Well-posedness of IBVP for 1D scalar non-local conservation laws

Paola Goatin, Elena Rossi

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Paola Goatin[1] Elena Rossi[1]

Abstract

We consider the initial boundary value problem (IBVP) for a non-local scalar conservation laws in one space dimension. The non-local operator in the flux function is not a mere convolution product, but it is assumed to be *aware of boundaries*. Introducing an adapted Lax-Friedrichs algorithm, we provide various estimates on the approximate solutions that allow to prove the existence of solutions to the original IBVP. The uniqueness follows from the Lipschitz continuous dependence on initial and boundary data, which is proved exploiting results available for the *local* IBVP.

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1 Introduction

We consider the following Initial Boundary Value Problem (IBVP) on the open bounded interval $]a,b[\subset\mathbb{R}$

$$\begin{cases} \partial_t\rho+\mathrm{d}_x\, f(t,x,\rho,\mathcal{J}\rho)=0, & (t,x)\in\mathbb{R}^+\times]a,b[,\\ \rho(0,x)=\rho_o(x), & x\in]a,b[,\\ \rho(t,a)=\rho_a(t), & t\in\mathbb{R}^+,\\ \rho(t,b)=\rho_b(t), & t\in\mathbb{R}^+, \end{cases} \tag{1.1}$$

where $\mathcal{J}$ denotes a non-local operator and we use the notation

$$\begin{aligned} \mathrm{d}_x\, f\left(t,x,\rho(t,x),\big(\mathcal{J}\rho(t)\big)(x)\right) = {} & \partial_x f\left(t,x,\rho(t,x),\big(\mathcal{J}\rho(t)\big)(x)\right)\\ & +\partial_\rho f\left(t,x,\rho(t,x),\big(\mathcal{J}\rho(t)\big)(x)\right)\,\partial_x\rho(t,x)\\ & +\partial_R f\left(t,x,\rho(t,x),\big(\mathcal{J}\rho(t)\big)(x)\right)\,\partial_x\left(\mathcal{J}\rho(t)\right)(x). \end{aligned} \tag{1.2}$$

The same problem was studied in [10]. In that case, the choice for $\mathcal{J}$ is the classical convolution product $\mathcal{J}\rho=\rho*\eta$, η being a smooth convolution kernel. However, in such formulation, the non-local term may exceed the boundaries of the spatial domain. The authors address this

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issue by extending the solution outside the spatial domain, setting it constantly equal to the corresponding boundary condition value.

Here, we propose a different approach. We follow the treatment of the boundary conditions proposed in [9], where a particular multi-dimensional system of conservation laws in bounded domains with zero boundary conditions is considered. More precisely, a non-local operator aware of the presence of boundaries is introduced. In the present one-dimensional setting, this reads

$$\big(\mathcal{J}\rho(t)\big)(x) = \frac{1}{W(x)} \int_a^b \rho(t,y)\,\omega(y-x)\,\mathrm{d}y\,, \quad \text{with} \quad W(x) = \int_a^b \omega(y-x)\,\mathrm{d}y\,, \tag{1.3}$$

for a suitable convolution kernel ω.

In recent years, the literature on non-local conservation laws has widely increased. These equations are indeed used to model various physical phenomena: from sedimentation models [4] to granular flow [1], from vehicular traffic [5] to crowd dynamics [6, 7, 8], from conveyor belts [11] to supply chains [2].
Although physically those models might be defined in a bounded domain and numerical integrations require it as well, they have been mostly studied in the whole space $\mathbb{R}$ or $\mathbb{R}^n$. The main difficulty lies indeed in the fact that the non-local operator may need to evaluate the unknown outside the boundaries of the spatial domain, where it is not defined.
The analysis of the non-local problem (1.1) is carried out exploiting the same strategy used in both [10] and [14]. As already mentioned, [10] studies the non local IBVP (1.1) where the non-local operator is the standard convolution product, while [14] considers the *local* problem for a balance law, i.e. a one dimensional IBVP where the flux function has the form $f(t,x,\rho)$ and there is also a source term. We remark that it could be possible to use the results of [14] to study the non-local problem (1.1): indeed, the link between the two problems is obtained by defining the *local* flux by $\tilde{f}(t,x,\rho) = f(t,x,\rho,\mathcal{J}\rho)$, where $\mathcal{J}\rho = \big(\mathcal{J}\rho(t)\big)(x)$. However, in this way the *a priori* estimates on the solution would be less precise than those presented in this work. Namely, a positivity result and an $\mathbf{L^1}$-bound on the solution are missing in [14]. Moreover, $\mathbf{L}^\infty$-estimate recovered here depends on the first derivatives of the flux function, see Theorem 2.3, while using the results of [14] yields an estimate depending on the mixed second derivatives of f.
Nevertheless, the result concerning the stability with respect to the flux function proved in [14], recalled below in Theorem A.4, is of crucial importance in this work, since it contributes significantly in the proof of the Lipschitz continuous dependence of solutions to (1.1) on initial and boundary data, see Proposition 4.1, and thus in the proof of the uniqueness of solution to (1.1). At this regard, we remark that the stability proof provided in [10] is wrong, but could be fixed following the same strategy proposed here.

The paper is organised as follows. Section 2 presents the assumptions needed on problem (1.1) and the main result of this paper, whose proof is postponed to Section 5. Section 3 is devoted to the introduction of the finite volume approximation of problem (1.1) and its analysis. The Lipschitz continuous dependence of solutions to (1.1) on initial and boundary data is proved in Section 4. The final appendix A recalls some results from [14] on the *local* IBVP, necessary throughout the paper.

2 Main results

We introduce the following notation:

$$\operatorname{sgn}^+(s) = \begin{cases} 1 & \text{if } s > 0, \\ 0 & \text{if } s \le 0, \end{cases} \qquad \operatorname{sgn}^-(s) = \begin{cases} 0 & \text{if } s \ge 0, \\ -1 & \text{if } s < 0, \end{cases} \qquad \begin{aligned} s^+ &= \max\{s, 0\}, \\ s^- &= \max\{-s, 0\}. \end{aligned}$$

In the rest of the paper, we will denote $\mathcal{I}(r,s) = [\min\{r,s\}, \max\{r,s\}]$, for any $r, s \in \mathbb{R}$. We make the following assumptions on the flux function f and on the convolution kernel ω:

(*f*) $f \in \mathbf{C^2}(\mathbb{R}^+ \times [a,b] \times \mathbb{R} \times \mathbb{R}; \mathbb{R})$ and there exist $L, C > 0$ such that

$$\begin{aligned} & f(t,x,0,R) = 0 && \text{for } t \in \mathbb{R}^+,\ x \in [a,b],\ R \in \mathbb{R}, \\ & \sup_{t,x,\rho,R} \left|\partial_\rho f(t,x,\rho,R)\right| < L, \\ & \sup_{t,x,R} \left|\partial_x f(t,x,\rho,R)\right| < C|\rho|, && \sup_{t,x,R} \left|\partial_R f(t,x,\rho,R)\right| < C|\rho|, \\ & \sup_{t,x,R} \left|\partial^2_{xx} f(t,x,\rho,R)\right| < C|\rho|, && \sup_{t,x,R} \left|\partial^2_{xR} f(t,x,\rho,R)\right| < C|\rho|, \\ & \sup_{t,x,R} \left|\partial^2_{RR} f(t,x,\rho,R)\right| < C|\rho|. \end{aligned}$$

($\boldsymbol{\omega}$) $\omega \in (\mathbf{C^2} \cap \mathbf{W}^{2,1} \cap \mathbf{W}^{2,\infty})(\mathbb{R};\mathbb{R})$ is such that

$$\int_{\mathbb{R}} \omega(y)\, \mathrm{d}y = 1$$

and there exists $K_\omega > 0$ such that for all $x \in [a,b]$

$$W(x) = \int_a^b \omega(y-x)\, \mathrm{d}y \ge K_\omega. \tag{2.1}$$

The requirement (2.1) guarantees that $\mathcal{J}$ in (1.3) is well defined for all $x \in]a,b[$.

We recall below two different definitions of solution to problem (1.1). Recall that the two definitions are equivalent for functions in $(\mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^+ \times]a,b[; \mathbb{R})$. We refer to [13] for further details on the link between this two definitions.

The first definition follows from [3].

Definition 2.1. *A function $\rho \in (\mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^+ \times]a,b[; \mathbb{R})$ is an entropy weak solution to problem* (1.1) *if, for all $\varphi \in \mathbf{C^1_c}(\mathbb{R}^2; \mathbb{R}^+)$ and $k \in \mathbb{R}$,*

$$\begin{aligned} \int_0^{+\infty} \int_a^b \Big\{ & |\rho - k| \partial_t \varphi(t,x) + \operatorname{sgn}(\rho - k) \Big[f\big(t,x,\rho,R(t,x)\big) - f\big(t,x,k,R(t,x)\big) \Big] \partial_x \varphi(t,x) \\ & - \operatorname{sgn}(\rho - k) \Big(\partial_x f\big(t,x,k,R(t,x)\big) + \partial_R f\big(t,x,k,R(t,x)\big)\, \partial_x R(t,x) \Big)\, \varphi(t,x) \Big\} \,\mathrm{d}x\,\mathrm{d}t \\ + \int_a^b & \left|\rho_o(x) - k\right| \varphi(0,x)\, \mathrm{d}x \\ + \int_0^{+\infty} & \operatorname{sgn}\big(\rho_a(t) - k\big) \Big[f\Big(t,a,\rho(t,a^+),R(t,a)\Big) - f\big(t,a,k,R(t,a)\big) \Big] \varphi(t,a)\, \mathrm{d}t \end{aligned} \tag{2.2}$$

$$
-\int_0^{+\infty} \operatorname{sgn}\left(\rho_b(t)-k\right) \left[f\left(t,b,\rho(t,b^-),R(t,b)\right) - f\left(t,b,k,R(t,b)\right)\right] \varphi(t,b)\,\mathrm{d}t \geq 0,
$$

where, for $x \in [a,b]$,

$$
R(t,x) = \big(\mathcal{J}\rho(t)\big)(x) = \frac{1}{W(x)} \int_a^b \rho(t,y)\,\omega(y-x)\,\mathrm{d}y\,, \tag{2.3}
$$

and W is as in (1.3)*.*

The second definition was introduced in [12, 15].

Definition 2.2. *A function $\rho \in \mathbf{L}^\infty(\mathbb{R}^+ \times]a,b[;\mathbb{R})$ is an entropy weak solution to problem* (1.1) *if, for all $\varphi \in \mathbf{C}^1_\mathbf{c}(\mathbb{R}^2;\mathbb{R}^+)$ and $k \in \mathbb{R}$,*

$$
\begin{aligned}
&\int_0^{+\infty}\!\!\int_a^b \Big\{ (\rho-k)^\pm\, \partial_t\varphi(t,x) + \operatorname{sgn}^\pm(\rho-k) \left[f\left(t,x,\rho,R(t,x)\right) - f\left(t,x,k,R(t,x)\right)\right] \partial_x\varphi(t,x) \\
&\qquad - \operatorname{sgn}^\pm(\rho-k) \Big(\partial_x f\left(t,x,k,R(t,x)\right) + \partial_R f\left(t,x,k,R(t,x)\right)\, \partial_x R(t,x)\Big) \varphi(t,x) \Big\}\,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_a^b \left(\rho_o(x)-k\right)^\pm \varphi(0,x)\,\mathrm{d}x \\
&+ \left\|\partial_\rho f\right\|_{\mathbf{L}^\infty} \left(\int_0^{+\infty} \left(\rho_a(t)-k\right)^\pm \varphi(t,a)\,\mathrm{d}t + \int_0^{+\infty} \left(\rho_b(t)-k\right)^\pm \varphi(t,b)\,\mathrm{d}t \right) \geq 0,
\end{aligned} \tag{2.4}
$$

where R is as in (2.3) *and $\left\|\partial_\rho f\right\|_{\mathbf{L}^\infty} = \sup_{(t,x)\in\mathbb{R}^+\times[a,b]} \left|\partial_\rho f\left(t,x,\rho(t,x),R(t,x)\right)\right|$.*

We can now state our main result.

Theorem 2.3. *Let **(f)** and **(ω)** hold. Let $\rho_o \in \mathbf{BV}(]a,b[;\mathbb{R}^+)$ and ρ_a, $\rho_b \in \mathbf{BV}(\mathbb{R}^+;\mathbb{R}^+)$. Then, for all $T>0$, problem* (1.1) *has a unique entropy weak solution $\rho \in (\mathbf{L^1} \cap \mathbf{L}^\infty \cap \mathbf{BV})([0,T]\times]a,b[;\mathbb{R}^+)$. Moreover, the following estimates hold: for any $t \in [0,T]$,*

$$
\left\|\rho(t)\right\|_{\mathbf{L^1}([a,b])} \leq \mathcal{R}_1(t),
$$

$$
\left\|\rho(t)\right\|_{\mathbf{L}^\infty(]a,b[)} \leq \mathcal{R}_\infty(t),
$$

$$
\operatorname{TV}\left(\rho(t)\right) \leq e^{t\,\mathcal{T}_1(t)} \left(\operatorname{TV}(\rho_o) + \operatorname{TV}(\rho_a;[0,t]) + \operatorname{TV}(\rho_b;[0,t])\right) + \frac{\mathcal{T}_2(t)}{\mathcal{T}_1(t)}(e^{t\,\mathcal{T}_1(t)}-1),
$$

and, for $\tau > 0$,

$$
\left\|\rho(t)-\rho(t-\tau)\right\|_{\mathbf{L^1}([a,b])} \leq \tau \Big(\mathcal{C}_t(t) + 3\,L\left(\operatorname{TV}(\rho_a;[t-\tau,t]) + \operatorname{TV}(\rho_b;[t-\tau,t])\right)\Big)\,,
$$

where

$$
\mathcal{R}_1(t) = \|\rho_o\|_{\mathbf{L^1}([a,b])} + \left\|\partial_\rho f\right\|_{\mathbf{L}^\infty} \Big(\|\rho_a\|_{\mathbf{L^1}([0,t])} + \|\rho_b\|_{\mathbf{L^1}([0,t])}\Big)\,, \tag{2.5}
$$

$$
\mathcal{R}_\infty(t) = e^{t\,C\,(1+\mathcal{L}\,\mathcal{R}_1(t))} \max\Big\{\|\rho_o\|_{\mathbf{L}^\infty(]a,b[)},\, \|\rho_a\|_{\mathbf{L}^\infty([0,t])},\, \|\rho_b\|_{\mathbf{L}^\infty([0,t])}\Big\}\,, \tag{2.6}
$$

$$
\mathcal{T}_1(t) = \left\|\partial^2_{\rho x} f\right\|_{\mathbf{L}^\infty([0,t]\times[a,b]\times\mathbb{R}^2)} + \mathcal{L}\,\mathcal{R}_1(t) \left\|\partial^2_{\rho R} f\right\|_{\mathbf{L}^\infty([0,t]\times[a,b]\times\mathbb{R}^2)}, \tag{2.7}
$$

$$
\mathcal{T}_2(t) = \mathcal{K}_2(t) + \frac{3}{2}\,C(1+\mathcal{L}\,\mathcal{R}_1(t))\,\mathcal{R}_\infty(t) + \left[\mathcal{K}_3(t) + \frac{C}{2}\left(1+\mathcal{L}\,\mathcal{R}_1(t)\right)\right] \|\rho_a\|_{\mathbf{L}^\infty([0,t])}, \tag{2.8}
$$

with $\mathcal{L}$ as in (3.12)*, $\mathcal{K}_2(t), \mathcal{K}_3(t)$ as in* (3.27)*, with $\mathcal{C}_1(t)$ substituted by $\mathcal{R}_1(t)$, and $\mathcal{C}_t(t)$ is as in* (3.33)*, with $\alpha = L$.*

3 Existence of weak entropy solutions

Fix $T > 0$. Fix a space step Δx such that $b - a = N\Delta x$, with $N \in \mathbb{N}$, and a time step Δt subject to a CFL condition, specified later. Introduce the following notation

$$\begin{aligned} y_k &:= (k - 1/2)\Delta x, & y_{k+1/2} &:= k\Delta x && \text{for } k \in \mathbb{Z}, \\ x_{j+1/2} &:= a + j\Delta x = a + y_{j+1/2}, &&&& \text{for } j = 0, \ldots, N, \\ x_j &:= a + (j - 1/2)\Delta x = a + y_j, &&&& \text{for } j = 1, \ldots, N, \end{aligned}$$

where $x_{j+1/2}$, $j = 0, \ldots, N$, are the cells interfaces and x_j, $j = 1, \ldots, N$, the cells centres. Moreover, set $N_T = \lfloor T/\Delta t \rfloor$ and, for $n = 0, \ldots, N_T$ let $t^n = n\Delta t$ be the time mesh. Set $\lambda = \Delta t/\Delta x$.

Approximate the initial datum ρ_o and the boundary data as follows:

$$\begin{aligned} \rho_j^0 &:= \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} \rho_o(x)\, \mathrm{d}x\,, \qquad j = 1, \ldots, N, \\ \rho_a^n &:= \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \rho_a(t)\, \mathrm{d}t\,, \quad \rho_b^n := \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \rho_b(t)\, \mathrm{d}t\,, \qquad n = 0, \ldots, N_T - 1. \end{aligned}$$

Introduce moreover the notation $\rho_0^n = \rho_a^n$ and $\rho_{N+1}^n = \rho_b^n$. For $n = 0, \ldots, N_T - 1$, set

$$\begin{aligned} \omega^k &:= \omega(y_k) = \omega\left((k - 1/2)\Delta x\right) && \text{for } k \in \mathbb{Z}, \\ W_{j+1/2} &:= \Delta x \sum_{k=1}^{N} \omega^{k-j} && \text{for } j = 0, \ldots, N, \\ R_{j+1/2}^n &:= \frac{\Delta x}{W_{j+1/2}} \sum_{k=1}^{N} \omega^{k-j}\, \rho_k^n && \text{for } j = 0, \ldots, N. \end{aligned} \tag{3.1}$$

Introduce the following modified Lax-Friedrichs flux adapted to the present setting: for $n = 0, \ldots, N_T - 1$ and $j = 0, \ldots, N$,

$$F_{j+1/2}^n(\rho_j^n, \rho_{j+1}^n) = \frac{1}{2}\left[f(t^n, x_{j+1/2}, \rho_j^n, R_{j+1/2}^n) + f(t^n, x_{j+1/2}, \rho_{j+1}^n, R_{j+1/2}^n) - \alpha\left(\rho_{j+1}^n - \rho_j^n\right)\right], \tag{3.2}$$

where $\alpha \geq 1$ is the viscosity coefficient.

We define a piecewise constant approximate solution ρ_Δ to (1.1) as

$$\rho_\Delta(t, x) = \rho_j^n \quad \text{for} \quad \begin{cases} t \in [t^n, t^{n+1}[\,, \\ x \in [x_{j-1/2}, x_{j+1/2}[\,, \end{cases} \quad \text{where} \quad \begin{array}{l} n = 0, \ldots, N_T - 1, \\ j = 1, \ldots, N, \end{array} \tag{3.3}$$

through the finite volume scheme

$$\rho_j^{n+1} = \rho_j^n - \lambda\left(F_{j+1/2}^n(\rho_j^n, \rho_{j+1}^n) - F_{j-1/2}^n(\rho_{j-1}^n, \rho_j^n)\right). \tag{3.4}$$

Remark 3.1. *Concerning the first formula in* (3.1)*, observe that a different (more accurate) choice for the approximation of the kernel function ω is possible: indeed, one may define*

$$\omega^k = \frac{1}{\Delta x} \int_{(k-1)\Delta x}^{k\Delta x} \omega(y)\, \mathrm{d}y\,,$$

which ensures that $W_{j+1/2} = \Delta x \sum_{k=1}^{N} \omega^{k-j} = W(x_{j+1/2})$. This choice wouldn't result in any relevant change in the estimates derived in this paper.

3.1 Positivity

In the case of positive initial and boundary data, we prove that under a suitable CFL condition the scheme (3.4) preserves the positivity.

Lemma 3.2. *Let $\rho_o \in \mathbf{L}^\infty(]a,b[;\mathbb{R}^+)$ and $\rho_a,\, \rho_b \in \mathbf{L}^\infty(\mathbb{R}^+;\mathbb{R}^+)$. Let* $\boldsymbol{(f)}$ *and* $\boldsymbol{(\omega)}$ *hold. Assume that*

$$\alpha \geq L, \qquad\qquad \lambda \leq \frac{1}{3}\min\left\{\frac{1}{\alpha}, \frac{1}{2\,L + C\,\Delta x}\right\}. \tag{3.5}$$

Then, for all $t > 0$ and $x \in]a,b[$, the piecewise constant approximate solution ρ_Δ (3.3) is such that $\rho_\Delta(t,x) \geq 0$.

