Admissible Generalizations of Examples as Rules

Philippe Besnard, Thomas Guyet, Véronique Masson

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And now . . .

Attribute-value rule learning

	(C) Price	(A_1) Area	(A_2) Rooms	(A_3) Energy	(A_4) Town	(A_5) District	(A_6) Exposure
1	low-priced	70	2	D	Toulouse	Minimes	
2	low-priced	75	4	D	Toulouse	Rangueil	
3	expensive	65	3		Toulouse	Downtown	
4	low-priced	32	2	D	Toulouse		SE
5	mid-priced	65	2	D	Rennes		SO
6	expensive	100	5	C	Rennes	Downtown	
7	low-priced	40	2	D	Betton		S

- Task: induce rules to predict the value of the class attribute (C)
- Rules extracted by Algorithm CN2

$$\pi_1^{CN2} : A_5 = \text{Downtown} \Rightarrow C = \text{expensive}$$
$$\pi_2^{CN2} : A_2 < 2.50 \wedge A_4 = \text{Toulouse} \Rightarrow C = \text{low-priced}$$
$$\pi_3^{CN2} : A_1 > 36.00 \wedge A_3 = D \Rightarrow C = \text{low-priced}$$

Interpretability of rules and rulesets

- The logical structure of a rule can be easily interpreted by users

  `IF` *conditions* `THEN` *class-label*

- Rule learning algorithms generate rules according to implicit or explicit principles[1]
  - are the generated rules the *interpretable* ones?
  - would it be possible to have different rulesets?
  - why a ruleset would be better than another one from the interpretability point of view?

$\Rightarrow$ We need ways to analyze the interpretability of the outputs of rule learning algorithms

[1] principles mainly based on statistical properties!

Analyzing the interpretability of rules

Analyzing the interpretativeness of ruleset

- Objective criteria on ruleset syntax [CZV13, BS15]
  - size of the rule (number of attributes)
  - size of the ruleset
- Intuitiveness of rules through the effects of cognitive biases [KBF18]

$\Rightarrow$ Our approach formalizes rule learning and some expected properties on rules to shed light on properties of some extracted ruleset

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In this talk

- We present the formalisation of rule learning, and we focus on the generalization of examples as a rule
- We introduce the notion of admissible rule that attempts to capture an intuitive generalization of the examples
- We develop the example of numerical attributes

Impact of examples generalization on rule interpretability

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Toward the notion of admissibility

	(C) Price	(A_1) Area
1	low-priced	70
2	low-priced	75
4	low-priced	32
7	low-priced	40

- Rote learning of a rule

$$A_1 = \{75\} \Rightarrow C = \text{low-priced}$$

- Most generalizing rule

$$A_1 = [32 : 75] \Rightarrow C = \text{low-priced}$$

- Would the following rule be better?

$$A_1 = [32 : 40] \cup [70 : 75] \Rightarrow C = \text{low-priced}$$

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The notion of admissibility has to capture an intuitive notion of generalization . . .

And now . . .

At a glance

Rule learning is formalized by two main functions

- ϕ: selects possible subsets of data
- f: generalizes examples as a rule (LearnOneRule process [Mit82])

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$$\xrightarrow{\phi} S \xrightarrow{f} \pi$$

