, i.e. $f_n : X_n \to Y_n$ is a weak equivalence in $\mathscr{C}$ for any $n > 0$. We are going to apply Lemma 6 when

$$U = F_1(T)$$

and

$$u = \zeta_0^{\mathbb{1}} : F_1(T) \longrightarrow F_0(\mathbb{1}) = \mathbb{1} \ .$$

Notice that U is cofibrant in $\mathscr{M}$ and, without loss of generality, we may assume that X and Y are fibrant objects in $\mathscr{M}$, because fibrant replacements in $\mathscr{M}$ are level equivalences and do not change neither the condition of the proposition, nor its conclusion. Then X and Y are fibrant in $\mathscr{M}^+$, too. As f is a weak equivalence in $\mathscr{M}^+$, by Lemma 5, the morphism $\Theta f = \underline{\mathrm{Hom}}(F_1(T), f)$ is a weak equivalence in $\mathscr{M}$. Then f is a weak equivalence in the model structure $\mathscr{M}_{(\zeta_0^{\mathbb{1}})}$ by Lemma 6. To complete the proof we need only to observe that, for any cofibrant object V in $\mathscr{M}$, the morphism $V \wedge \zeta_0^{\mathbb{1}}$ is a stable weak equivalence, so that $(\zeta_0^{\mathbb{1}})$ consists of weak equivalences in $\mathscr{M}_S$. Actually, $\mathscr{M}_{(\zeta_0^{\mathbb{1}})} = \mathscr{M}_S$, because $\zeta_n^X = F_n(X) \wedge \zeta_0^{\mathbb{1}}$. $\quad\square$

Recall that Q is a cofibrant replacement functor with respect to the model structure $\mathscr{M}$, and Q^+ is a cofibrant replacement functor with respect to the model structure $\mathscr{M}^+$. Replacing Q^+ by Q^+Q, we obtain a natural transformation

$$Q^+ \longrightarrow Q \ .$$

Corollary 8. *Let X and Z be two objects in $\mathscr{S}$, such that Z is S-local with respect to the projective model structure $\mathscr{M}$ in $\mathscr{S}$. Then the morphism*

$$\mathrm{map}(X, Z) \longrightarrow \mathrm{map}^+(X, Z) \ ,$$

induced by the above natural morphism $Q^+ X \to QX$, is a weak equivalence of simplicial sets.

Proof. As Z is S-local with respect to $\mathscr{M}$, it is fibrant in $\mathscr{M}$, and so in $\mathscr{M}^+$. Let

$$q : QX \longrightarrow X$$

be the cofibrant replacement in $\mathscr{M}$. The morphisms

$$q^* : \mathrm{map}(X, Z) \longrightarrow \mathrm{map}(QX, Z)$$

and

$$q^* : \mathrm{map}^+(X, Z) \longrightarrow \mathrm{map}^+(QX, Z)$$

are both weak equivalences of simplicial sets. Therefore, without loss of generality, one can assume that X is cofibrant in $\mathscr{M}$.

Let now

$$q^+ : Q^+ X \longrightarrow X$$

be the cofibrant replacement of X in $\mathscr{M}^+$. Then q^+ is a positive weak equivalence, hence a stable weak equivalence in $\mathscr{M}_S$, by Proposition 7. The objects X and $Q^+ X$ are cofibrant in $\mathscr{M}$, so in $\mathscr{M}_S$, and Z is fibrant in $\mathscr{M}_S$. Then the morphism

$$\operatorname{map}(X, Z) \sim \operatorname{Map}(X, Z) \xrightarrow{(q^+)^*} \operatorname{Map}(Q^+ X, Z) \sim \operatorname{map}^+(X, Z)$$

is a weak equivalence of simplicial sets because $\mathscr{S}$ is a simplicial model category with respect to the model structure $\mathscr{M}_S$. $\qquad\square$

Remark 9. For a natural n call an n-level weak equivalence (fibration) a morphism in $\mathscr{S}$ which is a level weak equivalence (fibration) for i-levels with $i \geq n$. These two classes of morphisms define a model structure $\mathscr{M}^{\geq n}$ on $\mathscr{S}$. Cofibrations in $\mathscr{M}^{\geq n}$ are cofibrations in $\mathscr{M}$ which are isomorphisms on i-levels with $i < n$ and n-level weak equivalences. By methods similar to those used above one can show that any n-level weak equivalence is a stable weak equivalence.

