THE EXPRESSIVITY OF FACTUAL CHANGE IN DYNAMIC EPISTEMIC LOGIC

LOUWE B. KUIJER

Abstract. A commonly used dynamic epistemic logic is one obtained by adding common knowledge and public announcements to a basic epistemic logic. It is known from [8] that adding public substitutions to such a logic adds expressivity over the class **K** of models. Here I show that substitutions also add expressivity over the classes **KD45**, **S4** and **S5** of models.

Since the combination of common knowledge, public announcements and substitutions was shown in [8] to be equally expressive to relativised common knowledge these results also show that relativized common knowledge is more expressive than common knowledge and public announcements over **KD45**, **S4** and **S5**. These results therefore extend the result from [3] that shows that relativized common knowledge is more expressive than common knowledge and public announcements over **K**.

§1. Introduction. Two operators often added to a basic epistemic logic are *common knowledge* and *public announcements*, as in [10, 1] and many subsequent publications such as [7, 5, 3, 9, 2]. Public announcements can be used to model information change. For example, in a card game an agent a can announce to the other agents what his card is. After this announcement the other agents will know what card a holds. Public announcements cannot however be used to model changes of basic facts. For example, public announcements cannot be used to model the event where an agent gives a card to another agent.

One way to incorporate factual change in a logic is to add an operator for public *substitutions*, see for example [6, 4, 3, 8]. A public substitution $[p := \varphi]$ changes the extension of a propositional variable p to the extension of a formula φ which allows us to model changes of basic facts. Suppose we use the propositional variables p to indicate that a holds a certain card and q to indicate that b holds that card. Then the event where a gives the card to b can be modeled by first changing the extension of q to that of $p \vee q$[1] and then changing the extension of p to that of $\bot$, so by the substitutions $[q := p \vee q][p := \bot]$.

One important question when considering related logics—such as the ones that differ only in whether they allow substitutions—is what their relative expressivity is. That is, is it possible to translate formulas from one logic to the other in such

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[1] The extension of q should be changed to that of $p \vee q$ in stead of simply $\top$ or p to account for the possibility that a does not have the card to begin with in which case the action fails. Agent b ends up holding the card if and only if either agent a held it and then gave it to him, or b himself held the card and the action failed.

a way that truth is preserved? If one logic is more expressive than another then the less expressive logic can be simulated in the more expressive one. When comparing logics that differ only in whether or not they allow one operator this means that the operator adds something fundamentally new to the logic if and only if the logic with the extra operator is more expressive than the one without it.[2]

It has been shown in [8] that if a logic $\mathcal{L}$ is one of basic epistemic logic, basic epistemic logic with common knowledge or basic epistemic logic with public announcements then the logic $\mathcal{L}_S$ obtained by adding substitutions is equally expressive as $\mathcal{L}$. So for any $\mathcal{L}_S$ formula φ there is an $\mathcal{L}$ formula φ' such that $\models \varphi \leftrightarrow \varphi'$. We can therefore simulate substitutions using only $\mathcal{L}$.

This changes if we start with a logic $\mathcal{L}_{CP}$ that uses both common knowledge and public announcements. In [8] it was shown that the logic $\mathcal{L}_{CPS}$ that additionally uses substitutions is strictly more expressive than $\mathcal{L}_{CP}$ over the class **K** of models.

However, considering that we are working in an *epistemic* logic it is a salient question whether the same holds if the class of models is taken to be one of **KD45**, **S4** or **S5**, as these classes are often used when modeling knowledge or belief.

The proof in [8] is by showing that over **K** the logic $\mathcal{L}_{CPS}$ is equally expressive to a logic $\mathcal{L}_R$ that is obtained by adding a different operator representing *relativized common knowledge*[3] to a basic epistemic logic. The logic $\mathcal{L}_R$ was shown in [3] to be more expressive than normal common knowledge with public announcements over **K**. So if $\mathcal{L}_{CPS}$ is equally expressive to $\mathcal{L}_R$ then $\mathcal{L}_{CPS}$ is more expressive than $\mathcal{L}_{CP}$ over **K**. However, the proof from [3] does not work for **KD45**, **S4** or **S5**, and up until now it has remained an open question whether substitutions add any expressivity to $\mathcal{L}_{CP}$ in **KD45**, **S4** or **S5**.

Here I show that substitutions do add expressivity to $\mathcal{L}_{CP}$ over **KD45** and **S4** if there are at least two agents and over **S5** if there are at least three agents. By the equal expressivity of $\mathcal{L}_{CPS}$ and $\mathcal{L}_R$ proven in [8] this also shows that relativized common knowledge is more expressive than the combination of common knowledge and public announcements over **KD45**, **S4** and **S5** if the set of agents is sufficiently large.

Note that the nonexistence of a translation from one logic to another that is truth-preserving for a class of models implies the nonexistence of a truth-preserving translation for a superclass of that class of models. As such, the fact that substitutions add expressivity to $\mathcal{L}_{CP}$ over **S5** implies that they add expressivity over **KD45**, **S4** and **K** as well. Likewise, the results that substitutions add expressivity over **KD45** and **S4** both imply that they add expressivity over

[2] Of course there can be reasons to use an operator that does not add expressivity to the logic. Such an operator might for example allow formulas to be expressed more succinctly or in a more natural way. Consider propositional logic. The operators $\wedge$ and $\rightarrow$ do not add expressivity to a logic that already has $\neg$ and $\vee$ but we still usually add $\wedge$ and $\rightarrow$, albeit often only as abbreviations.

[3] A relativized common knowledge formula $C_B(\varphi, \psi)$ can be read as "after announcing φ it is common knowledge among B that ψ used to be the case before the announcement" or "if we delete all $\neg\varphi$ worlds, then go to any B-reachable world and un-delete the previously deleted worlds φ will hold in that world".

K. However, the expressivity result for **S5** requires at least three agents, the expressivity results for **KD45** and **S4** require at least two agents and the result for **K** even works if there is only a single agent. As such it is useful to have separate proofs for each of the results.

In Section 2 I give definitions of the logics under consideration and in Section 3 I introduce the concepts needed to compare the logics on expressivity. In Section 4 I use reduction axioms to show the form $\mathcal{L}_{CPS}$ formulas must have in order to be untranslatable to $\mathcal{L}_{CP}$. In Section 5 I show that $\mathcal{L}_{CPS}$ is more expressive than $\mathcal{L}_{CP}$ over **KD45** and **S4**. Finally, in Section 6 I show that $\mathcal{L}_{CPS}$ is more expressive than $\mathcal{L}_{CP}$ over **S5**.

§2. Language, models and semantics. For technical reasons related to ease of notation it is convenient to define the logic $\mathcal{L}_{CPS}$ and then consider $\mathcal{L}_{CP}$ as a fragment of that logic. First, let us consider the formulas of $\mathcal{L}_{CPS}$. Let a finite nonempty set $\mathcal{A}$ of agents and a countably infinite set $\mathcal{P}$ of propositional variables be given.

Definition 1. The *formulas* of $\mathcal{L}_{CPS}$ are given by

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \vee \varphi) \mid \Box_a\varphi \mid C_B\varphi \mid [\varphi]\varphi \mid [p := \varphi]\varphi$$

where $p \in \mathcal{P}$, $B \subseteq \mathcal{A}$ and $a \in \mathcal{A}$.