$$\xrightarrow{\phi} S' \xrightarrow{f} \pi'$$

$$\xrightarrow{\phi} S'' \xrightarrow{f} \pi''$$

$$\ldots \qquad \ldots$$

$$\xrightarrow{\phi} S''' \xrightarrow{f} \pi'''$$

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$\dots \qquad \dots$

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ϕ $S \xrightarrow{f} \pi$

ϕ $S' \xrightarrow{f} \pi'$

ϕ $S'' \xrightarrow{f} \pi''$

$\dots \qquad \dots$

ϕ $S''' \xrightarrow{f} \pi'''$

The attribute-value model

	A_1	A_2	A_3	A_4	A_5	A_6	A_7	$\cdots$	A_n
item 1	$a_{1,1}$	$a_{1,2}$	$a_{1,3}$	$a_{1,4}$	$a_{1,5}$	$a_{1,6}$	$a_{1,7}$	$\cdots$	$a_{1,n}$
item 2	$a_{2,1}$	$a_{2,2}$	$a_{2,3}$	$a_{2,4}$	$a_{2,5}$	$a_{2,6}$	$a_{2,7}$	$\cdots$	$a_{2,n}$
item 3	$a_{3,1}$	$a_{3,2}$	$a_{3,3}$	$a_{3,4}$	$a_{3,5}$	$a_{3,6}$	$a_{3,7}$	$\cdots$	$a_{3,n}$
item 4	$a_{4,1}$	$a_{4,2}$	$a_{4,3}$	$a_{4,4}$	$a_{4,5}$	$a_{4,6}$	$a_{4,7}$	$\cdots$	$a_{4,n}$
item 5	$a_{5,1}$	$a_{5,2}$	$a_{5,3}$	$a_{5,4}$	$a_{5,5}$	$a_{5,6}$	$a_{5,7}$	$\cdots$	$a_{5,n}$
item 6	$a_{6,1}$	$a_{6,2}$	$a_{6,3}$	$a_{6,4}$	$a_{6,5}$	$a_{6,6}$	$a_{6,7}$	$\cdots$	$a_{6,n}$
item 7	$a_{7,1}$	$a_{7,2}$	$a_{7,3}$	$a_{7,4}$	$a_{7,5}$	$a_{7,6}$	$a_{7,7}$	$\cdots$	$a_{7,n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
item m	$a_{m,1}$	$a_{m,2}$	$a_{m,3}$	$a_{m,4}$	$a_{m,5}$	$a_{m,6}$	$a_{m,7}$	$\cdots$	$a_{m,n}$

$$\left.\begin{array}{ll}\text{Rows:} & \text{items } x_1, x_2, \ldots, x_m \\ \text{Columns:} & \text{attributes } A_1, A_2, \ldots, A_n\end{array}\right] \forall i,j \;\; a_{j,i} \in \mathrm{Rng}\, A_i$$

$\mathrm{Rng}\, A_i$ *denotes the set of possible values for attribute* A_i

Subsets of data to generalize

	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9
item 1	$a_{1,1}$	$a_{1,2}$	$a_{1,3}$	$a_{1,4}$	$a_{1,5}$	$a_{1,6}$	$a_{1,7}$	$a_{1,8}$	$a_{1,9}$
item 2	$a_{2,1}$	$a_{2,2}$	$a_{2,3}$	$a_{2,4}$	$a_{2,5}$	$a_{2,6}$	$a_{2,7}$	$a_{2,8}$	$a_{2,9}$
item 3	$a_{3,1}$	$a_{3,2}$	$a_{3,3}$	$a_{3,4}$	$a_{3,5}$	$a_{3,6}$	$a_{3,7}$	$a_{3,8}$	$a_{3,9}$
item 4	$a_{4,1}$	$a_{4,2}$	$a_{4,3}$	$a_{4,4}$	$a_{4,5}$	$a_{4,6}$	$a_{4,7}$	$a_{4,8}$	$a_{4,9}$
item 5	$a_{5,1}$	$a_{5,2}$	$a_{5,3}$	$a_{5,4}$	$a_{5,5}$	$a_{5,6}$	$a_{5,7}$	$a_{5,8}$	$a_{5,9}$
item 6	$a_{6,1}$	$a_{6,2}$	$a_{6,3}$	$a_{6,4}$	$a_{6,5}$	$a_{6,6}$	$a_{6,7}$	$a_{6,8}$	$a_{6,9}$
item 7	$a_{7,1}$	$a_{7,2}$	$a_{7,3}$	$a_{7,4}$	$a_{7,5}$	$a_{7,6}$	$a_{7,7}$	$a_{7,8}$	$a_{7,9}$
item 8	$a_{8,1}$	$a_{8,2}$	$a_{8,3}$	$a_{8,4}$	$a_{8,5}$	$a_{8,6}$	$a_{8,7}$	$a_{8,8}$	$a_{8,9}$
item 9	$a_{9,1}$	$a_{9,2}$	$a_{9,3}$	$a_{9,4}$	$a_{9,5}$	$a_{9,6}$	$a_{9,7}$	$a_{9,8}$	$a_{9,9}$