Proof. We closely follow [10, Lemma 1]. Fix j between 1 and N, n between 0 and $N_T - 1$. Suppose that $\rho_j^n \geq 0$ for all $j = 1, \ldots, N$. Rewrite (3.4) as follows:

$$\begin{aligned}
\rho_j^{n+1} &= \rho_j^n - \lambda\left[F_{j+1/2}^n(\rho_j^n, \rho_{j+1}^n) \pm F_{j+1/2}^n(\rho_j^n,\rho_j^n) \pm F_{j-1/2}^n(\rho_j^n,\rho_j^n) - F_{j-1/2}^n(\rho_{j-1}^n,\rho_j^n)\right]\\
&= (1-\beta_j^n - \gamma_j^n)\,\rho_j^n + \beta_j^n\,\rho_{j-1}^n + \gamma_j^n\,\rho_{j+1}^n - \lambda\left(F_{j+1/2}^n(\rho_j^n,\rho_j^n) - F_{j-1/2}^n(\rho_j^n,\rho_j^n)\right),
\end{aligned}$$

with

$$\beta_j^n = \begin{cases} \lambda\,\dfrac{F_{j-1/2}^n(\rho_j^n,\rho_j^n) - F_{j-1/2}^n(\rho_{j-1}^n,\rho_j^n)}{\rho_j^n - \rho_{j-1}^n} & \text{if } \rho_j^n \neq \rho_{j-1}^n,\\ 0 & \text{if } \rho_j^n = \rho_{j-1}^n, \end{cases} \tag{3.6}$$

$$\gamma_j^n = \begin{cases} -\lambda\,\dfrac{F_{j+1/2}^n(\rho_j^n,\rho_{j+1}^n) - F_{j+1/2}^n(\rho_j^n,\rho_j^n)}{\rho_{j+1}^n - \rho_j^n} & \text{if } \rho_j^n \neq \rho_{j+1}^n,\\ 0 & \text{if } \rho_j^n = \rho_{j+1}^n. \end{cases} \tag{3.7}$$

Using the explicit expression of the numerical flux (3.2) and $\boldsymbol{(f)}$, we obtain

$$\begin{aligned}
&\left|F_{j+1/2}^n(\rho_j^n,\rho_j^n) - F_{j-1/2}^n(\rho_j^n,\rho_j^n)\right|\\
&= \left|f(t^n, x_{j+1/2}, \rho_j^n, R_{j+1/2}^n) - f(t^n, x_{j-1/2},\rho_j^n, R_{j-1/2}^n) \pm f(t^n, x_{j-1/2}, \rho_j^n, R_{j+1/2}^n)\right|\\
&\leq \left|f(t^n, x_{j+1/2}, \rho_j^n, R_{j+1/2}^n) - f(t^n, x_{j-1/2}, \rho_j^n, R_{j+1/2}^n)\right|\\
&\quad + \left|f(t^n, x_{j-1/2}, \rho_j^n, R_{j+1/2}^n) - f(t^n, x_{j-1/2}, \rho_j^n, R_{j-1/2}^n)\right|\\
&\leq C\,\rho_j^n\,\Delta x + \left|f(t^n, x_{j-1/2}, \rho_j^n, R_{j+1/2}^n) - f(t^n, x_{j-1/2}, 0, R_{j+1/2}^n)\right|\\
&\quad + \left|f(t^n, x_{j-1/2}, \rho_j^n, R_{j-1/2}^n) - f(t^n, x_{j-1/2}, 0, R_{j-1/2}^n)\right|\\
&\leq C\,\rho_j^n\,\Delta x + 2\,L\,\rho_j^n.
\end{aligned}$$

Observe that, whenever $\rho_j^n \neq \rho_{j-1}^n$,

$$\beta_j^n = \frac{\lambda}{2\,(\rho_j^n - \rho_{j-1}^n)}\left[f(t^n, x_{j-1/2}, \rho_j^n, R_{j-1/2}^n) - f(t^n, x_{j-1/2}, \rho_{j-1}^n, R_{j-1/2}^n) + \alpha\,(\rho_j^n - \rho_{j-1}^n)\right]$$

$$= \frac{\lambda}{2} \left(\partial_\rho f(t^n, x_{j-1/2}, r^n_{j-1/2}, R^n_{j-1/2}) + \alpha \right),$$

with $r^n_{j-1/2} \in \mathcal{I}\left(\rho^n_{j-1}, \rho^n_j\right)$. Similarly, whenever $\rho^n_j \neq \rho^n_{j+1}$,

$$\begin{aligned}
\gamma^n_j =& -\frac{\lambda}{2\,(\rho^n_{j+1} - \rho^n_j)} \Big[f(t^n, x_{j+1/2}, \rho^n_{j+1}, R^n_{j+1/2}) - f(t^n, x_{j+1/2}, \rho^n_j, R^n_{j+1/2}) - \alpha\,(\rho^n_{j+1} - \rho^n_j) \Big] \\
=& \frac{\lambda}{2} \left(\alpha - \partial_\rho f(t^n, x_{j+1/2}, r^n_{j+1/2}, R^n_{j+1/2}) \right),
\end{aligned}$$

with $r^n_{j+1/2} \in \mathcal{I}\left(\rho^n_j, \rho^n_{j+1}\right)$. By the conditions (3.5), we get

$$\beta^n_j,\, \gamma^n_j \in \left[0, \frac{1}{3}\right], \quad (1 - \beta^n_j - \gamma^n_j) \in \left[\frac{1}{3}, 1\right], \quad \lambda \left| F^n_{j+1/2}(\rho^n_j, \rho^n_j) - F^n_{j-1/2}(\rho^n_j, \rho^n_j) \right| \leq \frac{1}{3}\,\rho^n_j,$$

which, using the inductive hypothesis, leads to

$$\rho^{n+1}_j \geq (1 - \beta^n_j - \gamma^n_j)\rho^n_j + \beta^n_j\,\rho^n_{j-1} + \gamma^n_j\,\rho^n_{j+1} - \frac{1}{3}\,\rho^n_j \geq \frac{1}{3}\,\rho^n_j - \frac{1}{3}\,\rho^n_j \geq 0.$$

□

3.2 $\mathbf{L^1}$ bound

Lemma 3.3. *Let $\rho_o \in \mathbf{L}^\infty(]a,b[;\mathbb{R}^+)$ and $\rho_a,\, \rho_b \in \mathbf{L}^\infty(\mathbb{R}^+;\mathbb{R}^+)$. Let **(f)**, **(ω)** and* (3.5) *hold. Then, for all $t > 0$, ρ_Δ in* (3.3) *satisfies*

$$\left\| \rho_\Delta(t, \cdot) \right\|_{\mathbf{L^1}(]a,b[)} \leq \mathcal{C}_1(t), \tag{3.8}$$

where

$$\mathcal{C}_1(t) = \|\rho_o\|_{\mathbf{L^1}(]a,b[)} + \alpha \left(\|\rho_a\|_{\mathbf{L^1}([0,t])} + \|\rho_b\|_{\mathbf{L^1}([0,t])} \right). \tag{3.9}$$

Proof. By Lemma 3.3, we know that the scheme (3.4) preserves the positivity. Therefore, for $n = 0, \ldots, N_T - 1$, compute

$$\begin{aligned}
\|\rho^{n+1}\|_{\mathbf{L^1}(]a,b[)} =& \Delta x \sum_{j=1}^{N} \rho^{n+1}_j \\
=& \Delta x \sum_{j=1}^{N} \left[\rho^n_j - \lambda \left(F^n_{j+1/2}(\rho^n_j, \rho^n_{j+1}) - F^n_{j-1/2}(\rho^n_{j-1}, \rho^n_j) \right) \right] \\
=& \Delta x \sum_{j=1}^{N} \rho^n_j - \lambda\,\Delta x \left(F^n_{N+1/2}(\rho^n_N, \rho^n_{N+1}) - F^n_{1/2}(\rho^n_0, \rho^n_1) \right) \\
=& \|\rho^n\|_{\mathbf{L^1}([a.b])} \\
&- \frac{\Delta t}{2} \Big[f(t^n, x_{N+1/2}, \rho^n_N, R^n_{N+1/2}) + f(t^n, x_{N+1/2}, \rho^n_b, R^n_{N+1/2}) - \alpha\big(\rho^n_b - \rho^n_N\big) \Big] \\
&+ \frac{\Delta t}{2} \left(f(t^n, x_{1/2}, \rho^n_a, R^n_{1/2}) + f(t^n, x_{1/2}, \rho^n_1, R^n_{1/2}) - \alpha\left(\rho^n_1 - \rho^n_a\right) \right)
\end{aligned}$$

$$
\begin{aligned}
&= \|\rho^n\|_{\mathbf{L^1}([a.b])} + \frac{\Delta t}{2}\left(-\alpha - \partial_\rho f(t^n, x_{N+1/2}, r^n_{N,0}, R^n_{N+1/2})\right)\rho^n_N \\
&\quad + \frac{\Delta t}{2}\left(\alpha - \partial_\rho f(t^n, x_{N+1/2}, r^n_{b,0}, R^n_{N+1/2})\right)\rho^n_b \\
&\quad + \frac{\Delta t}{2}\left(\alpha + \partial_\rho f(t^n, x_{1/2}, r^n_{a,0}, R^n_{1/2})\right)\rho^n_a \\
&\quad + \frac{\Delta t}{2}\left(-\alpha + \partial_\rho f(t^n, x_{1/2}, r^n_{1,0}, R^n_{1/2})\right)\rho^n_1,
\end{aligned}
$$

where $r^n_{N,0} \in \mathcal{I}\left(0, \rho^n_N\right)$, $r^n_{b,0} \in \mathcal{I}\left(0, \rho^n_b\right)$, $r^n_{a,0} \in \mathcal{I}\left(0, \rho^n_a\right)$ and $r^n_{1,0} \in \mathcal{I}\left(0, \rho^n_1\right)$. By **(*f*)** and the assumption (3.5) on α, the coefficients of ρ^n_N and ρ^n_1 are negative. Thus

$$
\|\rho^{n+1}\|_{\mathbf{L^1}(]a,b[)} \le \|\rho^n\|_{\mathbf{L^1}(]a.b[)} + \alpha\,\Delta t\left(\rho^n_a + \rho^n_b\right).
$$

An iterative argument yields the thesis. □

3.3 $\mathbf{L}^\infty$ bound

Lemma 3.4. *Let $\rho_o \in \mathbf{L}^\infty(]a,b[;\mathbb{R}^+)$ and ρ_a, $\rho_b \in \mathbf{L}^\infty(\mathbb{R}^+;\mathbb{R}^+)$. Let* **(*f*)**, **($\boldsymbol{\omega}$)** *and* (3.5) *hold. Then, for all $t>0$, ρ_Δ in* (3.3) *satisfies*

$$
\left\|\rho_\Delta(t,\cdot)\right\|_{\mathbf{L}^\infty(]a,b[)} \le \max\left\{\|\rho_o\|_{\mathbf{L}^\infty(]a,b[)},\ \|\rho_a\|_{\mathbf{L}^\infty([0,t])},\ \|\rho_b\|_{\mathbf{L}^\infty([0,t])}\right\} e^{\mathcal{C}_2(t)t}, \tag{3.10}
$$

where $\mathcal{C}_2(t)$ is given by (3.13).

Proof. Fix n between 0 and N_T-1. For $j=1,\ldots N$, rearrange (3.4) as in Lemma 3.2, with the notation (3.6)–(3.7):

$$
\rho^{n+1}_j = (1-\beta^n_j-\gamma^n_j)\,\rho^n_j + \beta^n_j\,\rho^n_{j-1} + \gamma^n_j\,\rho^n_{j+1} - \lambda\left(F^n_{j+1/2}(\rho^n_j,\rho^n_j) - F^n_{j-1/2}(\rho^n_j,\rho^n_j)\right). \tag{3.11}
$$

Compute

$$
\begin{aligned}
&\left|F^n_{j+1/2}(\rho^n_j,\rho^n_j) - F^n_{j-1/2}(\rho^n_j,\rho^n_j)\right| \\
={}& \left|f(t^n, x_{j+1/2}, \rho^n_j, R^n_{j+1/2}) - f(t^n, x_{j-1/2}, \rho^n_j, R^n_{j-1/2}) \pm f(t^n, x_{j-1/2}, \rho^n_j, R^n_{j+1/2})\right| \\
\le{}& \left|\partial_x f(t^n, \tilde{x}_j, \rho^n_j, R^n_{j+1/2})\right|\left|x_{j+1/2} - x_{j-1/2}\right| + \left|\partial_R f(t^n, x_{j-1/2}, \rho^n_j, \tilde{R}^n_j)\right|\left|R^n_{j+1/2} - R^n_{j-1/2}\right|.
\end{aligned}
$$

By (3.1), we have

$$
\begin{aligned}
&\left|R^n_{j+1/2} - R^n_{j-1/2}\right| \\
={}& \left|\frac{\Delta x}{W_{j+1/2}}\left(\sum_{k=1}^{N}\omega^{k-j}\,\rho^n_k\right) - \frac{\Delta x}{W_{j-1/2}}\left(\sum_{k=1}^{N}\omega^{k-j+1}\,\rho^n_k\right)\right| \\
\le{}& \frac{\Delta x}{|W_{j+1/2}|}\sum_{k=1}^{N}\left|\omega^{k-j} - \omega^{k-j+1}\right|\rho^n_k + \Delta x\left(\sum_{k=1}^{N}\left|\omega^{k-j+1}\right|\rho^n_k\right)\left|\frac{1}{W_{j+1/2}} - \frac{1}{W_{j-1/2}}\right|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{\Delta x}{K_\omega} \sum_{k=1}^{N} \rho_k^n \left| \int_{y_{k-j}}^{y_{k-j+1}} \omega'(y)\,\mathrm{d}y \right| + \frac{\Delta x^2}{K_\omega^2} \left\|\omega\right\|_{\mathbf{L}^\infty(\mathbb{R})} \left(\sum_{k=1}^{N} \rho_k^n \right) \left| \sum_{k=1}^{N} (\omega^{k-j+1} - \omega^{k-j}) \right| \\
&\leq \frac{\Delta x}{K_\omega} \left\|\omega'\right\|_{\mathbf{L}^\infty(\mathbb{R})} \left\|\rho^n\right\|_{\mathbf{L}^1(]a,b[)} + \frac{\Delta x}{K_\omega^2} \left\|\omega\right\|_{\mathbf{L}^\infty(\mathbb{R})} \left\|\rho^n\right\|_{\mathbf{L}^1(]a,b[)} \sum_{k=1}^{N} \left| \int_{y_{k-j}}^{y_{k-j+1}} \omega'(y)\,\mathrm{d}y \right| \\
&\leq \frac{\Delta x}{K_\omega} \left\|\omega'\right\|_{\mathbf{L}^\infty(\mathbb{R})} \left\|\rho^n\right\|_{\mathbf{L}^1(]a,b[)} + \frac{\Delta x}{K_\omega^2} \left\|\omega\right\|_{\mathbf{L}^\infty(\mathbb{R})} \left\|\omega'\right\|_{\mathbf{L}^1(\mathbb{R})} \left\|\rho^n\right\|_{\mathbf{L}^1(]a,b[)} \\
&\leq \Delta x \left(\frac{\left\|\omega'\right\|_{\mathbf{L}^\infty(\mathbb{R})}}{K_\omega} + \frac{\left\|\omega\right\|_{\mathbf{L}^\infty(\mathbb{R})} \left\|\omega'\right\|_{\mathbf{L}^1(\mathbb{R})}}{K_\omega^2} \right) \mathcal{C}_1(t^n),
\end{aligned}
$$

where $\mathcal{C}_1(t)$ is defined in (3.9). Setting

$$
\mathcal{L} = \left(\frac{\left\|\omega'\right\|_{\mathbf{L}^\infty(\mathbb{R})}}{K_\omega} + \frac{\left\|\omega\right\|_{\mathbf{L}^\infty(\mathbb{R})} \left\|\omega'\right\|_{\mathbf{L}^1(\mathbb{R})}}{K_\omega^2} \right), \tag{3.12}
$$

we obtain

$$
\left| F_{j+1/2}^n(\rho_j^n, \rho_j^n) - F_{j-1/2}^n(\rho_j^n, \rho_j^n) \right| \leq C\,\Delta x \left|\rho_n^n\right| \left(1 + \mathcal{L}\,\mathcal{C}_1(t^n)\right).
$$

Inserting the above estimate into (3.11) and exploiting the bounds on β_j^n and γ_j^n obtained in the proof of Lemma 3.2, we get

$$
\begin{aligned}
\rho_j^{n+1} &\leq (1 - \beta_j^n - \gamma_j^n)\,\rho_j^n + \beta_j^n\,\rho_{j-1}^n + \gamma_j^n\,\rho_{j+1}^n + \lambda \left| F_{j+1/2}^n(\rho_j^n, \rho_j^n) - F_{j-1/2}^n(\rho_j^n, \rho_j^n) \right| \\
&\leq (1 - \beta_j^n - \gamma_j^n) \left\|\rho^n\right\|_{\mathbf{L}^\infty(]a,b[)} + \beta_j^n \max\left\{ \left\|\rho^n\right\|_{\mathbf{L}^\infty(]a,b[)}, \, \rho_a^n \right\} + \gamma_j^n \max\left\{ \left\|\rho^n\right\|_{\mathbf{L}^\infty(]a,b[)}, \, \rho_b^n \right\} \\
&\quad + \lambda\,\Delta x\,C(1 + \mathcal{L}\,\mathcal{C}_1(t^n)) \left\|\rho^n\right\|_{\mathbf{L}^\infty(]a,b[)} \\
&\leq \max\left\{ \left\|\rho^n\right\|_{\mathbf{L}^\infty(]a,b[)}, \, \rho_a^n, \, \rho_b^n \right\} \left(1 + \Delta t\,C\left(1 + \mathcal{L}\,\mathcal{C}_1(t^n)\right)\right) \\
&\leq e^{\mathcal{C}_2(t^n)\Delta t} \max\left\{ \left\|\rho^n\right\|_{\mathbf{L}^\infty(]a,b[)}, \, \rho_a^n, \, \rho_b^n \right\},
\end{aligned}
$$

where

$$
\mathcal{C}_2(t) = C(1 + \mathcal{L}\,\mathcal{C}_1(t)), \tag{3.13}
$$

$\mathcal{L}$ being as in (3.12). An iterative argument, together with the fact that $\mathcal{C}_2(t^{n-1}) \leq \mathcal{C}_2(t^n)$ for all $n = 1, \ldots, N_T$, yields the thesis. □

3.4 BV estimates

Proposition 3.5. (BV estimate in space) *Let $\rho_o \in \mathbf{BV}(\,]a,b[;\mathbb{R}^+)$, ρ_a, $\rho_b \in \mathbf{BV}(\mathbb{R}^+;\mathbb{R}^+)$. Let* $(\boldsymbol{f})$, $(\boldsymbol{\omega})$ *and* (3.5) *hold. Then, for all $n = 1, \ldots, N_T$. the following estimate holds*

$$
\sum_{j=0}^{N} \left| \rho_{j+1}^n - \rho_j^n \right| \leq \mathcal{C}_x(t^n), \tag{3.14}
$$

where

$$\mathcal{C}_x(t^n) = e^{\mathcal{K}_1(t^n)\,t^n}\left[\sum_{j=0}^{N}\left|\rho_{j+1}^0-\rho_j^0\right| + \sum_{m=1}^{n}\left|\rho_a^m-\rho_a^{m-1}\right| + \sum_{m=1}^{n}\left|\rho_b^m-\rho_b^{m-1}\right|\right] + \frac{\mathcal{K}_4(t^n)}{\mathcal{K}_1(t^n)}\left(e^{\mathcal{K}_1(t^n)\,t^n}-1\right), \tag{3.15}$$

with $\mathcal{K}_1(t^n)$ *and* $\mathcal{K}_4(t^n)$ *are defined in* (3.27) *and* (3.30).

Remark 3.6. *Estimate* (3.14) *is defined also for* $n = 0$, *setting* $\sum_{m=1}^{0} a_m = 0$, *with some abuse of notation.*

Proof. Consider the inner terms and the boundary ones separately.