I use $\wedge, \rightarrow, \leftrightarrow, \top, \bot, \bigvee, \bigwedge$ and $\Diamond_a$ in the usual way as abbreviations and omit parentheses where this should not cause confusion. The intended reading of the non-Boolean operators is as follows:

- $\Box_a\varphi$ is read as “agent a knows that φ”,
- $C_B\varphi$ is read as “it is common knowledge among the group B of agents that φ”,
- $[\varphi]\psi$ is read as “after φ is publicly announced ψ holds” and
- $[p := \varphi]\psi$ is read as “after changing the extension of p to that of φ the formula ψ holds”.

Note that in this definition substitutions can only have one assignment like in [4], as opposed to the multiple simultaneous substitutions that are allowed in [6, 3, 8]. This restriction does not however limit the expressivity of the logic, as every formula with simultaneous substitutions can be translated to an equivalent one containing only single-assignment substitutions. To see why this is the case, consider a formula $\psi = [p_1 := \varphi_1, \cdots, p_n := \varphi_n]\varphi$ containing multiple simultaneous substitutions. The set $\mathcal{P}$ is infinite, so we can take n variables $q_1, \cdots, q_n \in \mathcal{P}$ that do not occur in ψ and use $q_1, \cdots, q_n$ to temporarily store the values of $\varphi_1, \cdots, \varphi_n$. This allows us to construct $\psi' = [q_1 := \varphi_1] \cdots [q_n := \varphi_n][p_1 := q_1] \cdots [p_n := q_n]\varphi$ that is equivalent to ψ but does not use simultaneous substitutions.

For some of the proofs we also need a concept of depth for formulas.

Definition 2. Let φ, ψ be any $\mathcal{L}_{CPS}$ formulas. The depth $d(\varphi)$ of φ is given inductively by

- $d(p) = 0$ for $p \in \mathcal{P}$
- $d(\neg\varphi) = d(\varphi)$

- $d(\varphi_1 \vee \varphi_2) = \max(d(\varphi), d(\psi))$
- $d(\Box_a \varphi) = d(\varphi) + 1$
- $d(C_B \varphi) = d(\varphi) + 1$
- $d([\varphi]\psi) = \max(d(\varphi), d(\psi)) + 1$
- $d([p := \varphi]\psi) = \max(d(\varphi), d(\psi)) + 1$

A formula φ is of *pure depth* n if $d(\varphi) = n$ and there is no subformula φ' of φ such that $d(\varphi') = n$.

Note that the formulas of pure depth are the formulas that have a non-Boolean main operator. The models for $\mathcal{L}_{CPS}$ are the usual multi-agent Kripke structures.

Definition 3. A *model* $\mathcal{M}$ is a triple $\mathcal{M} = (W, R, v)$ where W is a set of possible worlds, $R : \mathcal{A} \to \wp(W \times W)$ is an accessibility relation and $v : \mathcal{P} \to \wp(W)$ is a valuation.

A model $\mathcal{M} = (W, R, v)$ is a **KD45** *model* if for each $a \in \mathcal{A}$ the relation $R(a)$ is transitive, serial and euclidean.

A model $\mathcal{M} = (W, R, v)$ is an **S4** *model* if for each $a \in \mathcal{A}$ the relation $R(a)$ is reflexive and transitive.

A model $\mathcal{M} = (W, R, v)$ is an **S5** *model* if for each $a \in \mathcal{A}$ the relation $R(a)$ is an equivalence relation.

Now we can define the semantics of $\mathcal{L}_{CPS}$.

Definition 4. Given a model $\mathcal{M} = (W, R, v)$, a world w of $\mathcal{M}$ and φ, ψ formulas of $\mathcal{L}_{CPS}$ define the *satisfaction relation* $\models$ by

$$
\begin{array}{lll}
\mathcal{M}, w \models p & \Leftrightarrow & w \in v(p) \\
\mathcal{M}, w \models \neg\varphi & \Leftrightarrow & \mathcal{M}, w \not\models \varphi \\
\mathcal{M}, w \models \varphi \vee \psi & \Leftrightarrow & \mathcal{M}, w \models \varphi \text{ or } \mathcal{M}, w \models \psi \\
\mathcal{M}, w \models \Box_a \varphi & \Leftrightarrow & \mathcal{M}, w' \models \varphi \text{ for all } w' \in W \text{ such that } (w, w') \in R(a) \\
\mathcal{M}, w \models C_B \varphi & \Leftrightarrow & \mathcal{M}, w' \models \varphi \text{ for all } w' \in W \text{ such that } (w, w') \in R(B)^* \\
\mathcal{M}, w \models [\varphi]\psi & \Leftrightarrow & \mathcal{M}, w \models \varphi \text{ implies that } \mathcal{M}_{[\varphi]}, w \models \psi \\
\mathcal{M}, w \models [p := \varphi]\psi & \Leftrightarrow & \mathcal{M}_{[p:=\varphi]}, w \models \psi.
\end{array}
$$

where $R(B)^*$ is the reflexive transitive closure of $\bigcup_{a \in B} R(a)$. The *updated models* are given by

- $\mathcal{M}_{[\varphi]} = (W_{[\varphi]}, R_{[\varphi]}, v_{[\varphi]})$ with $W_{[\varphi]} = \{w \in W \mid \mathcal{M}, w \models \varphi\}$, $R_{[\varphi]}(a) = R(a) \cap (W_{[\varphi]} \times W_{[\varphi]})$ for all $a \in \mathcal{A}$ and $v_{[\varphi]}(p) = v(p) \cap W_{[\varphi]}$ for all $p \in \mathcal{P}$,
- $\mathcal{M}_{[p:=\varphi]} = (W, R, v_{[p:=\varphi]})$ with $v_{[p:=\varphi]}(p) = \{w \in W \mid \mathcal{M}, w \models \varphi\}$ and $v_{[p:=\varphi]}(q) = v(q)$ for all $q \in \mathcal{P} \setminus \{p\}$.

Write $\mathcal{M} \models \varphi$ if $\mathcal{M}, w \models \varphi$ for every $w \in W$, $\models \varphi$ if $\mathcal{M} \models \varphi$ for every model $\mathcal{M}$. Furthermore, for $\mathbf{I} \in \{\mathbf{K}, \mathbf{KD45}, \mathbf{S4}, \mathbf{S5}\}$ write $\models_{\mathbf{I}} \varphi$ if $\mathcal{M} \models \varphi$ for every $\mathbf{I}$ model $\mathcal{M}$.

Unfortunately the definition of the new models created by the model changing operators $[\varphi]\psi$ and $[p := \varphi]\psi$ requires some rather complicated notation. It is however possible to give a relatively simple informal description of the new models. The model $\mathcal{M}_{[\varphi]}$ is the result of removing all $\neg\varphi$ worlds from $\mathcal{M}$. The accessibility relations and valuation of $\mathcal{M}_{[\varphi]}$ are unchanged from $\mathcal{M}$ except that they are restricted to the new set of worlds. The model $\mathcal{M}_{[p:=\varphi]}$ is the result of changing the value of p in every world to the value φ has in that world in the model $\mathcal{M}$. The set of worlds and accessibility relations of $\mathcal{M}_{[p:=\varphi]}$ are unchanged from $\mathcal{M}$.

