How a single-factor CAPM works in a multi-currency world

By Robert Thomson, Şule Şahin and Taryn Reddy

Abstract

In this paper a single-factor multi-currency CAPM (SFM-CAPM) is developed. The advantage in using a single-factor model is that it does not treat currency risks as carrying different weight from investment risks; regardless of its source, risk is measured as variance and weighted accordingly. The aim of this paper is primarily to give actuaries a way ahead in the use of the single-factor CAPM in a multi-currency world for the purposes of the stochastic modelling of the assets and liabilities of long-term financial institutions such as pension funds, particularly for the purposes of liability-driven investments and market-consistent valuation, and the application of the model has been designed with that intention. However, it is envisaged that the model will also be of interest to other practitioners. The paper's major original contribution to the literature is its proof that, for a single-factor CAPM to work in a multi-currency world, there is a necessary condition. The theory is applied to two major currencies and two minor currencies, namely the USA dollar, the UK pound, the South African rand and the Turkish lira.

Keywords

International CAPM, single-factor multi-currency CAPM (SFM-CAPM)

Contact details

Robert Thomson (corresponding author)
School of Statistics and Actuarial Science
University of the Witwatersrand
Private Bag 3, WITS 2050, South Africa
Tel: +27 11 646 5332
Fax: +27 11 7176285
E-mail: rthomson@icon.co.za

Şule Şahin
Department of Actuarial Sciences
Hacettepe University
Ankara, Turkey
Tel.: +90 312 2976160
E-mail: sule@hacettepe.edu.tr

Taryn Reddy
School of Statistics and Actuarial Science
University of the Witwatersrand
Private Bag 3, WITS 2050, South Africa
Tel: +27 11 7176268
Fax: +27 11 7176285
E-mail: rthomson@icon.co.za

1. Introduction

As shown by Wilkie (unpublished), there is no unique single-factor capital-asset pricing model (CAPM) in a multi-currency world. As he points out, the standard CAPM assumes that all investors measure risk and return in the same currency.

In this paper a single-factor multi-currency CAPM (SFM-CAPM) is developed. The advantage in using a single-factor model is that it does not treat currency risks as carrying different weight from investment risks; regardless of its source, risk is measured by an investor as the variance of the return in the currency in which that investor measures risk, and weighted accordingly. This matter is further discussed in section 2.3. Unlike international CAPMs developed in the literature to date, and unlike the notion of a unique single-factor CAPM across all currencies (the notion rejected by Wilkie (op. cit.)), this paper assumes that, for every currency in which investors measure risk, there is a CAPM that is unique to those investors across all the markets in which they invest, but different from the CAPM of investors who measure their risks in other currencies. It also assumes that, regardless of the currency in which they measure risk, all investors have homogeneous expectations and all investors participate in the formation of equilibrium. It develops a theory for multi-currency CAPMs by developing a CAPM for each set of investors that measure their returns in a particular currency. In the development of this theory the meanings of 'homogeneous expectations' and of 'equilibrium' are reconsidered in the context of a multi-currency world.

In section 2 the literature on international versions of the CAPM is reviewed. In section 3 a single-factor multi-currency CAPM (SFM-CAPM) is developed and the necessary condition for the SFM-CAPM is derived. That condition is implemented by means of a penalty method in the specification of an objective function for the estimation of *ex-ante* expected returns. The method is applied to two major currencies and two minor currencies, namely the USA dollar, the UK pound, the South African rand and the Turkish lira. The data obtained for this purpose and the results of the application are described in section 4. The application illustrated in this paper is designed for use by actuaries in the modelling of the assets and liabilities of long-term financial institutions. To that end, the longest possible range of time periods is used and quarterly intervals are used rather than the relatively short time intervals typically used in the literature. However, the model could be applied to shorter periods and shorter time intervals. Section 5 concludes the paper with a summary of the findings and some indications of the way in which those findings will lead to further research for the purposes of such modelling.

2. Literature review

2.1 The domestic capital-asset pricing model and market segregation

In this paper, in order to distinguish between the CAPM used in a single domestic (segregated) market and the CAPM used in an international (integrated) market, the terms 'domestic CAPM' and 'international CAPM' are used respectively.

A market in which the price of an asset depends on where it is traded may be referred to as a 'segregated market' (Karolyi & Stulz, 2003). If markets are segregated then different domestic capital markets can be considered as independent entities and the international market consists of individual segregated markets (Solnik, 1974b). Jorion &

Schwartz (1986) pointed out that complete segregation implies that only domestic factors such as domestic systematic risk should enter the pricing of assets. Stulz (1981) stated that the widespread use, in all countries that have an equity market, of some proxy of the domestic market portfolio to determine how domestic assets are priced can be justified only by an assumption that markets are internationally segregated. This assumption implicitly underlies the original development of the CAPM by Treynor (unpublished), Sharpe (1964), Lintner (1965) and Mossin (1966).

In terms of the traditional CAPM, which uses the domestic market portfolio, and which, as stated above, is here referred to as the domestic CAPM, the equilibrium expected return of an asset may be expressed as follows:

$$E\left\{R_i\right\} = R_F + \beta_i\left[E\left\{R_M\right\} - R_F\right]; \tag{1}$$

where R_i , R_F and R_M are the returns on security i, on the risk-free asset F and on the domestic market portfolio M respectively, and β_i represents the sensitivity of the asset return to market movements; i.e.:

$$\beta_i = \frac{\sigma_{iM}}{\sigma_{MM}};\ \sigma_{iM} = \operatorname{cov}\left\{R_i, R_M\right\}; \text{and}$$

$$\sigma_{MM} = \operatorname{var}\left\{R_M\right\}.$$

One of the problems in applying the CAPM, whether in the domestic market or to the global market, is that, because *ex-post* means and covariances are generally used as *ex-ante* estimates, some anomalies may arise. For example, for some periods in some markets, market risk premiums may be negative. Whilst it is quite possible that *ex-post* mean returns may be lower than risk-free returns, a risk-averse world would not produce *ex-ante* mean returns that are lower than risk-free returns. This may be dealt with in various ways. For example, Thomson & Gott (2009) deal with it by imposing a minimum market risk premium in the application of the CAPM.

2.2 The Domestic Capital-asset Pricing Model and Market Integration

Karolyi & Stulz (op. cit.) defined an ‘integrated market’ as a market in which assets have the same price regardless of where they are traded. In such a market investors should earn the same risk-adjusted expected return on similar assets in different domestic markets, which is consistent with the definition of integration by Jorion & Schwartz (op. cit.). The latter argued that, with integration, the world market index should be mean–variance efficient, and as a result, the only priced risk should be systematic risk relative to the world market.

Because the domestic CAPM considers only domestic investment, it has important limitations (Solnik, 1974a). As Solnik (1974a) pointed out, because there is no universal risk-free asset, and because of exchange-rate risk, there is little intuitive reason to expect that the simple risk-pricing relation in the CAPM could be applied at the international level. Since then, certain authors (e.g. Stehle, 1977; Stulz, 1995a) have argued that a domestic CAPM is appropriate only for an asset that is traded in a closed, domestic financial market.

As observed above, Wilkie (op. cit.) showed that there is no unique CAPM in a multi-currency world. He showed that, if two investors measured risk and return in different currencies, the standard CAPM could not describe the pricing of capital assets for both investors.

Karolyi & Stulz (op. cit.) argued that there are systematic patterns in ownership of foreign equities that are hard to reconcile with models assuming perfect financial markets (such as the traditional CAPM) and therefore the only way to rationalise these patterns would be to argue that the gains from international diversification are too small to make it worthwhile to hold foreign assets. The inadequacies of the traditional CAPM in an international setting have therefore led to extensive debate and the development of equilibrium models (most of which are variations of the CAPM) to incorporate exchange-rate risk and global market portfolios.

2.3 The international capital-asset pricing model

As pointed out by Ng (2004), the starting point of the international CAPM literature is the observation that purchasing-power parity does not hold. This means that, in real terms, investors who measure their returns in different currencies earn different returns. This contravenes the standard CAPM assumption that investors have homogeneous expectations of returns, and it presents challenges for the aggregation of individual portfolios into a general asset-pricing equation. Wilkie (op. cit.) also concluded that, when different currencies exist, the traditional CAPM's assumption that all investors measure risk in the same currency breaks down. Stulz (1981) argued that, without a model showing how assets are priced in a world in which asset markets are fully integrated, it is impossible to determine whether asset markets are segregated internationally or not. In the international CAPM (ICAPM) of Solnik (1974a), Sercu (1980) and Stulz (1981), exchange-rate risk is priced by modifying the CAPM. The ICAPM contains risk premia that are based on the covariances of assets with exchange rates. There are different versions of the ICAPM. For the purpose of this paper, to illustrate the application of the single-factor CAPM in a multi-currency world, only the single-factor and multi-factor ICAPMs are considered.

The single-factor ICAPM, also referred to as the 'global CAPM' (GCAPM), as developed by Solnik (1974a), Grauer, Litzenberger & Stehle (op.cit.), Sercu (op. cit.), Adler & Dumas (op.cit.), Stulz (1981), and others, is expressed as follows:

$$E\left\{R_i\right\} = R_F + \beta_i^W \left[E\left\{R_W\right\} - R_F\right]; \tag{2}$$

where R_i, R_F and R_W are the nominal returns in domestic currency on security i, on the risk-free asset F and on the global market portfolio W respectively. β_i^W represents the sensitivity of the asset return to global market movements; i.e.:

$$\beta_i^W = \frac{\sigma_{iW}}{\sigma_{WW}};$$

$$\sigma_{iW} = \operatorname{cov}\left\{R_i, R_W\right\}; \text{and}$$

$$\sigma_{WW} = \operatorname{var}\left\{R_W\right\}.$$

Thus the GCAPM looks at the world from the perspective of one currency only—the currency in which the investor measures returns on the risk-free asset and on the global market portfolio. The drawback of this approach is that, in general, the implied price of a security to that investor is different from the price to an investor who measures returns in another currency. An exception occurs if strict purchasing-power parity applies and if returns are measured in real terms (or there is no inflation). Under these conditions, equation (2) applies regardless of the currency in which the investor measures returns. Under such circumstances the risk-free rate in each currency is equal to the risk-free rate in every other currency, the mean–variance optimal portfolio for each investor is equal to that of every other investor, regardless of the currency in which they measure returns, and the variances and covariances of returns—and therefore the beta of every asset—is similarly the same for every investor. (cf. e.g. Karolyi & Stulz, op.cit.)

In an attempt to determine the factors that affect share-price movements across the world, Solnik (1974b) determined the international market structure of asset prices. The resulting model is referred to as the 'multi-factor ICAPM'. The risk-pricing relation for the multi-factor ICAPM for asset i may be expressed as follows:

$$\begin{aligned} E\left\{R_i\right\} = R_F + \beta_i^W\left[E\left\{R_W\right\} - R_F\right] + \gamma_i^1\left[E\left\{R_M^1\right\} - R_F^1\right] + \gamma_i^2\left[E\left\{R_M^2\right\} - R_F^2\right] + \\ \ldots + \gamma_i^C\left[E\left\{R_M^C\right\} - R_F^C\right]; \end{aligned} \tag{3}$$

where R_i, R_F, R_W, R_M^c and R_F^c are the returns on security i (in the domestic currency), on the risk-free asset F (in the domestic currency), on the global market portfolio W, on the market portfolio in currency c and on the risk-free asset in currency c respectively. Here β_i^W represents the sensitivity of the asset return to global market movements and γ_i^1 to γ_i^C are the sensitivities of asset i to the currencies 1 to C (the number of exchange-rate factors can be as many as the number of currencies other than the numeraire currency); i.e.:

$$\beta_i^W = \frac{\sigma_{iW}}{\sigma_{WW}};$$

$$\sigma_{iW} = \operatorname{cov}\left\{R_i, R_W\right\}; \text{and}$$

$$\sigma_{WW} = \operatorname{var}\left\{R_W\right\}.$$

Under the assumptions of this model, a risk-free domestic asset is not risky for a domestic investor, but, because of currency risks, this same risk-free domestic asset is a risky asset for a foreign investor. The sensitivity parameters included in the ICAPM model capture these risks. Once again, that model looks at the world from the perspective of the currency in which the investor measures returns. Furthermore, there is no reason why, from the point of view of an investor who measures returns in that currency, investments in other currencies should not be diversified across currencies, nor why, for such an investor, the risks of investment in another currency should not be priced consistently with the risks of investment in the domestic currency. In equation (3) the values of γ_i^1 to γ_i^C are not necessarily related to betas and there is no underlying economic theory relating the sensitivity of expected returns to these values; there is merely an assumption of a linear relationship. As observed above, the advantage in using a single-factor model is that it does not treat currency risks as carrying different weight from investment risks; regardless of its source, risk is measured as variance and weighted accordingly.

Karolyi & Stulz (op. cit.) demonstrated that systematic mistakes are possible when one uses the domestic CAPM and when domestic investors have access to international markets. Other authors have demonstrated pricing errors from using the domestic CAPM in an integrated market such as Dolde et. al. (2011), Stulz (1995b), Stulz (1995c) and Koedijk et. al. (2002).

It is standard practice in the USA to use a long-term (say 30-year) yield on treasury bills as a proxy for the risk-free rate (Stulz, 1995c). However, Stulz (1995c) uses monthly returns and computes arithmetic means of returns per period over a long sample period during which markets were fairly integrated. Jorion & Schwartz (op. cit.) derive the risk-free rate from the yield on three-month treasury bills and use monthly returns. In principle one should measure the risk-free rate over a time interval up to the assumed time horizon of market participants, or the interval after which portfolio selection will be reconsidered. This time interval should be equal to the unit time interval adopted. It is inevitable that risk-free rates will have varied over the sample period. Unless one applies the CAPM using time series, the choice of a risk-free rate is problematic, even in a domestic CAPM.

It is accepted in this paper that the assumption of strict purchasing-power parity is untenable, so that, even in real terms, the GCAPM is untenable. Furthermore, it is assumed in this paper that exposure to variances of returns arising from currency exposure is no different from exposure to variances of returns arising from domestic sources, so that a multi-factor model is unnecessary.

In this paper, in the light of Wilkie's (op. cit.) conclusion and the literature reviewed, the authors develop a theory for a single-factor CAPM in a multi-currency world. They do so by specifying a CAPM for each set of investors that measures their returns in a particular currency. They assume that, for every currency in which investors measure risk, there is a unique CAPM across all the markets in which they invest.