$\Rightarrow$ "Square" = selection of rows and columns in the data

Rules

A rule π expresses constraints (for a generic item x) which lead to conclusion $C(x)$ (class which the item belongs to)

$$\pi : A_1(x) \in v_1^\pi \wedge \cdots \wedge A_n(x) \in v_n^\pi \to C(x) \in v_0^\pi \qquad (*)$$

where $\begin{cases} v_i^\pi \subseteq \mathrm{Rng}\, A_i \text{ pour } i = 1, \ldots, n, \\ v_0^\pi \subseteq \mathrm{Rng}\, C. \end{cases}$

attributes $i = 1, \ldots, n$ without constraints are such that $v_i^\pi = \mathrm{Rng}\, A_i$.

And now . . .

Eliciting a rule

- S being a square is supposed to capture a rule π requires that every item of S satisfies π
  - $\rightarrow$ generalisation does not capture the statistical representativeness of dataset, but only elicits a rule generalizing all its items

(C) Price	(A_1) Area	(A_2) #Rooms	(A_3) Energy	(A_4) Town	(A_5) District	(A_6) Exposure
low-priced	70	2	D	Toulouse	Minimes	
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expensive	100	5	C	Rennes	Downtown	
low-priced	40	2	D	Betton		S

$\downarrow f$

$$A_0 = 2 \wedge A_4 = \text{Toulouse} \Rightarrow C = \text{low-priced}$$

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$\downarrow f$

$$A_0 \in [2,4] \Rightarrow C \in \{\text{low-priced}, \text{expensive}\}$$

Eliciting a rule (f function)

	(A_0) *Price*	(A_1) *Area*	(A_2) *Rooms*
1	low-priced	70	2
2	low-priced	75	4
4	low-priced	32	2
7	low-priced	40	2

$S_0 = \{\text{low-priced}\}$

$S_2 = \{2, 4\}$

$S_1 = \{32, 40, 70, 75\}$

- For every attribute A_i, S_i is the set of values of A_i in items of S

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$\widehat{S_0}$ $\quad$ $\widehat{S_1}$ $\quad$ $\widehat{S_2}$

- For every attribute A_i, S_i is the set of values of A_i in items of S
- Each superset of S_i is, theoretically speaking, a generalization of S_i

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$S_0 = \{\text{low-priced}\}$ $\qquad S_2 = \{2, 4\}$

$S_1 = \{32, 40, 70, 75\}$

$S_0 \xrightarrow{f} \widehat{S}_0 \qquad S_1 \rightarrow \widehat{S}_1 \qquad S_2 \xrightarrow{f} \widehat{S}_2$

- For every attribute A_i, S_i is the set of values of A_i in items of S
- Each superset of S_i is, theoretically speaking, a generalization of S_i
- The generalisation process thus consists in selecting one of these supersets:
  - f choice function that is given as input a collection of supersets of S_i and picks one

We are looking for an appropriate $\widehat{\cdot}$ for (§) i.e.

$$A_1(x) \in \widehat{S}_1 \wedge \cdots \wedge A_n(x) \in \widehat{S}_n \rightarrow C(x) \in \widehat{S}_0 \qquad (§)$$

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$$\begin{tikzcd}[column sep=small]
\cdot \arrow[d] & \cdot \arrow[dd] & \cdot \arrow[d] \\
S_0 = \{\text{low-priced}\} \arrow[dd, "f"'] & & S_2 = \{2,4\} \arrow[dd, "f"] \\
& S_1 = \{32, 40, 70, 75\} \arrow[d] & \\
\widehat{S}_0 & \widehat{S}_1 & \widehat{S}_2
\end{tikzcd}$$

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Generalization of S_i : $\widehat{S}_i = f(\{Y \mid S_i \subseteq Y \subseteq \mathrm{Rng}\, A_i\})$

Notion of admissibility: propositions

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What collection $\mathcal{X} = \{\widehat{S}_i \mid S_i \subseteq \mathrm{Rng}\, A_i\}$ would do?

