Sequent systems for nondeterministic propositional logics without reflexivity

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Abstract. In order to deal with ambiguity in statements made in a natural language I introduce nondeterministic semantics for propositional logic with an arbitrary set C of connectives. The semantics are based on the idea that Γ entails Δ if and only if every possible deterministic disambiguation of Γ entails every possible deterministic disambiguation of Δ. I also introduce a cut-free sequent style proof system S_C that is sound and complete for the given semantics. Finally I show that while the semantics and proof system do not satisfy reflexivity they do allow certain kinds of substitution of equivalents.

1 Introduction

When attempting to apply logical methods to sentences in natural language we often run into problems related to ambiguity. This ambiguity could be caused by ambiguous predicates, but it could also be caused by ambiguous connectives. Take for example a sentence of the form "A or B". Such a statement is ambiguous; the "or" could be inclusively ($\vee$) or exclusively ($\oplus$).

One approach for dealing with such ambiguity is to require disambiguation before allowing the sentence to be phrased in a logic. So "A or B" would have to be represented by either the formula $A \vee B$ or by the formula $A \oplus B$. Unfortunately this approach sometimes cannot be used, as it is not always possible to determine which unambiguous sentence was meant by the speaker. Sometimes the speaker is unable or unwilling to clarify their utterance and it is even possible that the speaker is uncertain about what they meant.[1]

Another approach is to consider an ambiguous statement as nondeterministic, where all possible disambiguations of the statement could be the meaning of the statement. See for example [1] for an example of this approach applied to ambiguous predicates. This approach has the advantage of being applicable even if no good choice of disambiguation is available, at the cost of resulting in a weaker logic. This nondeterministic approach is the one I use in this paper.

It should be noted that this approach is also to some extent usable for connectives that are not merely ambiguous but not truth-functional. Consider a

[1] For an example of a situation where the speaker is uncertain about what he means consider the first paragraph of the introduction, where I state that an "or" can be interpreted inclusively ($\vee$) *or* exclusively ($\oplus$). I do not know whether this *or* should be considered inclusively or exclusively.

connective $\rightsquigarrow$ representing a causal implication. The causal part of the implication is not truth-functional; the truth value of $p \rightsquigarrow q$ is not known if either p doesn't hold or if both p and q hold. However, the truth value of $p \rightsquigarrow q$ is known if p holds and q does not, in that case it is false. Treating $p \rightsquigarrow q$ as a formula that is false if p is true and q is false and nondeterministically true or false in other cases allows some reasoning about $\rightsquigarrow$ in the framework of nondeterministic propositional logic.

In this paper I define a type of nondeterministic propositional semantics that allows all occurrences of nondeterministic connectives to be interpreted independently. In order to do this I start with deterministic semantics and then define the nondeterministic semantics based on all possible (deterministic) disambiguations. I also give a sound and complete proof system for the nondeterministic logic. Because of their pleasing properties with respect to automated reasoning I use a cut-free sequent style proof system. The proof system also turns out to be quite elegant, despite the complicated semantics.

The structure of the paper is as follows. First, in Section 1.1 I compare my approach to a somewhat similar existing approach using Nmatrices (see [2–4]). Then in Section 2 I define the deterministic logic $\mathcal{L}_{\overline{C}}$ and in Section 3 I define the nondeterministic logic $\mathcal{L}_C$. In Section 4 I give a sequent style proof system S_C which I prove to be complete for $\mathcal{L}_C$ in Section 5. Finally, in Section 6 I consider a few properties of $\mathcal{L}_C$ and S_C.

1.1 Comparison to Nmatrices

Semantics for nondeterministic propositional logic have been introduced in [2–4] using so-called *Nmatrices*. The semantics I use here for $\mathcal{L}_C$ are in many ways very similar to those using Nmatrices, but with one important difference. When using Nmatrices different occurrences of a single nondeterministic connective are allowed to have different interpretations, *but only insofar as identical (sub)formulas have the same interpretation everywhere.*

For example, let $*$ represent the "or" connective that can mean either $\vee$ or $\oplus$. Then in the Nmatrices approach the formula $(p * q) \wedge (p * q)$ can mean two things; either $(p \vee q) \wedge (p \vee q)$ or $(p \oplus q) \wedge (p \oplus q)$. The mixed disambiguations $(p \vee q) \wedge (p \oplus q)$ and $(p \oplus q) \wedge (p \vee q)$ are not allowed because $(p * q)$ and $(p * q)$ are the same formula and therefore must have the same choice of disambiguation.

The importance this approach gives to identity of formulas has a few unusual consequences, such as substitution of equivalents being unsound, even if the equivalents are provably equivalent. When using Nmatrices the formula $(p * q) \rightarrow (p * q)$ is a tautology but the formula $((p \wedge p) * q) \rightarrow (p * q)$ is not.

In the semantics for $\mathcal{L}_C$ I therefore allow different occurrences of a single nondeterministic connective to have different interpretations without restriction. The nondeterministic formula $(p * q) \wedge (p * q)$ then allows four disambiguations; $(p \vee q) \wedge (p \vee q)$, $(p \oplus q) \wedge (p \oplus q)$, $(p \vee q) \wedge (p \oplus q)$ or $(p \oplus q) \wedge (p \vee q)$.

The approach taken in this paper does lead to different unusual consequences. In particular, the inference relation for $\mathcal{L}_C$ is not reflexive since for example $p * q \models p * q$ might mean $p \vee q \models p \oplus q$.

2 Deterministic semantics

First we need a few notational preliminaries. Let a nonempty set $\mathcal{P}$ of propositional variables be given. Furthermore, let $\overline{C}$ be a set of truth-functional connectives. For each connective $\circ \in \overline{C}$ let $r_\circ \in \mathbb{N}$ be the arity of $\circ$ and $f_\circ : \{0,1\}^{r_\circ} \to \{0,1\}$ the truth function of $\circ$.

For binary connectives I use the standard infix notation. Nullary connectives are denoted $\top$ if the associated truth function is the constant function 1 and $\bot$ if the truth function is the constant function 0.

I use p, q as variables for propositional variables, lower case Greek letters for formulas and uppercase Greek letters for multisets of formulas. Now let us define the language $L_{\overline{C}}$ and the models for $\mathcal{L}_{\overline{C}}$.

Definition 1 (Language $L_{\overline{C}}$). *The language $L_{\overline{C}}$ using the connectives in $\overline{C}$ is the smallest set such that if $p \in \mathcal{P}$ then $p \in L_{\overline{C}}$ and if $\circ \in \overline{C}$ and $\varphi_1, \cdots, \varphi_{r_\circ} \in L_{\overline{C}}$ then $\circ(\varphi_1, \cdots, \varphi_{r_\circ}) \in L_{\overline{C}}$.*

Definition 2 (Models). *A model $\mathcal{M}$ is a valuation function $\mathcal{M} : \mathcal{P} \to \{0,1\}$ that assigns to each propositional variable the value 'true' (1) or 'false' (0).*

Now we can define the semantics for $\mathcal{L}_{\overline{C}}$.

