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Abstract

We show that any outerplanar graph admits a planar straight-line drawing such that the length ratio of the longest to the shortest edges is strictly less than 2. This result is tight in the sense that for any $\epsilon > 0$ there are outerplanar graphs that cannot be drawn with an edge-length ratio smaller than $2 - \epsilon$. We also show that this ratio cannot be bounded if the embeddings of the outerplanar graphs are given.

Keywords: Graph drawing, outerplanar graphs, edge-length ratio

1. Introduction

The problem of computing a planar straight-line drawing with prescribed edge lengths has been addressed by several authors, partly for its theoretical interest and partly for its application in different areas, including VLSI, wireless sensor networks, and computational geometry (see for example [6, 7, 10, 14]). Deciding whether a graph admits a straight-line planar drawing with prescribed edge lengths was shown to be NP-hard by Eades and Wormald for 3-connected planar graphs [8]. In the same paper, the authors show that it is NP-hard to determine whether a 2-connected planar graph has a *unit-length* planar straight-line drawing; that is, a drawing in which all edges have the same length. Cabello et al. extend this last result by showing that it is NP-hard to decide whether a 3-connected planar graph admits a unit-length planar straight-line drawing [3]. In addition, Bhatt and Cosmadakis prove that deciding whether a degree-4 tree has a planar drawing such that all edges have the same length and the vertices are at integer grid points is also NP-hard [2].

These hardness results have motivated the study of relaxations and variants of the problem of computing straight-line planar drawings with constraints on the edge lengths. For example, Aichholzer et al. [1] study the problem of computing straight-line planar drawings where, for each pair of edges of the input graph G, it is specified which edge must be longer. They characterize families of graphs that are *length universal*, i.e. that admit a planar straight-line drawing for any given total order of their edge lengths.

Perhaps one of the most natural variants of the problem is to compute planar straight-line drawings where the variance of the lengths of the edges is minimized. See for example [5], where this optimization goal is listed among the most relevant aesthetics that impact the readability of a drawing of a graph. Computing straight-line drawings where the ratio of the longest to the shortest edge is close to 1 also

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arises in the approximation of unit disk graph representations, a problem of interest in the area of wireless communication networks (see, e.g. [4, 12]).

Discouragingly, Eades and Wormald observe in their seminal paper that the NP-hardness of computing 2-connected planar straight-line drawings with unit edge lengths persists even when a small tolerance (independent of the problem size) in the length of the edges is allowed. To our knowledge, little progress has been made on bounding the ratio between the longest and shortest edge lengths in planar straight-line drawings. We recall the work of Hoffmann et al. [11], who compare different drawing styles according to different quality measures including the edge-length variance.

In this paper we study planar straight-line drawings of outerplanar graphs that bound the ratio of the longest to the shortest edge lengths from above by a constant. We define the *planar edge-length ratio* of a planar graph G as the smallest ratio between the longest and the shortest edge lengths over all planar straight-line drawings of G. The main result of the paper is the following.

Theorem 1. *The planar edge-length ratio of any outerplanar graph is strictly less than* 2*. Also, for any given real positive number* ϵ*, there exists an outerplanar graph whose planar edge-length ratio is greater than* $2 - \epsilon$*.*

Informally, Theorem 1 establishes that 2 is a tight bound for the planar edge-length ratio of outerplanar graphs. The upper bound is proved by using a suitable decomposition of an outerplanar graph into subgraphs called *strips*, then drawing the graph strip by strip. The lower bound is proved by taking into account all possible planar embeddings of a maximal outerplanar graph whose maximum vertex degree is a function of ϵ.

We note here that for any given outerplanar topological embedding of an outerplanar graph G, the algorithm of Theorem 1 computes an embedding preserving drawing of G whose planar edge length-ratio is strictly less than 2. Therefore, Theorem 1 naturally raises the question of whether a bound can be proven for every (not necessarily outerplanar) topological embedding of G. The next theorem answers this question in the negative. The *plane edge-length ratio* of a planar embedding $\mathcal{G}$ of a graph G is the minimum edge-length ratio taken over all embedding-preserving planar straight-line drawings of $\mathcal{G}$.

