A switching controller for a class of MIMO bilinear time-delay systems

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***Abstract*— In this paper we propose a switching controller for a class of MIMO bilinear systems with constant delays in both the state and the input. The motivation to consider such a class of systems is that, in the scalar case, it has shown to be suitable for modelling and controller design of some turbulent flow control systems.**

I. INTRODUCTION

Control of Turbulent Flows is a fundamental problem in several technology areas, e.g. transportation systems as road vehicles or aircraft. Advances in control of turbulent flows produce important benefits, e.g. reduction of fuel consumption in vehicles [2]. However, existing control techniques find several barriers to be applied on flow systems. On one hand, the main model for flow is the set of Navier-Stokes equations which can be very complex for design purposes. Besides, the computational cost for the implementation of a controller that requires to solve on-line the Navier-Stokes model is not realizable with the present-day technology [2]. On the other hand, model approximations, e.g. finite-dimensional linear systems, can be more appropriate for control design or implementation. Unfortunately, many relevant features of the flow are not well reproduced by such models. As a consequence, the obtained controllers only work satisfactorily in a narrow range of situations.

A model for separated flow control was proposed in [6]. With only few parameters, that model succeeded in reproducing the system behaviour, with a low margin error, in different experimental configurations [4]. Such a model consists in a bilinear differential equation with delays in the input and in the state. The argument to propose such a kind of equations as models for flow systems was presented in [4]. There, the author shows that a spatial discretization of the Burgers' equation (which is a unidimensional approximation of the Navier-Stokes equations) produces a bilinear time-delay differential equation.

For a particular case of the model introduced in [6], a sliding mode controller was proposed in [5]. That control technique was chosen due to the on-off characteristic of the actuators. A good experimental performance was obtained with such a controller[1]. Hence, it is worth continuing with the study of the class of bilinear delayed systems and to develop general schemes for analysis and control design.

Unfortunately, the controller proposed in [5] was designed only for SISO systems, and its generalization to the MIMO case is not obvious. In this paper we design a switching controller for a class of MIMO bilinear systems with constant delays in both the state and the input. Such systems can be rewritten as switched affine systems with delay in the state and in the switching signal. The switching strategy presented in this paper is an extension of a control technique developed for switched affine systems without delay, see for example [1], [3], [8].

Paper organization: In Section II we describe briefly the control problem. The controller design is explained in Section III. In Section IV a simulation example is given. Some final conclusions are stated in Section V.

Notation: $\mathbb{R}$ and $\mathbb{Z}$ denote the set of real and integer numbers, respectively. For any $a \in \mathbb{R}$, $\mathbb{R}_{\geq a}$ denotes the set $\{x \in \mathbb{R} \,:\, x \geq a\}$, and analogously for $\mathbb{R}_{>a}$. For a matrix $A \in \mathbb{R}^{n\times n}$, we mean by $A > 0$ ($A < 0$) that A is positive definite (negative definite).

II. PROBLEM STATEMENT

Consider the following bilinear time-delay system

$$\dot{x}(t) = A_0 x(t) + \left[\mathcal{A}(x(t-\tau)) + B\right] u(t-\varsigma)\,, \qquad (1)$$

$$\mathcal{A}(x(t-\tau)) = \left[\bar{A}_1 x(t-\tau)\,,\ \ldots\,,\ \bar{A}_m x(t-\tau)\right]\,,$$

where $x(t) \in \mathbb{R}^n$ is the (instantaneous) state, $u(t) \in \mathbb{R}^m$ is the control input, $A_0, \bar{A}_k \in \mathbb{R}^{n\times n}$, $k = 1,\ldots,m$, are constant matrices, and $\tau, \varsigma \in \mathbb{R}_{\geq 0}$ are constant delays. We consider (1) equipped with the output $y(t) = x(t)$. We also assume the following for (1).

Assumption 1: The matrix A_0 is Hurwitz, and $\tau \geq \varsigma > 0$.