Definition 3 (Satisfaction relation $\models$). *The satisfaction relation $\models$ between models and formulas is defined inductively by*

- *$\mathcal{M} \models p$ if and only if $\mathcal{M}(p) = 1$,*
- *for every $\circ \in \overline{C}$ we have $\mathcal{M} \models \circ(\varphi_1, \cdots, \varphi_{r_\circ})$ if and only if $f_\circ(v_1, \cdots, v_{r_\circ}) = 1$ where $v_i = 1$ if and only if $\mathcal{M} \models \varphi_i$.*

Furthermore, $\forall \gamma \in \Gamma : \mathcal{M} \models \gamma$ is denoted by $\mathcal{M} \models_\wedge \Gamma$ and $\exists \delta \in \Delta : \mathcal{M} \models \delta$ is denoted by $\mathcal{M} \models_\vee \Delta$.

From this satisfaction relation we define an entailment relation.

Definition 4 (Entailment relation $\models_{\overline{C}}$). *The entailment relation $\models_{\overline{C}}$ is a relation between multisets $\Gamma, \Delta \subseteq L_{\overline{C}}$ of formulas. We have $\Gamma \models_{\overline{C}} \Delta$ if and only if for every model $\mathcal{M}$ such that $\mathcal{M} \models_\wedge \Gamma$ it holds that $\mathcal{M} \models_\vee \Delta$.*

Lemma 1. *Let $\Gamma, \Delta \subseteq L_{\overline{C}}$ be any finite multisets of $\mathcal{L}_{\overline{C}}$ formulas such that $\Gamma \models_{\overline{C}} \Delta$, and let Φ be any set of propositional variables that contains all variables that occur in Γ or Δ. Then for any partition Φ_1, Φ_2 of Φ one of the following statements holds:*

1. *there is a $\gamma \in \Gamma$ such that $\Phi_1, \gamma \models_{\overline{C}} \Phi_2$,*
2. *there is a $\delta \in \Delta$ such that $\Phi_1 \models_{\overline{C}} \delta, \Phi_2$.*

Proof. All connectives in $\overline{C}$ are truth-functional, so the value of any formula in a model is fully determined by the values of the propositional variables that occur in the formula have in that model. Fix any partition Φ_1, Φ_2 of Φ and let $\mathfrak{M}$ be the set of models that satisfy all of Φ_1 and none of Φ_2. Then for every $\psi \in \Gamma \cup \Delta$ we either have $\mathcal{M} \models \psi$ for every model $\mathcal{M} \in \mathfrak{M}$ or $\mathcal{M} \not\models \psi$ for every $\mathcal{M} \in \mathfrak{M}$.

Suppose there is a $\gamma \in \Gamma$ such that the second possibility holds for that formula, so $\mathcal{M} \not\models \gamma$ for every $\mathcal{M} \in \mathfrak{M}$. Then every model that satisfies all of Φ_1 and none of Φ_2 does not satisfy γ, so every model that satisfies all of Φ_1 and γ must satisfy some of Φ_2. We therefore have $\Phi_1, \gamma \models_{\overline{C}} \Phi_2$.

Suppose then that for every $\gamma \in \Gamma$ the first possibility holds, so $\mathcal{M} \models \gamma$ for every $\mathcal{M} \in \mathfrak{M}$. We have $\Gamma \models_{\overline{C}} \Delta$ so for every $\mathcal{M} \in \mathfrak{M}$ there is a $\delta \in \Delta$ such that $\mathcal{M} \models \delta$. But then $\mathcal{M} \models \delta$ for every such model, as the value of δ is constant on $\mathfrak{M}$. This implies that every model that satisfies all of Φ_1 and none of Φ_2 also satisfies δ, so every model that satisfies all of Φ_1 must either satisfy δ or one of Φ_2. We therefore have $\Phi_1 \models_{\overline{C}} \delta, \Phi_2$.

Note that in particular Lemma 1 implies that if Φ contains all the propositional variables of φ then $\Phi_1, \varphi \models_{\overline{C}} \Phi_2$ or $\Phi_1 \models_{\overline{C}} \varphi, \Phi_2$ since $\varphi \models_{\overline{C}} \varphi$.

3 Nondeterministic semantics

Let us start by defining nondeterministic connectives.

Definition 5 (Nondeterministic connective). *A nondeterministic connective $\circ$ is a connective with arity $r_\circ$ and partial truth function $f_\circ : \{0,1\}^{r_\circ} \to \{0,1,?\}$.*

Note that besides $\top$ and $\bot$ there is now a third possible nondeterministic nullary connective that has the constant function ? as partial truth function. I denote this connective by ? as well.

Definition 6 (Language L_C). *The language L_C using the connectives in C is the smallest set such that if $p \in \mathcal{P}$ then $p \in L_C$ and if $\circ \in C$ and $\varphi_1, \cdots, \varphi_{r_\circ} \in L_C$ then $\circ(\varphi_1, \cdots, \varphi_{r_\circ}) \in L_C$.*

The logic $\mathcal{L}_C$ is then based on the idea that a nondeterministic connective $\circ$ can be disambiguated as one of several deterministic connectives $\overline{\circ}$.

Definition 7 (Disambiguation of a connective). *Let $\circ$ be a nondeterministic connective with arity $r_\circ$ and associated nondeterministic truth function $f_\circ$. A truth-functional connective $\overline{\circ}$ is a disambiguation of $\circ$ if $r_{\overline{\circ}} = r_\circ$ and for each $v \in \{0,1\}^{r_\circ}$ such that $f_\circ(v) \in \{0,1\}$ it holds that $f_\circ(v) = f_{\overline{\circ}}(v)$.*

So a disambiguation $\overline{\circ}$ of a connective $\circ$ is a truth-functional connective with a truth function $f_{\overline{\circ}}$ where every ? from $f_\circ$ is replaced by either a 0 or a 1.

Example 1. Suppose $*$ is the nondeterministic connective with the following truth table.

φ_1	φ_1	$\varphi_1 * \varphi_2$
1	1	?
1	0	1
0	1	1
0	0	0

Then there are two possible disambiguations $\overline{*}$ and $\overline{*}'$ of $*$, namely

φ_1	φ_1	$\varphi_1 \overline{*} \varphi_2$	φ_1	φ_1	$\varphi_1 \overline{*}' \varphi_2$
1	1	1	1	1	0
1	0	1	1	0	1
0	1	1	0	1	1
0	0	0	0	0	0

So the disambiguations of $*$ are a disjunction ($\vee$)and an "exclusive or" ($\oplus$).