Theorem 2. *For any given* $k \geqslant 1$*, there exists an embedded outerplanar graph whose plane edge-length ratio is at least* k*.*

The remainder of the paper is structured as follows. We prove Theorems 1 and 2 in Sections 2 and 3, respectively. We shall assume familiarity with basic definitions of graph planarity and of graph drawing [5] and introduce only the terminology and notation that is strictly needed for our proofs.

2. Edge-length ratio of outerplanar graphs

We prove in this section Theorem 1. It suffices to establish the result for maximal outerplanar graphs. To show that the edge-length ratio of a maximal outerplanar graph G is always less than 2, we consider any outerplanar topological embedding of G and decompose the dual G^* of this embedding into a set of disjoint paths, which we call *chains*. Each chain corresponds to some sequence of adjacent triangles of G. The set of chains inherits a tree structure from G^*, and we use this structure to direct an algorithm that draws each of the chains proceeding from the root of this tree down to its leaves. We formalize these concepts below.

Given an edge e on the outer face of some outerplanar topological embedding of G and an orientation for this edge, we label the source of e as v_0^- and the sink as v_0^+. The edge e is incident to a unique triangular face of G, which we label as T_0; we label the third vertex of T_0 as v_1. The orientation of e induces an orientation of the edges of G such that the edges of each face form an oriented cycle. Refer to Figure 1(a-b) (to reduce the visual clutter, not all edge orientation are shown). We define constructively the chain C_e as the maximal sequence $T_s, T_{s+1}, \ldots, T_0, \ldots, T_t$ of faces of G, with $s \leqslant 0 \leqslant t$, such that

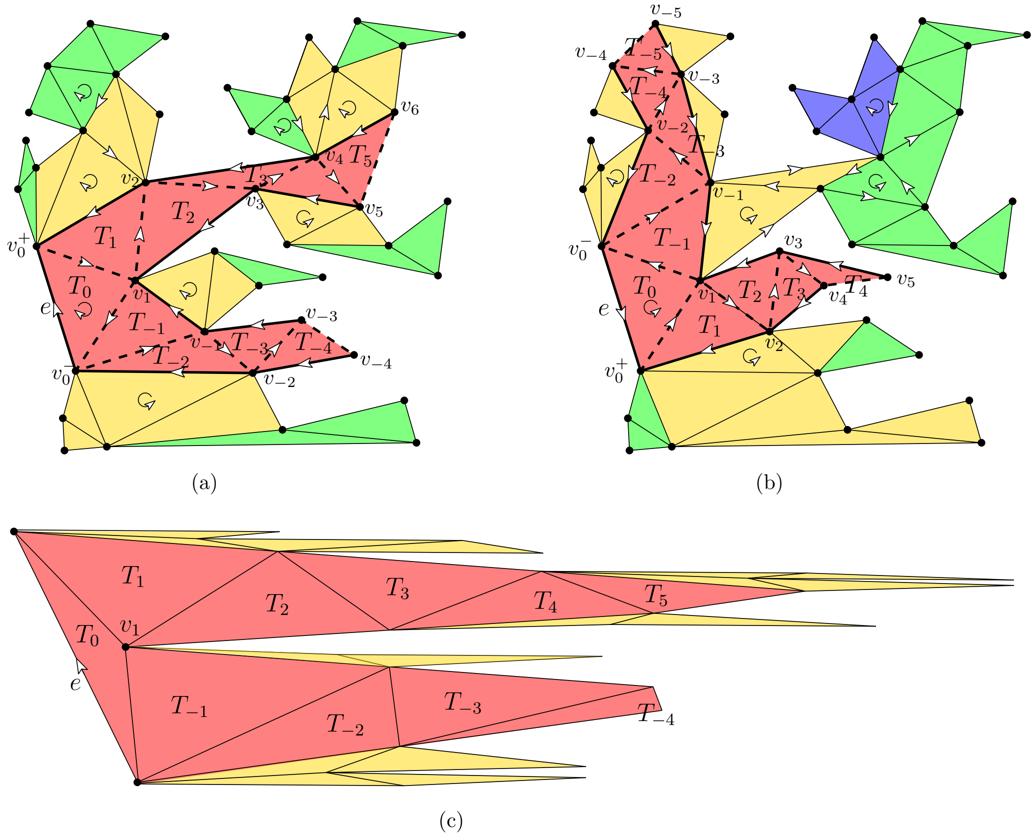


Figure 1: (a) An outerplanar graph with its chain decomposition determined by edge e. The chain C_e is clockwise and it is in red; the children chains of C_e are in yellow and their children are in green. The short edges of C_e are dashed. (b) Same as (a) but with the opposite orientation for e; C_e is counterclockwise (the blue chain is a child of a green chain). To reduce the visual clutter, not all edges of G have been oriented. (c) Schematic drawing of C_e and of the next level of chains in the chain decomposition of (a); the chains of subsequent levels are not drawn.