There is one thing that should be noted about the interaction between public announcements and the class **KD45** of models: **KD45** is not closed under public announcements, unlike the classes **K**, **S4** and **S5**. In a **KD45** model $\mathcal{M}$ there may be worlds w such that $\mathcal{M}, w \models \varphi \wedge \Box_a \neg\varphi$. In such a case the model $\mathcal{M}_{[\varphi]}$ is not a **KD45** model because there are no accessible worlds for a in w so $R_{[\varphi]}(a)$ is not serial.

In the semantics as given here this is not a problem, $\mathcal{M}_{[\varphi]}$ may not be a **KD45** model but it is a **K** model so it is defined whether $\mathcal{M}_{[\varphi]}, w \models \psi$. In a context where only **KD45** models are available different semantics for public announcements would have to be given. The simplest alternative semantics for such a context are obtained by requiring public announcements to be not only truthful but also consistent with the belief set of every agent. The result about **KD45** presented in this paper still holds under these alternative semantics.

Substitutions can be used to model a wide variety of factual changes. Examples found in the literature include washing a child in the muddy children problem [4], swapping or giving away cards in a card game [4, 3] and opening a window [3]. The property of substitutions that I use for the proofs in this paper is that they can be used to add *memory* to public announcements, as suggested in [8].

In general public announcements are 'destructive'. After a public announcement $[\varphi]$ has been made it is no longer possible to recover the epistemic state of the agents prior to the announcement. It is for example impossible to determine from an updated model $\mathcal{M}_{[p]}$ whether or not for a given world w of $\mathcal{M}_{[p]}$ it holds that $\mathcal{M}, w \models \Box_a p$. Substitutions can be used to store information about the epistemic state before the announcement. From the model $\mathcal{M}_{[q:=\Box_a p][p]}$ it can be determined whether or not for a given world w it holds that $\mathcal{M}, w \models \Box_a p$; we have $\mathcal{M}, w \models \Box_a p$ if and only if $\mathcal{M}_{[q:=\Box_a p][p]}, w \models q$. The substitution $[q := \Box_a p]$ in this case can be thought of as the change caused by agent a writing a note about whether she knows p. The announcement $[p]$ destroys all information about whether a knew that p before the announcement, but the agents can regain this information by looking at the note.

§3. Expressivity. As mentioned above the logic $\mathcal{L}_{CP}$ can be defined as a fragment of $\mathcal{L}_{CPS}$.

Definition 5. Let C stand for common knowledge, P for public announcements and S for substitutions. Then for any string X of characters the logic $\mathcal{L}_X$ is the fragment of $\mathcal{L}_{CPS}$ that uses only the operators $\vee, \neg, \Box_a$ and the operators corresponding to a letter that occurs in X.

We now have all we need to define expressivity.

Definition 6. Let X, Y be strings of characters and let $\mathbf{I} \in \{\mathbf{K}, \mathbf{KD45}, \mathbf{S4}, \mathbf{S5}\}$. We say that $\mathcal{L}_Y$ is *at least as expressive as* $\mathcal{L}_X$ over $\mathbf{I}$ if for every $\mathcal{L}_X$ formula φ there is an $\mathcal{L}_Y$ formula ψ such that $\models_{\mathbf{I}} \varphi \leftrightarrow \psi$. Denote this by $\mathcal{L}_X \preceq_{\mathbf{I}} \mathcal{L}_Y$.

Furthermore, $\mathcal{L}_Y$ is *more expressive than* $\mathcal{L}_X$, denoted by $\mathcal{L}_X \prec_{\mathbf{I}} \mathcal{L}_Y$, if $\mathcal{L}_X \preceq_{\mathbf{I}} \mathcal{L}_Y$ and $\mathcal{L}_Y \not\preceq_{\mathbf{I}} \mathcal{L}_X$. Finally, $\mathcal{L}_Y$ and $\mathcal{L}_X$ are *equally expressive*, denoted by $\mathcal{L}_X \equiv_{\mathbf{I}} \mathcal{L}_Y$, if $\mathcal{L}_X \preceq_{\mathbf{I}} \mathcal{L}_Y$ and $\mathcal{L}_Y \preceq_{\mathbf{I}} \mathcal{L}_X$.

Note that since $\mathcal{L}_X$ and $\mathcal{L}_Y$ are fragments of $\mathcal{L}_{CPS}$ the formula $\varphi \leftrightarrow \psi$ can always be seen as an $\mathcal{L}_{CPS}$ formula so it makes sense to write $\models_{\mathbf{I}} \varphi \leftrightarrow \psi$.

The next Lemma follows immediately from the definition.

Lemma 1. *For any* $\mathbf{I} \in \{\mathbf{K}, \mathbf{KD45}, \mathbf{S4}, \mathbf{S5}\}$ *we have* $\mathcal{L}_{CP} \preceq_{\mathbf{I}} \mathcal{L}_{CPS}$.

Proof. For any $\mathcal{L}_{CP}$ formula φ we have that φ is also an $\mathcal{L}_{CPS}$ formula and $\models_{\mathbf{I}} \varphi \leftrightarrow \varphi$. $\dashv$

The remaining questions are therefore whether $\mathcal{L}_{CP} \prec_{\mathbf{I}} \mathcal{L}_{CPS}$ or $\mathcal{L}_{CP} \equiv_{\mathbf{I}} \mathcal{L}_{CPS}$ for $\mathbf{I} = \mathbf{K}$, $\mathbf{I} = \mathbf{KD45}$, $\mathbf{I} = \mathbf{S4}$ and $\mathbf{I} = \mathbf{S5}$.

The first of these four questions was answered in [8] where it was shown that $\mathcal{L}_{CP} \prec_{\mathbf{K}} \mathcal{L}_{CPS}$. The other three questions are answered in this paper, where I show that $\mathcal{L}_{CP} \prec_{\mathbf{KD45}} \mathcal{L}_{CPS}$, $\mathcal{L}_{CP} \prec_{\mathbf{S4}} \mathcal{L}_{CPS}$ and $\mathcal{L}_{CP} \prec_{\mathbf{S5}} \mathcal{L}_{CPS}$.

§4. Reduction. One of the main tools in the study of expressivity is the use of so-called *reduction axioms*.[4] A reduction axiom for an operator X is a validity of the form $\models \varphi \leftrightarrow \psi$ where ψ either contains fewer instances of the operator X than φ, or the formulas inside the scope of X in ψ are less complex[5] than the formulas inside the scope of X in φ. Reduction axioms can be used to reduce the complexity of the formulas inside the scope of X and sometimes even remove instances of X entirely. The reduction axioms that are relevant to the logics under consideration are the following.