Definition 8 (Disambiguation of a formula). *The disambiguations of formulas are given inductively by the following.*

- *If $p \in \mathcal{P}$ then p is a disambiguation of p.*
- *If $\varphi = \circ(\varphi_1, \cdots, \varphi_{r_\circ})$, $\overline{\varphi_i}$ is a disambiguation of φ_i for each $1 \leq 1 \leq r_\circ$ and $\overline{\circ}$ is a disambiguation of $\circ$ then $\overline{\circ}(\overline{\varphi_1}, \cdots, \overline{\varphi_{r_\circ}})$ is a disambiguation of φ.*

If Γ is a multiset of formulas then $\overline{\Gamma}$ is a disambiguation of Γ if $\overline{\Gamma}$ can be obtained from Γ by replacing formulas with one of their disambiguations.

Definition 9 (Entailment relation $\models_C$). *Let $\overline{C}$ be the set of all connectives that are the disambiguation of a connective in C. The entailment relation $\models_C$ is a relation between multisets $\Gamma, \Delta \subseteq L_C$ of $\mathcal{L}_C$ formulas. We have $\Gamma \models_C \Delta$ if and only if $\overline{\Gamma} \models_{\overline{C}} \overline{\Delta}$ for every disambiguations $\overline{\Gamma}$ of Γ and $\overline{\Delta}$ of Δ.*

Note that this is a very strong standard for entailment. If we have $\Gamma \models_C \Delta$ and Γ holds under some disambiguation then Δ must hold under every disambiguation. Using the terminology of [1] the relation $\models_C$ not only preserves *truth on some disambiguation* (truth-osd) and *truth on every disambiguation* (truth-osd only), it also requires a true-osd antecedent to have a true-osd only consequent. This is a stronger condition than the ones discussed in [1], but it has a very clear game-theoretical or dialectical interpretation: $\Gamma \models_C \Delta$ if and only if you can conclude Δ from Γ without any possibility for an opponent to give a disambiguation that proves you wrong. Equivalently, $\Gamma \models_C \Delta$ if and only if one cannot rationally reject Δ while accepting Γ.

For the completeness theorems given later in the paper it is very useful to be able to work with finite multisets of formulas. I therefore give a compactness proof for $\models_C$ here. The compactness of deterministic propositional logic is quite well known, see for example [5] for two different versions of the proof. The proof for the compactness of nondeterministic propositional logic can be obtained by some small (but notationally complicated) modifications to the existing proofs. Here I give a topological proof using Tychonoff's theorem [6, 7], which states that every product of compact sets is compact.

Lemma 2 (Compactness of $\mathcal{L}_C$). *Let Γ, Δ be any multisets of $\mathcal{L}_C$ formulas. If $\Gamma \models_C \Delta$ then there are finite sub-multisets $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ such that $\Gamma' \models_C \Delta'$.*

Proof. In order to keep our notation simple I treat the multisets as sets in this proof. This is purely a matter of notation, it amounts to the same thing as adding a label to every occurrence of a formula in a multiset to make them distinct.

Let Γ, Δ be any sets of formulas such that $\Gamma \models_C \Delta$ and suppose towards a contradiction that there are no finite subsets $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ such that $\Gamma' \models_C \Delta'$. Let G and D be partitions of Γ and Δ respectively that contain only finite sets. By padding G or D with multiple copies of the empty set if necessary we can guarantee the existence of a bijection $f : G \to D$.

Consider the set $\{0, 1\}$ as a finite topological space (with the discrete topology). This topological space is compact. Then by Tychonoff's theorem the set $\{0, 1\}^{\mathcal{P}}$ (with the product topology) is compact. The elements of $\{0, 1\}^{\mathcal{P}}$ are exactly the models of $\mathcal{L}_{\overline{C}}$.

For any $\Phi \subseteq \Gamma$ and $\Psi \subseteq \Delta$ let $V(\Phi, \Psi)$ be the set of models $\mathcal{M} \in \{0, 1\}^{\mathcal{P}}$ such that for some disambiguations $\overline{\Phi}$ and $\overline{\Psi}$ of Φ and Ψ we have $\mathcal{M} \models_\wedge \overline{\Phi}$ and $\mathcal{M} \not\models_\vee \overline{\Psi}$. So $V(\Phi, \Psi)$ is the set of countermodels to $\Phi \models_C \Psi$. Take any $G' \subseteq G$ and let $D' = f(G')$. We will show that

$$\bigcap_{\Phi \in G'} V(\Phi, f(\Phi)) = V\left(\bigcup G', \bigcup D'\right). \tag{1}$$

The important step is that, since G and D are partitions of Γ and Δ, the elements of G' are mutually disjunct, as are those of D'. This implies that if for every $\Phi \in G'$ the set $\overline{\Phi}$ is a disambiguation of Φ then $\bigcup_{\Phi \in G'} \overline{\Phi}$ is a disambiguation of $\bigcup G'$, because the disambiguations of different elements of G' cannot disagree about how a formula should be disambiguated. We can also go the other way, every disambiguation $\overline{\bigcup G'}$ of G' induces unique disambiguations $\overline{\Phi}$ for every $\Phi \in G'$ such that $\overline{\bigcup G'} = \bigcup_{\Phi \in G'} \overline{\Phi}$.

The same holds for the disambiguations of $\bigcup D'$ and the disambiguations for every $f(\Phi) = \Psi \in D'$; if given $\overline{\Psi}$ for all $\Psi \in D'$ then $\bigcup_{\Psi \in D'} \overline{\Psi}$ is a disambiguation of $\bigcup D'$ and for each disambiguation $\overline{\bigcup D'}$ there are unique disambiguations $\overline{\Psi}$ for all $\Psi \in D'$ such that $\overline{\bigcup D'} = \bigcup_{\Psi \in D'} \overline{\Psi}$.