The problem we want to solve is stated as follows:

- to design a controller u to track asymptotically a desired constant reference $y^* \in \mathbb{R}^n$, taking into account that each control component $u_i : \mathbb{R} \to \mathbb{R}$ can only take a finite number of values.

Observe that, under the restriction on the controller, we can only have N different values of the input vector u, for some $N \in \mathbb{Z}_{>0}$. If we give a certain order to such N vectors, then we obtain the ordered set

$$U = \{u^1, \ldots, u^N\}\,, \qquad (2)$$

where each $u^j \in \mathbb{R}^m$, $j = 1,\ldots,N$, is a constant vector.

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[1] A video with some experiments, reported in [5], can be seen at https://www.youtube.com/watch?v=b5NnAV2qeno.

III. Controller design

In this section we develop the control law to solve the problem established in the previous section. To this end, we first state some basic preliminary considerations.

Observe that (1) can be rewritten as follows

$$\dot{x}(t) = A_0 x(t) + \sum_{k=1}^{m} u_k(t-\varsigma)\bar{A}_k x(t-\tau) + Bu(t-\varsigma)\,. \quad (3)$$

Now, for $j = 1, \ldots, N$, define the vectors $B_j = Bu^j$ and the matrices $A_j = \sum_{k=1}^{m} u_k^j \bar{A}_k$. Hence, we can consider (3) as an affine switched system given by

$$\dot{x}(t) = A_0 x(t) + A_{\sigma(t-\varsigma)} x(t-\tau) + B_{\sigma(t-\varsigma)}\,, \quad (4)$$

where $\sigma : \mathbb{R} \to \{1, \ldots, N\}$ is the switching signal. Obviously, the system's output remain the same, i.e. $y(t) = x(t)$. Thus we have changed the original problem to the problem of designing the switching rule for σ.

A. On the assignable references

As stated in Section II, the control objective is to track asymptotically a constant reference $y^* \in \mathbb{R}^n$. Since $y(t) = x(t)$, we define the change of coordinates $z(t) = x(t) - y^*$. The dynamics in variable z is given by

$$\begin{aligned}\dot{z}(t) = A_0 z(t) + A_{\sigma(t-\varsigma)} z(t-\tau) + B_{\sigma(t-\varsigma)} + \\ (A_0 + A_{\sigma(t-\varsigma)})y^*\,. \end{aligned} \quad (5)$$

Thus, we have to design σ such that $z(t) \to 0$ as $t \to \infty$. However, observe that in general $z = 0$ is not an equilibrium point of (5), i.e. the terms

$$B_{\sigma(t-\varsigma)} + (A_0 + A_{\sigma(t-\varsigma)})y^*\,, \quad (6)$$

can be different from zero. Note that to ask for the condition $B_{\sigma(t-\varsigma)} + (A_0 + A_{\sigma(t-\varsigma)})y^* = 0$, for any value of σ, could be very restrictive. Nevertheless, we can achieve that (in *average* [11], [1]) the terms in (6) do not influence the steady-state of (5). Thus, according to the switching nature of the control scheme designed below, we have to restrict the set of assignable references y^*. Define the set

$$\Gamma = \{\gamma \in \mathbb{R}^N \,:\, 0 \le \gamma_i \le 1,\, \sum_{j=1}^{N} \gamma_j = 1\}\,,$$

and the matrices

$$\bar{A}_\gamma = \sum_{j=1}^{N} \gamma_j A_j\,, \quad \bar{B}_\gamma = \sum_{j=1}^{N} \gamma_j B_j\,. \quad (7)$$

Thus, a necessary condition for y^* to be an admissible reference is that it is contained in the set

$$\{y^* \in \mathbb{R}^n \,:\, (A_0 + \bar{A}_\gamma)y^* + \bar{B}_\gamma = 0,\ \gamma \in \Gamma\}\,.$$

B. State-delay-independent control rule

Note that in (4), the controller σ is delayed in time, i.e., at time t the system receives the signal $\sigma(t-\varsigma)$. However, we can design a state dependent switching signal σ that drives the trajectories of (4) to a desired reference point y^*.