Lemma 2. *For any $\mathcal{L}_{CPS}$ formulas $\psi, \psi_1, \psi_2, \psi_3$, any $p \in \mathcal{P}$, any $a \in \mathcal{A}$ and any $B \subseteq \mathcal{A}$ the following statements hold:*

1. $\models [\psi]p \leftrightarrow (\psi \rightarrow p)$,
2. $\models [\psi_1]\neg\psi_2 \leftrightarrow (\psi_1 \rightarrow \neg[\psi_1]\psi_2)$,
3. $\models [\psi_1](\psi_2 \vee \psi_3) \leftrightarrow ([\psi_1]\psi_2 \vee [\psi_1]\psi_3)$,
4. $\models [\psi_1]\Box_a\psi_2 \leftrightarrow (\psi_1 \rightarrow \Box_a[\psi_1]\psi_2)$,
5. $\models [\psi_1][\psi_2]\psi_3 \leftrightarrow [\psi_1 \wedge [\psi_1]\psi_2]\psi_3$,
6. $\models [p := \psi]p \leftrightarrow \psi$,
7. $\models [p := \psi]q \leftrightarrow q$ *for* $q \neq p$,
8. $\models [p := \psi_1]\neg\psi_2 \leftrightarrow \neg[p := \psi_1]\psi_2$,
9. $\models [p := \psi_1](\psi_2 \vee \psi_3) \leftrightarrow ([p := \psi_1]\psi_2 \vee [p := \psi_1]\psi_3)$,
10. $\models [p := \psi_1]\Box_a\psi_2 \leftrightarrow \Box_a[p := \psi_1]\psi_2$ *and*
11. $\models [p := \psi_1]C_B\psi_2 \leftrightarrow C_B[p := \psi_1]\psi_2$.

The first five validities of the Lemma were introduced in [10], the latter six in [8]. They are also straightforward to verify using the semantics of $\mathcal{L}_{CPS}$.

All validities in Lemma 2 are reduction axioms for either public announcements or substitutions. It either holds that the formula on the right-hand side of the

[4]Comparing expressivity is by no means the only use of reduction axioms. In fact, they derive their name from the fact that if you add the reduction axioms for a certain operator X to a complete axiomatization of a logic $\mathcal{L}$ that does not use X the result is in many cases a complete axiomatization of the logic $\mathcal{L}_X$ obtained by adding X as an additional operator to $\mathcal{L}$.

[5]In the sense of requiring fewer steps in the inductive definition of formulas.

equivalence contains fewer public announcement or substitution operators, or that the formulas inside the scope of the public announcement or substitution operator on the right-hand side of the equivalence have a lower complexity than the ones on the left-hand side.

There are three combinations of operators for which there is no reduction axiom; there are no reduction axioms for the combination of a public announcement and common knowledge, the combination of a substitution and a public announcement and the combination of a substitution with another substitution. The lack of a reduction axiom for a substitution with another substitution is not important for this paper, but the fact that there are no reduction axioms for the other two combinations is important for the construction of an $\mathcal{L}_{CPS}$ formula that cannot be translated to $\mathcal{L}_{CP}$.

By repeatedly applying these validities it is possible to remove any public announcement or substitution operator $[X]$, as long a reduction axiom exists for $[X]$ and every operator inside the scope of $[X]$. For example, a formula $[p := \psi]\Box_a(p \vee q)$ is equivalent to $\Box_a[p := \psi](p \vee q)$, which is equivalent to $\Box_a([p := \psi]p \vee [p := \psi]q)$ which is equivalent to $\Box_a(\psi \vee q)$.

This allows us to say some things about the relative expressivity of certain extensions of a basic epistemic logic $\mathcal{L}$. The reduction axioms are used in [10] and [8] to show that

- $\mathcal{L} \equiv_{\mathbf{K}} \mathcal{L}_P$ because we can remove the public announcements from any $\mathcal{L}_P$ formula using the first four reduction axioms,
- $\mathcal{L} \equiv_{\mathbf{K}} \mathcal{L}_S$ because we can remove the substitutions from any $\mathcal{L}_S$ formula using the sixth to tenth reduction axioms,
- $\mathcal{L}_C \equiv_{\mathbf{K}} \mathcal{L}_{CS}$ because we can remove the substitutions from any $\mathcal{L}_{CS}$ formula using the sixth to eleventh reduction axioms,
- $\mathcal{L} \equiv_{\mathbf{K}} \mathcal{L}_{PS}$ because, even though we cannot change the order of a public announcement operator and a substitution operator, we can always remove the innermost public announcement or substitution operator using either the first four or the sixth to tenth reduction axioms.

The reduction axioms can also help determine where to look if we want to find an $\mathcal{L}_{CPS}$ formula that cannot be translated to $\mathcal{L}_{CP}$. Any $\mathcal{L}_{CPS}$ formula that cannot be translated to $\mathcal{L}_{CP}$ must obviously contain a substitution operator. Furthermore, since $\mathcal{L}_C \equiv_{\mathbf{K}} \mathcal{L}_{CS}$ the untranslatable formula must also contain at least one public announcement and since $\mathcal{L} \equiv_{\mathbf{K}} \mathcal{L}_{PS}$ it must contain at least one common knowledge operator.

Furthermore, using the reduction axioms we can move the substitution operator inward until it reaches a public announcement operator so we can assume without loss of generality that there is a substitution operator immediately preceding a public announcement. Likewise, the public announcement operator can be moved inward until it reaches a common knowledge operator. The untranslatable formula must therefore contain a subformula $[p := \varphi][\psi_1] \cdots [\psi_n]C_B\chi$. Finally we can collapse $[\psi_1] \cdots [\psi_n]$ into a single announcement ψ.

So if there is an untranslatable formula then there is an untranslatable formula of the form $[p := \varphi][\psi]C_B\chi$. This is a particular instance of the use of substitutions as memory; the propositional variable p is used to store the value of φ so that it is possible after the announcement of ψ to remember whether φ held before the announcement.[6]

The formula that is used to show that $\mathcal{L}_{CP} \prec_{\mathbf{K}} \mathcal{L}_{CPS}$ is indeed equivalent to a formula of such a form, namely $[q := \neg\Box_a p][p]C_{\mathcal{A}}q$. This formula cannot however be used to show that $\mathcal{L}_{CP} \prec_{\mathbf{S5}} \mathcal{L}_{CPS}$, since there is an $\mathcal{L}_{CP}$ formula that is equivalent to $[q := \neg\Box_a p][p]C_{\mathcal{A}}q$ on all **S5** models.[7] For the **S5** case we therefore need a slightly more complicated formula. The one I use is $[q := C_{\{a,b\}}p][p]C_{\mathcal{A}}\neg q$. For the **S4** case the simpler formula $[q := \neg\Box_a p][p]C_{\mathcal{A}}q$ of the **K** case could be used. However, in order to keep the proof for the **S4** case similar to the proof of the **KD45** and **S5** cases it is more convenient to use the formula $[q := C_{\mathcal{A}}p][p]C_{\mathcal{A}}\neg q$.

§5. The expressivity of factual change over KD45 and S4. With the preliminaries out of the way let us consider the proof that $\mathcal{L}_{CP} \prec_{\mathbf{KD45}} \mathcal{L}_{CPS}$ and $\mathcal{L}_{CP} \prec_{\mathbf{S4}} \mathcal{L}_{CPS}$. I first state the theorem, then give some auxiliary definitions and lemmas and finally give a proof of the theorem.