3. The necessary condition for the single-factor multi-currency CAPM

In this section an SFM-CAPM is developed. Section 3.1 is a preliminary discussion largely devoted to the definitions required for the following sections. In section 3.2 it is shown that, for a single-factor CAPM to work in a multi-currency world, there is a necessary condition. The SFM-CAPM is formulated in section 3.3. That formulation requires an optimisation process. Problems relating to local optima are discussed in section 3.4. Section 3.5 considers a special case.

3.1 Preliminary discussion

Suppose there are C currencies and that, in currency c, there is one risk-free asset and n_c risky capital assets have been issued. It is assumed that every investor measures investment returns in one of these currencies. Regardless of the currency in which an investor measures investment returns, the investor may invest in any currency. An ‘asset issued in currency *c*’ is a risky asset issued in that currency or the risk-free asset denominated in that currency. (For an investor who measures returns in another currency, the risk-free asset denominated in currency *c* is not risk-free; this matter is dealt with in greater detail below.)

The ‘return in currency *c*’ on an asset issued in currency *d* is the force of return (i.e. the ‘log return’) earned on that asset, over a unit interval, measured in currency *c*. Thus, for example, if the value in currency *d* of asset *i* issued in that currency changes over a unit interval from Y_{di0} to Y_{di1} then the return on that asset during that interval, measured in that currency, is:

$$X_{di} = \ln \frac{Y_{di1}}{Y_{di0}}.$$

If during that interval the exchange rate between currency *d* and a numeraire currency—say currency 1—changes from Y_{d0} units of currency 1 per unit of currency *d* to Y_{d1} such units then the increase in the exchange rate is:

$$X_d = \ln \frac{Y_{d1}}{Y_{d0}}.$$

X_d is thus a measure of the strengthening of currency *d* against currency 1. The value of the asset issued in currency *d*, measured in currency 1, changes from $Y_{di0}Y_{d0}$ to $Y_{di1}Y_{d1}$ and the value of that asset, measured in currency *c*, changes from $\frac{Y_{di0}Y_{d0}}{Y_{c0}}$ to $\frac{Y_{di1}Y_{d1}}{Y_{c1}}$. The return on that asset during that interval, measured in currency *c*, is:

$$\ln \frac{Y_{di1}Y_{d1}Y_{c0}}{Y_{di0}Y_{d0}Y_{c1}} = X_{di} + X_d - X_c.$$

Where exchange rates are expressed in units of currency *d* per unit of currency 1 (as where currency 1 is the US dollar), it should be noted that the exchange rate should be inverted so as to measure the strength of currency *d* against currency 1.

Whilst it is customary to measure returns as rates, there are substantial advantages to the use of forces. The implicit assumption of this approach is that assets are continuously rebalanced during the unit interval, so that the weightings of the respective forces remain constant. Exchange rates are measured per unit of currency 1. The rate of strengthening of currency *c* per unit of currency 1 is measured as a force over the unit interval, thus avoiding the need for compounding that would otherwise apply. Returns and rates of strengthening of currencies may be measured in real terms (relative to a price index) or in nominal terms. Again, the use of forces avoids the compounding effects of inflation.

We assume that the CAPM applies for investors in each currency. More specifically, we assume that:

1) investors who measure their investment returns in currency *c* (i.e. ‘currency-*c* investors’) have indifference curves in mean–variance space, the means and variances being those measured in that currency; and

2) all investors, regardless of the currency in which they measure returns, have homogeneous expectations of the means, variances and covariances of:
   (a) the returns in each currency on assets issued in that currency; and
   (b) rates of strengthening of each currency.

Note that, even after allowance for the strengthening of currencies, currency-*d* investors will not necessarily have the same indifference curves as currency-*c* investors.

First we consider returns in currency *c* on assets issued in that currency. For this purpose we define the following random variables, where, for $c = 1, \ldots, C$, $i = 1$ denotes the risk-free asset and $i = 2, \ldots, n_c$ the risky assets issued in that currency:

- X_{ci} is the return in currency *c* on asset *i* issued in that currency for $c = 1, \ldots, C;\ i = 1, \ldots, n_c$; and
- X_c is the rate of strengthening of currency *c* for $c = 2, \ldots, C.$

Here n_c is the number of assets issued in currency *c*. For $i = 1\ \ X_{c1} = r_c$, which is deterministic, being the return on the risk-free asset denominated in currency *c*. For $i = 2, \ldots, n_c\ \ X_{ci}$ is a random variable.

We define the following parameters, where, as above, for $c = 1, \ldots, C$, $i = 1$ denotes the risk-free asset denominated in that currency and $i = 2, \ldots, n_c$ denotes the risky assets issued in that currency:

- μ_{ci} is the expected return in currency *c* on risky asset *i* issued in that currency; i.e.: $\mu_{ci} = E\{X_{ci}\}$;
- $\sigma_{ci,dj}$ is the covariance of the return in currency *c* on risky asset *i* issued in that currency with the return in currency *d* on risky asset *j* issued in that currency; i.e.: $\sigma_{ci,dj} = \begin{cases} \mathrm{var}\{X_{ci}\} & \text{for } d = c, j = i; \\ \mathrm{cov}\{X_{ci}, X_{dj}\} & \text{otherwise;} \end{cases}$
- μ_c is the expected rate of strengthening of currency *c*; i.e.: $\mu_c = E\{X_c\}$;
- $\sigma_{c,di}$ is the covariance of the rate of strengthening of currency *c* with the return in currency *d* on risky asset *i* issued in that currency; i.e.: $\sigma_{c,di} = \mathrm{cov}\{X_c, X_{di}\}$;
- $\sigma_{c,c}$ is the variance of the rate of strengthening of currency *c*; i.e.: $\sigma_{c,c} = \mathrm{var}\{X_c\}$.

Because investors have homogeneous expectations (assumption (2) above), the means, variances and covariances defined above are the same for all investors, regardless of the currency in which they measure their returns. Because the expected values and the variances and covariances are those of forces rather than rates of return, the values will be different from those typically used. Whilst for the standard CAPM mean–variance analysis is expressed in terms of rates of return, here it is expressed in terms of forces. Utility functions—and therefore indifference curves—may similarly be expressed in terms of forces of return.

In the case of currency 1 the rate of strengthening is trivially zero. For that currency we therefore have:

$$\mu_1 = 0\,; \tag{4}$$

$$\sigma_{1,di} = 0\text{; and} \tag{5}$$

$$\sigma_{1,c} = 0\,. \tag{6}$$

Also, for the risk-free asset denominated in currency *c*, the return is deterministic, so we have:

$$\sigma_{c1,dj} = 0\,. \tag{7}$$

The variables defined above relate to returns in a particular currency as measured in that currency and to exchange rates between that currency and currency 1. Now we need to consider the returns to investors who measure their returns in other currencies, for example a currency-*c* investor. For this purpose we define the following:

– X_{di}^{c} is the return in currency *c* on asset *i* issued in currency *d* for $c,d = 1,\ldots,C;\ i = 1,\ldots,n_d$; i.e.:

$$X_{di}^{c} = X_{di} + X_d - X_c\,. \tag{8}$$

Note that subscripts are used to denote the currency in which an asset is issued and the category of that asset, whereas superscripts are used to denote the currency in which an investor measures returns; the former relates to the asset, whereas the latter relates to the investor.

Because we are working with forces of strengthening of currencies, the increases are additive. We may then determine the following:

– μ_{di}^{c} is the expected return in currency *c* on asset *i* issued in currency *d* for $c,d = 1,\ldots,C;\ i = 1,\ldots,n_d$; i.e.:

$$\mu_{di}^{c} = E\left\{X_{di} + X_d - X_c\right\} = \mu_{di} + \mu_d - \mu_c\,. \tag{9}$$

– $\sigma_{di,ej}^{c}$ is the covariance of the return in currency *c* on asset *i* issued in currency *d* with the return in currency *c* on risky asset *j* issued in currency *e*; i.e., from equation (8):

$$\begin{aligned}
\sigma_{di,ej}^{c} &= \operatorname{cov}\left\{X_{di} + X_d - X_c, X_{ej} + X_e - X_c\right\} \\
&= \operatorname{cov}\left\{X_{di,} X_{ej}\right\} + \operatorname{cov}\left\{X_{di}, X_e\right\} - \operatorname{cov}\left\{X_{di}, X_c\right\} \\
&\quad + \operatorname{cov}\left\{X_d X_{ej}\right\} + \operatorname{cov}\left\{X_d, X_e\right\} - \operatorname{cov}\left\{X_d X_c\right\} \\
&\quad - \operatorname{cov}\left\{X_c X_{ej}\right\} - \operatorname{cov}\left\{X_c X_e\right\} + \operatorname{var}\left\{X_c\right\} \\
&= \sigma_{di,ej} + \sigma_{e,di} - \sigma_{c,di} + \sigma_{d,ej} + \sigma_{d,e} - \sigma_{c,d} - \sigma_{c,ej} - \sigma_{c,e} + \sigma_{c,c}\,.
\end{aligned} \tag{10}$$

We refer to μ_{di} and μ_d as the ‘underlying expectations’ and to μ_{di}^{c} as the ‘expected returns to investors’.

Equation (10) is required in order to determine the variance of the return on the portfolio of risky assets to a currency-*c* investor as explained below (cf. equation (15)).

Let p_{di}^{c} denote the value in currency *c* of investments in asset *i* issued in currency *d* held by currency-*c* investors, per unit of the total value in that currency of the assets held by such investors. The value of p_{di}^{c} is unknown; it is estimated through an optimisation process explained below. We now define the portfolio of risky assets held by a currency-*c* investor as:

$$\left\{ p_{di}^{c} \mid (d,i) \in \Psi_c \right\} \tag{11}$$

where:

$\Psi_c = \left\{ (d,i) \mid d \in \{1,\ldots,C\}; i \in \Omega_d^c \right\}$; and

$$\Omega_d^c = \begin{cases} \{2,\ldots,n_d\} \text{ for } d = c; \\ \{1,\ldots,n_d\} \text{ for } d \neq c; \end{cases}.$$

The set Ω_d^c has $n_d + 1$ elements for $d \neq c$ or n_c for $d = c$. This is because, for currency $d \neq c$, the risk-free asset denominated in currency *d* is included (as p_{d1}^{c}) as a risky asset in this portfolio, whereas for currency $d = c$, the risk-free asset denominated in that currency is not included, as it is not a risky asset. By definition, the elements of the set $\left\{ p_{di}^{c} \mid (d,i) \in \Psi_c \right\}$ sum to 1; i.e.:

$$\sum_{(d,i)\in\Psi_c} p_{di}^{c} = 1. \tag{12}$$

Similarly, we define the returns on the risky assets held in currency *d* by a currency-*c* investor as:

$$\left\{ X_{di}^{c} \mid (d,i) \in \Psi_c \right\}; \tag{13}$$

where X_{di}^{c} is the return on risky asset *i* issued in currency *d* measured in currency *c* (equation (8)).

We similarly define the expected return on the risky assets held in currency *d* by a currency-*c* investor as:

$$\left\{ \mu_{di}^{c} \mid (d,i) \in \Psi_c \right\} \tag{14}$$

where μ_{di}^{c} is the expected return on risky asset *i* issued in currency *d* measured in currency *c* (equation (9)).

Also, we define the covariances of the returns on the risky assets held in currency *d* with those on the risky assets held in currency *e* by a currency-*c* investor as:

$$\left\{ \sigma_{di,ej}^{c} \mid (d,i),(e,j) \in \Psi_c \right\} \tag{15}$$

where $\sigma_{di,ej}^{c}$ is the covariance of the returns on risky assets *i* and *j* held in currencies *d* and *e* respectively by a currency-*c* investor (equation (10)).

Now, from the definitions in equations (11) and (13), we may express the return on the portfolio of risky assets held by a currency-*c* investor (i.e. on the 'market portfolio' of currency-*c* investors) as:

$$X_{\mathrm{M}}^{c} = \sum_{(d,i)\in\Psi_c} p_{di}^{c} X_{di}^{c} \,. \tag{16}$$

(We use the subscript M to denote that portfolio.) Similarly, from the definitions in equations (11) and (14), we may express the expected return on the portfolio of risky assets held by a currency-*c* investor as:

$$\mu_{\mathrm{M}}^{c} = E\left\{X_{\mathrm{M}}^{c}\right\} = \sum_{(d,i)\in\Psi_c} p_{di}^{c} \mu_{di}^{c} \,. \tag{17}$$

Also, from equations (11) and (15), we may express the variance of the return on the portfolio of risky assets held by a currency-*c* investor as:

$$\sigma_{\mathrm{M,M}}^{c} = \mathrm{var}\left\{X_{\mathrm{M}}^{c}\right\} = \sum_{(d,i),(e,j)\in\Psi_c} p_{di}^{c} p_{ej}^{c} \sigma_{di,ej}^{c} \,. \tag{18}$$

In terms of the CAPM, currency-*c* investors determine their portfolio of risky assets by maximising

$$k = \frac{\hat{\mu}_{\mathrm{M}}^{c} - r_c}{\sqrt{\hat{\sigma}_{\mathrm{M,M}}^{c}}}; \tag{19}$$

where:

$\hat{\mu}_{\mathrm{M}}^{c}$ is the *ex-ante* estimate of the expected return to a currency-*c* investor on her/his portfolio;

r_c is the return on the risk-free asset issued in currency *c*; and

$\hat{\sigma}_{\mathrm{M,M}}^{c}$ is the *ex-ante* estimate of the variance of the return to a currency-*c* investor on her/his portfolio during quarter *t*.

In practice the value of r_c will be known. For the purposes of this paper a neutral value was used, determined as the sample mean of the return on the risk-free asset in currency *c*:

$$r_c = \hat{\mu}_{c1}^{c} = \hat{\mu}_{c1} \,. \tag{20}$$

In order to avoid short positions in the market portfolio of currency-*c* investors, *k* is maximised subject to the constraints:

$$p_{di}^{c} \geq 0 \text{ for all } (d,i)\in\Psi_c \text{ and for all } c;$$

and, as in equation (12):

$$\sum_{(d,i)\in\Psi_c} p_{di}^{c} = 1.$$

This gives the tangency portfolio, i.e. the portfolio on the efficient frontier in mean–standard-deviation space at which the straight line intersecting the mean axis at r_c is tangential to the efficient frontier. Under the CAPM, the latter line is the capital-market line and the tangency portfolio is the market portfolio. *k*, the Sharpe ratio, is the slope of the capital-market line.