(i) for $1 \leqslant i \leqslant t$, T_i shares edge $v_{i-1}v_i$ with T_{i-1} (where v_0 refers here to v_0^+) and its third vertex is labelled v_{i+1} and (ii) for $s \leqslant i \leqslant -1$, T_i shares edge $v_{i+2}v_{i+1}$ with T_{i+1} (where v_0 refers here to v_0^-) and its third vertex is labelled v_i. Rougly speaking, this means that considering a walk from T_0 to T_t (resp. to T_s) in the chain of triangles, the exit edges *after* leaving T_0 alternate between right and left, starting with right if T_0 is oriented clockwise (as in Figure 1(a)) and starting with left otherwise (as in Figure 1(b)). If T_0 is oriented clockwise, we say that C_e is *a clockwise chain*; C_e is a *counterclockwise chain* otherwise.

The edges of C_e can be partitioned into two sets S_e and L_e which we refer to as *short edges* and *long edges*, respectively (long edges will be drawn with length 1 and short edges with length in $(\frac{1}{2}, 1)$). S_e consists of edges of C_e whose vertex indices differ by 1 and L_e consists of all other edges of C_e (i.e., those whose vertex indices differ by 2, along with e). See, for example Figure 1(a) where the short edges of C_e are dashed, whereas the long edges of C_e are solid. Note that all the (oriented) long edges of C_e are boundary edges of the union of the triangular faces of C_e and that they can be partitioned into edge e and four oriented paths, two of which ending at v_1 and the two others ending at v_0^+ and v_0^- respectively (see Figure 1(a-b)); the short edges of C_e are all the internal edges plus the (only) two boundary edges that are not long edges (e.g., edges $v_{-3}v_{-4}$ and v_5v_6 in Figure 1(a)).

Consider now the (not connected) outerplanar graph $G - S_e$ obtained by removing the edges of S_e

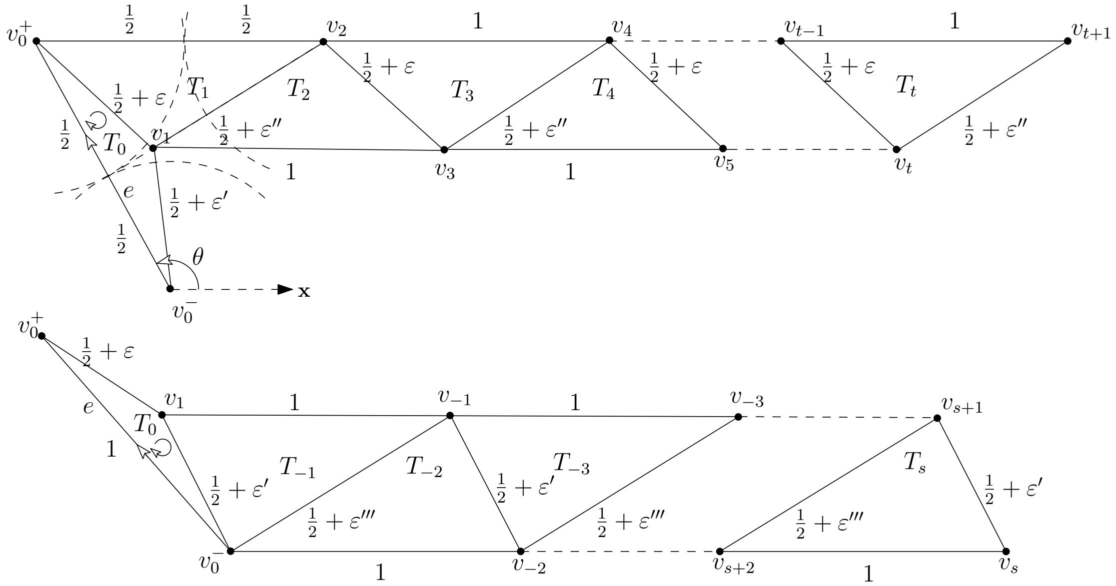


Figure 2: Illustration for the proof of Lemma 3.