For $j = 1, \ldots, N-1$ and $n = 0, \ldots, N_T - 1$, focus on the difference $\rho_{j+1}^{n+1} - \rho_j^{n+1}$, exploiting (3.4):

$$\begin{aligned}
&\rho_{j+1}^{n+1}-\rho_j^{n+1}\\
&= \rho_{j+1}^n-\rho_j^n\\
&\quad -\lambda\left[F_{j+3/2}^n(\rho_{j+1}^n,\rho_{j+2}^n)-F_{j+1/2}^n(\rho_j^n,\rho_{j+1}^n)-F_{j+1/2}^n(\rho_j^n,\rho_{j+1}^n)+F_{j-1/2}^n(\rho_{j-1}^n,\rho_j^n)\right]\\
&\quad \pm\lambda\,F_{j+3/2}^n(\rho_j^n,\rho_{j+1}^n)\pm\lambda F_{j+1/2}^n(\rho_{j-1}^n,\rho_j^n)\\
&= \rho_{j+1}^n-\rho_j^n\\
&\quad -\lambda\left[F_{j+3/2}^n(\rho_{j+1}^n,\rho_{j+2}^n)-F_{j+1/2}^n(\rho_j^n,\rho_{j+1}^n)+F_{j+1/2}^n(\rho_{j-1}^n,\rho_j^n)-F_{j+3/2}^n(\rho_j^n,\rho_{j+1}^n)\right]\\
&\quad -\lambda\left[F_{j+3/2}^n(\rho_j^n,\rho_{j+1}^n)-F_{j+1/2}^n(\rho_{j-1}^n,\rho_j^n)+F_{j-1/2}^n(\rho_{j-1}^n,\rho_j^n)-F_{j+1/2}^n(\rho_j^n,\rho_{j+1}^n)\right]\\
&= \mathcal{A}_j^n-\lambda\,\mathcal{B}_j^n,
\end{aligned}$$

where we set

$$\begin{aligned}
\mathcal{A}_j^n &= \rho_{j+1}^n-\rho_j^n\\
&\quad -\lambda\left[F_{j+3/2}^n(\rho_{j+1}^n,\rho_{j+2}^n)-F_{j+1/2}^n(\rho_j^n,\rho_{j+1}^n)+F_{j+1/2}^n(\rho_{j-1}^n,\rho_j^n)-F_{j+3/2}^n(\rho_j^n,\rho_{j+1}^n)\right],\\
\mathcal{B}_j^n &= F_{j+3/2}^n(\rho_j^n,\rho_{j+1}^n)-F_{j+1/2}^n(\rho_{j-1}^n,\rho_j^n)+F_{j-1/2}^n(\rho_{j-1}^n,\rho_j^n)-F_{j+1/2}^n(\rho_j^n,\rho_{j+1}^n).
\end{aligned}$$

Rearrange $\mathcal{A}_j^n$ as follows:

$$\begin{aligned}
\mathcal{A}_j^n &= \rho_{j+1}^n-\rho_j^n-\lambda\,\frac{F_{j+3/2}^n(\rho_{j+1}^n,\rho_{j+2}^n)-F_{j+3/2}^n(\rho_{j+1}^n,\rho_{j+1}^n)}{\rho_{j+2}^n-\rho_{j+1}^n}\left(\rho_{j+2}^n-\rho_{j+1}^n\right)\\
&\quad -\lambda\,\frac{F_{j+3/2}^n(\rho_{j+1}^n,\rho_{j+1}^n)-F_{j+3/2}^n(\rho_j^n,\rho_{j+1}^n)}{\rho_{j+1}^n-\rho_j^n}\left(\rho_{j+1}^n-\rho_j^n\right)\\
&\quad +\lambda\,\frac{F_{j+1/2}^n(\rho_j^n,\rho_{j+1}^n)-F_{j+1/2}^n(\rho_j^n,\rho_j^n)}{\rho_{j+1}^n-\rho_j^n}\left(\rho_{j+1}^n-\rho_j^n\right)\\
&\quad +\lambda\,\frac{F_{j+1/2}^n(\rho_j^n,\rho_j^n)-F_{j+1/2}^n(\rho_{j-1}^n,\rho_j^n)}{\rho_j^n-\rho_{j-1}^n}\left(\rho_j^n-\rho_{j-1}^n\right)\\
&= \delta_j^n\,(\rho_j^n-\rho_{j-1}^n)+\gamma_{j+1}^n\,(\rho_{j+2}^n-\rho_{j+1}^n)+(1-\gamma_j^n-\delta_{j+1}^n)(\rho_{j+1}^n-\rho_j^n),
\end{aligned}$$

where

$$\delta_j^n = \begin{cases} \lambda \dfrac{F_{j+1/2}^n(\rho_j^n, \rho_j^n) - F_{j+1/2}^n(\rho_{j-1}^n, \rho_j^n)}{\rho_j^n - \rho_{j-1}^n} & \text{if } \rho_j^n \neq \rho_{j-1}^n, \\ 0 & \text{if } \rho_j^n = \rho_{j-1}^n, \end{cases} \tag{3.16}$$

while γ_j^n is as in (3.7). It can be proven that $\delta_j^n \in \left[0, 1/3\right]$. Thus,

$$\begin{aligned}
&\sum_{j=1}^{N-1} \left|\mathcal{A}_j^n\right| \\
\leq &\sum_{j=1}^{N-1} \left|\rho_{j+1}^n - \rho_j^n\right| + \sum_{j=0}^{N-2} \delta_{j+1}^n \left|\rho_{j+1}^n - \rho_j^n\right| - \sum_{j=1}^{N-1} \delta_{j+1}^n \left|\rho_{j+1}^n - \rho_j^n\right| \\
&+ \sum_{j=2}^{N} \gamma_j^n \left|\rho_{j+1}^n - \rho_j^n\right| - \sum_{j=1}^{N-1} \gamma_j^n \left|\rho_{j+1}^n - \rho_j^n\right| \\
= &\sum_{j=1}^{N-1} \left|\rho_{j+1}^n - \rho_j^n\right| + \delta_1^n \left|\rho_1^n - \rho_a^n\right| - \delta_N^n \left|\rho_N^n - \rho_{N-1}^n\right| + \gamma_N^n \left|\rho_b^n - \rho_N^n\right| - \gamma_1^n \left|\rho_2^n - \rho_1^n\right|.
\end{aligned} \tag{3.17}$$

Focus now on $\mathcal{B}_j^n$:

$$\begin{aligned}
\mathcal{B}_j^n = &\frac{1}{2} \Big[f(t^n, x_{j+3/2}, \rho_j^n, R_{j+3/2}^n) + f(t^n, x_{j+3/2}, \rho_{j+1}^n, R_{j+3/2}^n) \\
&- f(t^n, x_{j+1/2}, \rho_j^n, R_{j+1/2}^n) - f(t^n, x_{j+1/2}, \rho_{j+1}^n, R_{j+1/2}^n) \\
&+ f(t^n, x_{j-1/2}, \rho_{j-1}^n, R_{j-1/2}^n) + f(t^n, x_{j-1/2}, \rho_j^n, R_{j-1/2}^n) \\
&- f(t^n, x_{j+1/2}, \rho_{j-1}^n, R_{j+1/2}^n) - f(t^n, x_{j+1/2}, \rho_j^n, R_{j+1/2}^n) \Big] \\
= &\frac{1}{2} \Big[f(t^n, x_{j+3/2}, \rho_{j+1}^n, R_{j+3/2}^n) - f(t^n, x_{j+1/2}, \rho_{j+1}^n, R_{j+1/2}^n) \Big] \\
&+ \frac{1}{2} \Big[f(t^n, x_{j-1/2}, \rho_{j-1}^n, R_{j-1/2}^n) - f(t^n, x_{j+1/2}, \rho_{j-1}^n, R_{j+1/2}^n) \Big] \\
&+ \frac{1}{2} \Big[f(t^n, x_{j+3/2}, \rho_j^n, R_{j+3/2}^n) - 2\, f(t^n, x_{j+1/2}, \rho_j^n, R_{j+1/2}^n) + f(t^n, x_{j-1/2}, \rho_j^n, R_{j-1/2}^n) \Big] \\
= &\frac{1}{2} \Big[\partial_R f(t^n, x_{j+3/2}, \rho_{j+1}^n, \tilde{R}_{j+1}^n)\, (R_{j+3/2}^n - R_{j+1/2}^n) + \Delta x\, \partial_x f(t^n, \tilde{x}_{j+1}, \rho_{j+1}^n, R_{j+1/2}^n) \Big] \\
&- \frac{1}{2} \Big[\partial_R f(t^n, x_{j-1/2}, \rho_{j-1}^n, \tilde{R}_j^n)\, (R_{j+1/2}^n - R_{j-1/2}^n) + \Delta x\, \partial_x f(t^n, \tilde{x}_j, \rho_{j-1}^n, R_{j+1/2}^n) \Big] \\
&+ \frac{1}{2} \Big[f(t^n, x_{j+3/2}, \rho_j^n, R_{j+1/2}^n) - 2\, f(t^n, x_{j+1/2}, \rho_j^n, R_{j+1/2}^n) + f(t^n, x_{j-1/2}, \rho_j^n, R_{j+1/2}^n) \Big] \\
&+ \frac{1}{2} \Big[f(t^n, x_{j+3/2}, \rho_j^n, R_{j+3/2}^n) - f(t^n, x_{j+3/2}, \rho_j^n, R_{j+1/2}^n) \Big] \\
&+ \frac{1}{2} \Big[f(t^n, x_{j-1/2}, \rho_j^n, R_{j-1/2}^n) - f(t^n, x_{j-1/2}, \rho_j^n, R_{j+1/2}^n) \Big] \\
= &\frac{\Delta x}{2} \Big[(\tilde{x}_{j+1} - \tilde{x}_j)\, \partial_{xx}^2 f(t^n, \hat{x}_{j+1/2}, \rho_{j+1}^n, R_{j+1/2}^n) + \partial_{\rho x}^2 f(t^n, \tilde{x}_j, \tilde{\rho}_j^n, R_{j+1/2}^n)\, (\rho_{j+1}^n - \rho_{j-1}^n) \Big] \\
&+ \frac{1}{2} \Big[2\, \Delta x\, \partial_{xR}^2 f(t^n, \check{x}_{j+1/2}, \rho_{j+1}^n, \tilde{R}_{j+1}^n)\, (R_{j+3/2}^n - R_{j+1/2}^n) \\
&\qquad + \partial_{\rho R}^2 f(t^n, x_{j-1/2}, \bar{\rho}_j^n, \tilde{R}_{j+1}^n)\, (R_{j+3/2}^n - R_{j+1/2}^n)\, (\rho_{j+1}^n - \rho_{j-1}^n)
\end{aligned}$$

$$
\begin{aligned}
&\quad + \partial^2_{RR} f(t^n, x_{j-1/2}, \rho^n_{j-1}, \hat R^n_{j+1/2})\,(R^n_{j+3/2} - R^n_{j+1/2})\,(\tilde R^n_{j+1} - \tilde R^n_j) \\
&\quad + \partial_R f(t^n, x_{j-1/2}, \rho^n_{j-1}, \tilde R^n_j)\,(R^n_{j+3/2} - R^n_{j+1/2} - R^n_{j+1/2} + R^n_{j-1/2})\Big] \\
&+ \frac{\Delta x}{2}\Big[\partial_x f(t^n, \bar x_{j+1}, \rho^n_j, R^n_{j+1/2}) - \partial_x f(t^n, \bar x_j, \rho^n_j, R^n_{j+1/2})\Big] \\
&+ \frac{1}{2}\Big[\partial_R f(t^n, x_{j+3/2}, \rho^n_j, \bar R^n_{j+1})\,(R^n_{j+3/2} - R^n_{j+1/2}) \\
&\quad - \partial_R f(t^n, x_{j-1/2}, \rho^n_j, \bar R^n_j)\,(R^n_{j+1/2} - R^n_{j-1/2})\Big] \\
= {} & \frac{\Delta x}{2}\Big[(\tilde x_{j+1} - \tilde x_j)\,\partial^2_{xx} f(t^n, \hat x_{j+1/2}, \rho^n_{j+1}, R^n_{j+1/2}) + \partial^2_{\rho x} f(t^n, \tilde x_j, \tilde\rho^n_j, R^n_{j+1/2})\,(\rho^n_{j+1} - \rho^n_{j-1})\Big] \\
&+ \frac{1}{2}\Big[2\,\Delta x\,\partial^2_{xR} f(t^n, \check x_{j+1/2}, \rho^n_{j+1}, \tilde R^n_{j+1})\,(R^n_{j+3/2} - R^n_{j+1/2}) \\
&\quad + \partial^2_{\rho R} f(t^n, x_{j-1/2}, \bar\rho^n_j, \tilde R^n_{j+1})\,(R^n_{j+3/2} - R^n_{j+1/2})\,(\rho^n_{j+1} - \rho^n_{j-1}) \\
&\quad + \partial^2_{RR} f(t^n, x_{j-1/2}, \rho^n_{j-1}, \hat R^n_{j+1/2})\,(R^n_{j+3/2} - R^n_{j+1/2})\,(\tilde R^n_{j+1} - \tilde R^n_j) \\
&\quad + \partial_R f(t^n, x_{j-1/2}, \rho^n_{j-1}, \tilde R^n_j)\,(R^n_{j+3/2} - R^n_{j+1/2} - R^n_{j+1/2} + R^n_{j-1/2})\Big] \\
&+ \frac{\Delta x}{2}\,(\bar x_{j+1} - \bar x_j)\,\partial^2_{xx} f(t^n, \bar x_{j+1/2}, \rho^n_j, R^n_{j+1/2}) \\
&+ \frac{1}{2}\Big[2\,\Delta x\,\partial^2_{xR} f(t^n, \bar{\bar x}_{j+1/2}, \rho^n_j, \bar R^n_{j+1})\,(R^n_{j+3/2} - R^n_{j+1/2}) \\
&\quad + \partial^2_{RR} f(t^n, x_{j-1/2}, \rho^n_j, \bar{\bar R}^n_{j+1/2})\,(R^n_{j+3/2} - R^n_{j+1/2})\,(\bar R^n_{j+1} - \bar R^n_j) \\
&\quad + \partial_R f(t^n, x_{j-1/2}, \rho^n_j, \bar R^n_j)\,(R^n_{j+3/2} - R^n_{j+1/2} - R^n_{j+1/2} + R^n_{j-1/2})\Big],
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde R^n_{j+1}, \bar R^n_{j+1} &\in \mathcal{I}\left(R^n_{j+1/2}, R^n_{j+3/2}\right), & \tilde x_{j+1}, \bar x_{j+1} &\in]x_{j+1/2}, x_{j+3/2}[, \\
\tilde R^n_j, \bar R^n_j &\in \mathcal{I}\left(R^n_{j-1/2}, R^n_{j+1/2}\right), & \tilde x_j, \bar x_j &\in]x_{j-1/2}, x_{j+1/2}[, \\
\hat R^n_{j+1/2} &\in \mathcal{I}\left(\tilde R^n_j, \tilde R^n_{j+1}\right), & \hat x_{j+1/2} &\in]\tilde x_j, \tilde x_{j+1}[, \\
\bar{\bar R}^n_{j+1/2} &\in \mathcal{I}\left(\bar R^n_j, \bar R^n_{j+1}\right), & \check x_{j+1/2}, \bar{\bar x}_{j+1/2} &\in]x_{j-1/2}, x_{j+3/2}[, \\
\tilde\rho^n_j, \bar\rho^n_j &\in \mathcal{I}\left(\rho^n_{j-1}, \rho^n_{j+1}\right), & \bar x_{j+1/2} &\in]\bar x_j, \bar x_{j+1}[.
\end{aligned}
$$

Notice that

$$
\left|\tilde x_{j+1} - \tilde x_j\right| \le 2\,\Delta x, \quad \left|\bar x_{j+1} - \bar x_j\right| \le 2\,\Delta x, \quad \left|R^n_{j+3/2} - R^n_{j+1/2}\right| \le \mathcal{L}\,\mathcal{C}_1(t^n)\,\Delta x,
$$

where $\mathcal{L}$ is as in (3.12). Moreover, by their very definition, for $\vartheta^n_{j+1}, \varepsilon^n_j \in [0,1]$,

$$
\begin{aligned}
\left|\tilde R^n_{j+1} - \tilde R^n_j\right| &= \left|\vartheta^n_{j+1} R^n_{j+3/2} + (1 - \vartheta^n_{j+1})\,R^n_{j+1/2} - \varepsilon^n_j\,R^n_{j+1/2} - (1 - \varepsilon^n_j)\,R^n_{j-1/2}\right| \\
&\le \left|R^n_{j+3/2} - R^n_{j+1/2}\right| + \left|R^n_{j+1/2} - R^n_{j-1/2}\right| \\
&\le 2\,\mathcal{L}\,\mathcal{C}_1(t^n)\,\Delta x,
\end{aligned}
$$

and similarly $\left|\bar{R}^n_{j+1} - \bar{R}^n_j\right| \leq 2\,\mathcal{L}\,\mathcal{C}_1(t^n)\,\Delta x$. Compute now

$$R^n_{j+3/2} - 2\,R^n_{j+1/2} + R^n_{j-1/2} = \Delta x \left[\sum_{k=1}^{N} \rho^n_k \left(\frac{\omega^{k-j-1}}{W_{j+3/2}} - \frac{\omega^{k-j}}{W_{j+1/2}} - \frac{\omega^{k-j}}{W_{j+1/2}} + \frac{\omega^{k-j+1}}{W_{j-1/2}}\right)\right]. \tag{3.18}$$

Observe that, for k fixed:

$$\begin{aligned}
&\frac{\omega^{k-j-1}}{W_{j+3/2}} - \frac{\omega^{k-j}}{W_{j+1/2}} \\
&= \frac{\Delta x}{W_{j+3/2}\,W_{j+1/2}} \left[\sum_{\ell=1}^{N} \left(\omega^{k-j-1}\,\omega^{\ell-j} - \omega^{k-j}\,\omega^{\ell-j-1}\right)\right] \\
&= \frac{\Delta x}{W_{j+3/2}\,W_{j+1/2}} \left[\sum_{\ell=1}^{N} \left(\omega^{k-j-1}\left(\omega^{\ell-j} - \omega^{\ell-j-1}\right) + \left(\omega^{k-j-1} - \omega^{k-j}\right)\omega^{\ell-j-1}\right)\right] \\
&= \frac{(\Delta x)^2}{W_{j+3/2}\,W_{j+1/2}} \left[\sum_{\ell=1}^{N} \left(\omega^{k-j-1}\,\omega'(\xi_{\ell-j-1/2}) - \omega^{\ell-j-1}\,\omega'(\xi_{k-j-1/2})\right)\right],
\end{aligned} \tag{3.19}$$

where $\xi_{\ell-j-1/2} \in]y_{\ell-j-1}, y_{\ell-j}[$, $\xi_{k-j-1/2} \in]y_{k-j-1}, y_{k-j}[$, and similarly

$$\frac{\omega^{k-j}}{W_{j+1/2}} - \frac{\omega^{k-j+1}}{W_{j-1/2}} = \frac{(\Delta x)^2}{W_{j+1/2}\,W_{j-1/2}} \left[\sum_{\ell=1}^{N} \left(\omega^{k-j}\,\omega'(\xi_{\ell-j+1/2}) - \omega^{\ell-j}\,\omega'(\xi_{k-j+1/2})\right)\right]. \tag{3.20}$$

In order to compute the difference between (3.19) and (3.20), which appears in (3.18), we add and subtract the term

$$\frac{(\Delta x)^2}{W_{j+3/2}\,W_{j+1/2}} \left[\sum_{\ell=1}^{N} \left(\omega^{k-j}\,\omega'(\xi_{\ell-j+1/2}) - \omega^{\ell-j}\,\omega'(\xi_{k-j+1/2})\right)\right].$$

The terms with common denominator yield

$$\begin{aligned}
&\frac{(\Delta x)^2}{W_{j+3/2}\,W_{j+1/2}} \left[\sum_{\ell=1}^{N} \left(\omega^{k-j-1}\,\omega'(\xi_{\ell-j-1/2}) - \omega^{\ell-j-1}\,\omega'(\xi_{k-j-1/2})\right.\right. \\
&\qquad\qquad \left.\left. -\omega^{k-j}\,\omega'(\xi_{\ell-j+1/2}) + \omega^{\ell-j}\,\omega'(\xi_{k-j+1/2})\right)\right] \\
&= \frac{(\Delta x)^2}{W_{j+3/2}\,W_{j+1/2}} \left[\sum_{\ell=1}^{N} \Big((\omega^{k-j-1} - \omega^{k-j})\,\omega'(\xi_{\ell-j-1/2}) + \omega^{k-j}\,(\omega'(\xi_{\ell-j-1/2}) - \omega'(\xi_{\ell-j+1/2}))\right. \\
&\qquad \left. -(\omega^{\ell-j-1} - \omega^{\ell-j})\,\omega'(\xi_{k-j-1/2}) - \omega^{\ell-j}\,(\omega'(\xi_{k-j-1/2}) - \omega'(\xi_{k-j+1/2}))\Big)\right]
\end{aligned}$$

$$= \frac{(\Delta x)^2}{W_{j+3/2}\, W_{j+1/2}} \left[\sum_{\ell=1}^{N} \left(-\Delta x\, \omega'(\xi_{k-j-1/2})\, \omega'(\xi_{\ell-j-1/2}) - \omega^{k-j} \int_{\xi_{\ell-j-1/2}}^{\xi_{\ell-j+1/2}} \omega''(y)\, \mathrm{d}y \right.\right.$$

$$\left.\left. + \Delta x\, \omega'(\xi_{\ell-j-1/2})\, \omega'(\xi_{k-j-1/2}) + \omega^{\ell-j} \int_{\xi_{k-j-1/2}}^{\xi_{k-j+1/2}} \omega''(y)\, \mathrm{d}y \right) \right]$$

$$= \frac{(\Delta x)^2}{W_{j+3/2}\, W_{j+1/2}} \left[\sum_{\ell=1}^{N} \left(\omega^{\ell-j} \int_{\xi_{k-j-1/2}}^{\xi_{k-j+1/2}} \omega''(y)\, \mathrm{d}y - \omega^{k-j} \int_{\xi_{\ell-j-1/2}}^{\xi_{\ell-j+1/2}} \omega''(y)\, \mathrm{d}y \right) \right]. \tag{3.21}$$