What choice function(s) can in practice capture these expected algebraic properties?

Notion of admissibility: propositions

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What collection $\mathcal{X} = \{\widehat{S_i} \mid S_i \subseteq \operatorname{Rng} A_i\}$ would do?

(i) $\operatorname{Rng} A_i \in \mathcal{X}$

(ii) if X and Y are in $\mathcal{X}$ then so $X \cap Y$.

- $\mathcal{X}$ is a closure system upon $\operatorname{Rng} A_i$.
- $\widehat{\cdot}$ is an operation enjoying weaker properties than closure operators; alternatives looked at:
  - pre-closure operator
  - capping operator

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What choice function(s) can in practice capture these expected algebraic properties?

- Proposal for some classes of choice functions generating specific types of operators
- Concrete examples of such functions for numerical rules

Weakening closure operators

- List of Kuratowski's axioms [Kur14] (closure system):

$$\begin{aligned} &\widehat{\emptyset} = \emptyset \\ &S \subseteq \widehat{S} \subseteq \operatorname{Rng} A_i \\ &\widehat{\widehat{S}} = \widehat{S} \\ &\widehat{S \cup S'} = \widehat{S} \cup \widehat{S'} \qquad \text{(pre-closure)} \end{aligned}$$

- Actually, we downgrade Kuratowski's axioms as follows

$$\begin{aligned} &\widehat{S} \subseteq \widehat{S'} \text{ whenever } S \subseteq S' && \text{(closure)} \\ &\widehat{S} = \widehat{S'} \text{ whenever } S \subseteq S' \subseteq \widehat{S} && \text{(cumulation)} \\ &\widehat{S \cup S'} \subseteq \widehat{S} \text{ whenever } S' \subseteq \widehat{S} && \text{(capping)} \end{aligned}$$

Lemma: Kuratowksi $\Rightarrow$ closure $\Rightarrow$ cumulation $\Rightarrow$ capping

Class of choice functions satisfying pre-closure

Theorem. *Given a set Z, let $f : 2^{2^Z} \to 2^Z$ be a function st for every upward closed $\mathcal{X} \subseteq 2^Z$ and every $\mathcal{Y} \subseteq 2^Z$:*

1. $f(2^Z) = \emptyset$
2. $f(\mathcal{X}) \in \mathcal{X}$
3. $f(\mathcal{X} \cap \mathcal{Y}) = f(\mathcal{X}) \cup f(\mathcal{Y})$

   whenever $\bigcup \min(\mathcal{X} \cap \mathcal{Y}) = \bigcup \min \mathcal{X} \cup \bigcup \min \mathcal{Y}$

Then, $\widehat{\cdot} : 2^Z \to 2^Z$ *as defined by*

$$\widehat{X} \stackrel{\mathrm{def}}{=} f(\{ Y \mid X \subseteq Y \subseteq Z \})$$

is a pre-closure operator upon Z.

Intuition: Z is $\mathrm{Rng}\, A_i$

$\mathcal{X}$ (and $\mathcal{Y}$, too) is a collection of intervals over $\mathrm{Rng}\, A_i$

moreover, $\mathcal{X}$ is a collection containing all super-intervals of an interval belonging to the collection

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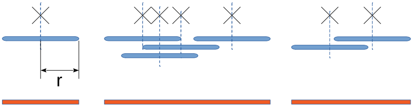
Numerical attributes: principle of single point (u) interpolation

$$A_i(x) \in [u - r : u + r] \to C(x) = c.$$



Class of choice functions satisfying capping

Theorem. *Given a set Z, let $f : 2^{2^Z} \to 2^Z$ be a function st for every $\mathcal{X} \subseteq 2^Z$ such that $\bigcap \mathcal{X} \in \mathcal{X}$ and for every $\mathcal{Y} \subseteq 2^Z$*