from G. Each 2-connected component of $G - S_e$ is either an edge of L_e or it is an outerplanar graph $G_{e'}$ having exactly one edge e' of L_e on its outer face. Note that the initial choice of orientation for e induces an orientation for all edges of $G_{e'}$. We recursively decompose each 2-connected component $G_{e'}$ of $G - S_e$ into chains by using the procedure described above, starting from the oriented edge e'. We call the set of chains so constructed a *chain decomposition of* G. This naturally defines a *chain-decomposition tree:* The root of the tree is the chain C_e and its children are the chains $C_{e'}$ for $e' \in L_e$; the next level of the tree consists of the chains $C_{e''}$ with $e'' \in L_{e'}$, and so forth. Note that the chain decomposition of G and its corresponding chain-decomposition tree are uniquely determined once edge e is oriented.

In order to compute a drawing of G with edge length ratio strictly less than 2, we perform a pre-order visit of the chain-decomposition tree, drawing each chain as we visit the correspond node of the tree. The correctness of the algorithm depends on a specific method for drawing a single chain.

Denote by $\mathbf{x}$ a vector oriented along the positive x-axis and, given an oriented segment s in the plane, denote by $\mathbf{s}$ its supporting vector. Define the *right strip of segment* s, denoted by $S(s, \mathbf{x})$, as the half-infinite strip bounded by s and by the two infinite rays emanating from the endpoints of s in the direction $\mathbf{x}$.

Lemma 3. *Given a clockwise (resp. counterclockwise) chain* C_e *with* n *vertices and an oriented segment* s *of length* 1 *such that the angle* $\theta = \angle(\mathbf{x}, \mathbf{s})$ *is in* $(\pi - \theta_0, \pi)$ *(resp. in* $(\pi, \pi + \theta_0)$*) with* $\theta_0 = \arccos(1/4) \approx 75.5°$*, there exists a planar straight-line drawing of* C_e *such that: (i) The oriented edge* e *is drawn as* s*, (ii) the drawing is contained in the strip* $S(s, \mathbf{x})$ *and lies outside (or on the boundary of) the right strips of all long edges of* C_e*; (ii) all long edges have length 1 and all short edges have length in* $(\frac{1}{2}, 1)$*; (iii) all long edges are drawn with an orientation such that their angle with* $\mathbf{x}$ *is in* $(\pi - \theta_0, \pi + \theta_0)$ *and distinct from* π*. Moreover, such a drawing can be computed in* $O(n)$*-time in the real RAM model.*

Proof. Assume that the chain C_e is clockwise (the proof for counterclockwise follow the same reasoning). Refer to Figure 2. Let v_0^+ and v_0^- be the vertices of edge e in the chain C_e, and let T_0 be its incident triangle. T_0 is either adjacent to zero, one, or two triangles of C_e. We handle these three cases in turn. If

T_0 is the only triangle in C_e, then we simply draw T_0 in $S(s, \mathbf{x})$ as an isosceles triangle with e drawn as s and with its third vertex drawn so that its two edges have length l, where $\frac{1}{2} < l < 1$.

Assume now that T_0 is adjacent to a triangle T_1 of C_e. In this case a more careful positioning of of v_1 is required. We first draw edge $v_0^+ v_2$ of T_1 as a unit-length segment in direction $\mathbf{x}$. As long as v_1 is positioned outside of the disks of radius $\frac{1}{2}$ centered at v_0^-, v_0^+, and v_2, the edges from each of these vertices to v_1 will have length greater than $\frac{1}{2}$ (see top half of Figure 2). By also placing v_1 close enough to the midpoint of e, the edges $v_1 v_0^-$ and $v_1 v_0^+$ will have lengths less than 1. Furthermore, by placing v_1 inside $S(s, \mathbf{x})$, the edge $v_1 v_2$ will have length less than 1 because $\angle v_2 v_0^+ v_1 < \theta_0 = \arccos(1/4)$, which is the largest angle in an isosceles triangle having side lengths 1, 1, and $\frac{1}{2}$.