3.2 A necessary condition

Let $\sigma^c_{di,\mathrm{M}}$ be the covariance of the return in currency *c* on asset *i* issued in currency *d* with the return in currency *c* on the market portfolio of a currency-*c* investor, i.e.:

$$\sigma^c_{di,\mathrm{M}} = \mathrm{cov}\left\{X^c_{di}, X^c_{\mathrm{M}}\right\} = \sum_{(e,j)\in\Psi_c} p^c_{ej}\sigma^c_{di,ej}\ ; \tag{21}$$

and, as in equation (18), let $\sigma^c_{\mathrm{M,M}}$ be the variance of the return in currency *c* on the market portfolio of a currency-*c* investor, i.e.:

$$\sigma^c_{\mathrm{M,M}} = \mathrm{var}\left\{X^c_{\mathrm{M}}\right\} = \sum_{(d,i),(e,j)\in\Psi_c} p^c_{di} p^c_{ej}\sigma^c_{di,ej}\ . \tag{22}$$

Now let:

$$\kappa^c_{di} = \frac{\sigma^c_{di,\mathrm{M}} - \sigma^c_{d1,\mathrm{M}}}{\sigma^c_{\mathrm{M,M}}}\left(\mu^c_{\mathrm{M}} - r_c\right); \tag{23}$$

where r_c is the risk-free rate in currency *c*.

Theorem *If the SFM-CAPM applies in a multi-currency world then, for any currencies c and e:*

$$\kappa^c_{di} = \kappa^e_{di}\ . \tag{24}$$

Proof

Since the CAPM applies for investors in each currency (assumption (1)), it follows that, for asset *i* issued in currency *d*, the expected return in currency *c* is:

$$\mu^c_{di} = r_c + \frac{\sigma^c_{di,\mathrm{M}}}{\sigma^c_{\mathrm{M,M}}}\left(\mu^c_{\mathrm{M}} - r_c\right)\ . \tag{25}$$

Similarly:

$$\mu^e_{di} = r_e + \frac{\sigma^e_{di,\mathrm{M}}}{\sigma^e_{\mathrm{M,M}}}\left(\mu^e_{\mathrm{M}} - r_e\right). \tag{26}$$

From equation (9) we have:

$$\mu^c_{di} = \mu_{di} + \mu_d - \mu_c\ . \tag{27}$$

Similarly:

$$\mu^e_{di} = \mu_{di} + \mu_d - \mu_e\ . \tag{28}$$

Making μ_{di} the subject of equation (28) we have:

$$\mu_{di} = \mu^e_{di} - \mu_d + \mu_e\ . \tag{29}$$

Substituting equation (29) into equation (27) we obtain:

$$\mu^c_{di} = \left(\mu^e_{di} - \mu_d + \mu_e\right) + \mu_d - \mu_c\,;$$

i.e.:

$$\mu^c_{di} + \mu_c = \mu^e_{di} + \mu_e\ . \tag{30}$$

Now we substitute equations (25) and (26) into equation (30) to give:

$$\left\{r_c + \frac{\sigma^c_{di,\mathrm{M}}}{\sigma^c_{\mathrm{M,M}}}\left(\mu^c_{\mathrm{M}} - r_c\right)\right\} + \mu_c = \left\{r_e + \frac{\sigma^e_{di,\mathrm{M}}}{\sigma^e_{\mathrm{M,M}}}\left(\mu^e_{\mathrm{M}} - r_e\right)\right\} + \mu_e\,;$$

i.e.:

$$r_c + \mu_c + \frac{\sigma^c_{di,\mathrm{M}}}{\sigma^c_{\mathrm{M,M}}}\left(\mu^c_{\mathrm{M}} - r_c\right) = r_e + \mu_e + \frac{\sigma^e_{di,\mathrm{M}}}{\sigma^e_{\mathrm{M,M}}}\left(\mu^e_{\mathrm{M}} - r_e\right). \tag{31}$$

From equations (27) and (28) we have, for i = 1:

$$\mu^c_{d1} = r_d + \mu_d - \mu_c\,;\ \text{and} \tag{32}$$

$$\mu^e_{d1} = r_d + \mu_d - \mu_e\,. \tag{33}$$

And from equations (25) and (26) we have, for i = 1:

$$\mu^c_{d1} = r_c + \frac{\sigma^c_{d1,\mathrm{M}}}{\sigma^c_{\mathrm{M,M}}}\left(\mu^c_{\mathrm{M}} - r_c\right)\ ;\ \text{and} \tag{34}$$

$$\mu^e_{d1} = r_e + \frac{\sigma^e_{d1,\mathrm{M}}}{\sigma^e_{\mathrm{M,M}}}\left(\mu^e_{\mathrm{M}} - r_e\right). \tag{35}$$

From equations (32) and (34) we have:

$$r_d + \mu_d - \mu_c = r_c + \frac{\sigma^c_{d1,\mathrm{M}}}{\sigma^c_{\mathrm{M,M}}}\left(\mu^c_{\mathrm{M}} - r_c\right);$$

i.e.:

$$r_c + \mu_c = r_d + \mu_d - \frac{\sigma^c_{d1,\mathrm{M}}}{\sigma^c_{\mathrm{M,M}}}\left(\mu^c_{\mathrm{M}} - r_c\right). \tag{36}$$

and similarly from equations (33) and (35) we have:

$$r_e + \mu_e = r_d + \mu_d - \frac{\sigma^e_{d1,\mathrm{M}}}{\sigma^e_{\mathrm{M,M}}}\left(\mu^e_{\mathrm{M}} - r_e\right). \tag{37}$$

Substituting (36) and (37) into (31) we obtain:

$$\left\{r_d + \mu_d - \frac{\sigma^c_{d1,\mathrm{M}}}{\sigma^c_{\mathrm{M,M}}}\left(\mu^c_{\mathrm{M}} - r_c\right)\right\} + \frac{\sigma^c_{di,\mathrm{M}}}{\sigma^c_{\mathrm{M,M}}}\left(\mu^c_{\mathrm{M}} - r_c\right)$$
$$= \left\{r_d + \mu_d - \frac{\sigma^e_{d1,\mathrm{M}}}{\sigma^e_{\mathrm{M,M}}}\left(\mu^e_{\mathrm{M}} - r_e\right)\right\} + \frac{\sigma^e_{di,\mathrm{M}}}{\sigma^e_{\mathrm{M,M}}}\left(\mu^e_{\mathrm{M}} - r_e\right);$$

i.e.:

$$\kappa^c_{di} = \frac{\sigma^c_{di,\mathrm{M}} - \sigma^c_{d1,\mathrm{M}}}{\sigma^c_{\mathrm{M,M}}}\left(\mu^c_{\mathrm{M}} - r_c\right) = \frac{\sigma^e_{di,\mathrm{M}} - \sigma^e_{d1,\mathrm{M}}}{\sigma^e_{\mathrm{M,M}}}\left(\mu^e_{\mathrm{M}} - r_e\right) = \kappa^e_{di}\,. \tag{38}$$

□

3.3 Formulation of the SFM-CAPM

Suppose that the sample values $\hat{\sigma}_{ci,dj}$ and $\hat{\sigma}_{c,di}$ are unbiased estimates of the *ex-ante* values of $\sigma_{ci,dj}$ and $\sigma_{c,di}$, both for the GCAPM and for the SFM-CAPM. For the GCAPM, suppose that the sample values $\hat{\mu}_{ci}$ and $\hat{\mu}_c$ are unbiased estimates of the *ex-ante* underlying expectations, but that for the SFM-CAPM they are not. Let $\hat{\mu}^{(\mathrm{G})}_{ci}$ and

$\hat{\mu}_c^{(\mathrm{G})}$ denote the sample values of the underlying expectations. Let $\mu_{ci}^{(\mathrm{S})}$ and $\mu_c^{(\mathrm{S})}$ denote the *ex-ante* values of the underlying expectations on the SFM-CAPM.

In principle we could determine $\hat{\mu}_{ci}^{(\mathrm{S})}$ and $\hat{\mu}_c^{(\mathrm{S})}$ so as to minimise:

$$D_\mu^2 = \frac{1}{Q_\mu}\left[\sum_{c=1}^{C}\left\{\sum_{i=2}^{n_c}\left(\hat{\mu}_{ci}^{(\mathrm{S})} - \hat{\mu}_{ci}^{(\mathrm{G})}\right)^2\right\} + \sum_{c=2}^{C}\left(\hat{\mu}_c^{(\mathrm{S})} - \hat{\mu}_c^{(\mathrm{G})}\right)^2\right] \tag{39}$$

where Q_μ is the number of terms in the summand, subject to the constraints:

$$\kappa_{di}^c = \frac{\hat{\sigma}_{di,\mathrm{M}}^c - \hat{\sigma}_{d1,\mathrm{M}}^c}{\hat{\sigma}_{\mathrm{M,M}}^c}\left(\hat{\mu}_{\mathrm{M}}^{c(\mathrm{S})} - r_c\right) = \frac{\hat{\sigma}_{di,\mathrm{M}}^e - \hat{\sigma}_{d1,\mathrm{M}}^e}{\hat{\sigma}_{\mathrm{M,M}}^e}\left(\hat{\mu}_{\mathrm{M}}^{e(\mathrm{S})} - r_e\right) = \kappa_{di}^e \text{ (equation (38)).}$$

The problem with this approach is that we have more constraints than unknowns.

Instead we effect a compromise between the GCAPM and the SFM-CAPM. Instead of treating the SFM-CAPM condition as a strict constraint, we can treat it as part of the objective by minimising:

$$D^2 = D_\mu^2 + hD_\kappa^2; \tag{40}$$

where:

$$D_\mu^2 = \frac{1}{Q_\mu}\left[\sum_{c=1}^{C}\left\{\sum_{i=2}^{n_c}\left(\hat{\mu}_{ci}^{(\mathrm{S})} - \hat{\mu}_{ci}^{(\mathrm{G})}\right)^2\right\} + \sum_{c=2}^{C}\left(\hat{\mu}_c^{(\mathrm{S})} - \hat{\mu}_c^{(\mathrm{G})}\right)^2\right] \text{ (equation (39));}$$

$$D_\kappa^2 = \frac{1}{Q_\kappa}\sum_{c,e=1}^{C}\sum_{(d,i)\in\Psi_c}\sum_{(f,j)\in\Psi_e}\left(\kappa_{di}^c - \kappa_{fj}^e\right)^2; \tag{41}$$

$$\kappa_{di}^f = \frac{\hat{\sigma}_{di,\mathrm{M}}^f - \hat{\sigma}_{d1,\mathrm{M}}^f}{\hat{\sigma}_{\mathrm{M,M}}^f}\left(\mu_{\mathrm{M}}^{f(\mathrm{S})} - r_f\right) \text{ (equation (23));} \tag{42}$$

Q_μ and Q_κ are the numbers of terms in the respective summands; and

h is a penalty coefficient.

This means that, whilst D_κ^2 will not generally be zero (which would be the case under the strict constraint) it can be reduced to an arbitrarily small value by increasing the penalty coefficient *h*. The estimates $\hat{\mu}_{ci}^{(\mathrm{S})}$ and $\hat{\mu}_c^{(\mathrm{S})}$ of the *ex-ante* underlying expectations will depend on *h*, as will the betas and the optimal portfolio. Bayesian credibility theory could be used to determine *h*. For $h = 0$ the model reduces to the GCAPM as the constraints are not applied.

In terms of equation (42) κ_{di}^f is a function of μ_{M}^f. From equation (17) we have:

$$\mu_{\mathrm{M}}^f = \sum_{(c,j)\in\Psi_f} p_{cj}^f \mu_{cj}^f .$$

This means that κ_{di}^f is a function both of $\left\{p_{cj}^f \mid (c,j)\in\Psi_f\right\}$ and of $\left\{\mu_{cj}^f \mid (c,j)\in\Psi_f\right\}$.

Now each element of $\left\{p_{cj}^f \mid (c,j)\in\Psi_f\right\}$ is also a function of the underlying expectations. This is because the former, being a currency-*f* investor's optimal exposure to a particular asset, is dependent on the latter. It involves finding that investor's market portfolio as explained in section 3.1. What we therefore need to do is to find the values of the underlying expectations that minimise D^2.

First, for each currency c, the GCAPM underlying expectations $\hat{\mu}_{ci}^{(\mathrm{G})}$ and $\hat{\mu}_{c}^{(\mathrm{G})}$, the optimal market portfolios p_{cj}^{f} and the betas are determined for each investor currency. Using these underlying expectations as initial values, we then calculate the SFM-CAPM underlying expectations $\hat{\mu}_{ci}^{(\mathrm{S})}$ and $\hat{\mu}_{c}^{(\mathrm{S})}$—and hence, for a currency-f investor:

- the expected returns on the assets available $\left\{\hat{\mu}_{cj}^{f} \mid (c,j) \in \Psi_f\right\}$;
- the tangency portfolio $\left\{p_{cj}^{f} \mid (c,j) \in \Psi_f\right\}$;
- the expected returns to investors $\hat{\mu}_{\mathrm{M}}^{f(\mathrm{S})}$ on the tangency portfolio; and
- the generalised market risk premium κ_{di}^{f};

so as to minimise D. We can then also calculate the betas $\left\{\beta_{cj}^{f} \mid (c,j) \in \Psi_f\right\}$ such that:

$$\beta_{cj}^{f} = \frac{\sigma_{cj,\mathrm{M}}^{f}}{\sigma_{\mathrm{M,M}}^{f}}. \tag{43}$$

3.4 Local optima

In applications of the method described above it was found that, for some values of h, the optimum value found by minimising D^2 was merely a local optimum, which did not conform to the theoretical requirements.

In the first place, the optimum value found should be independent of the initial value used for the iteration process followed in the optimisation function. It was found that this requirement was not invariably satisfied. For this reason, optimal values were found for a range of values of h. This range started with $h = 0$, which gives the GCAPM global optimum, $\left(\hat{\mu}_{ci}^{(\mathrm{G})}, \hat{\mu}_{c}^{(\mathrm{G})}\right)$ being the *ex-post* sample values of the underlying expectations. For each subsequent value of h two values of $\left(\hat{\mu}_{ci}^{(h)}, \hat{\mu}_{c}^{(h)}\right)$—and hence of $\left(\left(D_\mu^2\right)^{(h)}, \left(D_\kappa^2\right)^{(h)}\right)$—were calculated, the first using $\left(\hat{\mu}_{ci}^{(\mathrm{G})}, \hat{\mu}_{c}^{(\mathrm{G})}\right)$ as initial values and the second using the SFM-CAPM values $\left(\hat{\mu}_{ci}^{(h-)}, \hat{\mu}_{c}^{(h-)}\right)$ found for h–, the previous value of h. The results that gave the lower value of D^2 were selected.