Take any $\mathcal{M} \in \bigcap_{\Phi' \in G'} V(\Phi', f(\Phi'))$. This $\mathcal{M}$ has the property that for every $\Phi \in G'$ and $\Psi = f(\Phi)$ there are disambiguations $\overline{\Phi}$ and $\overline{\Psi}$ such that $\mathcal{M} \models_\wedge \overline{\Phi}$ and $\mathcal{M} \not\models_\vee \overline{\Psi}$. The sets $\bigcup_{\Phi \in G'} \overline{\Phi}$ and $\bigcup_{\Phi \in G'} \overline{\Phi}$ are disambiguations of $\bigcup G'$ and $\bigcup D'$. Furthermore, $\mathcal{M}$ satisfies all of $\bigcup_{\Phi \in G'} \overline{\Phi}$ and none of $\bigcup_{\Phi \in G'} \overline{\Phi}$ so $\mathcal{M} \in V(\bigcup G', \bigcup D')$. We therefore have

$$\bigcap_{\Phi \in G'} V(\Phi, f(\Phi)) \subseteq V\left(\bigcup G', \bigcup D'\right). \tag{2}$$

Now take any $\mathcal{M} \in V(\bigcup G', \bigcup D')$. This $\mathcal{M}$ has the property that there are disambiguations $\overline{\bigcup G'}$ and $\overline{\bigcup D'}$ of $\bigcup G'$ and $\bigcup D'$ such that $\mathcal{M}$ satisfies all of $\overline{\bigcup G'}$ and none of $\overline{\bigcup D'}$. For each $\Phi \in G'$ and $\Psi = f(\Phi) \in D'$ let $\overline{\Phi}$ and $\overline{\Psi}$ be the disambiguations such that $\overline{\bigcup G'} = \bigcup_{\Phi \in G'} \overline{\Phi}$ and $\overline{\bigcup D'} = \bigcup_{\Psi \in D'} \overline{\Psi}$. Then for each $\Phi \in G'$ and $\Psi = f(\Phi) \in D'$ the model $\mathcal{M}$ satisfies all of $\overline{\Phi}$ and none of $\overline{\Psi}$

so $\mathcal{M} \in \bigcap_{\Phi \in G'} V(\Phi, f(\Phi))$. We therefore have

$$V\left(\bigcup G', \bigcup D'\right) \subseteq \bigcap_{\Phi \in G'} V(\Phi, f(\Phi)) \tag{3}$$

which together with (2) implies (1).

For any $\Gamma' \in G$ and $\Delta' \in D$ the set $V(\Gamma', \Delta')$ is closed because all subsets of $\{0,1\}^p$ are clopen since we started with the discrete topology. Furthermore if follows from (1) that for any $n \in \mathbb{N}$ and any $\Gamma'_1, \cdots, \Gamma'_n \in G$ we have

$$\bigcap_{1 \le i \le n} V(\Gamma'_i, f(\Gamma'_i)) = V\left(\bigcup_{1 \le i \le n} \Gamma'_i, \bigcup_{1 \le i \le n} f(\Gamma'_i)\right).$$

The set $V(\bigcup_{1 \le i \le n} \Gamma'_i, \bigcup_{1 \le i \le n} f(\Gamma'_i))$ is nonempty because $\bigcup_{1 \le i \le n} \Gamma'_i$ and $\bigcup_{1 \le i \le n} f(\Gamma'_i)$ are finite subsets of Γ and Δ so by assumption $\bigcup_{1 \le i \le n} \Gamma'_i \not\models_C \bigcup_{1 \le i \le n} f(\Gamma'_i)$. Then $\bigcap_{\Gamma' \in G} V(\Gamma', f(\Gamma'))$ is an intersection of closed sets with the finite intersection property so it is nonempty by the compactness of $\{0,1\}^{\mathcal{P}}$. We have

$$\bigcap_{\Gamma' \in G} V(\Gamma', f(\Gamma')) = V\left(\bigcup_{\Gamma' \in G} \Gamma', \bigcup_{\Gamma' \in G} f(\Gamma')\right) = V(\Gamma, \Delta),$$

so $V(\Gamma, \Delta)$ is also nonempty. But this implies that $\Gamma \not\models_C \Delta$, which contradicts the choice of Γ and Δ. The assumption that there are no finite subsets $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ such that $\Gamma' \models_C \Delta'$ must therefore be false, which proves the lemma.

We also need a nondeterministic variant of Lemma 1.

Lemma 3. *Let Γ, Δ be any finite multisets of $\mathcal{L}_C$ formulas such that $\Gamma \models_C \Delta$, and let Φ be any set of propositional variables that contains all variables that occur in Γ or Δ. Then for any partition Φ_1, Φ_2 of Φ one of the following statements holds:*

1. *there is a $\gamma \in \Gamma$ such that $\Phi_1, \gamma \models_C \Phi_2$,*
2. *there is a $\delta \in \Delta$ such that $\Phi_1 \models_C \delta, \Phi_2$.*

Proof. Suppose towards a contradiction that $\Phi_1, \gamma \not\models_C \Phi_2$ for all $\gamma \in \Gamma$ and $\Phi_1 \not\models_C \delta, \Phi_2$ for all $\delta \in \Delta$. Then for each $\gamma_i \in \Gamma$ there is a disambiguation $\overline{\gamma}_i$ such that $\Phi_1, \overline{\gamma}_i \not\models_{\overline{C}} \Phi_2$ and for each $\delta_i \in \Delta$ there is a disambiguation $\overline{\delta}_i$ such that $\Phi_1 \not\models_{\overline{C}} \overline{\delta}_i, \Phi_2$.

Now let $\overline{\Gamma} = \{\overline{\gamma}_i \mid \gamma_i \in \Gamma\}$ and $\overline{\Delta} = \{\overline{\delta}_i \mid \delta_i \in \Delta\}$. These $\overline{\Gamma}$ and $\overline{\Delta}$ are disambiguations of Γ and Δ so from $\Gamma \models_C \Delta$ it follows that $\overline{\Gamma} \models_{\overline{C}} \overline{\Delta}$. But $\overline{\Gamma}$ and $\overline{\Delta}$ live in a deterministic logic, so from Lemma 1 it follows that there either is a $\overline{\gamma} \in \overline{\Gamma}$ such that $\Phi_1, \overline{\gamma}_i \models_{\overline{C}} \Phi_2$ or a $\overline{\delta} \in \overline{\Delta}$ such that $\Phi_1 \models_{\overline{C}} \overline{\delta}_i, \Phi_2$. This contradicts the choice of $\overline{\Gamma}$ and $\overline{\Delta}$, so the initial assumption must have been wrong, which proves the lemma.

4 Proof system

The proof system consists of a few structural rules and some rules that are based on the abbreviated partial truth tables of the connectives.

Definition 10 (Rules of $S_\varnothing$). *The rules of $S_\varnothing$ are the rules Axiom (Ax), Left Contraction (CL), Right Contraction (CR), Left Weakening (WL) and Right Weakening (WR), given by*

$$\frac{}{p \vdash p}\,Ax \qquad \frac{\Gamma_1, \Gamma_2, \Gamma_2 \vdash \Delta}{\Gamma_1, \Gamma_2 \vdash \Delta}\,CL \qquad \frac{\Gamma \vdash \Delta_1, \Delta_2, \Delta_2}{\Gamma \vdash \Delta_1, \Delta_2}\,CR$$

$$\frac{\Gamma_1 \vdash \Delta}{\Gamma_1, \Gamma_2 \vdash \Delta}\,WL \qquad \frac{\Gamma \vdash \Delta_1}{\Gamma \vdash \Delta_1, \Delta_2}\,WR$$

The formula p in Ax is called principal, as are all elements of Γ_2 in CL and WL and all elements of Δ_2 in CR and WR.