Assuming that $T_0, \ldots, T_{i-1}$ have been drawn for some $i > 1$, T_i is drawn by positioning v_{i+1} one unit distant from v_{i-1} in direction $\mathbf{x}$. The result is that each T_i is congruent to T_1 and so the edge-length ratio of C_e is less than 2. At this point, all of the unit-length segments, except for e, lie on the two rays in direction $\mathbf{x}$ emanating from v_0^+ and v_1. By rotating these rays a very small amount towards one another, we can preserve the lengths of the unit-length segments while ensuring that all of the remaining segments have lengths in the range $(\frac{1}{2}, 1)$.

Finally, suppose that T_0 has two adjacent triangles. Draw each $T_i, i > 0$ as in the previous case. Now draw the $T_i, i < 0$ in a similar fashion: Place vertex v_i one unit distant from v_{i+2} in direction $\mathbf{x}$. Then, as above, all of the unit length edges of the triangles $T_i, i < 0$ will lie on the two rays in direction $\mathbf{x}$ emanating from v_1 and v_0^-, and these two rays can be rotated slightly towards each other while maintaining the length of the unit-length edges and ensuring that the other edges still have lengths in the range $(\frac{1}{2}, 1)$. See the bottom of Figure 2.

However, we need to ensure that v_1 can be placed so that both triangle T_1 and triangle T_{-1} can simultaneously satisfy the required edge-length conditions: Namely, that edges $v_0^- v_0^+$, $v_1 v_{-1}$, and $v_0^+ v_2$ are all unit-length, while edges $v_1 v_2$, $v_0^+ v_1$, $v_1 v_0^-$, and $v_0^- v_{-1}$ all have lengths in the interval $(\frac{1}{2}, 1)$. The only one of these conditions that is not already guaranteed by the construction of $T_0, \ldots, T_t$ is that $v_0^- v_{-1}$ has length less than 1. However, placing v_1 sufficiently close to the midpoint of e also ensures that the length of $v_0^- v_{-1}$ is smaller than 1.

Finally, the computation of the locations of the vertices can each be computed in constant time in the real RAM model, giving a run-time linear in the size of the chain. □

We call the drawing defined by Lemma 3 a *U-strip drawing* of C_e and we now use it to prove the following lemma, which implies the upper bound of Theorem 1.

Lemma 4. *A maximal outerplanar graph with n vertices admits an outerplanar straight-line drawing such that the length ratio of the longest to the shortest edges is strictly less than 2. The drawing can be computed in $O(n)$ time assuming the real RAM model of computation.*

Proof. Let G be a maximal outerplanar graph with any outerplanar topological embedding. An embedding-preserving drawing of G is computed as follows. First, select and orient an edge e of the outer face of G, then compute the chain-decomposition tree T of G, from e. We will draw the chains of T in pre-order. Let C_e be the chain associated with the root of T. Choose arbitrarily an oriented line segment s of length 1 in the plane such that the angle $\angle(\mathbf{x}, \mathbf{s})$ is in $(\pi - \theta_0, \pi)$ if C_e is clockwise and in $(\pi, \pi + \theta_0)$ otherwise. Apply Lemma 3 to compute a U-strip drawing of C_e.

Now, consider a child $C_{e'}$ of C_e in T. Then e' is a long edge of C_e and thus, by Lemma 3, it is drawn as a segment $s_{e'}$ of length 1 that forms an angle with $\mathbf{x}$ that is in $(\pi - \theta_0, \pi + \theta_0) \setminus \{\pi\}$ and such that the drawing of C_e does not intersect the interior of the strip $S(s_{e'}, \mathbf{x})$. If the angle that $s_{e'}$ forms with $\mathbf{x}$ is in $(\pi - \theta_0, \pi)$ (resp. $(\pi, \pi + \theta_0)$), the triangle T_0' of $C_{e'}$ that contains e' is oriented clockwise (resp. counterclockwise); indeed, since the drawing of C_e does not intersect the interior of $S(s_{e'}, \mathbf{x})$ by Lemma 3, the triangle of C_e that contains e' is oriented counterclockwise (resp. clockwise). Hence, we can apply Lemma 3 to draw $C_{e'}$ in the strip $S(s_{e'}, \mathbf{x})$; in fact, the entire sub-tree of T rooted at $C_{e'}$ will