It was also found that, for different optimisation methods, different results were obtained. Again, this is due to different local optima. Again, the results that gave the lower value of D^2 were selected.

In theory, it may be shown that the locus of the solution in D_μ^2-D_κ^2 space should describe a monotonically decreasing function as h increases. For each pair of points $\left(\left(D_\mu^2\right)^{(h(1))}, \left(D_\kappa^2\right)^{(h(1))}\right)$ and $\left(\left(D_\mu^2\right)^{(h(2))}, \left(D_\kappa^2\right)^{(h(2))}\right)$ a check was made that they were monotonically decreasing. If that check failed, it was accepted that no global minimum of D^2 could be found for $h = h(2)$ and that value of h was ignored.

Not only should the locus of the solution in D_μ^2-D_κ^2 space describe a monotonically decreasing function, it should also describe a convex function. For each pair of points $\left(\left(D_{\mu}^{2}\right)^{(h(1))},\left(D_{\kappa}^{2}\right)^{(h(1))}\right)$ and $\left(\left(D_{\mu}^{2}\right)^{(h(3))},\left(D_{\kappa}^{2}\right)^{(h(3))}\right)$ a check was therefore made for the convexity of the values of $\left(\left(D_{\mu}^{2}\right)^{(h(1))},\left(D_{\kappa}^{2}\right)^{(h(1))}\right)$, $\left(\left(D_{\mu}^{2}\right)^{(h(2))},\left(D_{\kappa}^{2}\right)^{(h(2))}\right)$ and $\left(\left(D_{\mu}^{2}\right)^{(h(3))},\left(D_{\kappa}^{2}\right)^{(h(3))}\right)$ for $h(1)<h(2)<h(3)$. If that check failed, it was accepted that no global minimum of D^2 could be found for $h=h(2)$ and that value of *h* was ignored.

It is clear from the process described above that practitioners will not be able to predetermine a value of *h* and merely solve for that value. Instead, in order to avoid merely local optima, they will need to solve for a range of values of *h*. and then select a value of *h* that has not been rejected.

The resulting locus of the solution in D_{μ}^{2}-D_{κ}^{2} space will describe a monotonically decreasing, convex function as *h* increases. Whilst there is no guarantee that the resulting values of D^2 will be global minima, they are not obviously merely local minima. Practitioners may wish to explore the possibility of lower minima using global optimisation methods, but a comprehensive discussion of the application of such methods is considered to be beyond the scope of this paper.

3.5 A particular case

If strict purchasing-power parity holds then, in real terms (or if there is no inflation), the SFM-CAPM reduces to the GCAPM. In this case:

$$\sigma_{c,\mathrm{M}}^{c}=0;$$

and, for all currencies *c* and *e*:

$$\beta_{di}^{c}=\beta_{di}^{e}.$$

Equation (2) then applies regardless of the currency in which the investor measures returns.

4. Application

For illustrative purposes the method outlined in sections 3.3 and 3.4 was applied to a selection of currencies. In section 4.1 an overview of the data is given. The results are presented in section 4.2.

4.1 Data

The currencies selected are:

$$c=\begin{cases}1 \text{ for USA dollars;}\\ 2 \text{ for UK pounds;}\\ 3 \text{ for South African rands; and}\\ 4 \text{ for Turkish lira;}\end{cases} \tag{44}$$

so that C = 4. The selection is influenced by the authors' interests and the data most easily available to them, but the application is intended to be illustrative; care has been taken to specify the data sources and calculations so as to facilitate consistent applications to other currencies.

Quarterly forces of return and of strengthening of currencies were used. The methods were applied both to real returns and to nominal returns. The real returns and rates of strengthening of currencies were calculated as:

$$\tilde{x}_{ci}(t) = x_{ci}(t) - \theta_c(t); \text{ and} \quad (45)$$

$$\tilde{x}_c(t) = x_c(t) - \theta_c(t) \quad (46)$$

respectively; where:

$x_{ci}(t)$ is the return in currency *c* on asset *i* issued in that currency during quarter *t*;

$x_c(t)$ is the rate of strengthening currency *c* during that quarter; and

$\theta_c(t)$ is the force of inflation in currency *c* during quarter *t*.

Because we are working with forces, the relationships in equations (45) and (46) are linear.

The risky assets covered by the application comprised equities, and conventional and index-linked bonds (inflation-protected government bonds) of selected maturities. For nominal returns the risk-free asset was the conventional bond with a maturity of one quarter and for real returns it was the corresponding index-linked bond.

The period covered by the application was from 1975Q2 to 2012Q1. The calculation of $x_{ci}(t)$, $x_c(t)$ and $\theta_c(t)$ from the data available[1] entailed intensive work, some of which relied on assumptions and estimations by the authors. These calculations are dealt with in detail in a note entitled "How a single-factor CAPM works in a multi-currency world: information on the determination of data," which, together with a spreadsheet file showing the determination of the data required, is available free of charge from the authors. As explained in that note, data are not available for every series throughout the period and in some cases the data were not acceptable for the purposes of this paper. The quarters from which acceptable returns (or, in the case of inflation and exchange rates, acceptable forces of inflation and forces of strengthening of currencies respectively) could be calculated from the data available for the respective series are shown in Table 1.

The type of bond used is also shown in Table 1: 'coupon' denotes coupon-paying bonds and 'ZCBs' denotes zero-coupon bonds. These are not necessarily the type of bond in issue; where possible, returns were determined for zero-coupon bonds in order to avoid inaccuracies relating to the amount of the coupon on different bonds used for the determination of yield-curve data available. For each series of bonds, two terms to redemption were chosen: the short term of one quarter and a long term depending on the data available. The column headed 'long term (qtrs.)' shows the term to redemption chosen as the long term (in quarters). In Table 1, 'SA' refers to South Africa and 'TR' to Turkey.

Table 1. Periods for which acceptable returns could be calculated

[1] Sources: USA Federal Reserve Bank; Bureau of Labor Statistics, U.S. Department of Labor; Bank of England; INet; Central Bank of the Republic of Turkey; Republic of Turkey Prime Ministry, Undersecretariat of Treasury, Public Finance; Istanbul Stock Exchange

Currency		Series	Bonds		Available from
no.	country		type	long term (qtrs.)	
1	USA	conventional bonds	coupon	80	1975Q2
		index-linked bonds	ZCBs	40	2003Q2
		equities			1975Q2
		inflation rates			1975Q2
2	UK	conventional zero-coupon bonds	ZCBs	64	1975Q2
		index-linked bonds	ZCBs	64	1985Q2
		equities			1975Q2
		exchange rates			1975Q2
		inflation rates			1975Q2
3	SA	conventional bonds	coupon	80	1975Q2
		index-linked bonds	ZCBs	40	2005Q3
		equities			1975Q2
		exchange rates			1975Q2
		inflation rates			1975Q2
4	TR	conventional bonds	ZCBs	8	1985Q3
		index-linked bonds	ZCBs	40	2009Q4
		equities			1986Q1
		exchange rates			1975Q2
		inflation rates			1982Q2

In the light of the information in Table 1 it was decided to use various datasets for nominal returns and various datasets (not necessarily the same) for real returns. These datasets, comprising the periods, and the assets included in them, are shown in Table 2. In that table, 'e, cb' means equities and conventional bonds and 'ilb' means index-linked bonds.

Some of the periods in Table 2 are too short for the estimation of reliable parameters; they are included for the sake of inclusivity and to indicate how they may affect the results. On the other hand, it must be recognised that means and covariances may change over time, so the use of excessively long periods is inappropriate. However, long periods have been included for the purposes of illustration.

Table 2. Periods used

Data set	Period		USA		UK		SA		TR	
			e, cb	ilb	e, cb	ilb	e, cb	ilb	e, cb	ilb
nominal returns										
1	1975Q2	1985Q4	√		√		√			
2	1986Q1	1995Q4	√		√		√		√	
3	1996Q1	2005Q2	√		√		√		√	
4	2005Q3	2012Q1	√		√		√		√	
5	2005Q3	2012Q1	√	√	√	√	√	√		
6	2009Q4	2012Q1	√	√					√	√
7	1975Q2	2012Q1	√		√		√			

8	1986Q1	2012Q1	√		√		√		√	
real returns										
1	2003Q3	2009Q3	√	√	√	√				
2	2005Q3	2009Q3	√	√	√	√	√	√		
3	2009Q4	2012Q1	√	√					√	√
4	2005Q3	2012Q1	√	√	√	√	√	√		
5	2003Q3	2012Q1	√	√	√	√				

4.2 Results

For nominal and real returns, and for each dataset listed in Table 2, optimal ex-ante expected values and the corresponding portfolios were determined as explained in section 3 and values of the GCAPM and SFM-CAPM betas were calculated, for a range of values of *h*. The results of these calculations are set out in this section.

4.2.1 Analysis of results in D_μ - D_κ space

Figures 1 shows, for nominal and real returns respectively, and for the datasets as enumerated in the legend, the values of D_μ and D_κ for optimal values of D^2. On the horizontal axis D_μ shows the root-mean-square average of the differences between the SFM-CAPM and GCAPM expectations. On the vertical axis D_κ shows the root-mean-square average of the differences of the generalised market risk premiums between currencies. The intercept on the vertical axis shows the value of D_κ for $D_\mu = 0$; i.e. the GCAPM value of D_κ.

As explained in section 3.4, the locus of $\left(\left(D_\mu^2\right)^{(h)}, \left(D_\kappa^2\right)^{(h)}\right)$ describes a monotonically decreasing function as *h* increases. It may be observed that, for dataset 4 for example, the locus is not convex. This is because, whilst the figure is presented in $D_\mu^{(h)}$ - $D_\kappa^{(h)}$ space, convexity is required in $\left(D_\mu^{(h)}\right)^2$ - $\left(D_\kappa^{(h)}\right)^2$ space, since the objective function is expressed in terms of the latter; in that space all the loci are convex. Because of the rejection of local optima, some of the lines between one value and the next are quite long. (The lines themselves do not represent valid values; they merely connect the points at which successive values of *h* produce accepted results. These connections are important because they show that the locus does not double back on itself as *h* increases.)

The figure shows that, for nominal dataset 3, the value of D_κ for *h* = 0 (i.e. for the GCAPM) is 0.034. This means that the GCAPM results are far from the SFM-CAPM results. Nevertheless, it decreases quite rapidly as D_μ increases. In fact, for all datasets, D_κ ultimately becomes less than D_μ. However, whilst for some datasets it reaches below 0.001, in others it was not possible to find such low values. In practice it is not possible to obtain arbitrarily low values of D_κ; eventually D_μ—and therefore the level of the underlying expectations—becomes irrelevant.

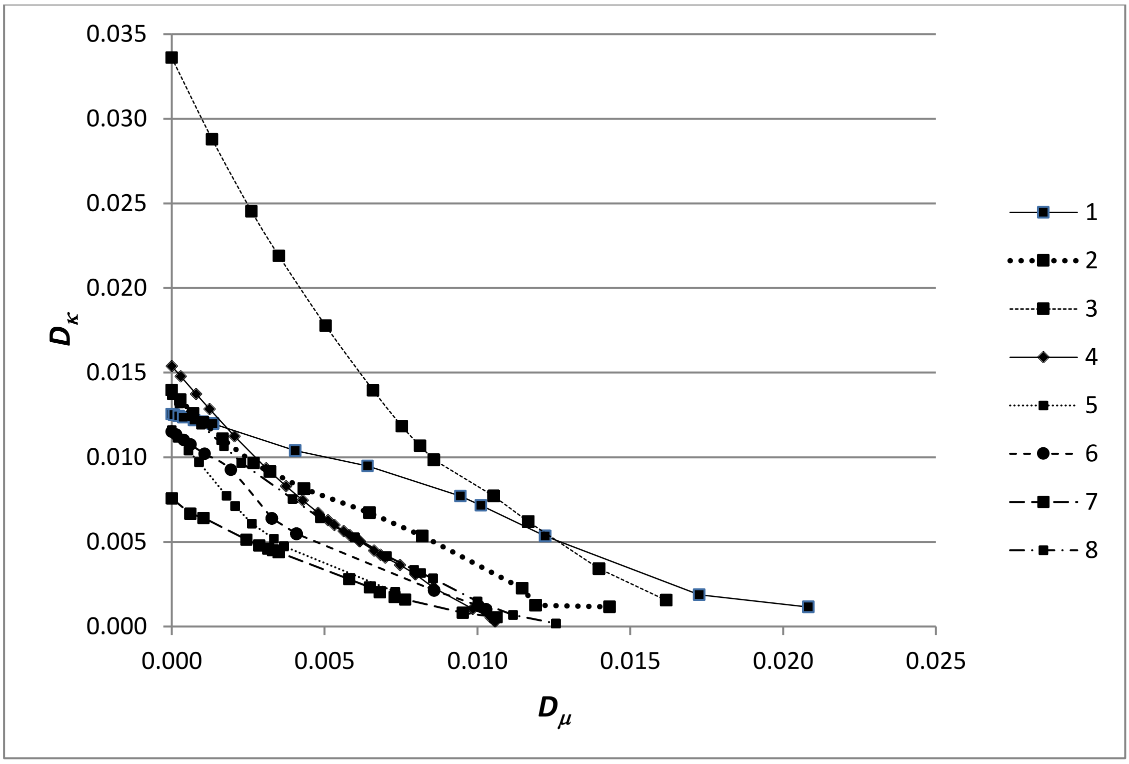

FIGURE 1: D_μ and D_κ for optimal values of D^2: nominal returns

Figure 2 shows the corresponding results for real returns. Similar effects were found.