6. Main theorem

Recall that $W_{T,S}$ is the class of weak equivalences in $\mathscr{M}_S$, and W_{T,S^+}^+ is the class of weak equivalences in $\mathscr{M}_{S^+}^+$. Let also $W_{T,S}^+$ be the class of weak equivalences in $\mathscr{M}_S^+$. Now we are ready to state and prove our main result.

Theorem 10. *Let $\mathscr{C}$ be a closed symmetric monoidal model category, whose model structure is left proper and cellular. Suppose, moreover, that the domains of the generating cofibrations I in $\mathscr{C}$ are cofibrant. Let T be an arbitrary cofibrant object in $\mathscr{C}$. Then, in the notation above,*

$$W_{T,S} = W_{T,S^+}^+ = W_{T,S}^+ \ .$$

In particular, the stable model structure $\mathscr{M}_S$ is Quillen equivalent to the positive stable model structure $\mathscr{M}_{S^+}^+$ via the identity functor on the category of spectra $\mathscr{S}$.

Proof. Let $f : X \to Y$ be a weak equivalence in $\mathscr{M}_S$. In order to prove that f is a weak equivalence in $\mathscr{M}_{S^+}^+$ we need to show that for any S^+-local object Z in $\mathscr{M}^+$ the morphism

$$\operatorname{map}^+(Y, Z) \longrightarrow \operatorname{map}^+(X, Z)$$

is a weak equivalence of simplicial sets. The morphism

$$\theta : Z \longrightarrow \Theta Z \ ,$$

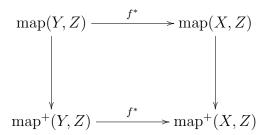together with the morphism f, give rise to the commutative square

$$
\begin{array}{ccc}
\mathrm{map}^+(Y,Z) & \xrightarrow{\ f^*\ } & \mathrm{map}^+(X,Z) \\
\downarrow{\scriptstyle\theta_*} & & \downarrow{\scriptstyle\theta_*} \\
\mathrm{map}^+(Y,\Theta Z) & \xrightarrow{\ f^*\ } & \mathrm{map}^+(X,\Theta Z)
\end{array}
$$

As Z is S^+-local in $\mathscr{M}^+$, Proposition 4 (i) gives that ΘZ is S-local in $\mathscr{M}$. Since f is a weak equivalence in $\mathscr{M}_S$, the morphism

$$
f^* : \mathrm{map}(Y,\Theta Z) \longrightarrow \mathrm{map}(X,\Theta Z)
$$

is a weak equivalence of simplicial sets. Applying Corollary 8 we obtain that the lower f^* in the above commutative square is also a weak equivalence of simplicial sets. Proposition 4 (ii) gives that the morphism θ is a weak equivalence in $\mathscr{M}^+$. It follows that the vertical morphisms in the above commutative square are weak equivalences of simplicial sets. Then the top horizontal morphism is a weak equivalence of simplicial sets, as required. Thus, $W_{T,S} \subset W^+_{T,S^+}$.

Let $f : X \to Y$ be a weak equivalence in $\mathscr{M}^+_{S^+}$. We want to show that f is a weak equivalence in $\mathscr{M}_S$. Take any S-local object Z in $\mathscr{M}$ and look at the commutative diagram

$$
\begin{array}{ccc}
\mathrm{map}(Y,Z) & \xrightarrow{\ f^*\ } & \mathrm{map}(X,Z) \\
\downarrow & & \downarrow \\
\mathrm{map}^+(Y,Z) & \xrightarrow{\ f^*\ } & \mathrm{map}^+(X,Z)
\end{array}
$$

As Z is S-local in $\mathscr{M}$, it is S^+-local in $\mathscr{M}^+$. Since f is a weak equivalence in $\mathscr{M}^+_{S^+}$, the lower horizontal morphism is a weak equivalence of simplicial sets. The vertical arrows in the diagrams are isomorphisms from Corollary 8. Then the top horizontal arrow is a weak equivalence of simplicial sets, for any S-local object Z in $\mathscr{M}$. It means that f is a weak equivalence in $\mathscr{M}_S$.

Thus, $W_{T,S} = W^+_{T,S^+}$. In particular, all morphisms in S are weak equivalences in $\mathscr{M}^+_{S^+}$. This implies that $(\mathscr{M}^+_{S^+})_S = \mathscr{M}^+_{S^+}$. On the other hand, $(\mathscr{M}^+_{S^+})_S = \mathscr{M}^+_S$, because $S^+ \subset S$. $\qquad\square$

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