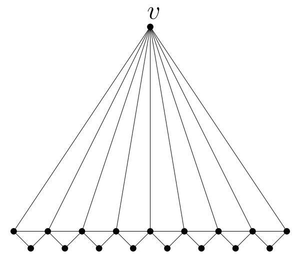


Figure 3: Example of graph G in the proof of Lemma 5 with $k = 7$.

be recursively drawn in this strip as the algorithm progresses. See Figure 1(c) for an illustration of the drawing of the first two levels of T.

The result is an outerplanar straight-line drawing in which all long edges in the chains of the decomposition are drawn with length 1 and all short edges have length in $(\frac{1}{2}, 1)$. If we assume that the input is provided to the algorithm in the form of a doubly-connected edge list [13], then a chain-decomposition tree for G can be computed in linear time. Also, since by Lemma 3 each chain can be drawn in time proportional to its length, the algorithm runs in $O(n)$ time in the real RAM model. $\square$

We now prove a lower bound on the edge length ratio of outerplanar graphs, which together with Lemma 4, proves Theorem 1.

Lemma 5. *For any $\epsilon > 0$ there exists a maximal outerplanar graph whose planar edge length ratio is greater than $2 - \epsilon$.*

Proof. Let the length of the longest edge be 1. We show that, for any value $\epsilon > 0$, there exists a maximal outerplanar graph G such that in any planar straight-line drawing of G, the length of the shortest edge must be smaller than $\frac{1}{2-\epsilon}$. Let us rewrite $\frac{1}{2-\epsilon}$ as $\frac{1}{2} + \delta$, where $\delta = \frac{\epsilon}{2(2-\epsilon)}$. In any planar straight-line drawing of a maximal outerplanar graph such that the longest edge has length 1 and the shortest edge has length at least $\frac{1}{2} + \delta$, the area of every triangular face cannot become arbitrarily small, and it has a lower bound that depends on the value δ. More precisely, the minimum area of such a triangle is obtained when one of its edge has length 1 while the other two have length $\frac{1}{2} + \delta$: Indeed, observe first that a triangle is trivially not of minimal area if at most one of its edges has minimum length. Now, if two of its edges have minimum length $\alpha = \frac{1}{2} + \delta$, with θ denoting the angle between these edges, the triangular area is $\alpha \cos(\theta/2) \cdot \alpha \sin(\theta/2) = \frac{1}{2}\alpha^2 \sin(\theta)$, which increases for θ in $[0, \pi/2]$ and decreases in $[\pi/2, \pi]$. So the minimum area is obtained when the third edge takes an extreme value, 1 or $\alpha = \frac{1}{2} + \delta$. Furthermore, if δ is small enough, the minimum area is obtained when the third edge has length 1 because, when δ tends to zero, the triangular area tends to zero if the third edge has length 1 and it tends to the area of the equilateral triangle of edge length $1/2$ otherwise. Thus, by Heron's formula (the area of a triangle whose edge lengths are a, b, c is $\sqrt{s(s-a)(s-b)(s-c)}$ with $s = \frac{a+b+c}{2}$), the area of any triangular face under these assumptions is at least $\frac{1}{2}\sqrt{\delta + \delta^2}$.

Let k be a positive integer and consider a fan graph F with a vertex v of degree $k+2$; G is constructed by adding $k + 1$ triangular faces to F as follows: for each edge e of F not incident to v, add a new vertex adjacent to both vertices of e. Refer to Figure 3. We call these non-fan triangles *pendant* triangles. Observe that in any planar straight-line drawing Γ of G, independently of whether all vertices of Γ appear on a common face or not, we have that the drawing has at least k area-disjoint pendant triangles, since any pendant triangle that contains another triangle must contain the entire graph. Also, since the longest edge in the drawing has length 1 and since every vertex has graph theoretic distance at most 2 from vertex v, we have that Γ lies inside a disk D of radius 2 centered at v. The number ν of area disjoint triangles that can be packed inside D such that every triangle has area at least $\frac{1}{2}\sqrt{\delta + \delta^2}$ must satisfy $\nu \leqslant \frac{8\pi}{\sqrt{\delta+\delta^2}}$.