We are left with

$$\left[\sum_{\ell=1}^{N} \left(\omega^{k-j}\, \omega'(\xi_{\ell-j+1/2}) - \omega^{\ell-j}\, \omega'(\xi_{k-j+1/2}) \right) \right] \left(\frac{(\Delta x)^2}{W_{j+3/2}\, W_{j+1/2}} - \frac{(\Delta x)^2}{W_{j+1/2}\, W_{j-1/2}} \right). \tag{3.22}$$

In particular, observe that

$$\begin{aligned} \frac{1}{W_{j+3/2}\, W_{j+1/2}} - \frac{1}{W_{j+1/2}\, W_{j-1/2}} &= \frac{W_{j-1/2} - W_{j+3/2}}{W_{j+3/2}\, W_{j+1/2}\, W_{j-1/2}} \\ &= \frac{\Delta x}{W_{j+3/2}\, W_{j+1/2}\, W_{j-1/2}} \left(\sum_{\beta=1}^{N} \left(\omega^{\beta-j+1} - \omega^{\beta-j-1} \right) \right) \\ &= \frac{\Delta x}{W_{j+3/2}\, W_{j+1/2}\, W_{j-1/2}} \left(\sum_{\beta=1}^{N} \int_{y_{\beta-j-1}}^{y_{\beta-j+1}} \omega'(y)\, \mathrm{d}y \right) \\ &\le \frac{2\, \Delta x\, \left\| \omega' \right\|_{\mathbf{L^1}}}{W_{j+3/2}\, W_{j+1/2}\, W_{j-1/2}}. \end{aligned} \tag{3.23}$$

Coming back to (3.18), exploiting (3.21), (3.22) and (3.23), we get

$$\begin{aligned} &\left| R^n_{j+3/2} - 2\, R^n_{j+1/2} + R^n_{j-1/2} \right| \\ &\le \frac{(\Delta x)^3}{\left| W_{j+3/2}\, W_{j+1/2} \right|} \left| \left(\sum_{k=1}^{N} \rho_k^n \int_{\xi_{k-j-1/2}}^{\xi_{k-j+1/2}} \omega''(y)\, \mathrm{d}y \right) \left(\sum_{\ell=1}^{N} \omega^{\ell-j} \right) \right. \\ &\qquad \left. - \left(\sum_{k=1}^{N} \rho_k^n\, \omega^{k-j} \right) \left(\sum_{\ell=1}^{N} \int_{\xi_{\ell-j-1/2}}^{\zeta_{\ell-j+1/2}} \omega''(y)\, \mathrm{d}y \right) \right| \\ &\quad + \frac{2\, (\Delta x)^4 \left\| \omega' \right\|_{\mathbf{L^1}}}{\left| W_{j+3/2}\, W_{j+1/2}\, W_{j-1/2} \right|} \left| \left(\sum_{k=1}^{N} \rho_k^n\, \omega^{k-j} \right) \left(\sum_{\ell=1}^{N} \omega'(\xi_{\ell-j+1/2}) \right) \right. \\ &\qquad \left. - \left(\sum_{k=1}^{N} \rho_k^n\, \omega'(\xi_{k-j+1/2}) \right) \left(\sum_{\ell=1}^{N} \omega^{\ell-j} \right) \right| \end{aligned}$$

$$
\begin{aligned}
&\le \frac{2\,(\Delta x)^2}{K_\omega}\,\left\|\omega''\right\|_{\mathbf{L}^\infty}\|\rho^n\|_{\mathbf{L}^1} + \frac{(\Delta x)^2}{K_\omega^2}\,\|\omega\|_{\mathbf{L}^\infty}\left\|\omega''\right\|_{\mathbf{L}^1}\|\rho^n\|_{\mathbf{L}^1}\\
&\quad + \frac{2\,(\Delta x)^2}{K_\omega^3}\|\omega\|_{\mathbf{L}^\infty}\left\|\omega'\right\|_{\mathbf{L}^1}^2\|\rho^n\|_{\mathbf{L}^1} + \frac{2\,(\Delta x)^2}{K_\omega^2}\left\|\omega'\right\|_{\mathbf{L}^\infty}\left\|\omega'\right\|_{\mathbf{L}^1}\|\rho^n\|_{\mathbf{L}^1}\\
&= \Delta x^2\,\mathcal{W}\,\|\rho^n\|_{\mathbf{L}^1},
\end{aligned}
\tag{3.24}
$$

where we set

$$
\mathcal{W} = \frac{2}{K_\omega}\left\|\omega''\right\|_{\mathbf{L}^\infty} + \frac{1}{K_\omega^2}\|\omega\|_{\mathbf{L}^\infty}\left\|\omega''\right\|_{\mathbf{L}^1} + \frac{2}{K_\omega^3}\|\omega\|_{\mathbf{L}^\infty}\left\|\omega'\right\|_{\mathbf{L}^1}^2 + \frac{2}{K_\omega^2}\left\|\omega'\right\|_{\mathbf{L}^\infty}\left\|\omega'\right\|_{\mathbf{L}^1}. \tag{3.25}
$$

Hence,

$$
\begin{aligned}
\left|\mathcal{B}_j^n\right| \le\; & (\Delta x)^2\, C\left|\rho_{j+1}^n\right| + \frac{\Delta x}{2}\left\|\partial^2_{\rho x} f\right\|_{\mathbf{L}^\infty}\left|\rho_{j+1}^n - \rho_{j-1}^n\right| + (\Delta x)^2\, C\,\mathcal{L}\,\mathcal{C}_1(t^n)\left|\rho_{j+1}^n\right|\\
& + \frac{\Delta x}{2}\,\mathcal{L}\,\mathcal{C}_1(t^n)\left\|\partial^2_{\rho R} f\right\|_{\mathbf{L}^\infty}\left|\rho_{j+1}^n - \rho_{j-1}^n\right| + (\Delta x)^2\, C\,\mathcal{L}^2\left(\mathcal{C}_1(t^n)\right)^2\left|\rho_{j-1}^n\right| + (\Delta x)^2\, C\left|\rho_j^n\right|\\
& + (\Delta x)^2\, C\,\mathcal{L}\,\mathcal{C}_1(t^n)\left|\rho_j^n\right| + (\Delta x)^2\, C\,\mathcal{L}^2\left(\mathcal{C}_1(t^n)\right)^2\left|\rho_j^n\right|\\
& + \frac{1}{2}\,(\Delta x)^2\, C\,\mathcal{W}\,\mathcal{C}_1(t^n)\,\left(\left|\rho_{j-1}^n\right| + \left|\rho_j^n\right|\right).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\sum_{j=1}^{N-1}\lambda\left|\mathcal{B}_j^n\right| \le\; & \frac{\Delta t}{2}\,\left(\left\|\partial^2_{\rho x} f\right\|_{\mathbf{L}^\infty} + \mathcal{L}\,\mathcal{C}_1(t^n)\left\|\partial^2_{\rho R} f\right\|_{\mathbf{L}^\infty}\right)\sum_{j=1}^{N-1}\left|\rho_{j+1}^n - \rho_{j-1}^n\right|\\
& + \Delta t\, C\left(1 + 2\,\mathcal{L}\,\mathcal{C}_1(t^n) + 2\,\mathcal{L}^2\left(\mathcal{C}_1(t^n)\right)^2 + \mathcal{W}\,\mathcal{C}_1(t^n)\right)\,\Delta x\sum_{j=1}^{N}\left|\rho_j^n\right|\\
& + \Delta t\,\Delta x\, C\left(\mathcal{L}^2\left(\mathcal{C}_1(t^n)\right)^2 + \frac{1}{2}\,\mathcal{W}\,\mathcal{C}_1(t^n)\right)\left|\rho_a^n\right|\\
\le\; & \mathcal{K}_1(t^n)\,\Delta t\left(\sum_{j=1}^{N-1}\left|\rho_{j+1}^n - \rho_j^n\right| + \frac{1}{2}\left|\rho_1^n - \rho_a^n\right|\right) + \mathcal{K}_2(t^n)\,\Delta t\, + \mathcal{K}_3(t^n)\,\Delta t\left|\rho_a^n\right|,
\end{aligned}
\tag{3.26}
$$

with

$$
\begin{aligned}
\mathcal{K}_1(t) &= \left\|\partial^2_{\rho x} f\right\|_{\mathbf{L}^\infty} + \mathcal{L}\,\mathcal{C}_1(t)\left\|\partial^2_{\rho R} f\right\|_{\mathbf{L}^\infty},\\
\mathcal{K}_2(t) &= C\,\mathcal{C}_1(t)\,\left(1 + 2\,\mathcal{L}\,\mathcal{C}_1(t) + 2\,\mathcal{K}_3(t)\right),\\
\mathcal{K}_3(t) &= C\,\mathcal{C}_1(t)\left(\mathcal{L}^2\,\mathcal{C}_1(t) + \frac{1}{2}\,\mathcal{W}\right),
\end{aligned}
\tag{3.27}
$$

$\mathcal{C}_1(t)$ is as in (3.9) and $\mathcal{L}$ is as in (3.12). Observe that the two norms of f appearing in $\mathcal{K}_1(t)$ are bounded due to **(*f*)**, Lemma 3.3 and Lemma 3.4, since they are evaluated on the compact set $[0,t]\times[a,b]\times[-\left\|\rho(t)\right\|_{\mathbf{L}^\infty},\left\|\rho(t)\right\|_{\mathbf{L}^\infty}]\times[-J(t),J(t)]$, with $J(t) = \frac{\|\omega\|_{\mathbf{L}^\infty}}{K_\omega}\,\mathcal{C}_1(t)$.

Focus now on the boundary terms. From the definition of the scheme (3.4), with the notation (3.6)–(3.7) and (3.16), we have

$$\begin{aligned}
&\rho_1^{n+1} - \rho_a^{n+1} \\
&= (1-\beta_1^n-\gamma_1^n)\rho_1^n + \beta_1^n\,\rho_a^n + \gamma_1^n\,\rho_2^n - \lambda\left[F_{3/2}^n(\rho_1^n,\rho_1^n) - F_{1/2}^n(\rho_1^n,\rho_1^n)\right] - \rho_a^{n+1} \pm \rho_a^n \\
&= \gamma_1^n(\rho_2^n-\rho_1^n) + (1-\beta_1^n)(\rho_1^n-\rho_a^n) + (\rho_a^{n+1}-\rho_a^n) - \lambda\left[F_{3/2}^n(\rho_1^n,\rho_1^n) - F_{1/2}^n(\rho_1^n,\rho_1^n)\right] \\
&= \gamma_1^n(\rho_2^n-\rho_1^n) + (1-\delta_1^n)(\rho_1^n-\rho_a^n) + (\rho_a^{n+1}-\rho_a^n) - \lambda\left[F_{3/2}^n(\rho_a^n,\rho_1^n) - F_{1/2}^n(\rho_a^n,\rho_1^n)\right],
\end{aligned}$$

since

$$\begin{aligned}
&\beta_1^n\,(\rho_1^n-\rho_a^n) + \lambda\left[F_{3/2}^n(\rho_1^n,\rho_1^n) - F_{1/2}^n(\rho_1^n,\rho_1^n)\right] \\
&= \lambda\left[F_{1/2}^n(\rho_1^n,\rho_1^n) - F_{1/2}^n(\rho_a^n,\rho_1^n) + F_{3/2}^n(\rho_1^n,\rho_1^n) - F_{1/2}^n(\rho_1^n,\rho_1^n) \pm F_{3/2}^n(\rho_a^n,\rho_1^n)\right] \\
&= \delta_1^n\,(\rho_1^n-\rho_a^n) + \lambda\left[F_{3/2}^n(\rho_a^n,\rho_1^n) - F_{1/2}^n(\rho_a^n,\rho_1^n)\right].
\end{aligned}$$

Observing that

$$\begin{aligned}
&\lambda\left[F_{3/2}^n(\rho_a^n,\rho_1^n) - F_{1/2}^n(\rho_a^n,\rho_1^n)\right] \\
&= \frac{\lambda}{2}\left[f(t^n,x_{3/2},\rho_a^n,R_{3/2}^n) + f(t^n,x_{3/2},\rho_1^n,R_{3/2}^n) - f(t^n,x_{1/2},\rho_a^n,R_{1/2}^n) - f(t^n,x_{1/2},\rho_1^n,R_{1/2}^n)\right] \\
&= \frac{\lambda}{2}\Big[\partial_x f(t^n,\tilde{x}_1,\rho_a^n,R_{3/2}^n)\,\Delta x + \partial_R f(t^n,x_{1/2},\rho_a^n,\tilde{R}_1^n)\,(R_{3/2}^n - R_{1/2}^n) \\
&\qquad + \partial_x f(t^n,\overline{x}_1,\rho_1^n,R_{3/2}^n)\,\Delta x + \partial_R f(t^n,x_{1/2},\rho_1^n,\bar{R}_1^n)\,(R_{3/2}^n - R_{1/2}^n)\Big],
\end{aligned}$$

where $\tilde{x}_1, \overline{x}_1 \in \left]x_{1/2}, x_{3/2}\right[$ and $\tilde{R}_1^n, \bar{R}_1^n \in \mathcal{I}(R_{1/2}^n, R_{3/2}^n)$, we conclude

$$\lambda\left|F_{3/2}^n(\rho_a^n,\rho_1^n) - F_{1/2}^n(\rho_a^n,\rho_1^n)\right| \le \Delta t\,\frac{C}{2}\,(1+\mathcal{L}\,\mathcal{C}_1(t^n))\,(|\rho_a^n| + |\rho_1^n|).$$

By the positivity of the coefficients involved, we obtain

$$\begin{aligned}
\left|\rho_1^{n+1} - \rho_a^{n+1}\right| \le{}& \gamma_1^n\,|\rho_2^n - \rho_1^n| + (1-\delta_1^n)\,|\rho_1^n-\rho_a^n| + |\rho_a^{n+1}-\rho_a^n| \\
&+ \Delta t\,\frac{C}{2}\,(1+\mathcal{L}\,\mathcal{C}_1(t^n))\,(|\rho_a^n| + |\rho_1^n|).
\end{aligned} \tag{3.28}$$

Concerning the other boundary term, we have

$$\begin{aligned}
\rho_b^{n+1} - \rho_N^{n+1} ={}& \rho_b^{n+1} \pm \rho_b^n - (1-\beta_N^n-\gamma_N^n)\rho_N^n + \beta_N^n\,\rho_{N-1}^n + \gamma_N^n\,\rho_b^n \\
&+ \lambda\left[F_{N+1/2}^n(\rho_N^n,\rho_N^n) - F_{N-1/2}^n(\rho_N^n,\rho_N^n)\right] \\
={}& (\rho_b^{n+1}-\rho_b^n) + (1-\gamma_N^n)(\rho_b^n-\rho_N^n) + \beta_N^n(\rho_N^n-\rho_{N-1}^n) \\
&+ \lambda\left[F_{N+1/2}^n(\rho_N^n,\rho_N^n) - F_{N-1/2}^n(\rho_N^n,\rho_N^n)\right] \\
={}& (\rho_b^{n+1}-\rho_b^n) + (1-\gamma_N^n)(\rho_b^n-\rho_N^n) + \delta_N^n(\rho_N^n-\rho_{N-1}^n) \\
&+ \lambda\left[F_{N+1/2}^n(\rho_{N-1}^n,\rho_N^n) - F_{N-1/2}^n(\rho_{N-1}^n,\rho_N^n)\right],
\end{aligned}$$

since

$$\begin{aligned}
&\beta_N^n(\rho_N^n - \rho_{N-1}^n) + \lambda \left[F_{N+1/2}^n(\rho_N^n, \rho_N^n) - F_{N-1/2}^n(\rho_N^n, \rho_N^n)\right] \\
&= \lambda \left[F_{N-1/2}^n(\rho_N^n, \rho_N^n) - F_{N-1/2}^n(\rho_{N-1}^n, \rho_N^n) + F_{N+1/2}^n(\rho_N^n, \rho_N^n) \right.\\
&\qquad \left. - F_{N-1/2}^n(\rho_N^n, \rho_N^n) \pm F_{N+1/2}^n(\rho_{N-1}^n, \rho_N^n)\right] \\
&= \delta_N^n(\rho_N^n - \rho_{N-1}^n) + \lambda \left[F_{N+1/2}^n(\rho_{N-1}^n, \rho_N^n) - F_{N-1/2}^n(\rho_{N-1}^n, \rho_N^n)\right].
\end{aligned}$$

Observing that

$$\begin{aligned}
&\lambda \left[F_{N+1/2}^n(\rho_{N-1}^n, \rho_N^n) - F_{N-1/2}^n(\rho_{N-1}^n, \rho_N^n)\right] \\
&= \frac{\lambda}{2} \left[f(t^n, x_{N+1/2}, \rho_{N-1}^n, R_{N+1/2}^n) + f(t^n, x_{N+1/2}, \rho_N^n, R_{N+1/2}^n) \right.\\
&\qquad \left. - f(t^n, x_{N-1/2}, \rho_{N-1}^n, R_{N-1/2}^n) - f(t^n, x_{N-1/2}, \rho_N^n, R_{N-1/2}^n)\right] \\
&= \frac{\lambda}{2} \left[\partial_x f(t^n, \tilde{x}_N, \rho_{N-1}^n, R_{N+1/2}^n)\, \Delta x + \partial_R f(t^n, x_{N-1/2}, \rho_{N-1}^n, \tilde{R}_N^n)\, (R_{N+1/2}^n - R_{N-1/2}^n) \right.\\
&\qquad \left. + \partial_x f(t^n, \overline{x}_N, \rho_N^n, R_{N+1/2}^n)\, \Delta x + \partial_R f(t^n, x_{N-1/2}, \rho_N^n, \bar{R}_N^n)\, (R_{N+1/2}^n - R_{N-1/2}^n)\right],
\end{aligned}$$

where $\tilde{x}_N, \overline{x}_N \in]x_{N-1/2}, x_{N+1/2}[$ and $\tilde{R}_N^n, \bar{R}_N^n \in \mathcal{I}(R_{N-1/2}^n, R_{N+1/2}^n)$, we conclude

$$\lambda \left|F_{N+1/2}^n(\rho_{N-1}^n, \rho_N^n) - F_{N-1/2}^n(\rho_{N-1}^n, \rho_N^n)\right| \le \Delta t\, \frac{C}{2}\, (1 + \mathcal{L}\, \mathcal{C}_1(t^n))\, (|\rho_{N-1}^n| + |\rho_N^n|).$$

By the positivity of the coefficients involved, we obtain

$$\begin{aligned}
\left|\rho_b^{n+1} - \rho_N^{n+1}\right| \le{}& |\rho_b^{n+1} - \rho_b^n| + (1 - \gamma_N^n)\, |\rho_b^n - \rho_N^n| + \delta_N^n\, |\rho_N^n - \rho_{N-1}^n| \\
&+ \Delta t\, \frac{C}{2}\, (1 + \mathcal{L}\, \mathcal{C}_1(t^n))\, (|\rho_{N-1}^n| + |\rho_N^n|).
\end{aligned} \tag{3.29}$$