Theorem 1: Consider (4) and a given $y^* \in \mathbb{R}^n$. If there exist $\gamma \in \Gamma$ and symmetric positive definite matrices $P, Q \in \mathbb{R}^{n\times n}$ such that the following holds,

$$(A_0 + \bar{A}_\gamma)y^* + \bar{B}_\gamma = 0\,, \quad \text{and} \quad \bar{M}_\gamma < 0\,, \quad (8)$$

where

$$\bar{M}_\gamma = \begin{bmatrix} PA_0 + A_0^\top P + Q & P\bar{A}_\gamma \\ \bar{A}_\gamma^\top P & -Q \end{bmatrix}\,,$$

then the output y of (4) in closed-loop with the switching controller

$$\begin{aligned}\sigma(t) &= \arg\min_j\, (w(t) - y^*)P[A_j y(t-\tau+\varsigma) + B_j]\,, \quad (9)\\ w(t) &= e^{A_0\varsigma} y(t) + \int_{-\varsigma}^{0} e^{-A_0\eta} \times \\ &\quad \left[A_{\sigma(t+\eta)} y(t+\eta-\tau+\varsigma) + B_{\sigma(t+\eta)}\right] d\eta\,,\end{aligned}$$

converges asymptotically to y^*.

The proof of Theorem 1 is given below, but first, let us enunciate the following remarks.

Remark 1 (On the solutions): Note that, for any $t \in \mathbb{R}_{\ge 0}$, the switching signal $\sigma(t)$ depends on $x(\vartheta)$, $\vartheta \in [t-\tau, t]$. Hence, the delayed switching signal $\sigma(t-\varsigma)$ depends on $x(\vartheta)$, $\vartheta \in [t-\tau-\varsigma, t-\varsigma]$. Thus, for $\varsigma > 0$, (4) in closed-loop with (9) can be seen as a linear time delay system with time varying coefficients. Therefore, the *method of steps*[2] is still valid to guarantee the existence and uniqueness of Carathéodory solutions of the closed-loop.

Remark 2: For the delay-free case ($\varsigma = 0$ and $\tau = 0$), Theorem 1 reduces to a result given in [Theorem 1, [3]], i.e.: If there exist $\gamma \in \Gamma$ and a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$ such that (8) holds, with $\bar{M}_\gamma = P(A_0 + \bar{A}_\gamma) + (A_0 + \bar{A}_\gamma)^\top P$, then the switching controller

$$\sigma(t) = \arg\min_j\, (y(t) - y^*)P[A_j y(t) + B_j]\,,$$

is such that the output y converges asymptotically to y^*.

C. Proof of Theorem 1

The proof consists in verifying, by means of a Lyapunov-Krasovskii functional, that the solutions of the closed-loop (5), (9) converge asymptotically to the origin. Consider the following standard Lyapunov-Krasovskii functional candidate (see e.g. [7])

$$V(\phi) = \phi^\top(0)P\phi(0) + \int_{-\tau}^{0} \phi^\top(\nu)Q\phi(\nu)\, d\nu\,, \quad (10)$$

[2] In this case, the method of steps consists in studying the systems in the intervals $[t_0, t_0+\varsigma], [t_0+\varsigma, t_0+2\varsigma], \ldots$, and so on, see e.g. [7].

whose derivative along (5) is given by

$$\dot{V}(z_t) = z^\top(t)(PA_0 + A_0^\top P + Q)z(t) - z^\top(t-\tau)Qz(t-\tau) + 2z^\top(t)P\left[B_{\sigma(t-\varsigma)} + A_{\sigma(t-\varsigma)}z(t-\tau) + (A_0 + A_{\sigma(t-\varsigma)})y^*\right], \quad (11)$$