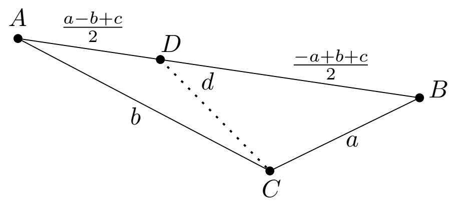


Figure 4: Illustration for the construction of Lemma 6.

Thus, for any $k > \frac{8\pi}{\sqrt{\delta+\delta^2}}$, any planar straight-line drawing of the maximal outerplanar graph G where the length of the longest edge is 1 requires at least one edge having length shorter than $\frac{1}{2} + \delta$. □

3. Embedded outerplanar graphs

We prove in this section Theorem 2, starting with a straightforward lemma. Refer to Figure 4. Consider a triangle $\triangle ABC$ whose longest edge is AB and let a, b, c be the lengths of the edges opposite to vertices A, B, C, respectively. Let D be the point on segment AB such that the distance from A to D and from D to B are respectively $\frac{a-b+c}{2}$ and $\frac{-a+b+c}{2}$, and let d be the length of segment CD. Then, the following holds.

Lemma 6. *Both triangles defined by D and, respectively, edges AC and BC have equal perimeter $\frac{a+b+c}{2} + d$, which is less than the perimeter of $\triangle ABC$ minus half the length of its smallest edge.*

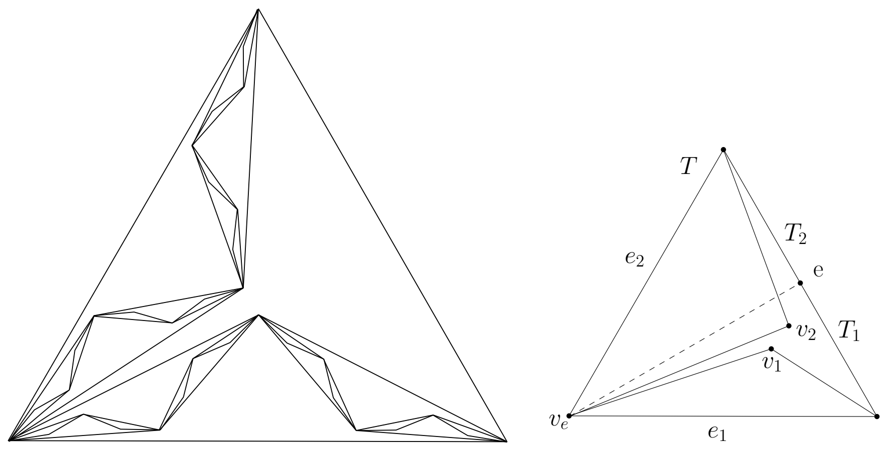
Proof. The first claim that both triangles have equal perimeter is trivial. Assuming without loss of generality that edge BC, of length a, is the shortest edge, the second claim states that $\frac{a+b+c}{2} + d < a+b+c-\frac{a}{2}$, which is equivalent to $2d < b + c$ and which follows from the fact that d is smaller than both b and c (to see this, note that the longest segment that is incident to C and inside $\triangle ABC$ is the edge CA, which implies that $d < b$, but $b \leqslant c$ by assumption). □

We now prove Theorem 2. We construct a family G_n of outerplanar graphs having planar embeddings $\mathcal{G}_n$ with unbounded plane edge-length ratio $\rho(\mathcal{G}_n)$, that is such that $\rho(\mathcal{G}_n) \to \infty$ as $n \to \infty$. Each $G_i, i \geqslant 0$, has a set of 2^{i+1} so-called distinguished edges, and $G_i, i > 0$, contains G_{i-1} as a subgraph. Refer to Figure 5(a). G_0 is a single triangle, with any two of its edges defined as distinguished. G_{i+1} is constructed from G_i by adding, for each distinguished edge e of G_i, a new vertex v_e that is adjacent to both vertices of e; the newly added edges are the distinguished edges of G_{i+1}. There is only 1 embedding of G_0; the embedding $\mathcal{G}_1$ of G_1 is obtained by putting the two vertices of $G_1 \setminus G_0$ in the inner face of $\mathcal{G}_0$. For $i > 1$, the embedding $\mathcal{G}_{i+1}$ of G_{i+1} is defined by placing each new vertex v_e in the (unique) triangular face of $\mathcal{G}_i$ having e on its boundary (see Figure 5(a)).