Theorem 1. *If* $|\mathcal{A}| \geq 2$ *then the logic* $\mathcal{L}_{CPS}$ *is more expressive than the logic* $\mathcal{L}_{CP}$ *over* **KD45** *and* **S4**.

For the rest of this section assume that $|\mathcal{A}| \geq 2$. It was already shown that $\mathcal{L}_{CP} \preceq_{\mathbf{KD45}} \mathcal{L}_{CPS}$ and $\mathcal{L}_{CP} \preceq_{\mathbf{KD45}} \mathcal{L}_{CPS}$ so in order to prove Theorem 1 it remains to be shown that $\mathcal{L}_{CPS} \not\preceq_{\mathbf{KD45}} \mathcal{L}_{CP}$ and $\mathcal{L}_{CPS} \not\preceq_{\mathbf{KD45}} \mathcal{L}_{CP}$.

In order to do this it is sufficient to show that there is an $\mathcal{L}_{CPS}$ formula φ with the property that there is no $\mathcal{L}_{CP}$ formula ψ that is equivalent to φ either on all **KD45** models or on all **S4** models.

For this purpose I define sets $\{\mathcal{M}_n = (W_n, R_n, v_n) \mid n \in \mathbb{N}_{>0}\}$ and $\{\mathcal{M}'_n = (W_n, R'_n, v_n) \mid n \in \mathbb{N}_{>0}\}$ of **KD45** and **S4** models respectively such that the $\mathcal{L}_{CPS}$ formula $\varphi = [q := C_{\mathcal{A}}p][p]C_{\mathcal{A}}\neg q$ can distinguish between the worlds $s_{2n-1} \in W_n$ and $t_{2n-1} \in W_n$ for all $n \in \mathbb{N}_{>0}$ but there is no single $\mathcal{L}_{CP}$ formula that for all $n \in \mathbb{N}_{>0}$ distinguishes the worlds.

[6] Those familiar with relativized common knowledge may also recognize formulas of this form as similar to how relativized common knowledge can be simulated in $\mathcal{L}_{CPS}$; if p does not occur in ψ then $[p := \varphi][\psi]C_B p$ is equivalent to the relativized common knowledge formula $C_B(\psi, \varphi)$. This is of course no coincidence, one of the interpretations given to $C_B(\psi, \varphi)$ in [3] is "if ψ is announced it becomes common knowledge among B that φ was the case before the announcement". This is also a good description of $[p := \varphi][\psi]C_B p$ if the substitution is seen as memory.

[7] The smallest $\mathcal{L}_{CP}$ formula that I know to be equivalent to $[q := \neg\Box_a p][p]C_{\mathcal{A}}q$ on all **S5** models grows exponentially with the size of $\mathcal{A}$ and is already quite complicated for small sets of agents. To give some idea of the form of this translation, consider the case $\mathcal{A} = \{a, b, c\}$. Then the translation is $\models_{\mathbf{S5}} [q := \neg\Box_a p][p]C_{\mathcal{A}}q \leftrightarrow (\neg p \vee [\neg p \rightarrow \Diamond_a(p \wedge \bigvee_{d\in\mathcal{A}} \Diamond_d\Box_a p)](\psi_a \wedge \psi_{ab} \wedge \psi_{abc} \wedge \psi_{ac} \wedge \psi_{acb}))$ where

- $\psi_a = [\bigvee_{d\in\mathcal{A}} \Diamond_d\neg p \rightarrow \Diamond_a\neg p]C_{\mathcal{A}}p$,
- $\psi_{ab} = [\neg p \rightarrow \Diamond_b(p \wedge \bigvee_{d\in\mathcal{A}} \Diamond_d(p \wedge \neg\psi_a))][\bigvee_{d\in\mathcal{A}} \Diamond_d\neg p \rightarrow \Diamond_b\neg p]C_{\mathcal{A}}p$,
- $\psi_{abc} = [\neg p \rightarrow \Diamond_c(p \wedge \bigvee_{d\in\mathcal{A}} \Diamond_d(p \wedge \neg\psi_{ab}))][\bigvee_{d\in\mathcal{A}} \Diamond_d\neg p \rightarrow \Diamond_c\neg p]C_{\mathcal{A}}p$,
- $\psi_{ac} = [\neg p \rightarrow \Diamond_c(p \wedge \bigvee_{d\in\mathcal{A}} \Diamond_d(p \wedge \neg\psi_a))][\bigvee_{d\in\mathcal{A}} \Diamond_d\neg p \rightarrow \Diamond_c\neg p]C_{\mathcal{A}}p$,
- $\psi_{abc} = [\neg p \rightarrow \Diamond_b(p \wedge \bigvee_{d\in\mathcal{A}} \Diamond_d(p \wedge \neg\psi_{ac}))][\bigvee_{d\in\mathcal{A}} \Diamond_d\neg p \rightarrow \Diamond_b\neg p]C_{\mathcal{A}}p$.

The proof of this equivalence is long, not very hard and outside the scope of this paper. I therefore omit it.

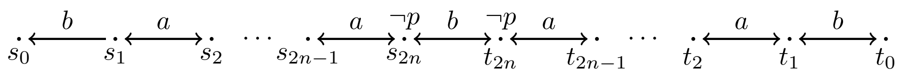

FIGURE 1. Model $\mathcal{M}_n$ or model $\mathcal{M}'_n$, depending on which worlds are accessible from themselves. In $\mathcal{M}'_n$ all worlds are accessible for themselves for all agents. In $\mathcal{M}_n$ all worlds other than s_1 are accessible from themselves for all agents and s_1 is accessible from itself for all agents other than agent b. Reflexive arrows are not drawn and p holds in every world except where it is stated otherwise. Note that the leftmost arrow is one-directional, unlike all other arrows.

The sets of models are defined as follows. Let $a, b \in \mathcal{A}$ be two distinct agents and let $p \in \mathcal{P}$. For $n \in \mathbb{N}_{>0}$ let $\mathcal{M}_n = (W_n, R_n, v_n)$ where

- $W_n = \{s_i \mid 0 \leq i \leq 2n\} \cup \{t_i \mid 0 \leq i \leq 2n\}$
- $R_n(a) = \{(s_{2i}, s_{2i-1}), (s_{2i-1}, s_{2i}), (t_{2i}, t_{2i-1}), (t_{2i-1}, t_{2i}) \mid 1 \leq i \leq n\} \cup \{(w, w) \mid w \in W_n\}$
- $R_n(b) = (\{(s_{2i}, s_{2i+1}), (s_{2i+1}, s_{2i}), (t_{2i}, t_{2i+1}), (t_{2i+1}, t_{2i}) \mid 0 \leq i \leq n-1\} \setminus \{(s_0, s_1)\})$
  $\cup \{(s_{2n}, t_{2n}), (t_{2n}, s_{2n})\} \cup \{(w, w) \mid w \in W_n \setminus \{s_1\}\}$
- $R_n(c) = \{(w, w) \mid w \in W_n\}$ for all $c \in \mathcal{A} \setminus \{a, b\}$
- $v_n(p) = W_n \setminus \{s_{2n}, t_{2n}\}$
- $v_n(q) = \emptyset$ for all $q \in \mathcal{P} \setminus \{p\}$

Furthermore, let

- $R'_n(a) = R_n(a)$
- $R'_n(b) = R_n(b) \cup \{(s_1, s_1)\}$
- $R'_n(c) = R_n(c)$ for all $c \in \mathcal{A} \setminus \{a, b\}$

and $\mathcal{M}'_n = (W_n, R'_n, v_n)$. See also Figure 1 for a visual representation of $\mathcal{M}_n$ and $\mathcal{M}'_n$. The proofs using the models $\mathcal{M}_n$ are completely analogous to those using the models $\mathcal{M}'_n$, so choose any $\mathcal{N} \in \{\mathcal{M}, \mathcal{M}'\}$. I give the proof for the models $\mathcal{N}_n$, thus simultaneously proving the result for **KD45** and **S4**.