Collect now the estimates (3.17), (3.26), (3.28) and (3.29):

$$\begin{aligned}
&\sum_{j=0}^{N} \left|\rho_{j+1}^{n+1} - \rho_j^{n+1}\right| \\
&= \left|\rho_1^{n+1} - \rho_a^{n+1}\right| + \sum_{j=1}^{N-1} \left|\rho_{j+1}^{n+1} - \rho_j^{n+1}\right| + \left|\rho_b^{n+1} - \rho_n^{n+1}\right| \\
&\le \gamma_1^n |\rho_2^n - \rho_1^n| + (1 - \delta_1^n) |\rho_1^n - \rho_a^n| + |\rho_a^{n+1} - \rho_a^n| + \Delta t\, \frac{C}{2}\, (1 + \mathcal{L}\, \mathcal{C}_1(t^n))\, (|\rho_a^n| + |\rho_1^n|) \\
&\quad + \sum_{j=1}^{N-1} \left|\rho_{j+1}^n - \rho_j^n\right| + \delta_1^n \left|\rho_1^n - \rho_a^n\right| - \delta_N^n \left|\rho_N^n - \rho_{N-1}^n\right| + \gamma_N^n \left|\rho_b^n - \rho_N^n\right| - \gamma_1^n \left|\rho_2^n - \rho_1^n\right| \\
&\quad + \mathcal{K}_1(t^n)\, \Delta t \left(\sum_{j=1}^{N-1} \left|\rho_{j+1}^n - \rho_j^n\right| + \frac{1}{2} \left|\rho_1^n - \rho_a^n\right|\right) + \mathcal{K}_2(t^n)\, \Delta t + \mathcal{K}_3(t^n)\, \Delta t \left|\rho_a^n\right| \\
&\quad + |\rho_b^{n+1} - \rho_b^n| + (1 - \gamma_N^n) |\rho_b^n - \rho_N^n| + \delta_N^n |\rho_N^n - \rho_{N-1}^n| + \Delta t\, \frac{C}{2}\, (1 + \mathcal{L}\, \mathcal{C}_1(t^n))\, (|\rho_{N-1}^n| + |\rho_N^n|)
\end{aligned}$$

$$\leq |\rho_a^{n+1} - \rho_a^n| + |\rho_b^{n+1} - \rho_b^n| + \left(1 + \mathcal{K}_1(t^n)\,\Delta t\right) \sum_{j=0}^{N} \left|\rho_{j+1}^n - \rho_j^n\right| + \mathcal{K}_2(t^n)\,\Delta t$$

$$+ \frac{3}{2}\,C(1 + \mathcal{L}\,\mathcal{C}_1(t^n))\,\|\rho^n\|_{\mathbf{L}^\infty(]a,b[)}\,\Delta t + \left(\mathcal{K}_3(t^n) + \frac{C}{2}\,(1 + \mathcal{L}\,\mathcal{C}_1(t^n))\right)\|\rho_a\|_{\mathbf{L}^\infty([0,t^n])}\,\Delta t.$$

Exploiting (3.10) and setting

$$\mathcal{K}_4(t^n) = \mathcal{K}_2(t^n) + \frac{3}{2}\,C(1 + \mathcal{L}\,\mathcal{C}_1(t^n))\,e^{\mathcal{C}_2(t^n)\,t^n} \max\left\{\|\rho_o\|_{\mathbf{L}^\infty(]a,b[)}, \|\rho_a\|_{\mathbf{L}^\infty([0,t])}, \|\rho_b\|_{\mathbf{L}^\infty([0,t])}\right\}$$
$$+ \left[\mathcal{K}_3(t^n) + \frac{C}{2}\,(1 + \mathcal{L}\,\mathcal{C}_1(t^n))\right]\|\rho_a\|_{\mathbf{L}^\infty([0,t^n])}, \tag{3.30}$$

we deduce from the previous estimate by a standard iterative procedure

$$\sum_{j=0}^{N} \left|\rho_{j+1}^n - \rho_j^n\right| \leq e^{\mathcal{K}_1(t^n)\,t^n} \left(\sum_{j=0}^{N} \left|\rho_{j+1}^0 - \rho_j^0\right| + \sum_{m=1}^{n} \left|\rho_a^m - \rho_a^{m-1}\right| + \sum_{m=1}^{n} \left|\rho_b^m - \rho_b^{m-1}\right|\right)$$
$$+ \frac{\mathcal{K}_4(t^n)}{\mathcal{K}_1(t^n)}\left(e^{\mathcal{K}_1(t^n)\,t^n} - 1\right),$$

concluding the proof. □

Corollary 3.7. (BV estimate in space and time) *Let $\rho_o \in \mathbf{BV}(\,]a,b[;\mathbb{R}^+)$ and ρ_a, $\rho_b \in \mathbf{BV}(\mathbb{R}^+;\mathbb{R}^+)$. Let **($f$)**, **($\omega$)** and* (3.5) *hold. Then for all $n = 1, \ldots, N_T$, the following estimate holds*

$$\sum_{m=0}^{n-1}\sum_{j=0}^{N} \Delta t \left|\rho_{j+1}^m - \rho_j^m\right| + \sum_{m=0}^{n-1}\sum_{j=0}^{N+1} \Delta x \left|\rho_j^{m+1} - \rho_j^m\right| \leq \mathcal{C}_{xt}(t^n), \tag{3.31}$$

where $\mathcal{C}_{xt}(n\,\Delta t)$ is given by (3.36).

Proof. By Proposition 3.5, we have

$$\sum_{m=0}^{n-1}\sum_{j=0}^{N} \Delta t \left|\rho_{j+1}^m - \rho_j^m\right| \leq n\,\Delta t\,\mathcal{C}_x(n\,\Delta t). \tag{3.32}$$

By the definition of the scheme (3.4), for $m \in \{0, \ldots, n-1\}$ and $j \in \{1, \ldots, N\}$, we have

$$\begin{aligned}
\left|\rho_j^{m+1} - \rho_j^m\right| \leq{}& \frac{\lambda\,\alpha}{2}\left(\left|\rho_{j+1}^m - \rho_j^m\right| + \left|\rho_j^m - \rho_{j-1}^m\right|\right)\\
&+ \frac{\lambda}{2}\Big|f(t^m, x_{j+1/2}, \rho_j^m, R_{j+1/2}^m) + f(t^m, x_{j+1/2}, \rho_{j+1}^m, R_{j+1/2}^m)\\
&\quad - f(t^m, x_{j-1/2}, \rho_{j-1}^m, R_{j-1/2}^m) - f(t^m, x_{j-1/2}, \rho_j^m, R_{j-1/2}^m)\Big|\\
\leq{}& \frac{\lambda\,\alpha}{2}\left(\left|\rho_{j+1}^m - \rho_j^m\right| + \left|\rho_j^m - \rho_{j-1}^m\right|\right)\\
&+ \frac{\lambda}{2}\Big[\left|\partial_x f(t^m, \tilde{x}_j, \rho_j^m, R_{j+1/2}^m)\right|\Delta x\\
&\quad + \left|\partial_\rho f(t^m, x_{j-1/2}, \tilde{\rho}_{j-1/2}^m, R_{j+1/2}^m)\right|\left|\rho_j^m - \rho_{j-1}^m\right|
\end{aligned}$$

$$
\begin{aligned}
&+ \left|\partial_R f(t^m, x_{j-1/2}, \rho^m_{j-1}, \tilde R^m_j)\right| \left|R^m_{j+1/2} - R^m_{j-1/2}\right| \\
&+ \left|\partial_x f(t^m, \tilde x_j, \rho^m_{j+1}, R^m_{j+1/2})\right| \Delta x \\
&+ \left|\partial_\rho f(t^m, x_{j-1/2}, \tilde\rho^m_{j+1/2}, R^m_{j+1/2})\right| \left|\rho^m_{j+1} - \rho^m_j\right| \\
&+ \left|\partial_R f(t^m, x_{j-1/2}, \rho^m_j, \tilde R^m_j)\right| \left|R^m_{j+1/2} - R^m_{j-1/2}\right| \Big] \\
\le{}& \frac{\lambda}{2} (\alpha + L) \left(\left|\rho^m_{j+1} - \rho^m_j\right| + \left|\rho^m_j - \rho^m_{j-1}\right| \right) \\
&+ \frac{\lambda}{2} C \Delta x \left[\left|\rho^m_{j+1}\right| + \left|\rho^m_j\right| + \mathcal{L}\, \mathcal{C}_1(t^m) \left(\left|\rho^m_j\right| + \left|\rho^m_{j-1}\right| \right) \right],
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde x_j &\in \,]x_{j-1/2}, x_{j+1/2}[, & \tilde R^m_j &\in \mathcal{I}(R^m_{j-1/2}, R^m_{j+1/2}), \\
\tilde\rho^m_{j-1/2} &\in \mathcal{I}(\rho^m_{j-1}, \rho^m_j), & \tilde\rho^m_{j+1/2} &\in \mathcal{I}(\rho^m_j, \rho^m_{j+1}).
\end{aligned}
$$

Therefore, by Lemma 3.3 and Proposition 3.5

$$
\begin{aligned}
\sum_{j=1}^{N} \Delta x \left|\rho^{m+1}_j - \rho^m_j\right| \le{}& \Delta t\, (\alpha + L) \sum_{j=0}^{N} \left|\rho^m_{j+1} - \rho^m_j\right| + \Delta t\, C \left(1 + \mathcal{L}\, \mathcal{C}_1(m\,\Delta t)\right) \Delta x \sum_{j=1}^{N} \left|\rho^m_j\right| \\
&+ \Delta t\, \Delta x\, \frac{C}{2} \left(\left|\rho^m_b\right| + \mathcal{L}\; \mathcal{C}_1(m\,\Delta t) \left|\rho^m_a\right| \right) \\
\le{}& \Delta t\, (\alpha + L)\, \mathcal{C}_x(m\,\Delta t) + \Delta t\, C \left(1 + \mathcal{L}\, \mathcal{C}_1(m\,\Delta t)\right) \mathcal{C}_1(m\,\Delta t) \\
&+ \Delta t\, \frac{C}{2} \left(\|\rho_b\|_{\mathbf{L}^\infty([0,t^m])} + \mathcal{L}\, \mathcal{C}_1(m\,\Delta t)\, \|\rho_a\|_{\mathbf{L}^\infty([0,t^m])} \right) \\
={}& \Delta t\, \mathcal{C}_t(m\,\Delta t),
\end{aligned}
$$

where we set

$$
\mathcal{C}_t(\tau) = (\alpha + L)\, \mathcal{C}_x(\tau) + C\, \mathcal{C}_1(\tau) \left(1 + \mathcal{L}\, \mathcal{C}_1(\tau)\right) + \frac{C}{2} \left(\|\rho_b\|_{\mathbf{L}^\infty([0,\tau])} + \mathcal{L}\, \mathcal{C}_1(\tau)\, \|\rho_a\|_{\mathbf{L}^\infty([0,\tau])} \right). \tag{3.33}
$$

In particular,

$$
\begin{aligned}
\sum_{j=0}^{N+1} \Delta x \left|\rho^{m+1}_j - \rho^m_j\right| ={}& \Delta x \left|\rho^{m+1}_a - \rho^m_a\right| + \Delta x \left|\rho^{m+1}_b - \rho^m_b\right| + \sum_{j=1}^{N} \Delta x \left|\rho^{m+1}_j - \rho^m_j\right| \\
\le{}& \Delta x \left|\rho^{m+1}_a - \rho^m_a\right| + \Delta x \left|\rho^{m+1}_b - \rho^m_b\right| + \Delta t\, \mathcal{C}_t(m\,\Delta t),
\end{aligned} \tag{3.34}
$$

which, summed over $m = 0, \ldots, n-1$, yields

$$
\sum_{m=0}^{n-1} \sum_{j=0}^{N+1} \Delta x \left|\rho^{m+1}_j - \rho^m_j\right| \le \Delta x \sum_{m=0}^{n-1} \left(\left|\rho^{m+1}_a - \rho^m_a\right| + \left|\rho^{m+1}_b - \rho^m_b\right| \right) + n\,\Delta t\, \mathcal{C}_t(n\,\Delta t). \tag{3.35}
$$

Summing (3.32) and (3.35) we obtain the desired estimate (3.31), with

$$
\mathcal{C}_{xt}(n\,\Delta t) = n\,\Delta t\, (1 + \alpha + L)\, \mathcal{C}_x(n\,\Delta t) + n\,\Delta t\, C\, \mathcal{C}_1(n\,\Delta t)(1 + \mathcal{L}\, \mathcal{C}_1(n\,\Delta t))
$$

$$
\begin{aligned}
&+ n\,\Delta t\,\frac{C}{2}\left(\|\rho_b\|_{\mathbf{L}^\infty([0,t^n])} + \mathcal{L}\,\mathcal{C}_1(n\,\Delta t)\,\|\rho_a\|_{\mathbf{L}^\infty([0,t^n])}\right) \\
&+ \Delta x \sum_{m=0}^{n-1}\left(\left|\rho_a^{m+1}-\rho_a^m\right| + \left|\rho_b^{m+1}-\rho_b^m\right|\right),
\end{aligned}
\tag{3.36}
$$

concluding the proof. Notice that the last sum in (3.36) is bounded by

$$
\Delta x\left(\mathrm{TV}\left(\rho_a;[0,T]\right) + \mathrm{TV}\left(\rho_b;[0,T]\right)\right).
$$

□

3.5 Discrete entropy inequality

We introduce the following notation: for $j = 1,\ldots,N$, $n = 0,\ldots,N_T-1$, $k \in \mathbb{R}$,

$$
\begin{aligned}
H_j^n(u,v,z) &= v - \lambda\left(F_{j+1/2}^n(v,z) - F_{j-1/2}^n(u,v)\right),\\
G_{j+1/2}^{n,k}(u,v) &= F_{j+1/2}^n(u\wedge k, v\wedge k) - F_{j+1/2}^n(k,k),\\
L_{j+1/2}^{n,k}(u,v) &= F_{j+1/2}^n(k,k) - F_{j+1/2}^n(u\vee k, v\vee k),
\end{aligned}
$$

where $F_{j+1/2}^n(u,v)$ is defined as in (3.2). Observe that, due to the definition of the scheme, $\rho_j^{n+1} = H_j^n(\rho_{j-1}^n,\rho_j^n,\rho_{j+1}^n)$. Notice moreover that the following equivalences hold true: $(s-k)^+ = s\wedge k - k$ and $(s-k)^- = k - s\vee k$.

Lemma 3.8. *Let* $(\boldsymbol{f})$, $(\boldsymbol{\omega})$ *and* (3.5) *hold. Then the approximate solution* ρ_Δ *in* (3.3) *satisfies the following discrete entropy inequalities: for* $j = 1,\ldots,N$, $n = 0,\ldots,N_T-1$ *and* $k\in\mathbb{R}$,

$$
\begin{aligned}
&(\rho_j^{n+1}-k)^+ - (\rho_j^n-k)^+ + \lambda\left(G_{j+1/2}^{n,k}(\rho_j^n,\rho_{j+1}^n) - G_{j-1/2}^{n,k}(\rho_{j-1}^n,\rho_j^n)\right)\\
&+\lambda\,\mathrm{sgn}^+(\rho_j^{n+1}-k)\left(f(t^n,x_{j+1/2},k,R_{j+1/2}^n) - f(t^n,x_{j-1/2},k,R_{j-1/2}^n)\right)\le 0,
\end{aligned}
\tag{3.37}
$$

and

$$
\begin{aligned}
&(\rho_j^{n+1}-k)^- - (\rho_j^n-k)^- + \lambda\left(L_{j+1/2}^{n,k}(\rho_j^n,\rho_{j+1}^n) - L_{j-1/2}^{n,k}(\rho_{j-1}^n,\rho_j^n)\right)\\
&+\lambda\,\mathrm{sgn}^-(\rho_j^{n+1}-k)\left(f(t^n,x_{j+1/2},k,R_{j+1/2}^n) - f(t^n,x_{j-1/2},k,R_{j-1/2}^n)\right)\le 0.
\end{aligned}
\tag{3.38}
$$

Proof. Consider the map $(u,v,z)\mapsto H_j^n(u,v,z)$. By the CFL condition (3.5), it holds

$$
\begin{aligned}
\frac{\partial H_j^n}{\partial u}(u,v,z) &= \frac{\lambda}{2}\left(\partial_\rho f(t^n,x_{j-1/2},u,R_{j-1/2}^n)+\alpha\right)\ge 0,\\
\frac{\partial H_j^n}{\partial v}(u,v,z) &= 1-\lambda\,\alpha - \frac{\lambda}{2}\left(\partial_\rho f(t^n,x_{j+1/2},v,R_{j+1/2}^n) - \partial_\rho f(t^n,x_{j-1/2},v,R_{j-1/2}^n)\right)\ge 0\\
\frac{\partial H_j^n}{\partial z}(u,v,z) &= \frac{\lambda}{2}\left(\alpha - \partial_\rho f(t^n,x_{j+1/2},z,R_{j+1/2}^n)\right)\ge 0.
\end{aligned}
$$

Notice that

$$
H_j^n(k,k,k) = k - \lambda\left(f(t^n,x_{j+1/2},k,R_{j+1/2}^n) - f(t^n,x_{j-1/2},k,R_{j-1/2}^n)\right).
$$

The monotonicity properties obtained above imply that

$$\begin{aligned}
& H_j^n(\rho_{j-1}^n \wedge k, \rho_j^n \wedge k, \rho_{j+1}^n \wedge k) - H_j^n(k,k,k) \\
\geq\ & H_j^n(\rho_{j-1}^n, \rho_j^n, \rho_{j+1}^n) \wedge H_j^n(k,k,k) - H_j^n(k,k,k) \\
=\ & \left(H_j^n(\rho_{j-1}^n, \rho_j^n, \rho_{j+1}^n) - H_j^n(k,k,k)\right)^+ \\
=\ & \left(\rho_j^{n+1} - k + \lambda\left(f(t^n, x_{j+1/2}, k, R_{j+1/2}^n) - f(t^n, x_{j-1/2}, k, R_{j-1/2}^n)\right)\right)^+ .
\end{aligned}$$

Moreover, we also have

$$\begin{aligned}
& H_j^n(\rho_{j-1}^n \wedge k, \rho_j^n \wedge k, \rho_{j+1}^n \wedge k) - H_j^n(k,k,k) \\
=\ & (\rho_j^n \wedge k) - k \\
& - \lambda\left[F_{j+1/2}^n(\rho_j^n \wedge k, \rho_{j+1}^n \wedge k) - F_{j-1/2}^n(\rho_{j-1}^n \wedge k, \rho_j^n \wedge k) - F_{j+1/2}^n(k,k) + F_{j-1/2}^n(k,k)\right] \\
=\ & (\rho_j^n - k)^+ - \lambda\left(G_{j+1/2}^{n,k}(\rho_j^n, \rho_{j+1}^n) - G_{j-1/2}^{n,k}(\rho_{j-1}^n, \rho_j^n)\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
& (\rho_j^n - k)^+ - \lambda\left(G_{j+1/2}^{n,k}(\rho_j^n, \rho_{j+1}^n) - G_{j-1/2}^{n,k}(\rho_{j-1}^n, \rho_j^n)\right) \\
\geq\ & \left(\rho_j^{n+1} - k + \lambda\left(f(t^n, x_{j+1/2}, k, R_{j+1/2}^n) - f(t^n, x_{j-1/2}, k, R_{j-1/2}^n)\right)\right)^+ \\
=\ & \operatorname{sgn}^+\left(\rho_j^{n+1} - k + \lambda\left(f(t^n, x_{j+1/2}, k, R_{j+1/2}^n) - f(t^n, x_{j-1/2}, k, R_{j-1/2}^n)\right)\right) \\
& \times\left(\rho_j^{n+1} - k + \lambda\left(f(t^n, x_{j+1/2}, k, R_{j+1/2}^n) - f(t^n, x_{j-1/2}, k, R_{j-1/2}^n)\right)\right) \\
\geq\ & \left(\rho_j^{n+1} - k\right)^+ + \lambda \operatorname{sgn}^+\left(\rho_j^{n+1} - k\right)\left(f(t^n, x_{j+1/2}, k, R_{j+1/2}^n) - f(t^n, x_{j-1/2}, k, R_{j-1/2}^n)\right),
\end{aligned}$$

proving (3.37), while (3.38) is proven in an entirely similar way. □

3.6 Convergence towards an entropy weak solution

The uniform $\mathbf{L}^\infty$-bound provided by Lemma 3.4 and the total variation estimate of Corollary 3.7 allow to apply Helly's compactness theorem, ensuring the existence of a subsequence of ρ_Δ, still denoted by ρ_Δ, converging in $\mathbf{L^1}$ to a function $\rho \in \mathbf{L}^\infty([0,T]\times]a,b[)$, for all $T > 0$. We need to prove that this limit function is indeed an entropy weak solution to (1.1), in the sense of Definition 2.2.

Lemma 3.9. *Let $\rho_o \in \mathbf{BV}(]a,b[;\mathbb{R}^+)$ and ρ_a, $\rho_b \in \mathbf{BV}(\mathbb{R}^+;\mathbb{R}^+)$. Let **(f)**, **(ω)** and* (3.5) *hold. Then the piecewise constant approximate solutions ρ_Δ in* (3.3) *resulting from the adapted Lax–Friedrichs scheme* (3.4) *converge, as $\Delta x \to 0$, towards an entropy weak solution of the initial boundary value problem* (1.1).