Note that the Axiom used here results in a very limited form of reflexivity. This is because $\models_C$ is not reflexive. The logical rules correspond to the abbreviated truth tables of the connectives and can be obtained by a multi-step procedure.

Definition 11 (Rules R_C). *The set R_C rules for the abbreviated truth tables are obtained using the following procedure.*

1. *Start with $R_C = \varnothing$.*
2. *For any $\circ \in C$, $v \in \{0,1\}^{r_\circ}$ and $1 \leq i \leq r_\circ$ let U_i be the sequent $\Gamma, \varphi_i \vdash \Delta$ if the i-th entry of v is 0 and let U_i be the sequent $\Gamma, \varphi_i \vdash \Delta$ if the i-th entry of v is 1. Now for every $\circ \in C$ and $v \in \{0,1\}^{r_\circ}$ add the rule*

   $$\frac{U_1 \qquad \cdots \qquad U_{r_\circ}}{\Gamma, \circ(\varphi_1, \cdots, \varphi_{r_\circ}) \vdash \Delta}\,R_{\circ,v}$$

   to R_C if $f_\circ(v) = 0$, add the rule

   $$\frac{U_1 \qquad \cdots \qquad U_{r_\circ}}{\Gamma \vdash \circ(\varphi_1, \cdots, \varphi_{r_\circ}), \Delta}\,R_{\circ,v}$$

   to R_C if $f_\circ(v) = 1$ and add no rule to R_C if $f_\circ(v) =?$.
3. *If there are two rules*

   $$\frac{U_1 \quad \cdots \quad U_{j-1} \qquad \Gamma, \varphi_i \vdash \Delta \qquad U_{j+1} \quad \cdots \quad U_k}{W}\,R_{\circ,v}$$

   and

   $$\frac{U_1 \quad \cdots \quad U_{j-1} \qquad \Gamma \vdash \varphi_i, \Delta \qquad U_{j+1} \quad \cdots \quad U_k}{W}\,R_{\circ,v'}$$

   in R_C then add the rule

   $$\frac{U_1 \quad \cdots \quad U_{j-1} \qquad U_{j+1} \quad \cdots \quad U_k}{W}\,R_{\circ,v''}$$

where $v'' \in \{0,1,?\}^{r_\circ}$ *is the vector with the same value as* v *and* v' *where they agree and ? where they do not agree. Repeat this step until no more rules can be added.*

4. *If there are two rules*

$$\frac{U_1 \quad \cdots \quad U_k}{W}\, R_1 \qquad \frac{U'_1 \quad \cdots \quad U'_{k'}}{W}\, R_2$$

in R_C *with* $\{U_1, \cdots, U_k\} \subset \{U'_1, \cdots, U'_{k'}\}$ *then remove the rule* R_2*. Repeat this step until no more rules can be removed.*

The rules for $\top, \bot$ and ? are degenerate cases, for $\top$ and $\bot$ we have the rules

$$\frac{}{\Gamma \vdash \top, \Delta}\, \top \qquad \frac{}{\Gamma, \bot \vdash \Delta}\, \bot$$

and for ? we have no rules at all. The procedure given in Definition 11 terminates for any finite set of connectives and gives a unique set of rules for a given set of connectives. Let us consider a simple example of how this procedure works.

Example 2. Let $*$ be the binary connective with the following partial truth table.

φ_1	φ_2	$\varphi_1 * \varphi_2$
0	0	?
0	1	1
1	0	1
1	1	1

so $*$ behaves either like a disjunction or like guaranteed truth. Then in step 2 of the procedure the following three rules are added.

$$\frac{\Gamma \vdash \varphi_1, \Delta \qquad \Gamma, \varphi_2 \vdash \Delta}{\Gamma \vdash \varphi_1 * \varphi_2, \Delta}\, \mathrm{R}_{*,(1,0)}$$

$$\frac{\Gamma, \varphi_1 \vdash \Delta \qquad \Gamma \vdash \varphi_2, \Delta}{\Gamma \vdash \varphi_1 * \varphi_2, \Delta}\, \mathrm{R}_{*,(0,1)} \quad \frac{\Gamma \vdash \varphi_1, \Delta \qquad \Gamma \vdash \varphi_2, \Delta}{\Gamma \vdash \varphi_1 * \varphi_2, \Delta}\, \mathrm{R}_{*,(1,1)}$$

In step 3 we then combine the rule $\mathrm{R}_{*,(1,1)}$ with both the rule $\mathrm{R}_{*,(1,0)}$ and the rule $\mathrm{R}_{*,(0,1)}$ to obtain the following two rules

$$\frac{\Gamma \vdash \varphi_2, \Delta}{\Gamma \vdash \varphi_1 * \varphi_2, \Delta}\, \mathrm{R}_{*,(?,1)} \quad \frac{\Gamma \vdash \varphi_1, \Delta}{\Gamma \vdash \varphi_1 * \varphi_2, \Delta}\, \mathrm{R}_{*,(1,?)}$$

Finally, in step 4 we remove the rules $\mathrm{R}_{*,(1,0)}$, $\mathrm{R}_{*,(0,1)}$ and $\mathrm{R}_{*,(1,1)}$ because their premises are supersets of those of $\mathrm{R}_{*,(?,1)}$ and $\mathrm{R}_{*,(1,?)}$. In the end the rules for $*$ are therefore only the rules $\mathrm{R}_{*,(?,1)}$ and $\mathrm{R}_{*,(1,?)}$. These rules represent all we know about $*$, namely that $\varphi_1 * \varphi_2$ is true if at least one of φ_1 and φ_2 is true.

Definition 12 (Rule Cut). *The rule Cut is given by*

$$\frac{\Gamma_1 \vdash \varphi, \Delta_1 \qquad \Gamma_2, \varphi \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\, Cut$$

Definition 13 (Proof system S_C). *The proof system S_C consists of the rules $S_\varnothing$ together with the rules R_C. The proof system S_C+Cut consists of the rules of S_C together with the rule Cut.*

Definition 14 (Derivation). *A derivation in a proof system S is a finite labeled tree T such that:*

- *every node of T is labeled by either a sequent or an empty label,*
- *if a node s of T with label V has child nodes $t_1, \cdots, t_n$ with labels $U_1, \cdots U_n$ then*

$$\frac{U_1 \qquad \cdots \qquad U_n}{V}$$

  is an instance of a rule of S.

The non-empty labels of nodes that do not have child nodes are called the premises of the derivation and the label of the root is called the conclusion of the derivation.