An important thing to note about these models is that the arrow from s_1 to s_0 is one-directional, unlike all other arrows. As a result of this one-directional arrow we have $\mathcal{N}_n, s_0 \models C_{\mathcal{A}}p$ even though $\mathcal{N}_n, w \not\models C_{\mathcal{A}}p$ for all other worlds w. We can use this to distinguish between s_{2n-1} and t_{2n-1} by first using the substitution $[q := C_{\mathcal{A}}p]$ to store this difference between s_0 and all other worlds, then removing the $\neg p$ worlds that connect s_{2n-1} to t_{2n-1} and finally checking whether there is a q world reachable. Written as a formula this means that $\mathcal{N}_n, s_{2n-1} \not\models [q := C_{\mathcal{A}}p][p]C_{\mathcal{A}}\neg q$ while $\mathcal{N}_n, t_{2n-1} \models [q := C_{\mathcal{A}}p][p]C_{\mathcal{A}}\neg q$.

Another key property of these models is that there is no single $\mathcal{L}_{CP}$ formula that for every $n \in \mathbb{N}_{>0}$ distinguishes s_{2n-1} from t_{2n-1}. This is shown by induction, but in order to keep the induction hypothesis relatively simple it is convenient to restrict to a certain subset of formulas.

Definition 7. An $\mathcal{L}_{CP}$ formula φ is in *ab-form* if all C_B operators in φ have $B = \{a, b\}$ and are immediately preceded by a $[\psi]$ operator and all $[\psi]$ operators in φ are immediately succeeded by a C_B operator.

The following lemma shows that we can restrict to formulas in ab-form without loss of generality.

Lemma 3. *For every $\mathcal{L}_{CP}$ formula φ there is an $\mathcal{L}_{CP}$ formula φ' such that φ' is in ab-form and for all $n \in \mathbb{N}_{>0}$ and all $w \in W_n$ it holds that $\mathcal{N}_n, w \models \varphi \Leftrightarrow \mathcal{N}_n, w \models \varphi'$.*

Proof. Let φ be any $\mathcal{L}_{CP}$ formula. The only agents for which there are arrows from one world to another in $\mathcal{N}_n$ are a and b. We can therefore replace any operator C_B in φ by $C_{B'}$ with $B' = B \cap \{a, b\}$ without changing the truth value on $\mathcal{N}_n$. If this B' is a singleton $C_{B'}$ can be replaced by either $\Box_a$ or $\Box_b$ and if $B' = \emptyset$ it can be removed entirely, again without changing the truth value on $\mathcal{N}_n$. Call the formula obtained by modifying φ in these ways ϑ.

Recall the reduction axioms

- $\models [\psi]p \leftrightarrow (\psi \to p)$
- $\models [\psi_1]\neg\psi_2 \leftrightarrow (\psi_1 \to \neg[\psi_1]\psi_2)$
- $\models [\psi_1](\psi_2 \vee \psi_3) \leftrightarrow ([\psi_1]\psi_2 \vee [\psi_1]\psi_3)$
- $\models [\psi_1]\Box_a\psi_2 \leftrightarrow (\psi_1 \to \Box_a[\psi_1]\psi_2)$
- $\models [\psi_1][\psi_2]\psi_3 \leftrightarrow [\psi_1 \wedge [\psi_1]\psi_2]\psi_3$

that were shown to hold in [10]. If a public announcement occurs immediately preceding any $\mathcal{L}_{CP}$ operator other than a common knowledge operator they allow us to either change the order of the public announcement and the other operator or collapse two announcements into one. We can therefore find a formula ϑ' that is equivalent to ϑ and that only contains public announcements immediately preceding $C_{\{a,b\}}$ operators. Finally, we can take φ' to be the formula obtained by adding a $[\top]$ operator before every $C_{\{a,b\}}$ operator in ϑ' that is not yet immediately preceded by a public announcement. This φ' is in ab-form and satisfies $\mathcal{N}_n, w \models \varphi \Leftrightarrow \mathcal{N}_n, w \models \varphi'$ for all $n \in \mathbb{N}_{>0}$ and all $w \in W_n$. $\dashv$

It is relatively easy to show that formulas in ab-form cannot distinguish between worlds s_i and t_i of $\mathcal{N}_n$ unless they are of sufficient depth.

Lemma 4. *Let $n \in \mathbb{N}_{>0}$ be given. Then for any $k \leq n$ and for any $2k < i \leq 2n$ there is no $\mathcal{L}_{CP}$ formula φ that is in ab-form and of depth at most k such that φ distinguishes between s_i and t_i in $\mathcal{N}_n$.*

Proof. By induction on k. The statement trivially holds for $k = 0$. Now suppose as induction hypothesis that $k > 0$ and that the statement holds for all $k' < k$. Fix any i with $2k \leq i \leq 2n$ and let φ be any $\mathcal{L}_{CP}$ formula in ab-form with depth $\leq k$. To show is that φ does not distinguish between s_i and t_i.

Suppose towards a contradiction that φ does in fact distinguish between s_i and t_i. If a Boolean combination of formulas distinguishes two worlds then at least one of the combined formulas also distinguishes between them, so we can assume without loss of generality that φ is of pure depth $\leq k$. Since φ is in ab-form this implies that φ is either of the form $\Box_x\varphi_1$ or of the form $[\varphi_1]C_{\{a,b\}}\varphi_2$ where $x \in \mathcal{A}$, φ_1 and φ_2 are in ab-form and φ_1 and φ_2 are of depth $\leq k - 1$. Suppose

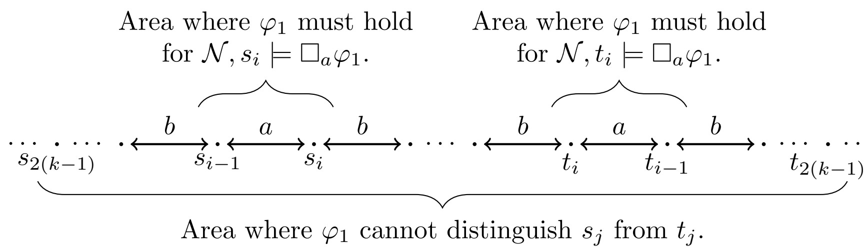

Figure 2. The formula $\varphi = \Box_a\varphi_1$ cannot distinguish between s_i and t_i, since $\mathcal{N}_n, w_i \models \varphi \Leftrightarrow (\mathcal{N}_n, w_i \models \varphi_1$ and $\mathcal{N}_n, w_{i-1} \models \varphi_1)$ for $w \in \{s, t\}$, and by the induction hypothesis φ_1 cannot distinguish either s_i from t_i or s_{i-1} from t_{i-1}.