Proof. We consider the discrete entropy inequality (3.37), for the positive semi-entropy, and we follow [10], see also [15]. Add and subtract $G_{j+1/2}^{n,k}(\rho_j^n, \rho_j^n)$ in (3.37) and rearrange it as follows

$$0 \geq (\rho_j^{n+1} - k)^+ - (\rho_j^n - k)^+ + \lambda\left(G_{j+1/2}^{n,k}(\rho_j^n, \rho_{j+1}^n) - G_{j+1/2}^{n,k}(\rho_j^n, \rho_j^n)\right)$$

$$+ \lambda \left(G^{n,k}_{j+1/2}(\rho^n_j, \rho^n_j) - G^{n,k}_{j-1/2}(\rho^n_{j-1}, \rho^n_j) \right)$$
$$+ \lambda \operatorname{sgn}^+(\rho^{n+1}_j - k) \left(f(t^n, x_{j+1/2}, k, R^n_{j+1/2}) - f(t^n, x_{j-1/2}, k, R^n_{j-1/2}) \right)$$

Let $\varphi \in \mathbf{C^1_c}([0,T[\times[a,b];\mathbb{R}^+)$ for some $T > 0$, multiply the inequality above by $\Delta x\, \varphi(t^n, x_j)$ and sum over $j = 1, \ldots, N$ and $n \in \mathbb{N}$, so to get

$$0 \geq \Delta x \sum_{n=0}^{+\infty} \sum_{j=1}^{N} \left[(\rho^{n+1}_j - k)^+ - (\rho^n_j - k)^+ \right] \varphi(t^n, x_j) \tag{3.39}$$
$$+ \Delta t \sum_{n=0}^{+\infty} \sum_{j=1}^{N} \left[\left(G^{n,k}_{j+1/2}(\rho^n_j, \rho^n_{j+1}) - G^{n,k}_{j+1/2}(\rho^n_j, \rho^n_j) \right) \right. \tag{3.40}$$
$$\left. - \left(G^{n,k}_{j-1/2}(\rho^n_{j-1}, \rho^n_j) - G^{n,k}_{j+1/2}(\rho^n_j, \rho^n_j) \right) \right] \varphi(t^n, x_j) \tag{3.41}$$
$$+ \Delta t \sum_{n=0}^{+\infty} \sum_{j=1}^{N} \operatorname{sgn}^+(\rho^{n+1}_j - k) \left(f(t^n, x_{j+1/2}, k, R^n_{j+1/2}) - f(t^n, x_{j-1/2}, k, R^n_{j-1/2}) \right) \varphi(t^n, x_j). \tag{3.42}$$

Summing by parts, we obtain

$$[(3.39)] = -\Delta x \sum_{j=1}^{N} (\rho^0_j - k)^+ \varphi(0, x_j) - \Delta x\, \Delta t \sum_{n=1}^{+\infty} \sum_{j=1}^{N} (\rho^n_j - k)^+ \frac{\varphi(t^n, x_j) - \varphi(t^{n-1}, x_j)}{\Delta t}$$
$$\underset{\Delta x \to 0^+}{\longrightarrow} - \int_a^b (\rho_o(x) - k)^+ \varphi(0, x)\, \mathrm{d}x - \int_0^{+\infty} \int_a^b (\rho(t,x) - k)^+ \partial_t \varphi(t,x)\, \mathrm{d}x\, \mathrm{d}t\,,$$

and

$$[(3.42)]$$
$$= \Delta x\, \Delta t \sum_{n=0}^{+\infty} \sum_{j=1}^{N} \operatorname{sgn}^+(\rho^{n+1}_j - k) \frac{f(t^n, x_{j+1/2}, k, R^n_{j+1/2}) - f(t^n, x_{j-1/2}, k, R^n_{j-1/2})}{\Delta x} \varphi(t^n, x_j)$$
$$\underset{\Delta x \to 0^+}{\longrightarrow} \int_0^{+\infty} \int_a^b \operatorname{sgn}^+(\rho(t,x) - k) \frac{\mathrm{d}}{\mathrm{d}x} f(t, x, k, R(t,x))\, \varphi(t,x)\, \mathrm{d}x\, \mathrm{d}t\,,$$

by the Dominated Convergence Theorem. Concerning (3.40)–(3.41), we get

$$[(3.40) - (3.41)] = \Delta t \sum_{n=0}^{+\infty} \sum_{j=1}^{N} \left(G^{n,k}_{j+1/2}(\rho^n_j, \rho^n_{j+1}) - G^{n,k}_{j+1/2}(\rho^n_j, \rho^n_j) \right) \varphi(t^n, x_j)$$
$$- \Delta t \sum_{n=0}^{+\infty} \sum_{j=0}^{N-1} \left(G^{n,k}_{j+1/2}(\rho^n_j, \rho^n_{j+1}) - G^{n,k}_{j+3/2}(\rho^n_{j+1}, \rho^n_{j+1}) \right) \varphi(t^n, x_{j+1})$$
$$= \Delta t \sum_{n=0}^{+\infty} \sum_{j=1}^{N-1} \left[\left(G^{n,k}_{j+1/2}(\rho^n_j, \rho^n_{j+1}) - G^{n,k}_{j+1/2}(\rho^n_j, \rho^n_j) \right) \varphi(t^n, x_j) \right.$$
$$\left. - \left(G^{n,k}_{j+1/2}(\rho^n_j, \rho^n_{j+1}) - G^{n,k}_{j+3/2}(\rho^n_{j+1}, \rho^n_{j+1}) \right) \varphi(t^n, x_{j+1}) \right]$$

$$
\begin{aligned}
&+ \Delta t \sum_{n=0}^{+\infty} \Big[\Big(G^{n,k}_{N+1/2}(\rho^n_N, \rho^n_b) - G^{n,k}_{N+1/2}(\rho^n_N, \rho^n_N) \Big) \; \varphi(t^n, x_N) \\
&\qquad - \Big(G^{n,k}_{1/2}(\rho^n_a, \rho^n_1) - G^{n,k}_{3/2}(\rho^n_1, \rho^n_1) \Big) \; \varphi(t^n, x_1) \Big] \\
&= T^{int} + T^b = T,
\end{aligned}
$$

where we set

$$
\begin{aligned}
T^{int} &= \Delta t \sum_{n=0}^{+\infty} \sum_{j=1}^{N-1} \Big[\Big(G^{n,k}_{j+1/2}(\rho^n_j, \rho^n_{j+1}) - G^{n,k}_{j+1/2}(\rho^n_j, \rho^n_j) \Big) \; \varphi(t^n, x_j) \\
&\qquad - \Big(G^{n,k}_{j+1/2}(\rho^n_j, \rho^n_{j+1}) - G^{n,k}_{j+3/2}(\rho^n_{j+1}, \rho^n_{j+1}) \Big) \; \varphi(t^n, x_{j+1}) \Big], \\
T^b &= \Delta t \sum_{n=0}^{+\infty} \Big[\Big(G^{n,k}_{N+1/2}(\rho^n_N, \rho^n_b) - G^{n,k}_{N+1/2}(\rho^n_N, \rho^n_N) \Big) \; \varphi(t^n, x_N) \\
&\qquad - \Big(G^{n,k}_{1/2}(\rho^n_a, \rho^n_1) - G^{n,k}_{3/2}(\rho^n_1, \rho^n_1) \Big) \; \varphi(t^n, x_1) \Big].
\end{aligned}
$$

Define

$$
\begin{aligned}
S = &\; - \Delta x \, \Delta t \sum_{n=0}^{+\infty} \sum_{j=1}^{N} G^{n,k}_{j+1/2}(\rho^n_j, \rho^n_j) \, \frac{\varphi(t^n, x_{j+1}) - \varphi(t^n, x_j)}{\Delta x} \\
&- \alpha \, \Delta t \sum_{n=0}^{+\infty} \Big((\rho^n_a - k)^+ \, \varphi(t^n, a) + (\rho^n_b - k)^+ \, \varphi(t^n, b) \Big).
\end{aligned} \tag{3.43}
$$

Observe that

$$
\begin{aligned}
G^{n,k}_{j+1/2}(\rho^n_j, \rho^n_j) &= F^n_{j+1/2}(\rho^n_j \wedge k, \rho^n_j \wedge k) - F^n_{j+1/2}(k, k) \\
&= f(t^n, x_{j+1/2}, \rho^n_j \wedge k, R^n_{j+1/2}) - f(t^n, x_{j+1/2}, k, R^n_{j+1/2}) \\
&= \operatorname{sgn}^+(\rho^n_j - k) \Big(f(t^n, x_{j+1/2}, \rho^n_j, R^n_{j+1/2}) - f(t^n, x_{j+1/2}, k, R^n_{j+1/2}) \Big).
\end{aligned}
$$

It follows then easily that

$$
\begin{aligned}
S \underset{\Delta x \to 0^+}{\longrightarrow} &\; - \int_0^{+\infty} \int_a^b \operatorname{sgn}^+(\rho(t,x) - k) \; \big(f(t, x, \rho, R(t,x)) - f(t, x, k, R(t,x)) \big) \; \partial_x \varphi(t,x) \, \mathrm{d}x \, \mathrm{d}t \\
&- \alpha \left(\int_0^{+\infty} (\rho_a(t) - k)^+ \, \varphi(t, a) \, \mathrm{d}t + \int_0^{+\infty} (\rho_b(t) - k)^+ \, \varphi(t, b) \, \mathrm{d}t \right).
\end{aligned}
$$

Let us rewrite S (3.43) as follows:

$$
\begin{aligned}
S = &\; - \Delta t \sum_{n=0}^{+\infty} \sum_{j=1}^{N} G^{n,k}_{j+1/2}(\rho^n_j, \rho^n_j) \, \big(\varphi(t^n, x_{j+1}) - \varphi(t^n, x_j) \big) \\
&- \alpha \, \Delta t \sum_{n=0}^{+\infty} \Big((\rho^n_a - k)^+ \, \varphi(t^n, a) + (\rho^n_b - k)^+ \, \varphi(t^n, b) \Big)
\end{aligned}
$$

$$
\begin{aligned}
&= \Delta t \sum_{n=0}^{+\infty} \left(\sum_{j=1}^{N} G^{n,k}_{j+1/2}(\rho^n_j, \rho^n_j)\, \varphi(t^n, x_{j+1}) - \sum_{j=0}^{N-1} G^{n,k}_{j+3/2}(\rho^n_{j+1}, \rho^n_{j+1})\, \varphi(t^n, x_{j+1}) \right) \\
&\quad - \alpha\, \Delta t \sum_{n=0}^{+\infty} \Big((\rho^n_a - k)^+ \,\varphi(t^n, a) + (\rho^n_b - k)^+\, \varphi(t^n, b) \Big) \\
&= \, - \Delta t \sum_{n=0}^{+\infty} \sum_{j=1}^{N-1} \Big(G^{n,k}_{j+1/2}(\rho^n_j, \rho^n_j) - G^{n,k}_{j+3/2}(\rho^n_{j+1}, \rho^n_{j+1}) \Big)\; \varphi(t^n, x_{j+1}) \\
&\quad - \Delta t \sum_{n=0}^{+\infty} \Big(G^{n,k}_{N+1/2}(\rho^n_N, \rho^n_N)\, \varphi(t^n, x_{N+1}) - G^{n,k}_{3/2}(\rho^n_1, \rho^n_1))\, \varphi(t^n, x_1) \Big) \\
&\quad - \alpha\, \Delta t \sum_{n=0}^{+\infty} \Big((\rho^n_a - k)^+ \,\varphi(t^n, a) + (\rho^n_b - k)^+\, \varphi(t^n, b) \Big) \\
&= S^{int} + S^b,
\end{aligned}
$$

where we set

$$
\begin{aligned}
S^{int} = &\, - \Delta t \sum_{n=0}^{+\infty} \sum_{j=1}^{N-1} \Big(G^{n,k}_{j+1/2}(\rho^n_j, \rho^n_j) - G^{n,k}_{j+3/2}(\rho^n_{j+1}, \rho^n_{j+1}) \Big)\; \varphi(t^n, x_{j+1}), \\
S^b = &\, - \Delta t \sum_{n=0}^{+\infty} \Big(G^{n,k}_{N+1/2}(\rho^n_N, \rho^n_N)\, \varphi(t^n, x_{N+1}) - G^{n,k}_{3/2}(\rho^n_1, \rho^n_1))\, \varphi(t^n, x_1) \Big) \\
&\, - \alpha\, \Delta t \sum_{n=0}^{+\infty} \Big((\rho^n_a - k)^+ \,\varphi(t^n, a) + (\rho^n_b - k)^+\, \varphi(t^n, b) \Big)\,.
\end{aligned}
$$

Focus on S^{int}: by adding and subtracting $G^{n,k}_{j+1/2}(\rho^n_j, \rho^n_j)$ in the brackets, we can rewrite it as

$$
\begin{aligned}
S^{int} = &\, - \Delta t \sum_{n=0}^{+\infty} \sum_{j=1}^{N-1} \Big(G^{n,k}_{j+1/2}(\rho^n_j, \rho^n_{j+1}) - G^{n,k}_{j+1/2}(\rho^n_j, \rho^n_j) \Big)\; \varphi(t^n, x_{j+1}) \\
&\, - \Delta t \sum_{n=0}^{+\infty} \sum_{j=1}^{N-1} \Big(G^{n,k}_{j+1/2}(\rho^n_j, \rho^n_{j+1}) - G^{n,k}_{j+3/2}(\rho^n_{j+1}, \rho^n_{j+1}) \Big)\; \varphi(t^n, x_{j+1}),
\end{aligned}
$$

We evaluate now the distance between T^{int} and S^{int}:

$$
\left| T^{int} - S^{int} \right| \le \Delta t \sum_{n=0}^{+\infty} \sum_{j=1}^{N-1} \left| G^{n,k}_{j+1/2}(\rho^n_j, \rho^n_{j+1}) - G^{n,k}_{j+1/2}(\rho^n_j, \rho^n_j) \right| \left| \varphi(t^n, x_{j+1}) - \varphi(t^n, x_j) \right|.
$$

Observe that

$$
\begin{aligned}
&\left| G^{n,k}_{j+1/2}(\rho^n_j, \rho^n_{j+1}) - G^{n,k}_{j+1/2}(\rho^n_j, \rho^n_j) \right| \\
&= \left| F^n_{j+1/2}(\rho^n_j \wedge k, \rho^n_{j+1} \wedge k) - F^n_{j+1/2}(\rho^n_j \wedge k, \rho^n_j) \wedge k \right| \\
&= \frac{1}{2} \left| f(t^n, x_{j+1/2}, \rho^n_j \wedge k, R^n_{j+1/2}) + f(t^n, x_{j+1/2}, \rho^n_{j+1} \wedge k, R^n_{j+1/2}) \right.
\end{aligned}
$$

$$
\begin{aligned}
&\qquad -2\, f(t^n, x_{j+1/2}, \rho_j^n \wedge k, R_{j+1/2}^n) - \alpha\,(\rho_{j+1}^n \wedge k - \rho_j^n \wedge k)\Big| \\
&= \frac{1}{2}\left| f(t^n, x_{j+1/2}, \rho_{j+1}^n \wedge k, R_{j+1/2}^n) - f(t^n, x_{j+1/2}, \rho_j^n \wedge k, R_{j+1/2}^n) \right. \\
&\qquad\quad \left. -\alpha\,(\rho_{j+1}^n \wedge k - \rho_j^n \wedge k)\right| \\
&\le \frac{1}{2}\,(L+\alpha)\left|\rho_{j+1}^n \wedge k - \rho_j^n \wedge k\right| \\
&\le \alpha\left|\rho_{j+1}^n - \rho_j^n\right|.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\left|T^{int} - S^{int}\right| &\le \alpha\,\Delta x\,\Delta t\,\|\partial_x\varphi\|_{\mathbf{L}^\infty}\sum_{n=0}^{+\infty}\sum_{j=1}^{N-1}\left|\rho_{j+1}^n - \rho_j^n\right| \\
&\le \alpha\,\Delta x\,T\,\|\partial_x\varphi\|_{\mathbf{L}^\infty}\max_{0\le n\le T/\Delta t}\mathrm{TV}\left(\rho_\Delta(t^n,\cdot)\right) = \mathcal{O}(\Delta x),
\end{aligned} \tag{3.44}
$$

thanks to the uniform BV estimate (3.14). Pass now to the terms T^b and S^b:

$$
\begin{aligned}
S^b - T^b = &-\Delta t\sum_{n=0}^{+\infty}\Big(G_{N+1/2}^{n,k}(\rho_N^n,\rho_N^n)\,\varphi(t^n,x_{N+1}) - G_{3/2}^{n,k}(\rho_1^n,\rho_1^n))\,\varphi(t^n,x_1)\Big) \\
&-\alpha\,\Delta t\sum_{n=0}^{+\infty}\Big((\rho_a^n - k)^+\,\varphi(t^n,a) + (\rho_b^n - k)^+\,\varphi(t^n,b)\Big) \\
&-\Delta t\sum_{n=0}^{+\infty}\Big(G_{N+1/2}^{n,k}(\rho_N^n,\rho_b^n) - G_{N+1/2}^{n,k}(\rho_N^n,\rho_N^n)\Big)\ \varphi(t^n,x_N) \\
&+\Delta t\sum_{n=0}^{+\infty}\Big(G_{1/2}^{n,k}(\rho_a^n,\rho_1^n) - G_{3/2}^{n,k}(\rho_1^n,\rho_1^n)\Big)\ \varphi(t^n,x_1) \\
= &\ \Delta t\sum_{n=0}^{+\infty}\Big(G_{1/2}^{n,k}(\rho_a^n,\rho_1^n)\,\varphi(t^n,x_1) - \alpha\,(\rho_a^n - k)^+\,\varphi(t^n,a)\Big) \qquad (3.45)\\
&-\Delta t\sum_{n=0}^{+\infty}\Big(\alpha\,(\rho_b^n - k)^+\,\varphi(t^n,b) + G_{N+1/2}^{n,k}(\rho_N^n,\rho_b^n)\,\varphi(t^n,x_N)\Big) \qquad (3.46)\\
&-\Delta t\sum_{n=0}^{+\infty}G_{N+1/2}^{n,k}(\rho_N^n,\rho_N^n)\,\big(\varphi(t^n,x_{N+1}) - \varphi(t^n,x_N)\big)\,. \qquad (3.47)
\end{aligned}
$$

Observe that

$$
\begin{aligned}
\frac{\partial F_{j+1/2}^n}{\partial u}(u,v) &= \frac{1}{2}\,\Big(\partial_\rho f(t^n,x_{j+1/2},u,R_{j+1/2}^n) + \alpha\Big) \ge \frac{1}{2}\,(-L+\alpha) \ge 0, \\
\frac{\partial F_{j+1/2}^n}{\partial v}(u,v) &= \frac{1}{2}\,\Big(\partial_\rho f(t^n,x_{j+1/2},v,R_{j+1/2}^n) - \alpha\Big) \le \frac{1}{2}\,(L-\alpha) \le 0,
\end{aligned}
$$

meaning that the numerical flux is increasing with respect to the first variable and decreasing with respect to the second one. Thus,

$$
G_{j+1/2}^{n,k}(u,v) = F_{j+1/2}^n(u \wedge k, v \wedge k) - F_{j+1/2}^n(k,k)
$$

$$
\begin{aligned}
&\geq F^n_{j+1/2}(k, v\wedge k) - F^n_{j+1/2}(k,k)\\
&= \frac{1}{2}\,\Big(f(t^n, x_{j+1/2}, v\wedge k, R^n_{j+1/2}) - f(t^n, x_{j+1/2}, k, R^n_{j+1/2}) - \alpha(v\wedge k - k)\Big)\\
&\geq -\frac{L+\alpha}{2}\,|v\wedge k - k|\\
&\geq -\alpha\,(v-k)^+
\end{aligned}
$$

and

$$
\begin{aligned}
G^{n,k}_{j+1/2}(u,v) &= F^n_{j+1/2}(u\wedge k, v\wedge k) - F^n_{j+1/2}(k,k)\\
&\leq F^n_{j+1/2}(u\wedge k, k) - F^n_{j+1/2}(k,k)\\
&= \frac{1}{2}\,\Big(f(t^n, x_{j+1/2}, u\wedge k, R^n_{j+1/2}) - f(t^n, x_{j+1/2}, k, R^n_{j+1/2}) - \alpha(k - u\wedge k)\Big)\\
&\leq \frac{L+\alpha}{2}\,|u\wedge k - k|\\
&\leq \alpha\,(u-k)^+.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
[(3.45)] &= \Delta t\sum_{n=0}^{+\infty} G^{n,k}_{1/2}(\rho^n_a, \rho^n_1)\,\big(\varphi(t^n, x_1) - \varphi(t^n, a)\big)\\
&\quad + \Delta t\sum_{n=0}^{+\infty}\Big(G^{n,k}_{1/2}(\rho^n_a,\rho^n_1) - \alpha\,(\rho^n_a - k)^+\Big)\,\varphi(t^n, a)\\
&\leq \alpha\, T\,\Delta x\,\|\partial_x\varphi\|_{\mathbf{L}^\infty}\sup_{0\leq n\leq T/\Delta t}(\rho^n_a - k)^+\\
&\quad + \alpha\,\Delta t\sum_{n=0}^{+\infty}\Big((\rho^n_a - k)^+ - (\rho^n_a - k)^+\Big)\,\varphi(t^n, a)\\
&\leq \alpha\, T\,\Delta x\,\|\partial_x\varphi\|_{\mathbf{L}^\infty}\|\rho_a\|_{\mathbf{L}^\infty} = \mathcal{O}(\Delta x),\\
[(3.46)] &= -\Delta t\sum_{n=0}^{+\infty}\Big(\alpha\,(\rho^n_b - k)^+ + G^{n,k}_{N+1/2}(\rho^n_N, \rho^n_b)\Big)\,\varphi(t^n, b)\\
&\quad - \Delta t\sum_{n=0}^{+\infty} G^{n,k}_{N+1/2}(\rho^n_N, \rho^n_b)\,\big(\varphi(t^n, x_N) - \varphi(t^n, b)\big)\\
&\leq -\alpha\,\Delta t\sum_{n=0}^{+\infty}\Big((\rho^n_b - k)^+ - (\rho^n_b - k)^+\Big)\,\varphi(t^n, b)\\
&\quad + \alpha\, T\,\Delta x\,\|\partial_x\varphi\|_{\mathbf{L}^\infty}\sup_{0\leq n\leq T/\Delta t}(\rho^n_b - k)^+\\
&\leq \alpha\, T\,\Delta x\,\|\partial_x\varphi\|_{\mathbf{L}^\infty}\|\rho_b\|_{\mathbf{L}^\infty} = \mathcal{O}(\Delta x),\\
[(3.47)] &= \Delta t\left|\sum_{n=0}^{+\infty} G^{n,k}_{N+1/2}(\rho^n_N, \rho^n_N)\,\big(\varphi(t^n, x_{N+1}) - \varphi(t^n, x_N)\big)\right|\\
&\leq \Delta t\,\Delta x\,\|\partial_x\varphi\|_{\mathbf{L}^\infty}\sum_{n=0}^{+\infty}\left|G^{n,k}_{N+1/2}(\rho^n_N, \rho^n_N)\right|
\end{aligned}
$$