Definition 15 (Derivable). *A sequent U is derivable in a proof system S if there is a derivation in S that has no premises and U as conclusion.*

Definition 16 (Admissible). *A rule*

$$\frac{U_1 \qquad \cdots \qquad U_n}{V}\, R$$

is admissible in S if every sequent that is derivable in S+R is derivable in S.

For most of the rules of S_C+Cut it should be immediately clear that they are sound for $\models_C$. The only rules for which there could be some doubt about the soundness are the contraction and Cut rules. For these rules it can also quite easily be seen that they are sound. Consider for example left contraction.

If we have $\Gamma_1, \Gamma_2, \Gamma_2 \models_C \Delta$ then for any disambiguations $\overline{\Gamma_1}$ of Γ_1, $\overline{\Gamma_2}$ of Γ_2, $\overline{\Gamma_2}'$ of Γ_2 and $\overline{\Delta}$ of Δ we have $\overline{\Gamma_1}, \overline{\Gamma_2}, \overline{\Gamma_2}' \models_{\overline{C}} \overline{\Delta}$. In particular this is the case if $\overline{\Gamma_2} = \overline{\Gamma_2}'$, so we have $\overline{\Gamma_1}, \overline{\Gamma_2}, \overline{\Gamma_2} \models_{\overline{C}} \overline{\Delta}$. CL is sound for deterministic propositional logic so $\overline{\Gamma_1}, \overline{\Gamma_2} \models_{\overline{C}} \overline{\Delta}$. This holds for any disambiguations $\overline{\Gamma_1}, \overline{\Gamma_2}$ and $\overline{\Delta}$ so $\Gamma_1, \Gamma_2 \models_C \Delta$. Soundness for CR and Cut is obtained in the same way.

5 Completeness

I prove the completeness of S_C by showing the completeness of S_C+Cut and showing that Cut is admissible in S_C. I start with a very limited form of completeness and then use it to show full completeness.

Lemma 4. *Let φ be any $\mathcal{L}_C$ formula and let Φ be a set of propositional variables that includes all the variables that occur in φ and let Φ_1, Φ_2 be any partition of Φ. Then $\Phi_1, \varphi \models_C \Phi_2$ implies that $\Phi_1, \varphi \vdash \Phi_2$ is derivable in S_C+Cut and $\Phi_1 \models_C \varphi, \Phi_2$ implies that $\Phi_1 \vdash \varphi, \Phi_2$ is derivable in S_C+Cut.*

Proof. I give the proof for the case where $\Phi_1, \varphi \models_C \Phi_2$. The other case is analogous by the duality of the left and right side. Suppose that φ, Φ, Φ_1 and Φ_2 are as in the lemma and $\Phi_1, \varphi \models_C \Phi_2$. To show is that $\Phi_1, \varphi \vdash \Phi_2$ is derivable.

The proof now proceeds by induction on the construction of φ. First suppose φ is atomic, so $\varphi = p$ for some $p \in \mathcal{P}$. Then $\varphi \in \Phi_2$ so $\Phi_1, \varphi \vdash \Phi_2$ is derivable by using Ax to obtain $p \vdash p$ and subsequently weakening.

Suppose therefore as induction hypothesis that φ is not atomic, and that the lemma holds for all subformulas of φ. Then $\varphi = \circ(\varphi_1, \cdots, \varphi_{r_\circ})$ for some $\circ \in C$ and $\mathcal{L}_C$ formulas $\varphi_1, \cdots, \varphi_{r_\circ}$. Let $N_1 = \{i \in \{1, \cdots, r_\circ\} \mid \Phi_1, \varphi_i \models \Phi_2\}$, $N_2 = \{i \in \{1, \cdots, r_\circ\} \mid \Phi_1 \models \varphi_i, \Phi_2\}$ and $N_3 = \{1, \cdots, r_\circ\} \setminus (N_1 \cup N_2)$.

For $i \in N_1$ let U_i be the sequent $\Phi_1, \varphi_i \vdash \Phi_2$ and for $i \in N_2$ let U_i be the sequent $\Phi_1 \vdash \varphi_i, \Phi_2$. Then by the induction hypothesis U_i is derivable for $i \in N_1 \cup N_2$. Now take any $i \in N_3$. Then the value of φ_i under the partition Φ_1, Φ_2 depends on the chosen disambiguation of φ_i. But for every disambiguation $\overline{\varphi}$ of φ we have $\Phi_1, \overline{\varphi} \models_{\overline{C}} \Phi_2$. This implies that, given the (fixed) values of φ_j with $j \in (N_1 \cup N_2)$ the value of φ is determinate and independent of the values of φ_i with $i \in N_3$.

Let $v = (v_1, \cdots, v_{r_\circ}) \in \{0,1\}^{r_\circ}$ be any vector such that $v_i = 1$ if $i \in N_1$ and $v_i = 0$ if $i \in N_2$ and $v' = (v'_1, \cdots, v'_{r_\circ}) \in \{0, 1, ?\}^{r_\circ}$ the vector such that $v'_i = 1$ if $i \in N_1$, $v'_i = 0$ if $i \in N_2$ and $v'_i = ?$ if $i \in N_3$. The value of φ is independent of the values of φ_i with $i \in N_3$ so the rule

$$\frac{\{U_i \mid 1 \leq i \leq r_\circ\}}{\Phi_1, \varphi \vdash \Phi_2} \, \mathrm{R}_{\circ,v}$$

was added in step 2 of Definition 11. This is true regardless of the choice of v_i for $i \in N_3$, so in step 3 of Definition 11 a rule

$$\frac{\{U_i \mid i \in N_1 \cup N_2\}}{\Phi_1, \varphi \vdash \Phi_2} \, \mathrm{R}_{\circ,v'}$$

is generated. It is possible that $\mathrm{R}_{\circ,v'}$ is removed in step 4, but then there is a rule $\mathrm{R}_{\circ,v''}$ that takes a subset of $\{U_i \mid i \in N_1 \cup N_2\}$ as premises and has $\Phi_1, \varphi \vdash \Phi_2$ as conclusion. So whether or not $\mathrm{R}_{\circ,v'}$ gets removed in step 4 it follows from the fact that U_i is derivable for $i \in N_1 \cup N_2$ that $\Phi_1, \varphi \vdash \Phi_2$ is derivable. This completes the induction step and thereby the proof.