$\varphi = \Box_x\varphi_1$. Then φ_1 cannot distinguish between s_j and t_j for $2(k-1) \leq j \leq 2n$, so in particular it cannot distinguish between s_i and t_i, between s_{i-1} and t_{i-1} or between s_{i+1} and t_{i+1}. From this it follows that φ cannot distinguish between s_i and t_i. See Figure 2 for a visual representation of the case where $x = a$ and i is even.

Suppose then that $\varphi = [\varphi_1]C_{\{a,b\}}\varphi_2$. The worlds s_i and t_i are $\{a, b\}$-reachable from each other. In order for φ to distinguish between the two points it is therefore necessary that φ_1 fails to hold for some s_j or t_j with $j \geq i$. But by the induction hypothesis φ_1 cannot distinguish s_j from t_j so φ_1 must fail to hold on both worlds. But since $j \geq i$ this implies that after such an update neither of the $\neg p$ worlds is reachable from either s_i or t_i. As a result s_i and t_i are bisimilar after the update, and therefore indistinguishable by *any* $\mathcal{L}_{CP}$ formula. So in particular they are not distinguishable by φ_2. The formula φ therefore does not distinguish between the two worlds.

In both possible forms for φ we arrive at a contradiction with the assumption that φ distinguishes between s_i and t_i. The assumption must therefore be false, so φ does not distinguish between s_i and t_i. This completes the induction step and thereby the proof. ⊣

Note that in particular Lemma 4 implies that an $\mathcal{L}_{CP}$ formula in *ab*-form must be of length at least n to distinguish between the worlds s_{2n-1} and t_{2n-1} of $\mathcal{N}_n$. The proof of Theorem 1 now follows easily.

Proof of Theorem 1. From Lemma 1 it follows that $\mathcal{L}_{CP} \preceq_{\mathbf{KD45}} \mathcal{L}_{CPS}$ and $\mathcal{L}_{CP} \preceq_{\mathbf{S4}} \mathcal{L}_{CPS}$.

Let $\varphi = [q := C_{\mathcal{A}}p][p]C_{\mathcal{A}}\neg q$ and take any $\mathcal{N} \in \{\mathcal{M}, \mathcal{M}'\}$. For any $n \in \mathbb{N}_{>0}$ we have $\mathcal{N}_n, s_{2n-1} \not\models \varphi$ and $\mathcal{N}_n, t_{2n-1} \models \varphi$. The $\mathcal{L}_{CPS}$ formula φ therefore distinguishes between s_{2n} and t_{2n} for every $n \in \mathbb{N}_{>0}$. From Lemmas 3 and 4 it follows that there is no single $\mathcal{L}_{CP}$ formula ψ that for every $n \in \mathbb{N}_{>0}$ distinguishes between s_{2n-1} and t_{2n-1}.

There is therefore no $\mathcal{L}_{CP}$ formula that is equivalent to φ on all **KD45** models or on all **S4** models, so $\mathcal{L}_{CPS} \not\preceq_{\mathbf{KD45}} \mathcal{L}_{CP}$ and $\mathcal{L}_{CPS} \not\preceq_{\mathbf{S4}} \mathcal{L}_{CP}$. ⊣

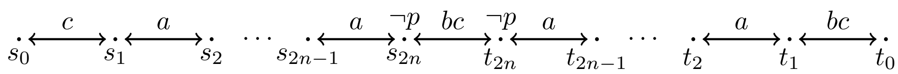

FIGURE 3. Model $\mathcal{M}''_n$. Reflexive arrows are omitted and p holds everywhere except where it is stated otherwise.

§6. The expressivity of factual change over S5. What remains to be shown is that substitutions add expressivity over **S5**. The models $\mathcal{M}_n$ and $\mathcal{M}'_n$ are not **S5** models so they cannot be used for expressivity results over **S5**. The models can be modified to **S5** models, but this requires the use of an extra agent.

THEOREM 2. *If $|\mathcal{A}| \geq 3$ then the logic $\mathcal{L}_{CPS}$ is more expressive than the logic $\mathcal{L}_{CP}$ over* **S5**.

Apart from some technicalities that are necessary due to the extra agent, the proofs of Theorem 2 and its auxiliary lemmas are analogous to that of Theorem 1 and its auxiliary lemmas. I therefore give only a sketch of the proofs here. First, let us define the set $\{\mathcal{M}''_n \mid n \in \mathbb{N}_{>0}\}$ of **S5** models. Let $a, b, c \in \mathcal{A}$ be three distinct agents and let $p \in \mathcal{P}$. For $n \in \mathbb{N}_{>0}$ let $\mathcal{M}''_n = (W_n, R''_n, v_n)$ where

- $R'_n(a) = R_n(a)$
- $R'_n(b) = R_n(b) \setminus \{(s_1, s_0)\}$
- $R'_n(c) = R'_n(b) \cup \{(s_0, s_1), (s_1, s_0)\}$
- $R'_n(d) = \{(w, w) \mid w \in W_n\}$ for all $d \in \mathcal{A} \setminus \{a, b, c\}$

See also Figure 3 for a visual representation of $\mathcal{M}''_n$.

Like in the **S4** case we can restrict to formulas in a certain form.

DEFINITION 8. An $\mathcal{L}_{CP}$ formula φ is in *abc-form* if all C_B operators in φ have $B \in \{\{a, b\}, \{a, c\}, \{a, b, c\}\}$ and are immediately preceded by a $[\psi]$ operator and all $[\psi]$ operators in φ are immediately succeeded by a C_B operator.

LEMMA 5. *For every $\mathcal{L}_{CP}$ formula φ there is an $\mathcal{L}_{CP}$ formula φ' such that φ' is in abc-form and for all $n \in \mathbb{N}_{>0}$ and all $w \in W'_n$ it holds that $\mathcal{M}''_n, w \models \varphi \Leftrightarrow \mathcal{M}''_n, w \models \varphi'$.*

SKETCH OF PROOF. The proof is analogous to that of Lemma 3; every C_B operator with $B \notin \{\{a, b\}, \{a, c\}, \{a, b, c\}\}$ can be replaced by either $C_{B'}$ with $B' \in \{\{a, b\}, \{a, c\}, \{a, b, c\}\}$ or by a normal knowledge operator. The reduction axioms can then be used to make $[\psi]$ and C_B only occur together. $\dashv$

Formulas in *abc*-form cannot distinguish between s_{2n} and t_{2n} in $\mathcal{M}''_n$ unless they are of sufficient depth.

LEMMA 6. *Let $n \in \mathbb{N}_{>0}$ be given. Then for any $k \leq n$ and for any $2k < i \leq 2n$ there is no $\mathcal{L}_{CP}$ formula φ that is in abc-form and of depth at most k such that φ distinguishes between s_i and t_i in $\mathcal{M}''_n$.*

SKETCH OF PROOF. The induction on k is analogous to that in the proof of Lemma 4. Suppose the lemma holds for all $k' < k$. A formula $\Box_x \varphi$ with $d(\varphi) \leq k - 1$ can only distinguish between s_i and t_i if φ distinguishes between some s_j and t_j with $j \geq i - 1$ but that is impossible under the induction hypothesis.