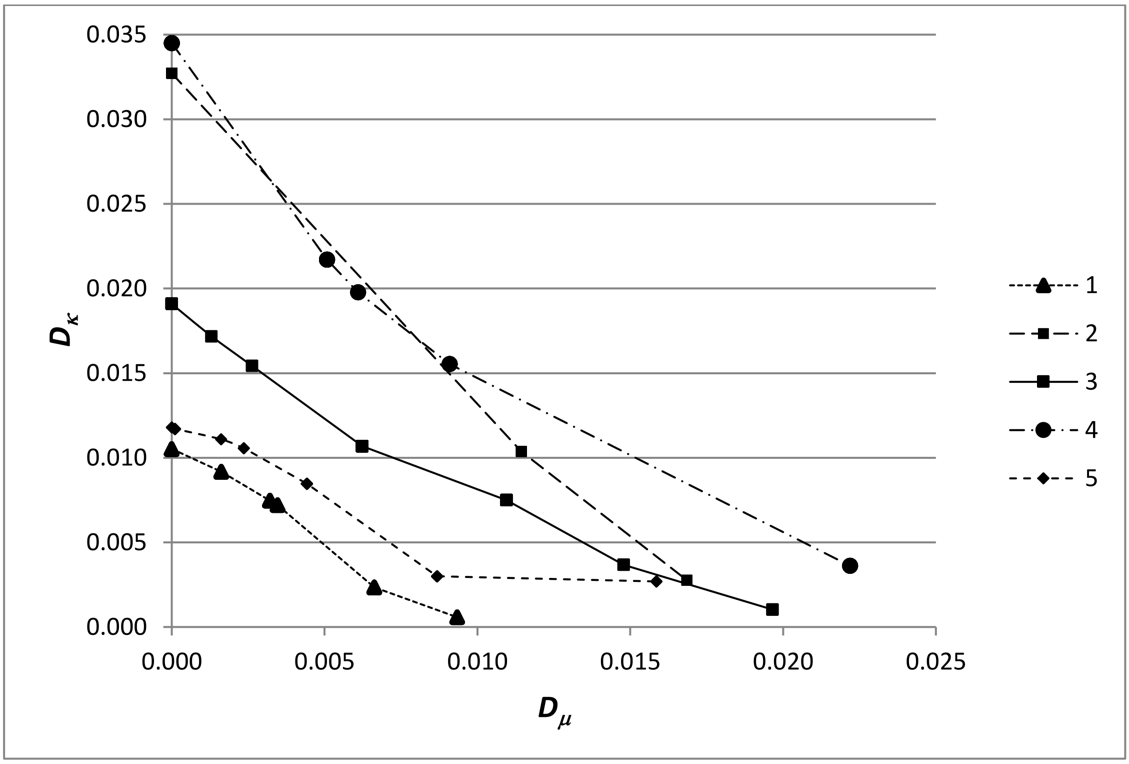

FIGURE 2. D_μ and D_κ for optimal values of D^2: real returns

4.2.2 Underlying expectations

Figure 3 shows, for nominal returns, the values of the SFM-CAPM underlying expectations $\hat{\mu}_{di}^{(S)}$ and $\hat{\mu}_{d}^{(S)}$ against the GCAPM underlying expectations $\hat{\mu}_{di}^{(G)}$ and $\hat{\mu}_{d}^{(G)}$. For the purposes of this and subsequent figures, values of *h* have been selected from the results so as to show the effects of different penalty coefficients on the optimal values of the variables concerned. For nominal returns the values were selected from the results shown in Figure 1 to give $D_\mu \approx 0.005, 0.010, 0.015$, representing low, medium and high levels of departure of the *ex-ante* estimates of the expected values from the *ex-post* estimates, i.e. low, medium and high levels of credibility of the SFM-CAPM. These values are referred to as 'low D.mu', 'medium D.mu' and 'high D.mu' respectively. They may be compared with the line referred to as 'zero D.mu', which represents the GCAPM value.

As observed in the preceding section, not all datasets have values of D_μ reaching 0.015, and medium and high levels are omitted where necessary.

The SFM-CAPM values are largely clustered around the GCAPM (zero D.mu) line. Nevertheless, in relation to investment-management decision-making the differences are material. As might be expected, the relatively few values for $D_\mu \approx 0.015$ (high D.mu) frequently appear outside of the rest. The values are more-or-less clustered into three groups: the higher group is for expected returns on assets issued in Turkish lira, whilst the lower group is for negative expected strengthening of the Turkish lira. The high nominal returns on Turkish assets (both bonds and equities) are offset by the expected weakening of that currency.

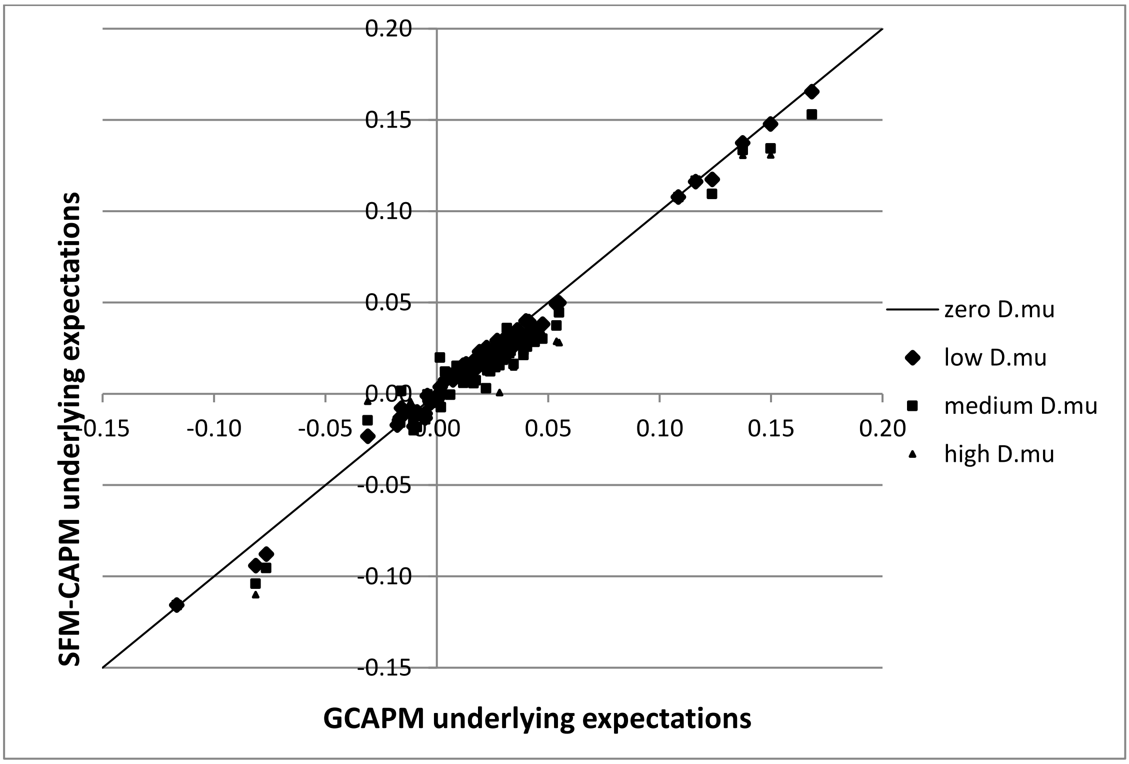

FIGURE 3: Underlying expectations: nominal returns

Because of the large number of points in Figure 3, some of the detail is lost. By way of illustration, Figure 4 gives the same information for nominal-returns dataset 1 only. In that figure, for the sake of clarity, a larger scale has been used. Here it may be seen that, for each value of GCAPM underlying expectations, there are three values of SFM-CAPM underlying expectations, the low D.mu value being the closest to the GCAPM and the high D.mu value the furthest.

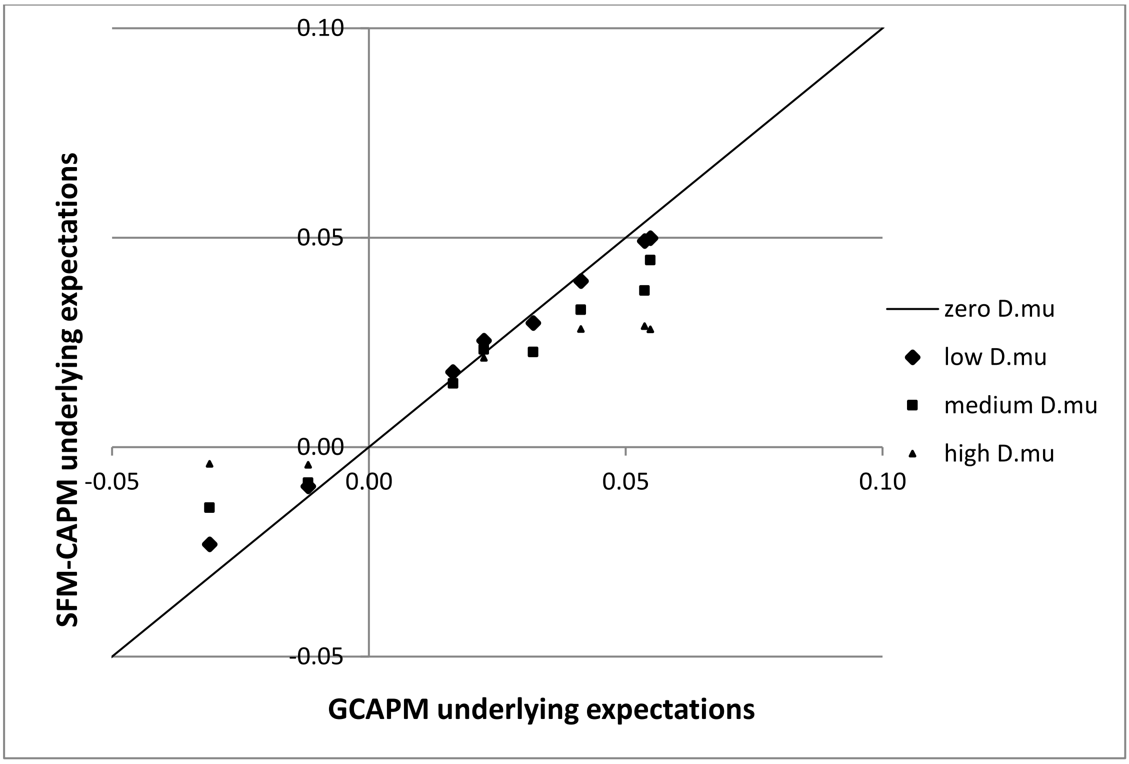

FIGURE 4: Underlying expectations: nominal returns dataset 1

Figure 5 gives, for real returns, information corresponding to Figure 3. For the sake of comparability the same scale has been used. Here the high and low groups do not occur; high expected returns on Turkish assets and expected weakening of the Turkish lira are offset by high inflation. For high values of *h* (and therefore of D_μ) expected real returns and expected real rates of strengthening of currencies under the SFM-CAPM may be negative, even for positive GCAPM values.

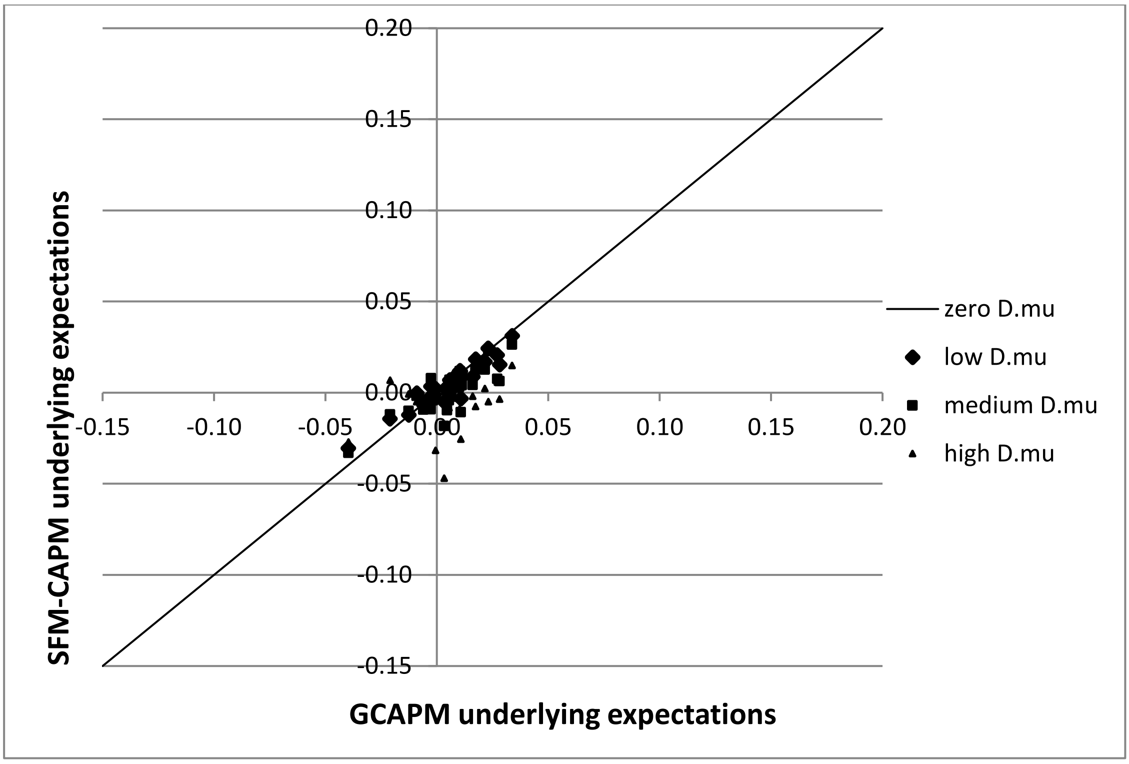

FIGURE 5: Underlying expectations: real returns

4.2.3 Expected returns to investors

Figure 6 shows, for nominal returns, the values of $\hat{\mu}_{di}^{c}$, i.e. the expected returns to investors. In comparison with Figure 3, two features are noteworthy: first, the SFM-CAPM values are more dispersed around the GCAPM values and secondly, the negative SFM-CAPM values are substantially less material than those of underlying expectations. This is because:

$$\mu_{di}^{c} = \mu_{di} + \mu_{d} - \mu_{c} \text{ (equation (9)).}$$

Firstly, to the extent that μ_c and μ_d vary independently of each other, the variability of μ_{di}^{c} will be greater than that of μ_{di}. And secondly, if $\mu_d - \mu_c >> 0$ then $\mu_{di}^{c} >> \mu_{di}$ and vice versa.

Here the points are more-or-less segregated into two groups: the upper group represents Turkish investors, for whom high nominal expected returns are not offset either by the weakening lira or by inflation. The lower group displays some curvature; high *ex-post* real returns as reflected in the GCAPM are not necessarily reflected in high *ex-ante* real returns as reflected in the SFM-CAPM.