$$
\begin{aligned}
&= \Delta t\, \Delta x\, \|\partial_x \varphi\|_{\mathbf{L}^\infty} \sum_{n=0}^{+\infty} \left| F^n_{N+1/2}(\rho^n_N \wedge k, \rho^n_N \wedge k) - F^N_{N+1/2}(k,k) \right| \\
&= \Delta t\, \Delta x\, \|\partial_x \varphi\|_{\mathbf{L}^\infty} \sum_{n=0}^{+\infty} \left| f(t^n, x_{N+1/2}, \rho^n_N \wedge k, R^n_{N+1/2}) - f(t^n, x_{N+1/2}, k, R^n_{N+1/2}) \right| \\
&\le L\, \Delta t\, \Delta x\, \|\partial_x \varphi\|_{\mathbf{L}^\infty} \sum_{n=0}^{+\infty} \left| \rho^n_N \wedge k - k \right| \\
&= L\, \Delta t\, \Delta x\, \|\partial_x \varphi\|_{\mathbf{L}^\infty} \sum_{n=0}^{+\infty} (\rho^n_N - k)^+ \\
&\le L\, T\, \Delta x\, \|\partial_x \varphi\|_{\mathbf{L}^\infty} \sup_{0 \le n \le T/\Delta t} \|\rho^n\|_{\mathbf{L}^\infty} = \mathcal{O}(\Delta x),
\end{aligned}
$$

thanks to the uniform $\mathbf{L}^\infty$ estimate (3.10). Hence, $S^b - T^b \le \mathcal{O}(\Delta x)$, so that we finally get

$$
\begin{aligned}
0 &\ge [(3.39) \ldots (3.42)] \\
&= [(3.39)] + [(3.42)] + T \pm S \\
&\ge [(3.39)] + [(3.42)] + S - \mathcal{O}(\Delta x),
\end{aligned}
$$

concluding the proof. □

4 Lipschitz continuous dependence on initial and boundary data

Proposition 4.1. *Fix $T > 0$. Let **(f)**, **(ω)** and (3.5) hold. Assume moreover $\partial^2_{x\rho} f$, $\partial^2_{\rho R} f \in \mathbf{L}^\infty([0,T] \times [a,b] \times \mathbb{R}^2; \mathbb{R})$. Let $\rho_o, \sigma_o \in \mathbf{BV}(\,]a,b[; \mathbb{R}^+)$ and ρ_a, ρ_b, σ_a, $\sigma_b \in \mathbf{BV}(\,]0,T[; \mathbb{R}^+)$. Call ρ and σ the corresponding solutions to* (1.1)*. Then the following estimate holds*

$$
\begin{aligned}
&\left\| \rho(T) - \sigma(T) \right\|_{\mathbf{L}^1(]a,b[)} \\
&\le \left(\|\rho_o - \sigma_o\|_{\mathbf{L}^1(]a,b[)} + L \left(\|\rho_a - \sigma_a\|_{\mathbf{L}^1([0,T])} + \|\rho_b - \sigma_b\|_{\mathbf{L}^1([0,T])} \right) \right) \left(1 + B(T)\, T\, e^{B(T)\, T} \right),
\end{aligned}
$$

and $B(T)$ is defined in (4.25)*.*

Proof. Introduce the following notation:

$$
\begin{aligned}
R(t,x) &= \frac{1}{W(x)} \int_a^b \rho(t,y)\, \omega(y-x)\, \mathrm{d}y\,, \qquad g(t,x,u) = f(t,x,u,R(t,x)), \\
S(t,x) &= \frac{1}{W(x)} \int_a^b \sigma(t,y)\, \omega(y-x)\, \mathrm{d}y\,, \qquad h(t,x,u) = f(t,x,u,S(t,x)).
\end{aligned}
\tag{4.1}
$$

Observe that, due to **(ω)** and Theorem 2.3, for any $t \in [0,T]$ and $x \in [a,b]$,

$$
R(t,x) \le \frac{\|\omega\|_{\mathbf{L}^\infty}}{K_\omega} \mathcal{R}_1(t), \qquad\qquad S(t,x) \le \frac{\|\omega\|_{\mathbf{L}^\infty}}{K_\omega} \mathcal{S}_1(t),
$$

where $\mathcal{S}_1(t)$ is defined analogously to $\mathcal{R}_1(t)$ in (2.5) for σ. For later use, set

$$J(t) = \frac{\|\omega\|_{\mathbf{L}^\infty}}{K_\omega} \max\left\{\mathcal{R}_1(t),\, \mathcal{S}_1(t)\right\},$$

so that $R(t,x),\ S(t,x) \in [-J(t), J(t)]$. Compute for later use

$$\left|\partial_x R(t,x)\right| \leq \left(\frac{\left\|\omega'\right\|_{\mathbf{L}^1}\|\omega\|_{\mathbf{L}^\infty}}{K_\omega^2} + \frac{\left\|\omega'\right\|_{\mathbf{L}^\infty}}{K_\omega}\right)\left\|\rho(t)\right\|_{\mathbf{L}^1(]a,b[)} \leq \mathcal{L}\,\mathcal{R}_1(t), \tag{4.2}$$

$$\begin{aligned}
&\left|\partial^2_{xx} R(t,x)\right| \\
\leq & \left(\frac{2\left\|\omega'\right\|^2_{\mathbf{L}^1}\|\omega\|_{\mathbf{L}^\infty}}{K_\omega^3} + \frac{\left\|\omega''\right\|_{\mathbf{L}^1}\|\omega\|_{\mathbf{L}^\infty}}{K_\omega^2} + \frac{2\left\|\omega'\right\|_{\mathbf{L}^1}\left\|\omega'\right\|_{\mathbf{L}^\infty}}{K_\omega^2} + \frac{\left\|\omega''\right\|_{\mathbf{L}^\infty}}{K_\omega}\right)\left\|\rho(t)\right\|_{\mathbf{L}^1(]a,b[)} \\
\leq & \,\mathcal{W}\,\mathcal{R}_1(t),
\end{aligned} \tag{4.3}$$

with $\mathcal{L}$ defined exactly as in (3.12) and $\mathcal{W}$ defined as in (3.25). Observe that $\mathcal{L}$ and $\mathcal{W}$ are finite thanks to $\boldsymbol{(\omega)}$. Compute also

$$\left|R(t,x) - S(t,x)\right| \leq \frac{\|\omega\|_{\mathbf{L}^\infty}}{K_\omega}\int_a^b \left|\rho(t,y) - \sigma(t,y)\right| \mathrm{d}y\,, \tag{4.4}$$

$$\left|\partial_x R(t,x) - \partial_x S(t,x)\right| \leq \mathcal{L}\int_a^b \left|\rho(t,y) - \sigma(t,y)\right| \mathrm{d}y\,. \tag{4.5}$$

We can think of ρ and σ as solutions to the following IBVPs

$$\begin{cases}
\partial_t \rho + \partial_x g(t,x,\rho) = 0,\\
\rho(0,x) = \rho_o(x),\\
\rho(t,a) = \rho_a(t),\\
\rho(t,b) = \rho_b(t),
\end{cases}
\qquad
\begin{cases}
\partial_t \sigma + \partial_x h(t,x,\sigma) = 0, & (t,x) \in]0,T[\times]a,b[,\\
\sigma(0,x) = \sigma_o(x), & x \in]a,b[,\\
\sigma(t,a) = \sigma_a(t), & t \in]0,T[,\\
\sigma(t,b) = \sigma_b(t), & t \in]0,T[.
\end{cases}$$

Moreover, consider also the following IBVP:

$$\begin{cases}
\partial_t \pi + \partial_x g(t,x,\pi) = 0, & (t,x) \in]0,T[\times]a,b[,\\
\pi(0,x) = \sigma_o(x), & x \in]a,b[,\\
\pi(t,a) = \sigma_a(t), & t \in]0,T[,\\
\pi(t,b) = \sigma_b(t), & t \in]0,T[.
\end{cases} \tag{4.6}$$

Thanks to $\boldsymbol{(f)}$ and to the additional assumptions on f, the flux functions g and h defined in (4.1) satisfy the hypotheses of Theorem A.2, Proposition A.3 and Theorem A.4. Indeed, focusing on g, compute:

$$\partial^2_{xu} g(t,x,u) = \partial^2_{x\rho} f(t,x,u,R(t,x) + \partial^2_{uR} f(t,x,u,R(t,x))\,\partial_x R(t,x).$$

Thus, thanks also to (4.2), $\partial^2_{xu} g$ is finite. Therefore, we can use the results of [14], recalled in Appendix A: by Theorem A.2, problem (4.6) admits a unique solution in $(\mathbf{L}^\infty \cap \mathbf{BV})(]0,T[\times]a,b[;\mathbb{R})$, which satisfies, for all $t \in [0,T[$

$$\left\|\pi(t)\right\|_{\mathbf{L}^\infty(]a,b[)} \leq \mathcal{P}(t),$$

$$\mathrm{TV}\,(\pi(t)) \leq \left(\mathrm{TV}\,(\sigma_o) + \mathrm{TV}\,(\sigma_a;[0,t]) + \mathrm{TV}\,(\sigma_b;[0,t]) + \mathcal{K}(t)\,t\right) e^{\mathcal{C}_4(t)\,t},$$

with

$$\begin{aligned}
\mathcal{P}(t) &:= \left(\max\left\{\|\sigma_o\|_{\mathbf{L}^\infty(]a.b[)},\,\|\sigma_a\|_{\mathbf{L}^\infty([0,t])},\,\|\sigma_b\|_{\mathbf{L}^\infty([0,t])}\right\} + \mathcal{C}_3(t)\,t\right) e^{\mathcal{C}_4(t)\,t},\\
\mathcal{C}_3(t) &:= \left\|\partial_x g(\cdot,\cdot,0)\right\|_{\mathbf{L}^\infty([0,t]\times[a,b])},\\
\mathcal{C}_4(t) &:= \left\|\partial^2_{xu} g\right\|_{\mathbf{L}^\infty([0,t]\times[a,b]\times\mathbb{R})},\\
\mathcal{K}(t) &:= 2\,\mathcal{C}_3(t) + 2\,(b-a)\left\|\partial^2_{xx} g\right\|_{\mathbf{L}^\infty([0,t]\times[a,b]\times[-\mathcal{P}(t),\,\mathcal{P}(t)])}\\
&\quad + \frac{1}{2}\left(3\,\mathcal{P}(t) + \|\sigma_a\|_{\mathbf{L}^\infty([0,t])}\right)\left\|\partial^2_{xu} g\right\|_{\mathbf{L}^\infty([0,t]\times[a,b]\times[-\mathcal{P}(t),\,\mathcal{P}(t)])}.
\end{aligned}$$

Due to the definition of g in (4.1) and $(\boldsymbol{f})$, we obtain

$$\begin{aligned}
\left|\partial_x g(t,x,0)\right| &= \left|\partial_x f(t,z,0,R(t,x)) + \partial_R f(t,x,0,R(t,x))\,\partial_x R(t,x)\right| = 0,\\
\left\|\partial^2_{xu} g\right\|_{\mathbf{L}^\infty([0,t]\times[a,b]\times\mathbb{R})} &\leq \left\|\partial^2_{x\rho} f\right\|_{\mathbf{L}^\infty([0,t]\times[a,b]\times\mathbb{R}\times[-J(t),J(t)])}\\
&\quad + \left\|\partial^2_{\rho R} f\right\|_{\mathbf{L}^\infty([0,t]\times[a,b]\times\mathbb{R}\times[-J(t),J(t)])}\mathcal{L}\,\mathcal{R}_1(t),
\end{aligned}$$

so that $\mathcal{C}_3(t) = 0$, $\mathcal{C}_4(t) \leq \mathcal{C}_5(t)$, where we set

$$\mathcal{C}_5(t) = \left\|\partial^2_{x\rho} f\right\|_{\mathbf{L}^\infty([0,t]\times[a,b]\times\mathbb{R}\times[-J(t),J(t)])} + \left\|\partial^2_{\rho R} f\right\|_{\mathbf{L}^\infty([0,t]\times[a,b]\times\mathbb{R}\times[-J(t),J(t)])}\mathcal{L}\,\mathcal{R}_1(t) \tag{4.7}$$

Hence

$$\left\|\pi(t)\right\|_{\mathbf{L}^\infty(]a,b[)} \leq \mathcal{P}_\infty(t) = e^{\mathcal{C}_5(t)\,t}\max\left\{\|\sigma_o\|_{\mathbf{L}^\infty(]a.b[)},\,\|\sigma_a\|_{\mathbf{L}^\infty([0,t])},\,\|\sigma_b\|_{\mathbf{L}^\infty([0,t])}\right\}. \tag{4.8}$$

Moreover,

$$\begin{aligned}
&\left\|\partial^2_{xx} g\right\|_{\mathbf{L}^\infty([0,t]\times[a,b]\times[-\mathcal{P}(t),\,\mathcal{P}(t)])}\\
\leq{}& \left\|\partial^2_{xx} f\right\|_{\mathbf{L}^\infty([0,t]\times[a,b]\times[-\mathcal{P}(t),\,\mathcal{P}(t)]\times[-J(t),J(t)])}\\
&+ 2\left\|\partial^2_{xR} f\right\|_{\mathbf{L}^\infty([0,t]\times[a,b]\times[-\mathcal{P}(t),\,\mathcal{P}(t)]\times[-J(t),J(t)])}\|\partial_x R\|_{\mathbf{L}^\infty([0,t]\times[a,b])}\\
&+ \left\|\partial^2_{RR} f\right\|_{\mathbf{L}^\infty([0,t]\times[a,b]\times[-\mathcal{P}(t),\,\mathcal{P}(t)]\times[-J(t),J(t)])}\|\partial_x R\|^2_{\mathbf{L}^\infty([0,t]\times[a,b])}\\
&+ \|\partial_R f\|_{\mathbf{L}^\infty([0,t]\times[a,b]\times[-\mathcal{P}(t),\,\mathcal{P}(t)]\times[-J(t),J(t)])}\left\|\partial^2_{xx} R\right\|_{\mathbf{L}^\infty([0,t]\times[a,b])}\\
\leq{}& C\,\mathcal{P}_\infty(t)\left(1 + \mathcal{R}_1(t)\left(2\,\mathcal{L} + \mathcal{L}^2\,\mathcal{R}_1(t) + \mathcal{W}\right)\right),
\end{aligned}$$

so that $\mathcal{K}(t) \leq \hat{\mathcal{K}}(t)$ where we set

$$\hat{\mathcal{K}}(t) = 2\,(b-a)\,C\,\mathcal{P}_\infty(t)\left[1 + \mathcal{R}_1(t)\left(2\,\mathcal{L} + \mathcal{L}^2\,\mathcal{R}_1(t) + \mathcal{W}\right)\right] + \frac{1}{2}\left(3\,\mathcal{P}_\infty(t) + \|\sigma_a\|_{\mathbf{L}^\infty([0,t])}\right)\mathcal{C}_3(t), \tag{4.9}$$

and we then obtain

$$\mathrm{TV}\left(\pi(t)\right) \leq \Big(\mathrm{TV}\left(\sigma_o\right) + \mathrm{TV}\left(\sigma_a; [0,t]\right) + \mathrm{TV}\left(\sigma_b; [0,t]\right) + \hat{\mathcal{K}}(t)\, t\Big)\, e^{\mathcal{C}_5(t)\, t}. \tag{4.10}$$

For $t > 0$, compute

$$\left\|\rho(t) - \sigma(t)\right\|_{\mathbf{L}^1(]a,b[)} \leq \left\|\rho(t) - \pi(t)\right\|_{\mathbf{L}^1(]a,b[)} + \left\|\pi(t) - \sigma(t)\right\|_{\mathbf{L}^1(]a,b[)}. \tag{4.11}$$

The first term on the right hand side of (4.11) evaluates the distance between solutions to IBVPs of the type considered in the Appendix A with the same flux function, but different initial and boundary data. Therefore, we can apply Proposition A.3, to get

$$\begin{aligned}\left\|\rho(t) - \pi(t)\right\|_{\mathbf{L}^1(]a,b[)} \leq{} & \|\rho_o - \sigma_o\|_{\mathbf{L}^1(]a,b[)} \\ & + \|\partial_u g\|_{\mathbf{L}^\infty([0,t]\times[a,b]\times\mathbb{R})} \Big(\|\rho_a - \sigma_a\|_{\mathbf{L}^1([0,t])} + \|\rho_b - \sigma_b\|_{\mathbf{L}^1([0,t])}\Big).\end{aligned}$$

Due to the definition of g in (4.1), we obtain

$$\|\partial_u g\|_{\mathbf{L}^\infty([0,t]\times[a,b]\times\mathbb{R})} = \left\|\partial_\rho f\right\|_{\mathbf{L}^\infty([0,t]\times[a,b]\times\mathbb{R}\times\mathbb{R})} < L,$$

hence

$$\left\|\rho(t) - \pi(t)\right\|_{\mathbf{L}^1(]a,b[)} \leq \|\rho_o - \sigma_o\|_{\mathbf{L}^1(]a,b[)} + L\Big(\|\rho_a - \sigma_a\|_{\mathbf{L}^1([0,t])} + \|\rho_b - \sigma_b\|_{\mathbf{L}^1([0,t])}\Big). \tag{4.12}$$

On the other hand, the second term on the right hand side of (4.11) evaluates the distance between solutions to IBVPs of the type considered in Appendix A with different flux functions, but same initial and boundary data. We apply Theorem A.4 to obtain

$$\begin{aligned}&\left\|\pi(t) - \sigma(t)\right\|_{\mathbf{L}^1(]a,b[)} \\ \leq{} & \int_0^t \int_a^b \left\|\partial_x (g-h)(s,x,\cdot)\right\|_{\mathbf{L}^\infty(U(s))} \mathrm{d}x\, \mathrm{d}s & (4.13)\\ & + \int_0^t \left\|\partial_u (g-h)(s,\cdot,\cdot)\right\|_{\mathbf{L}^\infty(]a,b[\times U(s))} \min\Big\{\mathrm{TV}\left(\sigma(s)\right), \mathrm{TV}\left(\pi(s)\right)\Big\}\, \mathrm{d}s & (4.14)\\ & + 2\int_0^t \left\|(g-h)(s,a,\cdot)\right\|_{\mathbf{L}^\infty(U(s))} \mathrm{d}s + 2\int_0^t \left\|(g-h)(s,b,\cdot)\right\|_{\mathbf{L}^\infty(U(s))} \mathrm{d}s\,, & (4.15)\end{aligned}$$

where $U(s) = [-\mathcal{U}(s), \mathcal{U}(s)]$, with $\mathcal{U}(s) = \max\left\{\left\|\pi(s)\right\|_{\mathbf{L}^\infty(]a,b[)}, \left\|\sigma(s)\right\|_{\mathbf{L}^\infty(]a,b[)}\right\}$. Let us now estimate all the terms appearing in (4.13)–(4.15). First of all, by Theorem 2.3,

$$\left\|\sigma(t)\right\|_{\mathbf{L}^\infty(]a,b[)} \leq \mathcal{S}_\infty(t) = e^{t\, C\,(1+\mathcal{L}\,\mathcal{S}_1(t))} \max\left\{\|\sigma_o\|_{\mathbf{L}^\infty(]a.b[)}, \|\sigma_a\|_{\mathbf{L}^\infty([0,t])}, \|\sigma_b\|_{\mathbf{L}^\infty([0,t])}\right\}, \tag{4.16}$$

so that, comparing (4.16) with (4.8) we obtain

$$\mathcal{U}(t) \leq \max\left\{\|\sigma_o\|_{\mathbf{L}^\infty(]a.b[)}, \|\sigma_a\|_{\mathbf{L}^\infty([0,t])}, \|\sigma_b\|_{\mathbf{L}^\infty([0,t])}\right\} \exp\Big(t\, C\, \left(1 + \mathcal{L}\,\mathcal{S}_1(t)\right) + t\, \mathcal{C}_5(t)\Big). \tag{4.17}$$