for almost all $t \in \mathbb{R}_{\geq 0}$. Define the vector

$$\zeta(t) = [z^\top(t),\ z^\top(t-\tau)]^\top,$$

and let us rewrite (11) as $\dot{V}(z_t) = W(\zeta; \sigma(t-\varsigma))$ where

$$W(\zeta; \sigma(t-\varsigma)) = \zeta^\top(t)M(\sigma(t-\varsigma))\zeta(t) + 2z^\top(t)P \times \left[B_{\sigma(t-\varsigma)} + (A_0 + A_{\sigma(t-\varsigma)})y^*\right], \quad (12)$$

$$M(\sigma(t-\varsigma)) = \begin{bmatrix} PA_0 + A_0^\top P + Q & PA_{\sigma(t-\varsigma)} \\ A_{\sigma(t-\varsigma)}^\top P & -Q \end{bmatrix}.$$

Now define the function $W_\gamma : \mathbb{R}^{2n} \to \mathbb{R}$, given by

$$W_\gamma(\zeta) = \zeta^\top(t)\bar{M}_\gamma\zeta(t) + 2z^\top(t)P\left[B_\gamma + (A_0 + \bar{A}_\gamma)y^*\right].$$

Observe that, according to the hypotheses of the theorem, W_γ is negative definite and can be rewritten as

$$W_\gamma(\zeta) = \sum_{j=1}^{N} \gamma_j W(\zeta; j).$$

Since W_γ is negative definite, and each γ_i is nonnegative, we can assure that for each $\zeta \in \mathbb{R}^{2n} \setminus \{0\}$, there exists (at least) one $j \in \{1, \ldots, N\}$ such that $W(\zeta; j) < 0$. Therefore, the switching law

$$\sigma(t-\varsigma) = \arg\min_j W(\zeta; j)$$

guarantees that (12) is negative definite. Now consider (11) to note that

$$\min_j W(\zeta; j) = z^\top(t)(PA_0 + A_0^\top P + Q)z(t) - z^\top(t-\tau)Qz(t-\tau) + 2\min_j\left(z^\top(t)P \times [A_j z(t-\tau) + (A_0 + A_j)y^* + B_j]\right),$$

nonetheless, by using the change of coordinates for z we obtain the equality

$$\min_j\left(z^\top(t)P\left[A_j z(t-\tau) + (A_0 + A_j)y^* + B_j\right]\right) = \min_j\left((x(t) - y^*)^\top P\left[A_j x(t-\tau) + B_j\right]\right) + z^\top(t)PA_0 y^*.$$

Hence we can assert that the switching law $\sigma(t-\varsigma) = \arg\min_j W(\zeta; j)$ is equivalent to the delayed switching law

$$\sigma(t-\varsigma) = \arg\min_j (x(t) - y^*)P[A_j x(t-\tau) + B_j].$$

By advancing in time this equation, we obtain

$$\sigma(t) = \arg\min_j (x(t+\varsigma) - y^*)P[A_j x(t-\tau+\varsigma) + B_j], \quad (13)$$

but this switching law requires the prediction of the state at $t+\varsigma$. Since the solution of (4) can be written as

$$x(t) = \int_{t_0}^{t} e^{A_0(t-\nu)}\left[A_{\sigma(\nu-\varsigma)}x(\nu-\tau) + B_{\sigma(\nu-\varsigma)}\right] d\nu + e^{A_0(t-t_0)}x(t_0),$$

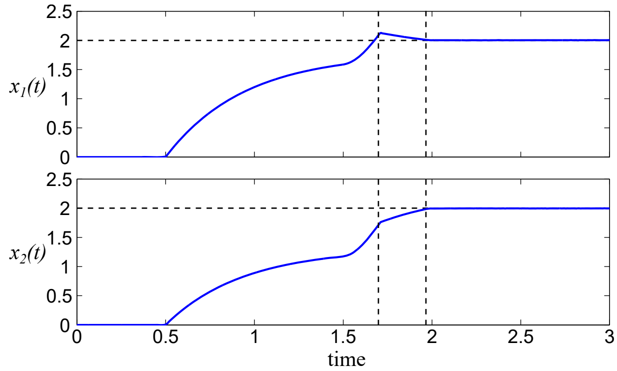

Fig. 1. States of the system.

the prediction of the state at $t+\varsigma$ is given by

$$x(t+\varsigma) = e^{A_0\varsigma}x(t) + \int_t^{t+\varsigma} e^{A_0(t+\varsigma-\nu)} \times \left[A_{\sigma(\nu-\varsigma)}x(\nu-\tau) + B_{\sigma(\nu-\varsigma)}\right] d\nu.$$

Observe that in (9), $w(t) = x(t+\varsigma)$ by means of the change of variable $\eta = -t - \varsigma + \nu$. This completes the proof of Theorem 1.