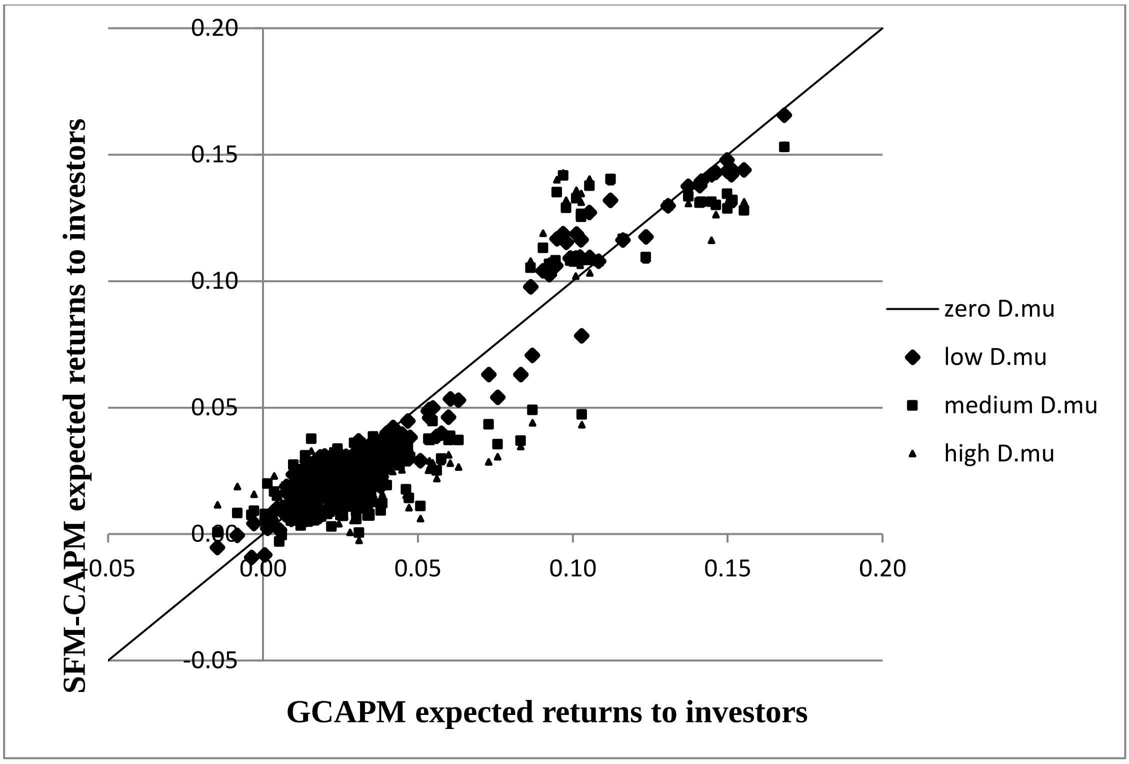

FIGURE 6: Expected returns to investors: nominal returns

Figure 7 gives, for real returns, information corresponding to Figure 6. Here the negative values are more evident. This is because, being net of inflation, they are generally lower. Also, the range of values is lower. This is because, when expected returns are high, they may to a large extent be offset by high expected rates of inflation. Here the curvature of the results is more noticeable.

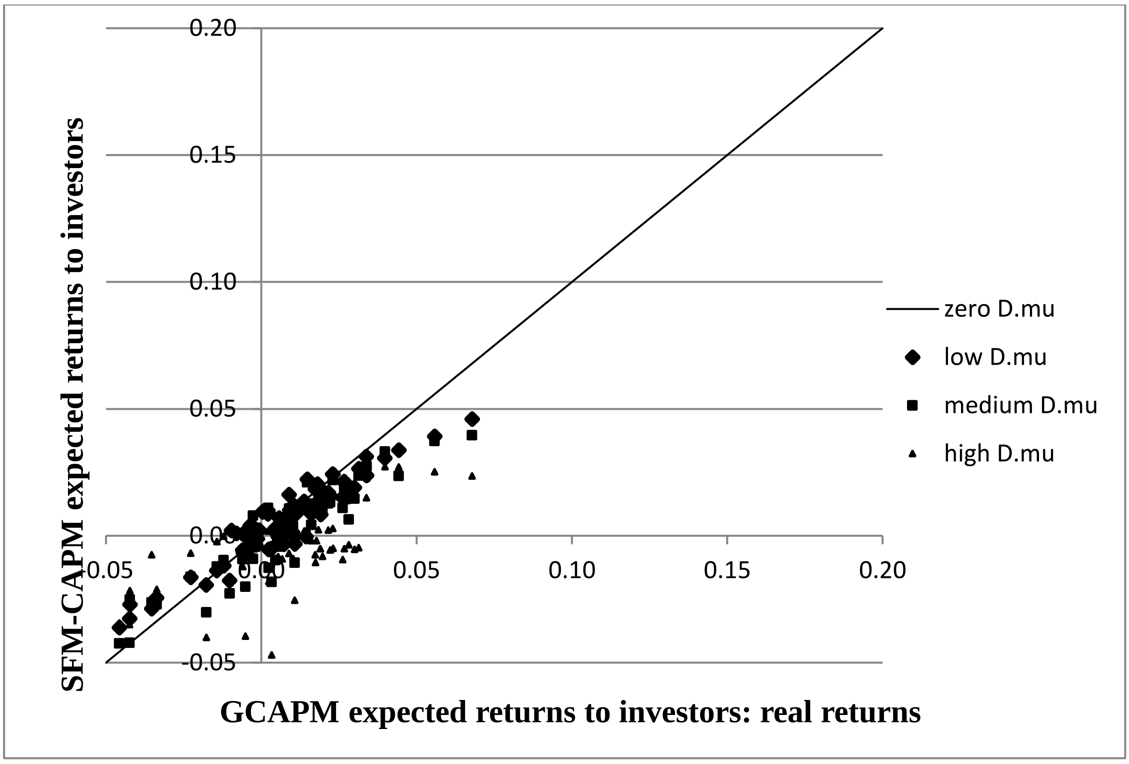

FIGURE 7: Expected returns to investors: real returns

4.2.4 Optimal Portfolios

Figures 8 to 10 show, for nominal returns, the values of $p_{di}^{c(S)}$, i.e. the optimal portfolio weightings of a currency-*c* investor in asset *i* issued in currency *d*. Figure 8 shows investments in the investor's home currency, Figure 9 shows investments by USA and UK investors in assets issued in South Africa and Turkey. Figure 10 shows other investments.

There are numerous points at the origin representing zero exposure under both models. In each figure there are relatively few points in the upper right half of the chart; in general, the higher the exposure to an asset under the GCAPM, the less likely it is to attract high exposure under the SFM-CAPM, and conversely, the higher the exposure to an asset under the SFM-CAPM, the less likely it is to attract high exposure under the GCAPM.

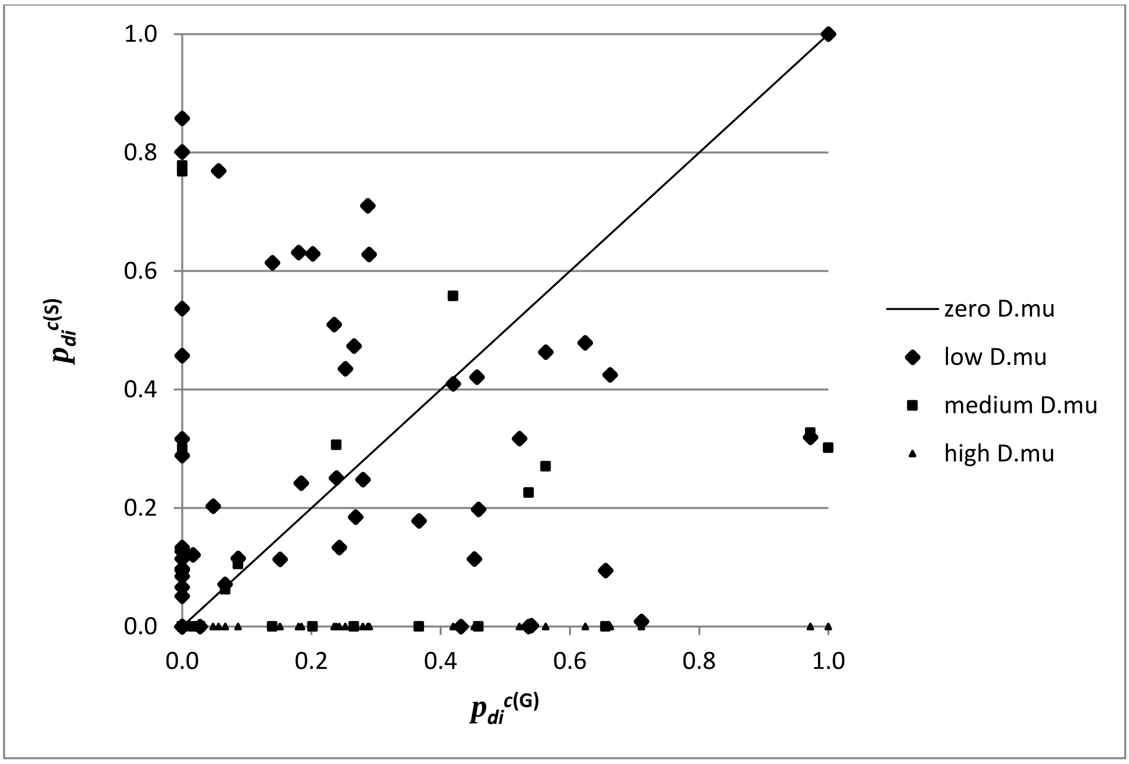

FIGURE 8: $p_{di}^{c(S)}$: nominal returns: investments in home currency

The sheer size of the major currencies makes it impossible that substantial proportions of their investments are in assets of minor currencies. It should be recognised that the optimisation of the portfolio choice makes no allowance for home bias. However, apart from two triplets of outliers, Figure 9 shows that the exposure of USA and UK investors to smaller currencies is relatively low, particularly under the SFM-CAPM. A higher proportion of the points are at the origin than for home investment. These results suggest that home bias is better justified under the SFM-CAPM than under the GCAPM and that, for assets in demand by USA and UK investors, the smaller supply of assets in the South African and Turkish markets results in higher prices, thus reducing the holdings of those investors. Nevertheless, some of the points represent exposures that would be unattainable in practice. In practice, it would be necessary to limit exposures to attainable proportions. The results of this research must be qualified by the understanding that they represented the results that would have obtained if the major-currency investors could have invested substantial proportions of their assets in minor currencies.

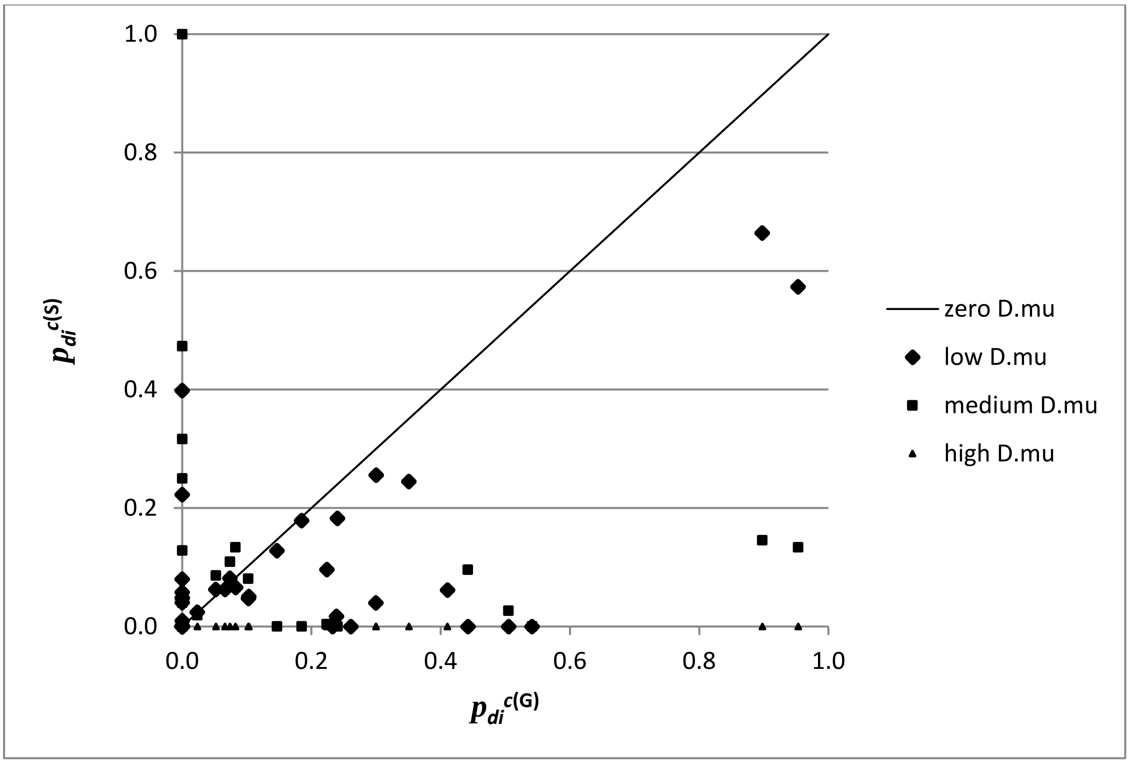

FIGURE 9: $p_{di}^{c(S)}$: nominal returns: investments in smaller currencies

As might be expected, Figure 10 shows results intermediate between those of Figures 8 and 9.