1. $f(\mathcal{X}) \in \mathcal{X}$
2. *if* $\mathcal{Y} \subseteq \mathcal{X}$ *and* $\exists W \in \mathcal{Y}$, $W \subseteq f(\mathcal{X})$ *then* $f(\mathcal{Y}) \subseteq f(\mathcal{X})$

Then, $\widehat{\cdot} : 2^Z \to 2^Z$ *as defined by*

$$\widehat{X} \stackrel{\text{def}}{=} f(\{Y \mid X \subseteq Y \subseteq Z\})$$

is a capping operator upon Z.

Intuition: Z is $\mathrm{Rng}\, A_i$
$\mathcal{X}$ (and $\mathcal{Y}$, too) is a collection of intervals over $\mathrm{Rng}\, A_i$
moreover, $\mathcal{X}$ is a collection whose intersection
is itself a member of the collection

Class of choice functions satisfying capping

Theorem. *Given a set Z, let $f : 2^{2^Z} \to 2^Z$ be a function st for every $\mathcal{X} \subseteq 2^Z$ such that $\bigcap \mathcal{X} \in \mathcal{X}$ and for every $\mathcal{Y} \subseteq 2^Z$*

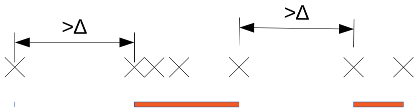
1. $f(\mathcal{X}) \in \mathcal{X}$
2. *if* $\mathcal{Y} \subseteq \mathcal{X}$ *and* $\exists W \in \mathcal{Y}$, $W \subseteq f(\mathcal{X})$ *then* $f(\mathcal{Y}) \subseteq f(\mathcal{X})$

Then, $\widehat{\cdot} : 2^Z \to 2^Z$ *as defined by*

$$\widehat{X} \stackrel{\text{def}}{=} f(\{Y \mid X \subseteq Y \subseteq Z\})$$

is a capping operator upon Z.

Numerical attributes: principle of pairwise point interpolation



And now . . .

Illustrations of the behaviour of CN2

Generation of synthetic data:

- Data with 2 dimensions: a numerical attribute and a symbolic class attribute
- Data with two classes (green and blue)

Objective:

- Illustrate the behaviour of the rule learning algorithm in terms of characteristics of generalisation of examples
  - ▷ based on a pre-closure (*interpolation over single points*)
  - ▷ based on a capping (*interpolation over pairs of points*)

Using Algorithm CN2 [CN89]

Separating interesting intervals

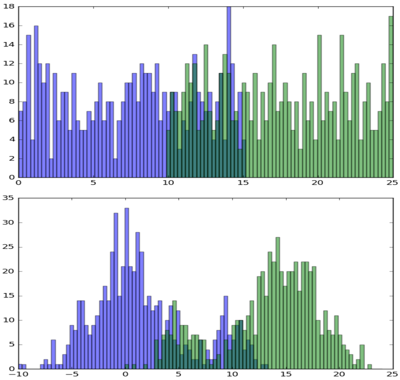


Figure: Distributions of the data for the classes `blue` and `green`.

Comparison of rules obtained out of uniform distributions vs. normal distributions

- distance between two successive values are small wrt the range of the attribute
- mixing normal distributions causes disparate average distances (pairwise distance between examples)
- the second dataset can be viewed as a super set of the first dataset (add of examples in between examples)

Separating interesting intervals

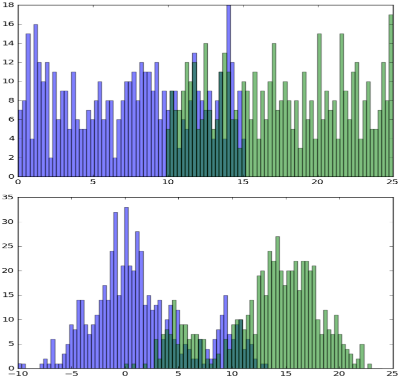


Figure: Distributions of the data for the classes `blue` and `green`.