For Remark 2, the proof is analogous but considering the Lyapunov function candidate $V(z) = z^\top Pz$.

IV. EXAMPLE

Consider (1) with $n = 2$, $m = 2$, $\tau = 1$, $\varsigma = 1/2$, and

$$A_0 = \begin{bmatrix} -3 & 1 \\ 2 & -5 \end{bmatrix}, \quad \bar{A}_1 = \begin{bmatrix} 4 & 2 \\ 1 & 6 \end{bmatrix}, \quad \bar{A}_2 = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}.$$

The control components are assumed to be on-off, i.e. $u_i : \mathbb{R} \to \{0, 1\}$, $i = 1, 2$, hence $U = \{u^1, \ldots, u^4\}$ where $u^1 = [0\ 0]^\top$, $u^2 = [1\ 0]^\top$, $u^3 = [0\ 1]^\top$, and $u^4 = [1\ 1]^\top$. It is important to mention that the trajectories of this model diverge by maintaining any of the inputs on. If we choose $y^* = [2\ 2]^\top$, then for $\gamma = \frac{1}{30}[14\ 4\ 12\ 0]^\top$ we have that (8) hold with

$$P = \begin{bmatrix} 4.8581 & 1.1108 \\ 1.1108 & 2.6861 \end{bmatrix}, \quad Q = \begin{bmatrix} 1.1122 & -0.0054 \\ -0.0054 & 1.1990 \end{bmatrix}.$$

The LMI in (8) was solved in MATLAB by using the solver SeDuMi [10] version 1.3, and the toolbox YALMIP [9].

For the simulation, we used Simulink with the fixed step Euler integration method. The initial conditions were chosen as $x(t) = 0$, $u(t) = 0$ for all $t \leq 0$, and the integration step was equal to $0.1ms$.

Fig. 1 shows the system's states converging to the reference point. The vertical doted lines indicate the time when the delayed inputs start to switch in a high-frequency regime. Such a switching can be appreciated in the nondelayed control signals shown in Fig. 2.

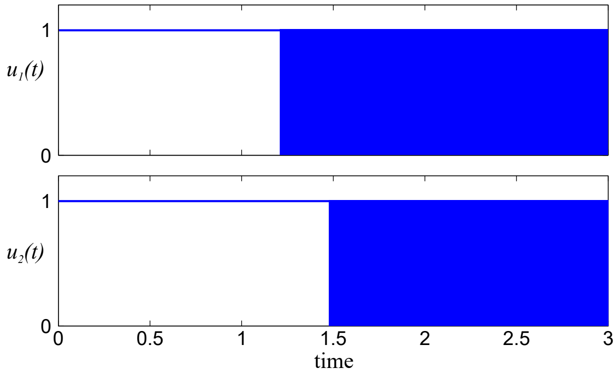

Fig. 2. Control signals.

V. Conclusion

A standard technique to control affine switched systems was extended to a class of bilinear time-delay systems. The importance of this controller lies on its potential applicability to turbulent flow control systems. The conceptual simplicity of the controller allows us to use it for both SISO and MIMO systems. Interestingly the proposed controller is very suitable for the kind of on-off actuators used in several flow control applications, see e.g. [2], [5].

Future work: There are several aspects that have to be considered to improve the control approach considered in this paper. We can mention, e.g. reduction of computational cost in the predictor implementation, relaxations or modification of the system's assumptions, robustness analysis, and generalization to a wider set of systems.

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