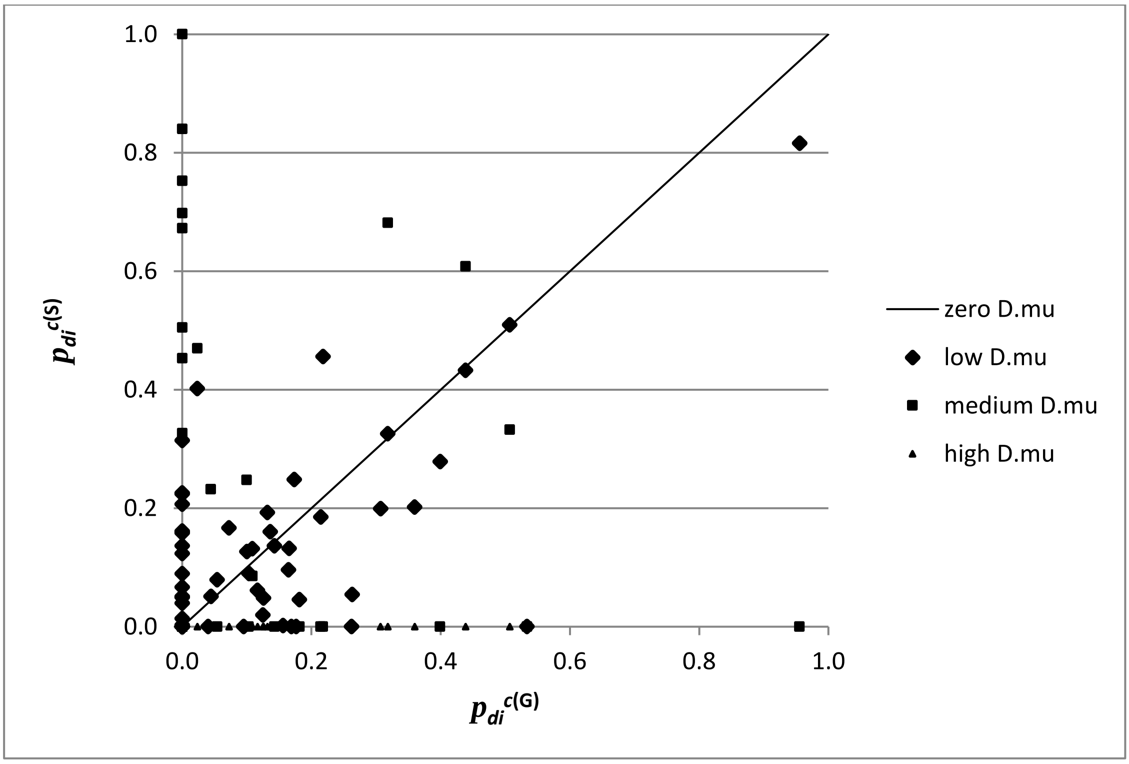

FIGURE 10: $p_{di}^{c(S)}$: nominal returns: investments in other currencies

Figure 11 shows corresponding results for real returns. There were insufficient points to warrant separate consideration of home and foreign investments, so in this figure they are combined. Otherwise the results show no material differences from those for nominal returns.

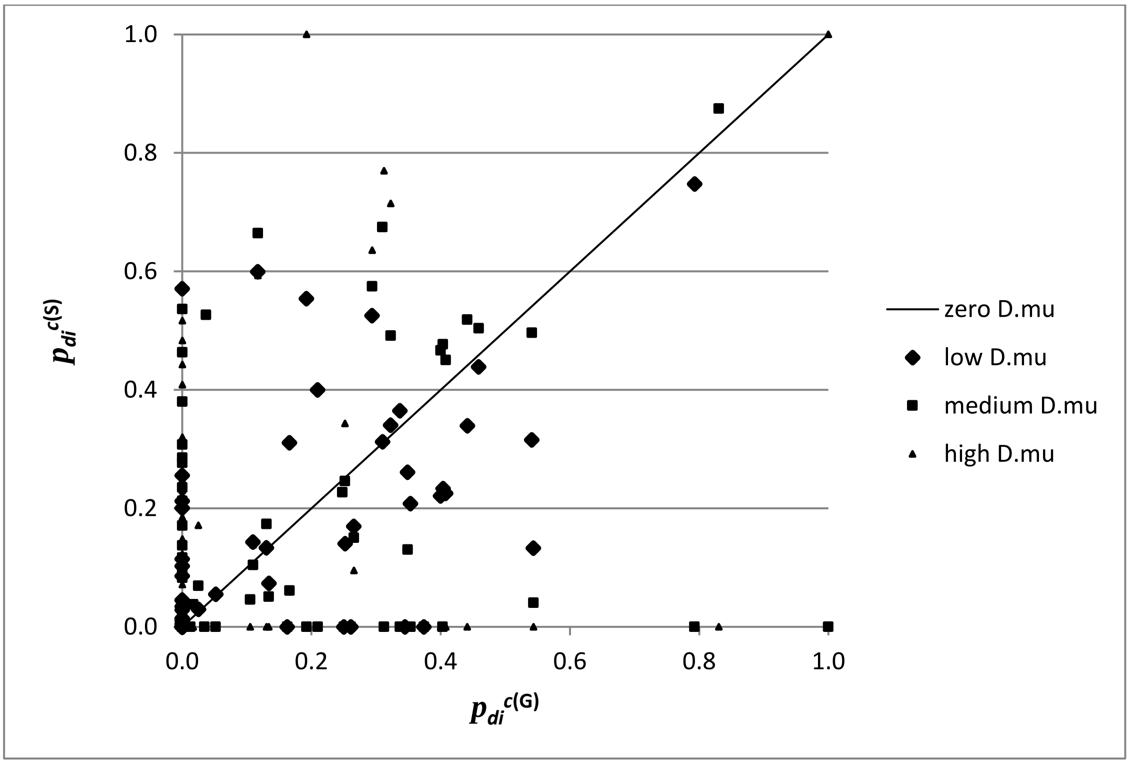

FIGURE 11: $p_{di}^{c(S)}$: real returns

4.2.5 Beta

It was expected that the SFM-CAPM estimates of beta would be approximately equal to the GCAPM estimates, with some shift to represent the effects of the SFM-CAPM requirements and some noise. An upward shift would indicate that the sample betas are understated and a downward shift would indicate that they are overstated. The shifts indicate the corrections required to the sample betas by the underlying assumptions of the SFM-CAPM.

Figure 12 shows the relationship of the SFM-CAPM estimates of the betas to the GCAPM estimates for nominal returns. The SFM-CAPM estimates show considerable shifts from the GCAPM values. A number of outliers are evident. These are for Turkish assets held by foreign investors. In each case the unrealistically high SFM-CAPM betas arise from relatively high GCAPM betas. These in turn arise from high covariances of nominal returns on investments in Turkish assets with those on other assets relative to the variances of the latter. This suggests that, for nominal returns on assets issued in a weak currency, care should be taken in the application of the SFM-CAPM where GCAPM betas are above 2.

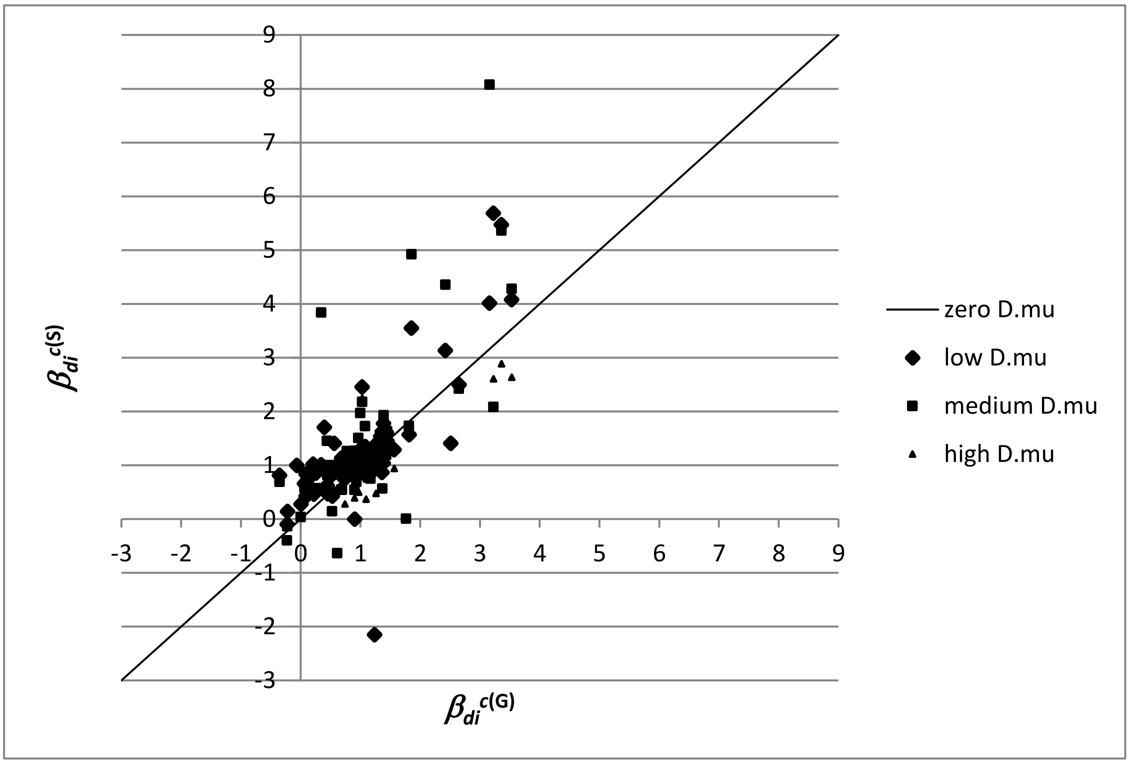

FIGURE 12. SFM-CAPM beta versus GCAPM beta: nominal returns

In Figure 13 values outside of the range [–1, 3] have been omitted so as to give a clearer impression of the relationship between the values within that range. In that figure it is evident that the SFM-CAPM betas tend to be greater than the GCAPM betas. The difference is frequently substantial. This suggests that the GCAPM tends to understate the betas of individual assets.

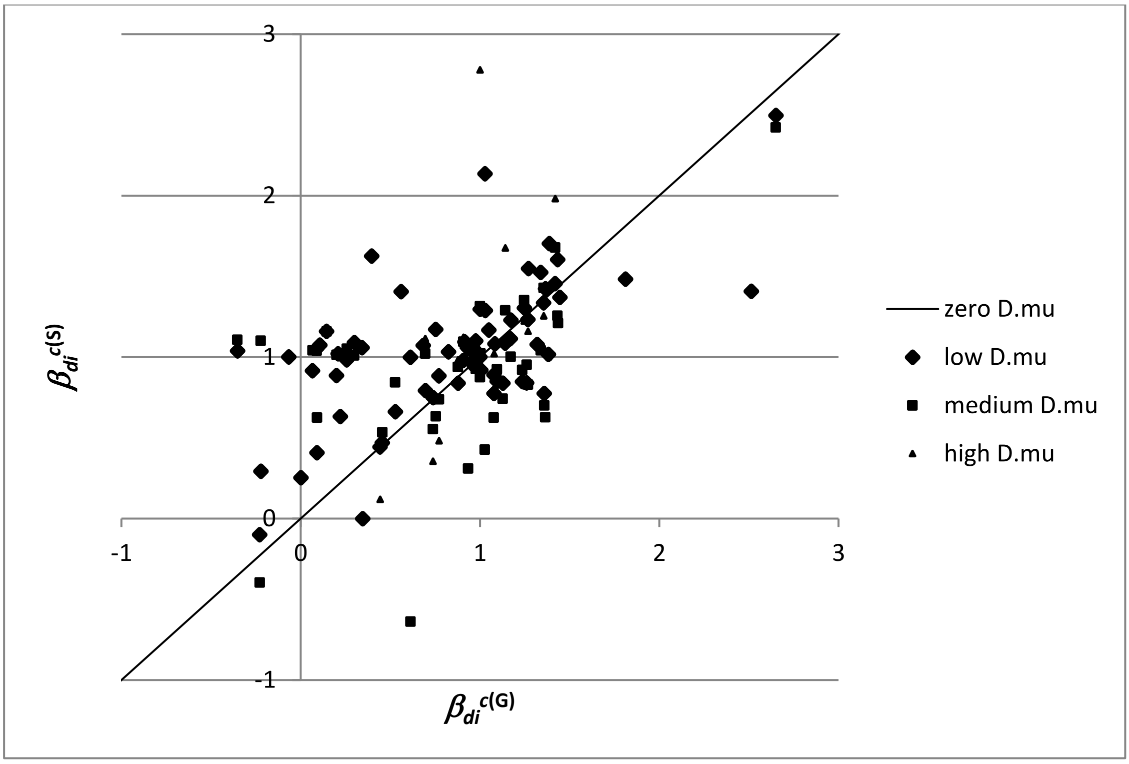

FIGURE 13. SFM-CAPM beta versus GCAPM beta: nominal returns excluding outliers

Figure 14 shows the relationship of the SFM-CAPM betas to the GCAPM betas for real returns. These results exhibit some upward curvature. For low values of GCAPM betas the SFM-CAPM betas are close, whereas for higher values they are even greater. There are no unrealistic outliers.

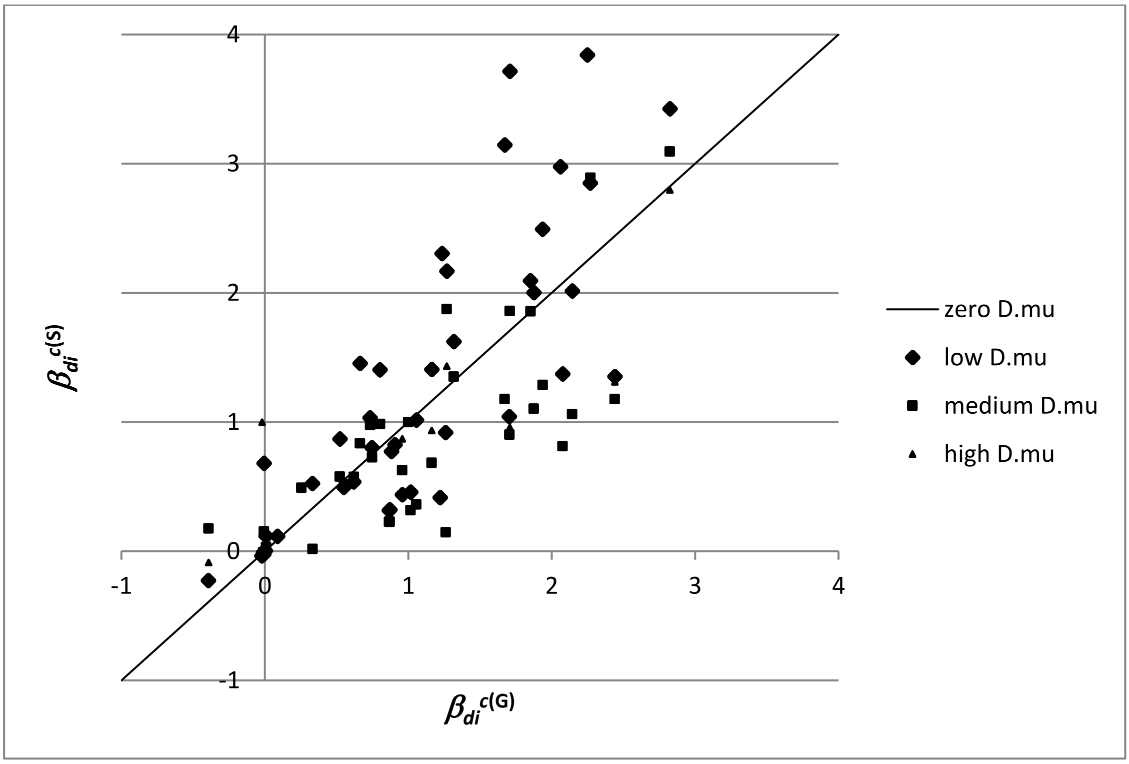

FIGURE 14. SFM-CAPM beta versus GCAPM beta: real returns

5. Conclusions

5.1 Summary

It is shown above that, for a single-factor CAPM to work in a multi-currency world, there is a necessary and sufficient condition. That condition applies to the *ex-ante* variances and covariances of returns. The resulting SFM-CAPM model developed in this paper may be specified as:

$$\mu_{di}^{c} = r_c + \beta_{di}^{c}\left(\mu_{\mathrm{M}}^{c} - r_c\right);$$

where:

$\mu_{\mathrm{M}}^{c} = E\left\{X_{\mathrm{M}}^{c}\right\} = \sum_{(d,i)\in\Psi_c} p_{di}^{c}\mu_{di}^{c}$

$X_{\mathrm{M}}^{c} =$ is the return in currency *c* on the tangency portfolio of a currency-*c* investor;

$\mu_{di}^{c} = E\left\{X_{di}^{c}\right\}$

$X_{di}^{c} = X_{di} + X_d - X_c$ is the return in currency *c* on asset *i* issued in currency *d* for $c = 1,\ldots,C$; $(d,i)\in\Psi_c$;

X_{di} is the return in currency *d* on asset *i* issued in that currency for $d = 1,\ldots,C$; $(d,i)\in\Psi_d$, so that $X_{di} = X_{di}^{d}$, $X_{d1} = r_d$;

X_e is the rate of strengthening of currency *e* relative to an arbitrarily chosen currency 1;

$\Psi_c = \left\{(d,i) \mid d\in\{1,\ldots,C\}; i\in\Omega_d^{c}\right\}$;

$$\Omega_d^{c} = \begin{cases} \{2,\ldots,n_d\} & \text{for } d = c; \\ \{1,\ldots,n_d\} & \text{for } d \neq c; \end{cases}$$

$i = 1$ denotes the risk-free asset in currency *d* and $i > 1$ a risky asset;

r_c is the return on the risk-free asset denominated in currency *c*;

$\sigma_{di,\mathrm{M}}^{c} = \mathrm{cov}\left\{X_{di}^{c}, X_{\mathrm{M}}^{c}\right\} = \sum_{(e,j)\in\Psi_c} p_{ej}^{c}\sigma_{di,ej}^{c}$;

$\sigma_{\mathrm{M,M}}^{c} = \mathrm{var}\left\{X_{\mathrm{M}}^{c}\right\} = \sum_{(d,i),(e,j)\in\Psi_c} p_{di}^{c} p_{ej}^{c}\sigma_{di,ej}^{c}$;

$\sigma_{di,ej}^{c} = \mathrm{cov}\left\{X_{di}^{c}, X_{\mathrm{ej}}^{c}\right\}$; and

$\left\{p_{cj}^{f} \mid (c,j)\in\Psi_f\right\}$ is the tangency portfolio of a currency-*c* investor, which maximises:

$$k = \frac{\mu_{\mathrm{M}}^{c} - r_c}{\sqrt{\sigma_{\mathrm{M,M}}^{c}}};$$

subject to the constraints:

$p_{di}^{c} \geq 0$ for all $(d,i)\in\Psi_c$ and for all *c*; and

$$\sum_{(d,i)\in\Psi_c} p_{di}^c = 1.$$

All returns on assets and rates of strengthening of currencies are expressed as forces.

In practice this model generally has more constraints than unknowns, so the constraints cannot be applied rigorously. However, the condition may be applied by means of a penalty method by finding $\hat{\mu}_{ci}^{(\mathrm{S})}$ and $\hat{\mu}_c^{(\mathrm{S})}$ so as to minimise:

$$D^2 = D_\mu^2 + hD_\kappa^2;$$

where:

$$D_\mu^2 = \frac{1}{Q_\mu}\left[\sum_{c=1}^{C}\left\{\sum_{i=2}^{n_c}\left(\hat{\mu}_{ci}^{(\mathrm{S})} - \hat{\mu}_{ci}^{(\mathrm{G})}\right)^2\right\} + \sum_{c=2}^{C}\left(\hat{\mu}_c^{(\mathrm{S})} - \hat{\mu}_c^{(\mathrm{G})}\right)^2\right];$$

$$D_\kappa^2 = \frac{1}{Q_\kappa}\sum_{c,e=1}^{C}\sum_{(d,i)\in\Psi_c}\sum_{(f,j)\in\Psi_e}\left(\kappa_{di}^c - \kappa_{fj}^e\right)^2;$$

$$\kappa_{di}^f = \frac{\hat{\sigma}_{di,\mathrm{M}}^f - \hat{\sigma}_{d1,\mathrm{M}}^f}{\hat{\sigma}_{\mathrm{M,M}}^f}\left(\hat{\mu}_{\mathrm{M}}^{f(\mathrm{S})} - r_f\right);$$

Q_μ and Q_κ are the numbers of terms in the respective summands; and

h is a penalty coefficient.

In the definition of D_μ^2, the superscripts (S) and (G) refer to the SFM-CAPM and the GCAPM respectively. Under the GCAPM, the underlying expectations $\hat{\mu}_{ci}^{(\mathrm{G})}$ and $\hat{\mu}_c^{(\mathrm{G})}$ are the *ex-post* sample means of X_{ci} and X_c respectively.

This means that, whilst D_κ^2 will not generally be zero, it can theoretically be reduced to an arbitrarily small value by increasing the penalty coefficient *h*. However, in practice, it is not possible to obtain an arbitrarily small value of D_κ^2. The estimates $\hat{\mu}_{ci}^{(\mathrm{S})}$ and $\hat{\mu}_c^{(\mathrm{S})}$ of the *ex-ante* underlying expectations will depend on *h*, as will the betas and the optimal portfolio. Bayesian credibility theory could be used to determine *h*. Otherwise it is a matter to which professional judgement should be applied.

The resulting SFM-CAPM developed in this paper may be applied as:

$$\hat{\mu}_{di}^c = r_c + \beta_{di}^c\left(\hat{\mu}_{\mathrm{M}}^c - r_c\right);$$

where:

$$\beta_{di}^c = \frac{\hat{\sigma}_{di,\mathrm{M}}^c}{\hat{\sigma}_{\mathrm{M,M}}^c}.$$

The model was applied to major categories of assets issued in the USA, the UK, South Africa and Turkey.

The SFM-CAPM values of expected returns and expected rates of strengthening of currencies are quite close to the GCAPM values. Nevertheless, in relation to investment-management decision-making the differences are material. For high values of D_μ expected real returns and expected real rates of strengthening of currencies under the SFM-CAPM may be negative, even for positive GCAPM values. When expected returns measured in different currencies are considered the differences between the GCAPM and the SFM-CAPM become more substantial.

With regard to optimal portfolios it was found that, in general, the higher the exposure to an asset under the GCAPM, the less likely it was to attract high exposure under the SFM-CAPM, and conversely, the higher the exposure to an asset under the SFM-CAPM, the less likely it was to attract high exposure under the GCAPM. The results suggest that home bias is better justified under the SFM-CAPM than under the GCAPM. Some of the optimal exposures would be unattainable in practice. The results must be interpreted as those that would have obtained if the major-currency investors could have invested substantial proportions of their assets in minor currencies. In practice, it would be necessary to limit exposures to attainable proportions.

The SFM-CAPM estimates show considerable shifts from the GCAPM values. for nominal returns on assets issued in a weak currency, care should be taken in the application of the SFM-CAPM where GCAPM betas are above 2. The GCAPM tends to understate the betas of individual assets. The betas of real returns appeared to be more reliable than those of nominal returns.

The findings of this paper give adequate grounds for the implementation of the SFM-CAPM. It is preferable to multi-factor models in that it does not treat currency risks as carrying different weight from investment risks; regardless of its source, risk is measured as variance in returns in the investor's currency and weighted accordingly. It is preferable to the GCAPM in that the implied price of a security to an investor who measures returns in a particular currency is the same as the price to an investor who measures returns in another currency.

The paper suggests that, if this model is to be applied, it would be better to apply it to real returns than to nominal returns. Nevertheless, even in nominal terms, where the portfolios produced by the optimisation procedure are credible and the market risk premiums are realistic, the results of this paper show that the SFM-CAPM produces reasonable results.

5.2 Further Research

In practice the stochastic modelling of the assets and liabilities of a long-term financial institution requires the use of time series in which the expected returns on assets and the expected forces of inflation, and perhaps their variances and covariances, may vary over time. This means that the application of the SFM-CAPM to such modelling will necessitate the use of the method in a time series. For that purpose the time-series model may be used to simulate, at the start of each year, estimates of expected returns on the market portfolio during the forthcoming year and of the variances and covariances (for example in terms of a GARCH model)—and hence the betas—of each asset category. The SFM-CAPM may then be used to estimate expected returns during that year, conditional on the simulated estimates, of the returns on each asset category. These estimates may then be used to simulate returns on each asset category. For the SFM-CAPM the distribution of the return on the market portfolio and on each asset category may (as for other versions of the CAPM) be taken as any elliptically symmetric distribution. In the single-currency case the theory of this process has been developed in Thomson & Gott (op. cit.) and an application has been demonstrated in Thomson (2011). Its application in the multi-currency case awaits further research. Because the SFM-

CAPM is an equilibrium model, it is well suited to such applications. The advantage of equilibrium models for such purposes is that they do not assume that the investor can outperform the market on a risk-adjusted basis, thus allowing market-consistency, and that, unlike more general no-arbitrage models, they do not assume complete markets, thus allowing for the fact that the liabilities of a financial institution cannot be replicated in the market. Whilst the assumption of equilibrium is inappropriate for an investment manager whose mandate is to outperform a benchmark portfolio, it is appropriate for the formulation of such a portfolio, and the SFM-CAPM may be used for such purposes.

The application of the principles explored in this paper to the use of the CAPM for a domestic market with different time horizons and different definitions of the risk-free asset requires further research.

The further testing of hypotheses regarding the SFM-CAPM is another field of research. For the purposes of such testing, however, it must be recognised that our tests should, in principle, be based on *ex-ante* expectations, not necessarily *ex-post* estimations of those expectations. The use of the rational-expectations hypothesis to argue that the latter is an unbiased estimate of the former is at best only asymptotically true.

The principal interest of the authors is in the development of stochastic models for actuarial use. Nevertheless, the SFM-CAPM clearly does have wider application—for example in determining cost of capital. For such applications it is not necessarily envisaged that this model will replace other models, but subject to the results of the further research suggested here, there is no reason why the SFM-CAPM should not take its place alongside other models in informing subjective assessment by practitioners of the expected returns on assets in a multi-currency world.

Acknowledgements

The financial assistance of the Actuarial Society of South Africa is acknowledged. Opinions expressed and conclusions drawn are those of the authors and are not to be attributed to the Society. The authors also acknowledge the assistance of Dr. Ömür Uğur of the Institute of Applied Mathematics at the Middle East Technical University with regard to the use of alternative optimisation methods. Valuable criticism by anonymous referees is also acknowledged.

References

Adler, M. & Dumas, B., 1983. International portfolio choice and corporate finance: a synthesis. Journal of Finance 38 (3), 925–984

Dolde, W., Giaccotto, C., Mishra, D.R. & O'Brien, T., 2011. Foreign exchange exposure and cost of equity for US companies: local versus global CAPM. Journal of Applied Finance 1 (1), 78–86

Grauer, F.L.A., Litzenberger, R.H. & Stehle, R.E., 1976. Sharing rules and equilibrium in an international capital market under uncertainty. Journal of Financial Economics 3 (3), 233–56

Jorion, P. & Schwartz, E., 1986. Integration vs. segmentation in the Canadian stock market. Journal of Finance 41 (3), 603–14

Karolyi, G.A. & Stulz, R.M., 2003. Are financial assets priced locally or globally? Handbook of the Economics of Finance 1, 975–1020

Koedijk, K.G., Kool, C.J.M., Schotman, P.C. & van Dijk, M.A., 2002. The cost of capital in international financial markets: local or global? Journal of International Money and Finance 21 (6), 905–29

Lintner, J. (1965) The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets. *Review of Economics and Statistics* **47(1),** 13–37

Mossin, J. (1966) Equilibrium in a Capital Asset Market. *Econometrica* 34(4), 768–783

Sercu, P., 1980. A generalization of the international asset pricing model. Journal of Finance 8, 500–24

Sharpe, W.F. (1964) Capital Asset Prices: A Theory of Market Equilibrium Under Conditions of Risk. *Journal of Finance* 19(3), 425–442

Solnik, B.H., 1974a. An equilibrium model of the international capital market. Journal of Economic Theory 8, 500–24

Solnik, B.H., 1974b. The international pricing of risk: an empirical investigation of the world capital market structure. Journal of Finance 29 (2), 365–78

Stehle, R., 1977. An empirical test of the alternative hypotheses of national and international pricing of risky assets. Journal of Finance 32 (2), 493–502

Stulz, R.M., 1981. International asset pricing. Journal of Financial Economics 9, 383–406

Stulz, R.M., 1995a. International portfolio choice and asset pricing: an integrative survey. National Bureau of Economic Research, http://www.nber.org/papers/w4645

Stulz, R.M., 1995b. The cost of capital in internationally integrated markets: the case of Nestle. European Financial Management 1 (1), 11–22

Stulz, R.M., 1995c. Globalization of the capital markets and the cost of capital: the case of Nestle. Journal of Applied Corporate Finance 8 (3), 30–8

Thomson, R.J., 2011. The arbitrage-free equilibrium pricing of liabilities in an incomplete market: application to a South African retirement fund. *South African Actuarial Journal*, 11, 1–41

Thomson, R.J. & Gott, D.V., 2009. Stochastic models for actuarial use: the equilibrium modelling of local markets. ASTIN Bulletin 39 (1), 339–70

Treynor, J.L. (unpublished) Toward a Theory of Market Value of Risky Assets. Unpublished manuscript. 1962

Wilkie, A.D., (unpublished). Why the capital asset pricing model fails in a multi-currency world. Proceedings of the 7th International AFIR Colloquium, 1997, 951–60