Expected rules assuming capping or pre-closure for each class:

- topmost dataset:
  - $v \in [-\infty : 15] \Rightarrow A_0 = \texttt{blue}$
  - $v \in [10 : +\infty] \Rightarrow A_0 = \texttt{green}$
- bottom dataset:
  - $v \in [-\infty : 15] \Rightarrow A_0 = \texttt{blue}$
  - $v \in [0 : +\infty] \Rightarrow A_0 = \texttt{green}$

Separating interesting intervals

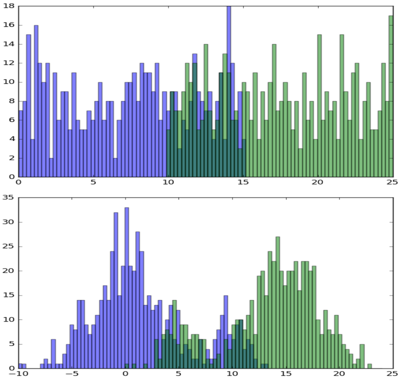


Figure: Distributions of the data for the classes `blue` and `green`.

Rules learned by CN2, topmost dataset:

- $v \in [-\infty : 10.03] \Rightarrow A_0 = \texttt{blue}$
- $v \in [12.73 : 14.83] \Rightarrow A_0 = \texttt{blue}$
- $v \in [10.65 : 12.81] \Rightarrow A_0 = \texttt{green}$
- $v \in [15.01 : +\infty] \Rightarrow A_0 = \texttt{green}$

Rules learned by CN2, bottom dataset:

- $v \in [-\infty : 0.96] \Rightarrow A_0 = \texttt{blue}$
- $v \in [0.97 : 2.57] \Rightarrow A_0 = \texttt{blue}$
- $v \in [3.09 : 10.04] \Rightarrow A_0 = \texttt{blue}$
- $v \in [3.50 : 7.18] \Rightarrow A_0 = \texttt{green}$
- $v \in [11.55 : 13.14] \Rightarrow A_0 = \texttt{green}$
- $v \in [13.15 : +\infty] \Rightarrow A_0 = \texttt{green}$

Does density influence the choice of boundaries?

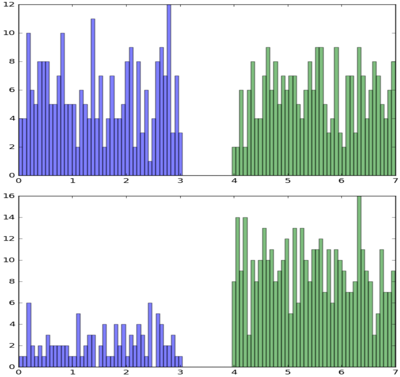


Figure: Distributions of the data for the classes `blue` and `green`.

Comparison of rules obtained out of well-separated uniform distributions, for two similar situations

- topmost dataset: same number of examples in both classes
- bottom dataset: the `blue` class is under-represented as compared to the `green` class

Does density influence the choice of boundaries?

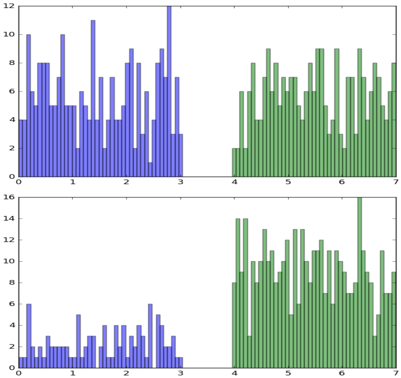


Figure: Distributions of the data for the classes `blue` and `green`.

Observed behaviour:

- no difference between the extracted rules for either dataset
- CN2 systematically chooses the boundary to be the middle of the limits in between the two classes

Does density influence the choice of boundaries?