Assume for a contradiction that for each $\mathcal{G}_n$ there is a planar straight-line drawing Γ_n of $\mathcal{G}_i$ that preserves the embedding described above, and that, for some $\rho^* \geqslant 1, \rho(\Gamma_n) \leqslant \rho^*$ for all $n \geqslant 0$. By scaling the drawing, we can assume without loss of generality that the longest edge of Γ_0 has length 1; thus the edges in any Γ_n have lengths at most 1 and at least $\frac{1}{\rho^*}$.

Consider a triangular face T created in the construction of $\mathcal{G}_i$ for some $i > 0$ and refer to Figure 5(b). It consists of a distinguished edge e of G_{i-1} along with the vertex v_e in $G_i \setminus G_{i-1}$ adjacent to e. The two edges e_1 and e_2 incident with v_e are distinguished edges of G_i, so in $\mathcal{G}_{i+1}$ each of them will form a triangle with some new vertex, say v_1 and v_2, respectively, and both new vertices will be in the face T. We consider two cases depending on whether e is the longest edge of T.

If e is the longest edge of T, consider the partition of T in two triangles T_1 and T_2, as in Lemma 6. At least one of T_1 and T_2 contains the triangle (e_1, v_1) or (e_2, v_2); say T_1 contains triangle (e_1, v_1) as in



(a) (b)

Figure 5: For the proof of Theorem 2: (a) Embedding $\mathcal{G}_n$ of graph G_n. (b) Bounding from above the perimeters of some triangles in $\mathcal{G}_n$.

Figure 5(b). Then the perimeter of triangle (e_1, v_1) is less than that of T_1, which is less than the perimeter of T minus half the length of the shortest edge of T, by Lemma 6. But all edges of the Γ_i have lengths at least $\frac{1}{\rho^*}$, so the perimeter of triangle (e_1, v_1) is smaller than that of T minus $\frac{1}{2\rho^*}$.

Now, if e is not the longest edge of T, then one of e_1 or e_2 must be. Let e' denote this longest edge, then after the construction of $\mathcal{G}_{i+1}$, e' must be the longest edge of the (new) triangle T' formed by e' and $v_{e'}$ because the edges of T' are contained in T, and thus their lengths are at most the length of the longest edge of T, which is e'. Thus, in the construction of $\mathcal{G}_{i+2}$, one of the new triangles formed, by similar application of Lemma 6, has a perimeter that is at least $\frac{1}{2\rho^*}$ shorter than the perimeter of T', which itself is shorter than the perimeter of T.

In each of the two cases we have identified a triangle, either in $\mathcal{G}_{i+1}$ or in $\mathcal{G}_{i+2}$, whose perimeter is shorter than that of T minus $\frac{1}{2\rho^*}$. Since the perimeter of T is at most 3, repeating this process $\lceil 6\rho^* \rceil$ times results in a triangle whose perimeter is negative, which is a contradiction and concludes the proof of Theorem 2.

4. Conclusion

We conclude this paper by listing some open questions. One is whether better bounds on the planar edge-length ratio can be established for subfamilies of outerplanar graphs; for instance, the planar edge-length ratio of a *bipartite* outerplanar graph is 1 and this actually holds for the larger class of the duals of weak pseudo line arrangements [9, Thm. 3]. One other interesting class is that of triangle-free outerplanar graphs: it is not hard to see that if all faces of an outerplanar graph have five vertices, a unit edge length drawing may not exist; however, the planar edge length ratio for this family of graphs could nonetheless be smaller than the one established in Theorem 1.

Another problem is to extend the result of Theorem 1 to families of non-outerplanar graphs. For example it would be interesting to determine whether the planar edge-length ratio of 2-trees is bounded by a constant. We conjecture that this is, in fact, not the case, but it is not clear how to establish this using our current techniques.

Finally, a natural question is to determine the complexity of deciding whether an outerplanar graph admits a straight-line drawing where the ratio of the longest to the shortest edge is within a given range. This problem is also interesting in the special case where all edges are required to be of unit length.

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