Theorem 1 (Weak completeness of S_C+Cut). *For every finite multisets Γ, Δ of $\mathcal{L}_C$ formulas we have that $\Gamma \models_C \Delta$ implies that $\Gamma \vdash \Delta$ is derivable in S_C+Cut.*

Proof. Let $\Phi = \{p_1, \cdots, p_n\}$ be the set of propositional variables that occur in either Γ or Δ, and Φ_1, Φ_2 any partition of Φ. Then from Lemma 3 it follows that there either is a $\gamma \in \Gamma$ such that $\Phi_1, \gamma \models_C \Phi_2$ or a $\delta \in \Delta$ such that $\Phi_1 \models_C \delta, \Phi_2$.

From Lemma 4 it follows that in the first case the sequent $\Phi_1, \gamma \vdash \Phi_2$ is derivable and in the second case the sequent $\Phi_1 \vdash \delta, \Phi_2$ is derivable. In either case the sequent $\Gamma, \Phi_1 \vdash \Phi_2, \Delta$ can then be derived by weakening.

So for every partition Φ_1, Φ_2 of Φ the sequent $\Gamma, \Phi_1 \vdash \Phi_2, \Delta$ is derivable. For $0 \leq m \leq n$ let $\Phi^m = \{p_1, \cdots, p_{n-m}\}$. The proof now proceeds by induction on m. I just showed that if $m = 0$ then for every partition Φ_1^m, Φ_2^m of Φ^m the sequent $\Gamma, \Phi_1^m \vdash \Phi_2^m, \Delta$ is derivable.

Suppose then as induction hypothesis that $m > 0$ and that for partitions $\Phi_1^{m-1}, \Phi_2^{m-1}$ of Φ^{m-1} the sequent $\Gamma, \Phi_1^{m-1} \vdash \Phi_2^{m-1}, \Delta$ is derivable. Let Φ_1^m, Φ_2^m be any partition of Φ^m. Then both $\Phi_1^m \cup \{p_{n-m+1}\}, \Phi_2$ and $\Phi_1^m, \Phi_2 \cup \{p_{n-m+1}\}$ are partitions of Φ^{m-1} so $\Gamma, \Phi_1^m, p_{n-m+1} \vdash \Phi_2^m, \Delta$ and $\Gamma, \Phi_1^m \vdash p_{n-m+1}, \Phi_2^m, \Delta$ are both derivable. By using Cut (followed by CL and CR to get rid of extra copies of Γ, Δ, Φ_1^m and Φ_2^m) the sequent $\Gamma, \Phi_1^m \vdash \Phi_2^m, \Delta$ is then also derivable. This completes the induction step, so $\Gamma, \Phi_1^m \vdash \Phi_2^m, \Delta$ is derivable for any m and any partition Φ_1^m, Φ_2^m of Φ^m. Taking $m = n$ we then get $\Gamma \vdash \Delta$ being derivable, which is what was to be shown.

Left to show now is that Cut is admissible in S_C. The proof I give here is very similar to existing proofs for Cut-elimination as given in for example [8–10].

Theorem 2 (Cut elimination). *The rule Cut is admissible in S_C.*

Proof. The proof is by a case distinction on the rule R preceding the application of Cut. In all possible cases the application of Cut could be “moved up”; that is, it would have been possible to either apply Cut before R or to eliminate the Cut entirely. Since Cut cannot be applied before the first step of a proof this implies that at some point the Cut must be removed, so Cut is admissible.

Most of the cases are as in the existing Cut-elimination proofs. I omit those cases, for details see the proofs in for example [8–10]. The case that is different from the existing proofs is if both premises for the Cut rule are obtained using a $R_{\circ,v}$ rule where the Cut formula is principal. The last few steps of T are then

$$\dfrac{\dfrac{\Gamma_1, \pm\varphi_1 \vdash \mp\varphi_1, \Delta_1 \qquad \cdots \qquad \Gamma_1, \pm\varphi_{r_\circ} \vdash \mp\varphi_{r_\circ}, \Delta_1}{\Gamma_1 \vdash \varphi, \Delta_1}\,\mathrm{R}_{\circ,v} \qquad \dfrac{\Gamma_2, \pm\varphi_1 \vdash \mp\varphi_1, \Delta_2 \qquad \cdots \qquad \Gamma_2, \pm\varphi_{r_\circ} \vdash \mp\varphi_{r_\circ}, \Delta_2}{\Gamma_2, \varphi \vdash \Delta_2}\,\mathrm{R}_{\circ,v'}}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\,\mathrm{Cut}$$

where $\varphi = \circ(\varphi_1, \cdots, \varphi_{r_\circ})$. The application of $\mathrm{R}_{\circ,v}$ adds a φ on the right side of the $\vdash$, the application of $\mathrm{R}_{\circ,v'}$ adds a φ on the left side of the $\vdash$. The rules $\mathrm{R}_{\circ,v}$ and $\mathrm{R}_{\circ,v'}$ must therefore be different, so $v \neq v'$. This implies that there is at least one i such that φ_i occurs on one side of the $\vdash$ in a premise $\Gamma_1, \pm\varphi_i \vdash \mp\varphi_1, \Delta_1$ and on the other side in a premise $\Gamma_2, \mp\varphi_1 \vdash \pm\varphi_1, \Delta_2$. So

$$\dfrac{\Gamma_1, \pm\varphi_i \vdash \mp\varphi_1, \Delta_1 \qquad \Gamma_2, \mp\varphi_1 \vdash \pm\varphi_1, \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\,\mathrm{Cut}$$

is an alternative derivation of $\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2$. The Cut could therefore have been applied before the $\mathrm{R}_{\circ,v}$ rules, which is what was to be shown.

Corollary 1 (Strong completeness of S_C). *For every multisets Γ, Δ of $\mathcal{L}_C$ formulas we have that $\Gamma \models_C \Delta$ implies that $\Gamma \vdash \Delta$ is derivable in S_C.*

Proof. By the compactness Lemma 2 there are finite multisets $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ such that $\Gamma' \models_C \Delta'$, so by Theorems 1 and 2 the sequent $\Gamma' \vdash \Delta'$ is derivable in S_C. Then $\Gamma \vdash \Delta$ is also derivable in S_C by weakening from $\Gamma' \vdash \Delta'$.

6 Properties and applications of S_C

Let us consider a few of the properties of S_C. Reflexivity is not admissible in S_C, because it is not sound for $\models_C$. Likewise, if $\leftrightarrow$ is a classical bi-implication and $[\varphi/\psi]$ represents the substitution of ψ for φ the rule

$$\frac{\Gamma \vdash \Delta}{\varphi \leftrightarrow \psi, \Gamma[\varphi/\psi] \vdash \Delta[\varphi/\psi]}$$

representing a very strong kind of substitution of equivalents is not admissible. Two weaker kinds of substitution of equivalents are admissible though.