	K	**KD45**	**S4**	**S5**
$\lvert\mathcal{A}\rvert = 1$	$\prec$	$\equiv$	$\equiv$	$\equiv$
$\lvert\mathcal{A}\rvert = 2$	$\prec$	$\prec$	$\prec$	?
$\lvert\mathcal{A}\rvert \geq 3$	$\prec$	$\prec$	$\prec$	$\prec$

TABLE 1. The relative expressivity of $\mathcal{L}_{CP}$ and $\mathcal{L}_{CPS}$ with different numbers of agents in different classes of models. A $\prec$ entry means that $\mathcal{L}_{CPS}$ is more expressive than $\mathcal{L}_{CP}$ in that case, a $\equiv$ entry means that $\mathcal{L}_{CPS}$ and $\mathcal{L}_{CP}$ are equally expressive.

A formula $[\varphi]C_B\psi$ with $d(\varphi) \leq k-1$ must have $B \in \{\{a,b\},\{a,c\},\{a,b,c\}\}$ so s_i and t_i are B-reachable form each other. In order for $[\varphi]C_B\psi$ to distinguish between the worlds it is therefore necessary that $[\varphi]$ remove at least one world s_j or t_j with $j \geq i$. But by the induction hypothesis it then also removes the other world, making s_i and t_i completely indistinguishable in the remaining part of the model. $\dashv$

The proof of Theorem 2 then follows easily.

PROOF OF THEOREM 2. From Lemma 1 it follows that $\mathcal{L}_{CP} \preceq_{\mathbf{S5}} \mathcal{L}_{CPS}$. Let $\varphi = [q := C_{\{a,b\}}p][p]C_{\mathcal{A}}\neg q$. For any $n \in \mathbb{N}_{>0}$ we have $\mathcal{M}''_n, s_{2n-1} \not\models \varphi$ and $\mathcal{M}''_n, t_{2n-1} \models \varphi$. The $\mathcal{L}_{CPS}$ formula φ therefore distinguishes between s_{2n-1} and t_{2n-1} for every $n \in \mathbb{N}_{>0}$. From Lemmas 5 and 6 it follows that there is no single $\mathcal{L}_{CP}$ formula ψ that for every $n \in \mathbb{N}_{>0}$ distinguishes between s_{2n-1} and t_{2n-1}.

There is therefore no $\mathcal{L}_{CP}$ formula that is equivalent to φ on all **S5** models, so $\mathcal{L}_{CPS} \not\preceq_{\mathbf{S5}} \mathcal{L}_{CP}$. $\dashv$

§7. Conclusion.

I showed that $\mathcal{L}_{CPS}$ is more expressive than $\mathcal{L}_{CP}$ over **KD45**, **S4** and **S5**. Since the logic $\mathcal{L}_R$ obtained by adding relativized common knowledge to a basic epistemic logic was shown to be equally expressive to $\mathcal{L}_{CPS}$ in [8] this also shows that $\mathcal{L}_R$ is more expressive than $\mathcal{L}_{CP}$ over **KD45**, **S4** and **S5**. The results presented here can therefore also be seen as an extension of the results in [3] that $\mathcal{L}_R$ is more expressive than $\mathcal{L}_{CP}$ over **K**.

One limitation of the proofs of $\mathcal{L}_{CP} \prec_{\mathbf{KD45}} \mathcal{L}_{CPS}$, $\mathcal{L}_{CP} \prec_{\mathbf{S4}} \mathcal{L}_{CPS}$ and $\mathcal{L}_{CP} \prec_{\mathbf{S5}} \mathcal{L}_{CPS}$ given in this paper is that they require a minimum number of agents. For the proofs of $\mathcal{L}_{CP} \prec_{\mathbf{KD45}} \mathcal{L}_{CPS}$ and $\mathcal{L}_{CP} \prec_{\mathbf{S4}} \mathcal{L}_{CPS}$ we need $|\mathcal{A}| \geq 2$ and for the proof of $\mathcal{L}_{CP} \prec_{\mathbf{S5}} \mathcal{L}_{CPS}$ we need $|\mathcal{A}| \geq 3$. If $|\mathcal{A}| = 1$ then common knowledge reduces to normal knowledge in **KD45**, **S4** and **S5** due to the transitivity of the accessibility relation. This implies that $\mathcal{L}_{CP} \equiv_{\mathbf{KD45}} \mathcal{L}_{CPS}$, $\mathcal{L}_{CP} \equiv_{\mathbf{S4}} \mathcal{L}_{CPS}$ and $\mathcal{L}_{CP} \equiv_{\mathbf{S5}} \mathcal{L}_{CPS}$ if $|\mathcal{A}| = 1$, as it has been shown in [8] that $\mathcal{L}_P \equiv_{\mathbf{K}} \mathcal{L}_{PS}$.

It was also shown in [8] that $\mathcal{L}_{CP} \prec_{\mathbf{K}} \mathcal{L}_{CPS}$ for any number of agents, so the relative expressivity of $\mathcal{L}_{CP}$ and $\mathcal{L}_{CPS}$ can be summarized as in Table 1.

One obvious question for further research is whether $\mathcal{L}_{CPS}$ is more expressive than $\mathcal{L}_{CP}$ in **S5** with exactly two agents. I do not have a conjecture about this question, as there seem to be good reasons for assuming either position.

On the one hand, the two-agents **S5** case is unlike the other cases in several important ways. In particular, consider $\mathcal{L}_{CPS}$ formulas of the form $[q := \varphi][\psi]C_Bq$.

This is the only combination of operators that is hard to translate to $\mathcal{L}_{CP}$. This formula holds in $\mathcal{M}, w$ if and only if φ holds in every world that is reachable from w by passing over only ψ worlds.

In the two-agents **S5** case it holds that if two adjacent worlds w_1 and w_2 of $\mathcal{M}$ satisfy $\mathcal{M}, w_1 \models \varphi$ and $\mathcal{M}, w_2 \not\models \varphi$ then there must be some propositional variable r with the property that $\mathcal{M}, w_1 \models \Diamond^{d(\varphi)+1} r \wedge \Diamond^{d(\varphi)+1} \neg r$. In other words, there must be nearby worlds that are distinguishable by an atomic formula. Since distinguishability by an atomic formula is retained under public announcements this difference can often be used as a witness after the announcement for the existence of a world that did not satisfy φ before the announcement. This technique can be used to translate several large fragments of $\mathcal{L}_{CPS}$ to $\mathcal{L}_{CP}$ for two-agent **S5**.

On the other hand, this method is not immediately applicable if ψ contains common knowledge operators. And while some auxiliary tricks can be used for simple ψ containing common knowledge such as $\psi = C_{\mathcal{A}} p$ it is not clear whether this technique generalizes to all $\mathcal{L}_{CPS}$ formulas.

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FACULTY OF PHILOSOPHY
UNIVERSITY OF GRONINGEN
OUDE BOTERINGESTRAAT 52, 9712 GL GRONINGEN, THE NETHERLANDS
E-mail: l.b.kuijer@rug.nl