Then, by Theorem 2.3,

$$\mathrm{TV}\left(\sigma(t)\right) \leq e^{t\,\mathcal{T}_1(t)}\left(\mathrm{TV}\left(\sigma_o\right) + \mathrm{TV}\left(\sigma_a; [0,t]\right) + \mathrm{TV}\left(\sigma_b; [0,t]\right)\right) + \frac{\mathcal{T}_2(t)}{\mathcal{T}_1(t)}(e^{t\,\mathcal{T}_1(t)} - 1) \tag{4.18}$$

with

$$\begin{aligned}
\mathcal{T}_1(t) &= \left\|\partial^2_{\rho x} f\right\|_{\mathbf{L}^\infty([0,t]\times[a,b]\times\mathbb{R}^2)} + \mathcal{L}\,\mathcal{S}_1(t)\left\|\partial^2_{\rho R} f\right\|_{\mathbf{L}^\infty([0,t]\times[a,b]\times\mathbb{R}^2)},\\
\mathcal{T}_2(t) &= \mathcal{K}_2(t) + \frac{3}{2}C(1+\mathcal{L}\,\mathcal{S}_1(t))\,\mathcal{S}_\infty(t) + \left[\mathcal{K}_3(t) + \frac{C}{2}\left(1+\mathcal{L}\,\mathcal{S}_1(t)\right)\right]\|\sigma_a\|_{\mathbf{L}^\infty([0,t])},\\
\mathcal{K}_2(t) &= C\,\mathcal{S}_1(t)\,\left(1+2\,\mathcal{L}\,\mathcal{S}_1(t)+2\,\mathcal{K}_3(t)\right),\\
\mathcal{K}_3(t) &= C\,\mathcal{S}_1(t)\left(\mathcal{L}^2\,\mathcal{S}_1(t)+\frac{1}{2}\,\mathcal{W}\right).
\end{aligned}$$

Hence, comparing (4.10) and (4.18), we get

$$\begin{aligned}
\min\left\{\mathrm{TV}\left(\sigma(s)\right),\,\mathrm{TV}\left(\pi(s)\right)\right\} &\le e^{t\,\mathcal{T}_3(t)}\left(\mathrm{TV}\left(\sigma_o\right)+\mathrm{TV}\left(\sigma_a;[0,t]\right)+\mathrm{TV}\left(\sigma_b;[0,t]\right)\right)\\
&\qquad + \min\left\{\hat{\mathcal{K}(t)}\,e^{\mathcal{C}_5(t)\,t},\,\frac{\mathcal{T}_2(t)}{\mathcal{T}_1(t)}(e^{t\,\mathcal{T}_1(t)}-1)\right\}\\
&=: \mathcal{T}_4(t),
\end{aligned} \tag{4.19}$$

with

$$\mathcal{T}_3(t) = \left\|\partial^2_{\rho x} f\right\|_{\mathbf{L}^\infty([0,t]\times[a,b]\times\mathbb{R}^2)} + \mathcal{L}\,\min\left\{\mathcal{R}_1(t),\,\mathcal{S}_1(t)\right\}\left\|\partial^2_{\rho R} f\right\|_{\mathbf{L}^\infty([0,t]\times[a,b]\times\mathbb{R}^2)}. \tag{4.20}$$

Focus on (4.13): by $(\boldsymbol{f})$, (4.4) and (4.5),

$$\begin{aligned}
&\left|\partial_x\left(g(t,x,u)-h(t,x,u)\right)\right|\\
&= \left|\frac{\mathrm{d}}{\mathrm{d}x}\left(f\left(t,x,u,R(t,x)\right)-f\left(t,x,u,S(t,x)\right)\right)\right|\\
&\le \left|\partial_x f(t,x,u,R(t,x))-\partial_x f(t,x,u,S(t,x))\right|\\
&\quad + \left|\partial_R f(t,x,u,R(t,x))\,\partial_x R(t,x)-\partial_R f(t,x,u,S(t,x))\,\partial_x S(t,x)\right|\\
&\le \left|\partial^2_{xR} f(t,x,u,R_1(t,x))\right|\left|R(t,x)-S(t,x)\right|\\
&\quad + \left|\partial^2_{RR} f(t,x,u,R_2(t,x))\right|\left|\partial_x R(t,x)\right|\left|R(t,x)-S(t,x)\right|\\
&\quad + \left|\partial_R f(t,x,u,S(t,x))\right|\left|\partial_x R(t,x)-\partial_x S(t,x)\right|\\
&\le C\,|u|\left[\frac{\|\omega\|_{\mathbf{L}^\infty}}{K_\omega}\left(1+\mathcal{L}\,\mathcal{R}_1(t)\right)+\mathcal{L}\right]\int_a^b\left|\rho(t,y)-\sigma(t,y)\right|\mathrm{d}y\,,
\end{aligned}$$

with $R_i(t,x)\in\mathcal{I}\left(R(t,x),S(t,x)\right)$, $i=1,2$. Hence,

$$\left\|\partial_x(g-h)(s,x,\cdot)\right\|_{\mathbf{L}^\infty(U(s))} \le C\,\mathcal{U}(s)\left[\frac{\|\omega\|_{\mathbf{L}^\infty}}{K_\omega}\left(1+\mathcal{L}\,\mathcal{R}_1(s)\right)+\mathcal{L}\right]\int_a^b\left|\rho(s,y)-\sigma(s,y)\right|\mathrm{d}y\,. \tag{4.21}$$

Pass to (4.14): by $(\boldsymbol{f})$, (4.4) and (4.5),

$$\begin{aligned}
\left|\partial_u\left(g(t,x,u)-h(t,x,u)\right)\right| &= \left|\partial_\rho\left(f\left(t,x,u,R(t,x)\right)-f\left(t,x,u,S(t,x)\right)\right)\right|\\
&= \left|\partial^2_{\rho R} f\left(t,x,u,R_3(t,x)\right)\right|\left|R(t,x)-S(t,x)\right|,
\end{aligned}$$

and therefore

$$\left\|\partial_u(g-h)(s,\cdot,\cdot)\right\|_{\mathbf{L}^\infty(]a,b[\times U(s))} \leq \left\|\partial^2_{\rho R} f\right\|_{\mathbf{L}^\infty([0,s]\times[a,b]\times U(s)\times\mathbb{R})} \frac{\|\omega\|_{\mathbf{L}^\infty}}{K_\omega}\int_a^b \left|\rho(s,y)-\sigma(s,y)\right| \mathrm{d}y\,. \tag{4.22}$$

Finally, consider the first integral in (4.15): by **(*f*)** and (4.4)

$$\begin{aligned}\left|g(t,a,u)-h(t,a,u)\right| &= \left|f\left(t,a,u,R(t,a)\right)-f\left(t,a,u,S(t,a)\right)\right|\\ &\leq \left|\partial_R f\left(t,a,u,\tilde{R}(t,a)\right)\right|\left|R(t,a)-S(t,a)\right|\\ &\leq C|u|\,\frac{\|\omega\|_{\mathbf{L}^\infty}}{K_\omega}\int_a^b \left|\rho(t,y)-\sigma(t,y)\right| \mathrm{d}y\end{aligned}$$

with $\tilde{R}(t,a)\in\mathcal{I}\left(R(t,a),S(t,a)\right)$, so that

$$\left\|(g-h)(s,a,\cdot)\right\|_{\mathbf{L}^\infty(U(s))} \leq C\,\mathcal{U}(s)\,\frac{\|\omega\|_{\mathbf{L}^\infty}}{K_\omega}\int_a^b \left|\rho(s,y)-\sigma(s,y)\right| \mathrm{d}y\,, \tag{4.23}$$

and similarly for the second integral in (4.15).

Collecting together (4.21), (4.22) and (4.23), exploiting (4.17) and (4.19), we obtain

$$\begin{aligned}&\left\|\pi(t)-\sigma(t)\right\|_{\mathbf{L}^1(]a,b[)}\\ \leq &\left\{(b-a)\,C\,\mathcal{U}(t)\left[\frac{\|\omega\|_{\mathbf{L}^\infty}}{K_\omega}\left(1+\mathcal{L}\,\mathcal{R}_1(t)\right)+\mathcal{L}\right]+\left\|\partial^2_{\rho R} f\right\|_{\mathbf{L}^\infty([0,t]\times[a,b]\times U(t)\times\mathbb{R})}\frac{\|\omega\|_{\mathbf{L}^\infty}}{K_\omega}\,\mathcal{T}_4(t)\right.\\ &\left.\quad+4\,C\,\mathcal{U}(t)\,\frac{\|\omega\|_{\mathbf{L}^\infty}}{K_\omega}\right\}\int_0^t\int_a^b \left|\rho(s,y)-\sigma(s,y)\right| \mathrm{d}y\,\mathrm{d}s\,.\end{aligned} \tag{4.24}$$

Insert now (4.12) and (4.24) into (4.11):

$$\left\|\rho(t)-\sigma(t)\right\|_{\mathbf{L}^1(]a,b[)} \leq A(t)+B(t)\int_0^t \left\|\rho(s)-\sigma(s)\right\|_{\mathbf{L}^1(]a,b[)} \mathrm{d}s\,,$$

where

$$\begin{aligned}A(t) &= \|\rho_o-\sigma_o\|_{\mathbf{L}^1(]a,b[)}+L\left(\|\rho_a-\sigma_a\|_{\mathbf{L}^1([0,t])}+\|\rho_b-\sigma_b\|_{\mathbf{L}^1([0,t])}\right),\\ B(t) &= (b-a)\,C\,\mathcal{U}(t)\left[\frac{\|\omega\|_{\mathbf{L}^\infty}}{K_\omega}\left(1+\mathcal{L}\,\mathcal{R}_1(t)\right)+\mathcal{L}\right]+4\,C\,\mathcal{U}(t)\,\frac{\|\omega\|_{\mathbf{L}^\infty}}{K_\omega}\\ &\quad+\left\|\partial^2_{\rho R} f\right\|_{\mathbf{L}^\infty([0,t]\times[a,b]\times U(t)\times\mathbb{R})}\frac{\|\omega\|_{\mathbf{L}^\infty}}{K_\omega}\,\mathcal{T}_4(t)\end{aligned} \tag{4.25}$$

with $\mathcal{U}(t)$ as in (4.17), $\mathcal{L}$ as in (3.12), $\mathcal{R}_1$ as in (2.5), $\mathcal{T}_4(t)$ as in (4.19). An application of Gronwall's Lemma yields the desired estimate:

$$\begin{aligned}\left\|\rho(t)-\sigma(t)\right\|_{\mathbf{L}^1(]a,b[)} &\leq A(t)+\int_0^t A(s)B(s)e^{\int_s^t B(\tau)\mathrm{d}\tau}\,\mathrm{d}s\\ &\leq A(t)+B(t)\int_0^t A(s)e^{B(t)(t-s)}\,\mathrm{d}s\\ &\leq A(t)\,\left(1+B(t)\,t\,e^{B(t)\,t}\right).\end{aligned}$$

□

5 Proof of Theorem 2.3

Proof of Theorem 2.3. The existence of solutions to problem (1.1) follows from the results of Section 3, in particular § 3.6. The uniqueness is ensured by the Lipschitz continuous dependence of solutions to (1.1) on initial and boundary data, see Section 4.

The estimates on the solution to (1.1) are obtained from the corresponding discrete estimates passing to the limit. In particular, the $\mathbf{L^1}$ bound follows from (3.8), the $\mathbf{L}^\infty$ bound from (3.10), the total variation bound from (3.14) and the Lipschitz continuity in time from (3.34), since $\Delta x = \frac{\Delta t}{\lambda}$ and taking $\lambda = \frac{1}{3L}$. □

Appendix A The local 1D IBVP

We recall below some results concerning the classical (local) one dimensional initial boundary value problem for a scalar conservation laws. Detailed proofs can be found in [14], which deals with the more general case of a balance law.

Fix $T > 0$, set $I =]0, T[$ and consider the IBVP:

$$\begin{cases} \partial_t u + \mathrm{d}_x f(t,x,u) = 0, & (t,x) \in I \times]a,b[, \\ u(0,x) = u_o(x), & x \in]a,b[, \\ u(t,a) = u_a(t), & t \in I, \\ u(t,b) = u_b(t), & t \in I. \end{cases} \tag{A.1}$$

Above, the notation for $\mathrm{d}_x f\left(t,x,u(t,x)\right)$ follows closely that introduced in (1.2), that is:

$$\mathrm{d}_x f\left(t,x,u(t,x)\right) = \partial_x f\left(t,x,u(t,x)\right) + \partial_u f\left(t,x,u(t,x)\right)\, \partial_x u(t,x).$$

Recall the definition of solution to (A.1). In particular, we focus on the adaptation to the present one dimensional setting of the definition of solution provided by Bardos, le Roux and Nédélec [3, p. 1028].

Definition A.1. *A function $u \in (\mathbf{L}^\infty \cap \mathbf{BV})(I \times]a,b[;\mathbb{R})$ is an entropy weak solution to problem* (A.1) *if for all test function $\varphi \in \mathbf{C_c^1}(]-\infty,T[\times\mathbb{R};\mathbb{R}^+)$ and $k \in \mathbb{R}$*

$$\begin{aligned}
&\int_0^T \int_a^b \Big\{ \left|u(t,x)-k\right| \partial_t \varphi(t,x) + \mathrm{sgn}\left(u(t,x)-k\right) \left[f\left(t,x,u(t,x)\right) - f\left(t,x,k\right)\right] \partial_x \varphi(t,x) \\
&\qquad\qquad - \mathrm{sgn}\left(u(t,x)-k\right) \partial_x f\left(t,x,k\right)\, \varphi(t,x) \Big\}\, \mathrm{d}x\, \mathrm{d}t \\
&+ \int_a^b \left|u_o(x)-k\right| \varphi(0,x)\, \mathrm{d}x \\
&+ \int_0^T \mathrm{sgn}\left(u_a(t)-k\right) \left[f\left(t,a,u(t,a^+)\right) - f\left(t,a,k\right)\right] \varphi(t,a)\, \mathrm{d}t \\
&- \int_0^T \mathrm{sgn}\left(u_b(t)-k\right) \left[f\left(t,b,u(t,b^-) - f\left(t,b,k\right)\right)\right] \varphi(t,b)\, \mathrm{d}t \geq 0.
\end{aligned}$$

The well-posedness of problem (A.1), some *a priori* estimates on its solution and the stability of its solution with respect to variations in the flux function are proved in [14]. We report the results below, adapted to the present setting without source term.

Theorem A.2. *[14, Theorem 2.4] Let* $f \in \mathbf{C^2}([0,T] \times [a,b] \times \mathbb{R};\mathbb{R})$*, with* $\partial_u f$*,* $\partial^2_{xu} f \in \mathbf{L}^\infty([0,T] \times [a,b] \times \mathbb{R};\mathbb{R})$*. Let* $u_o \in \mathbf{BV}(\,]a,b[;\mathbb{R}^+)$*,* u_a*,* $u_b \in \mathbf{BV}(I;\mathbb{R}^+)$*.*

Then the IBVP (A.1) *has a unique solution* $u \in (\mathbf{L}^\infty \cap \mathbf{BV})(I \times]a,b[;\mathbb{R})$*, satisfying*

$$\left\|u(t)\right\|_{\mathbf{L}^\infty(]a,b[)} \leq \mathcal{U}(t), \tag{A.2}$$

$$\mathrm{TV}\left(u(t)\right) \leq e^{\mathcal{C}_4(t)\,t} \left(\mathrm{TV}\left(u_o\right) + \mathrm{TV}\left(u_a;[0,t]\right) + \mathrm{TV}\left(u_b;[0,t]\right) + \mathcal{K}(t)\,t\right), \tag{A.3}$$

for any $t \in [0,T[$*, with*

$$\begin{aligned}
\mathcal{U}(t) = {} & \left(\max\left\{\|u_o\|_{\mathbf{L}^\infty(]a,b[)}, \|u_a\|_{\mathbf{L}^\infty([0,t])}, \|u_b\|_{\mathbf{L}^\infty([0,t])}\right\} + \mathcal{C}_3(t)\,t\right) e^{\mathcal{C}_4(t)\,t},\\
\mathcal{C}_3(t) = {} & \left\|\partial_x f(\cdot,\cdot,0)\right\|_{\mathbf{L}^\infty([0,t]\times[a,b])},\\
\mathcal{C}_4(t) = {} & \left\|\partial^2_{xu} f\right\|_{\mathbf{L}^\infty([0,t]\times[a,b]\times\mathbb{R})},\\
\mathcal{K}(t) = {} & 2\,\mathcal{C}_3(t) + 2\,(b-a)\left\|\partial^2_{xx} f\right\|_{\mathbf{L}^\infty([0,t]\times[a,b]\times[-\mathcal{U}(t),\mathcal{U}(t)])}\\
& + \frac{1}{2}\left(3\mathcal{U}(t) + \|u_a\|_{\mathbf{L}^\infty([0,t])}\right)\left\|\partial^2_{xu} f\right\|_{\mathbf{L}^\infty([0,t]\times[a,b]\times[-\mathcal{U}(t),\mathcal{U}(t)])}.
\end{aligned}$$

Proposition A.3. *[14, Proposition 3.7] Let* $f \in \mathbf{C^2}([0,T] \times [a,b] \times \mathbb{R};\mathbb{R})$*, with* $\partial_u f$*,* $\partial^2_{xu} f \in \mathbf{L}^\infty([0,T] \times [a,b] \times \mathbb{R};\mathbb{R})$*. Let* u_o*,* $v_o \in \mathbf{BV}(\,]a,b[;\mathbb{R}^+)$*,* u_a*,* u_b*,* v_a*,* $v_b \in \mathbf{BV}(I;\mathbb{R}^+)$*. Call* u *and* v *the corresponding solutions to the IBVP* (A.1)*. Then, for all* $t > 0$*, the following estimate holds*

$$\begin{aligned}
\left\|u(t) - v(t)\right\|_{\mathbf{L^1}(]a,b[)} \leq {} & \|u_o - v_o\|_{\mathbf{L^1}(]a,b[)}\\
& + \|\partial_u f\|_{\mathbf{L}^\infty([0,t]\times[a,b]\times\mathbb{R})}\left(\|u_a - v_a\|_{\mathbf{L^1}([0,t])} + \|u_b - v_b\|_{\mathbf{L^1}([0,t])}\right).
\end{aligned}$$

Theorem A.4. *[14, Theorem 2.6] Let* $f_1, f_2 \in \mathbf{C^2}([0,T] \times [a,b] \times \mathbb{R};\mathbb{R})$*, with* $\partial_u f_i$*,*$\partial^2_{xu} f_i \in \mathbf{L}^\infty([0,T] \times [a,b] \times \mathbb{R};\mathbb{R})$ *for* $i = 1,2$*. Let* $u_o \in \mathbf{BV}(\,]a,b[;\mathbb{R}^+)$*,* u_a*,* $u_b \in \mathbf{BV}(I;\mathbb{R}^+)$*. Call* u_1 *and* u_2 *the corresponding solutions to the IBVP* (A.1)*. Then, for* $t \in [0,T[$*, the following estimate holds*

$$\begin{aligned}
& \left\|u_1(t) - u_2(t)\right\|_{\mathbf{L^1}(]a,b[)}\\
\leq {} & \int_0^t \int_a^b \left\|\partial_x(f_2 - f_1)(s,x,\cdot)\right\|_{\mathbf{L}^\infty(U(s))} \mathrm{d}x\,\mathrm{d}s\\
& + \int_0^t \left\|\partial_u(f_2 - f_1)(s,\cdot,\cdot)\right\|_{\mathbf{L}^\infty(]a,b[\times U(s))} \min\left\{\mathrm{TV}\left(u_1(s)\right), \mathrm{TV}\left(u_2(s)\right)\right\} \mathrm{d}s\\
& + 2\int_0^t \left\|(f_2 - f_1)(s,a,\cdot)\right\|_{\mathbf{L}^\infty(U(s))} \mathrm{d}s + 2\int_0^t \left\|(f_2 - f_1)(s,b,\cdot)\right\|_{\mathbf{L}^\infty(U(s))} \mathrm{d}s\,,
\end{aligned}$$

where, with the notation introduced in Theorem A.2, $\left\|u_i(s)\right\|_{\mathbf{L}^\infty(]a,b[)} \leq \mathcal{U}_i(s)$*, for* $i = 1,2$*, and*

$$U(s) = [-\mathcal{U}(s), \mathcal{U}(s)], \qquad \text{with} \quad \mathcal{U}(s) = \max_{i=1,2} \mathcal{U}_i(s).$$

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