Lemma 5. *If $\leftrightarrow$ is the classical bi-implication and $\leftrightarrow \in C$ the rule Substitution of Deterministic Equivalents (EqDet) given by*

$$\frac{\varphi \vdash \varphi \qquad \psi \vdash \psi \qquad \Gamma \vdash \Delta}{\varphi \leftrightarrow \psi, \Gamma[\varphi/\psi] \vdash \Delta[\varphi/\psi]}\ \textit{EqDet}$$

and the rule Substitution of Provably Equivalents (EqPr) given by

$$\frac{\varphi \vdash \psi \qquad \psi \vdash \varphi \qquad \Gamma \vdash \Delta}{\Gamma[\varphi/\psi] \vdash \Delta[\varphi/\psi]}\ \textit{EqPr}$$

are admissible.

Proof. The easiest way to see that these rules are admissible is to use the soundness and completeness of S_C. A rule is admissible in S_C if the conclusion of the rule is derivable in S_C if all the premises are. Let us first consider the rule EqDet. Suppose that $\varphi \vdash \varphi$, $\psi \vdash \psi$ and $\Gamma \vdash \Delta$ are derivable. To show is that $\varphi \leftrightarrow \psi, \Gamma[\varphi/\psi] \vdash \Delta[\varphi/\psi]$ is derivable.

By the soundness of S_C we know that $\varphi \models_C \varphi$, $\psi \models_C \psi$ and $\Gamma \models_C \Delta$. Now let $\mathcal{M}$ be a model such that for some disambiguations $\overline{\varphi}$ of φ, $\overline{\psi}$ of ψ and $\overline{\Gamma[\varphi/\psi]}$ of $\Gamma[\varphi/\psi]$ we have $\mathcal{M} \models \overline{\varphi} \leftrightarrow \overline{\psi}$ and $\mathcal{M} \models_\wedge \overline{\Gamma[\varphi/\psi]}$. From $\varphi \models_C \varphi$ it follows that for every model all disambiguations of φ have the same value. Likewise, from $\psi \models_C \psi$ it follows that all disambiguations of ψ have the same value.

Since some disambiguations of φ and ψ have the same value of $\mathcal{M}$ this implies that every disambiguation of φ has the same value as every disambiguation of ψ in $\mathcal{M}$. The disambiguations live in a deterministic truth-functional logic so we can replace any occurrence of any disambiguation of ψ by any disambiguation of φ without changing the value on $\mathcal{M}$. So from $\mathcal{M} \models_\wedge \overline{\Gamma[\varphi/\psi]}$ it follows that $\mathcal{M} \models_\wedge \overline{\Gamma}$ for some disambiguation $\overline{\Gamma}$ of Γ.

Then by $\Gamma \vdash \Delta$ we know that $\mathcal{M} \models_\vee \overline{\Delta}$ for each disambiguation $\overline{\Delta}$ of Δ. We can replace any disambiguation of φ by any disambiguation of ψ without changing the value on $\mathcal{M}$ so $\mathcal{M} \models_\vee \overline{\Delta[\varphi/\psi]}$ for any disambiguation $\overline{\Delta[\varphi/\psi]}$.

We started with any model $\mathcal{M}$ satisfying $\overline{\varphi} \leftrightarrow \overline{\psi}$ and $\overline{\Gamma[\varphi/\psi]}$ for some disambiguations and found that $\mathcal{M}$ satisfies $\overline{\Delta[\varphi/\psi]}$ for all disambiguations, so $\varphi \leftrightarrow \psi, \Gamma[\varphi/\psi] \models_C \Delta[\varphi/\psi]$. By completeness this implies that $\varphi \leftrightarrow \psi, \Gamma[\varphi/\psi] \vdash \Delta[\varphi/\psi]$ is derivable which is what was to be shown.

Left to show is that ExPr is admissible. Suppose towards a contradiction that $\varphi \vdash \psi$, $\psi \vdash \varphi$ and $\Gamma \vdash \Delta$ are derivable in S_C but $\Gamma[\varphi/\psi] \vdash \Delta[\varphi/\psi]$ is not. Then by the soundness of S_C we have $\varphi \models_C \psi$, $\psi \models_C \varphi$ and $\Gamma \models_C \Delta$ while by the completeness of S_C we have $\Gamma[\varphi/\psi] \not\models_C \Delta[\varphi/\psi]$.

So for some disambiguations $\overline{\Gamma[\varphi/\psi]}$ of $\Gamma[\varphi/\psi]$ and $\overline{\Delta[\varphi/\psi]}$ of $\Delta[\varphi/\psi]$ we have $\overline{\Gamma[\varphi/\psi]} \not\models_{\overline{C}} \overline{\Delta[\varphi/\psi]}$. There are disambiguations $\overline{\Gamma}$ of Γ, $\overline{\Delta}$ of Δ, $\overline{\varphi}$ of φ and $\overline{\psi}$ of ψ such that $\overline{\Gamma[\varphi/\psi]} = \overline{\Gamma}[\overline{\varphi}/\overline{\psi}]$ and $\overline{\Delta[\varphi/\psi]} = \overline{\Delta}[\overline{\varphi}/\overline{\psi}]$.

From $\varphi \models_C \psi$ and $\psi \models_C \varphi$ it follows that any disambiguations of φ and ψ are equivalent, so in particular $\overline{\varphi}$ and $\overline{\psi}$ are equivalent. But from $\Gamma \models_C \Delta$ it follows that $\overline{\Gamma} \models_{\overline{C}} \overline{\Delta}$ and by substitution of equivalents in deterministic propositional logic this implies that $\overline{\Gamma}[\overline{\varphi}/\overline{\psi}] \models_{\overline{C}} \overline{\Delta}[\overline{\varphi}/\overline{\psi}]$, which contradicts $\overline{\Gamma[\varphi/\psi]} \not\models_{\overline{C}} \overline{\Delta[\varphi/\psi]}$. Our initial assumption that $\varphi \vdash \psi$, $\psi \vdash \varphi$ and $\Gamma \vdash \Delta$ are derivable and $\Gamma[\varphi/\psi] \vdash \Delta[\varphi/\psi]$ is not must therefore be false. So if $\varphi \vdash \psi$, $\psi \vdash \varphi$ and $\Gamma \vdash \Delta$ are derivable then so is $\Gamma[\varphi/\psi] \vdash \Delta[\varphi/\psi]$.

7 Conclusion

I introduced nondeterministic semantics for propositional logic that do not satisfy reflexivity. The main idea of the semantics is to use deterministic disambiguations of nondeterministic formulas and to say that $\Gamma \models_C \Delta$ if and only if $\overline{\Gamma} \models_{\overline{C}} \overline{\Delta}$ for all possible disambiguations $\overline{\Gamma}$ of Γ and $\overline{\Delta}$ of Δ. I also introduced a sequent-style proof system S_C that is sound and complete for $\models_C$ and showed that S_C allows some types of substitution of equivalents.

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