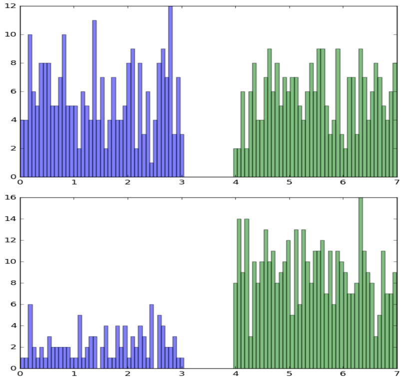
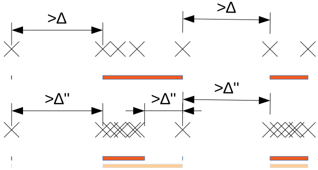


Figure: Distributions of the data for the classes `blue` and `green`.

Behaviour from *capping* :

- adding extra examples can alter boundaries



⇒ to be insensitive to density of examples corresponds to a *cumulation* operator

And now . . .

Conclusion (1)

- The logical structure of rules makes them easy to read but ...
- The interpretability of rules learned from examples requires, in particular, to take care of the way examples are generalized
  - Example of numerical attributes, but also symbolic attributes with structures (e.g. orders)
- Qualifying the interpretable nature of rule learning outputs is challenging

What can our approach do for rule interpretability

- Our work contributes by giving a way to do such analysis
  - A proposal of a general framework for rule learning
  - A topological study of *admissible generalisations* of examples

Conclusion (2)

Formalisation of rule learning

- ϕ: selects possible subsets of data
- $\widehat{\cdot}$: elicits the rule

$\rightarrow$ offer a framework for analysing rule learning algorithms

(C) *Price*	(A_1) *Area*	(A_2) *#Rooms*	(A_3) *Energy*	(A_4) *Town*	(A_5) *District*	(A_6) *Exposure*
low-priced	70	2	D	Toulouse	Minimes	
low-priced	75	4	D	Toulouse	Rangueil	
expensive	65	3		Toulouse	Downtown	
low-priced	32	2	D	Toulouse		SE
mid-priced	65	2	D	Rennes		SW
expensive	100	5	C	Rennes	Downtown	
low-priced	40	2	D	Betton		S

$\xrightarrow{\phi} S \xrightarrow{f} \pi$

$\xrightarrow{\phi} S' \xrightarrow{f} \pi'$

$\xrightarrow{\phi} S'' \xrightarrow{f} \pi''$

$\dots \qquad \dots$

$\xrightarrow{\phi} S''' \xrightarrow{f} \pi'''$

Conclusion (3)

Admissible generalisation of examples

- Admissible generalisations resulting of a choice among the supersets of the examples
- Proposed topological property of the choice: closure-like operators (pre-closure, capping)
- Definition of classes of choice functions
  - Proposal of concrete choice functions upon numerical attributes
  - Can be generalized to symbolic attributes, including attributes with structure (e.g. total order)

Perspectives

- Long term objective: study the characteristics of extracted rulesets
    - Comparing the *set* of rules extracted by machine learning
- Need a formalism to represent a set of rules
    - A formalism that enables to represent
        - → rules actually extracted by machine learning algorithms (e.g., Ripper, CN2, etc)
        - → rules selected using a selection criteria (interestingness measures, etc)
    - Formalize essential notions of rule learning
    - The formalism will be a way to reason about the machine learning algorithms

Bibliography I

Fernando Benites and Elena Sapozhnikova, *Hierarchical interestingness measures for association rules with generalization on both antecedent and consequent sides*, Pattern Recognition Letters **65** (2015), 197–203.

Peter Clark and Tim Niblett, *The CN2 induction algorithm*, Machine Learning **3** (1989), no. 4, 261–283.

Alberto Cano, Amelia Zafra, and Sebastián Ventura, *An interpretable classification rule mining algorithm*, Information Sciences **240** (2013), 1–20.

Tomás Kliegr, Stepán Bahník, and Johannes Fürnkranz, *A review of possible effects of cognitive biases on interpretation of rule-based machine learning models*, CoRR **abs/1804.02969** (2018).

Kazimierz Kuratowski, *Topology*, vol. 1, Elsevier, 2014.

Tom M Mitchell, *Generalization as search*, Artificial Intelligence **18** (1982), 203–226.