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VORTICITY AND STREAM FUNCTION FORMULATIONS FOR THE 2D NAVIER-STOKES EQUATIONS IN A BOUNDED DOMAIN

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Abstract. The main purpose of this work is to provide a Hilbertian functional framework for the analysis of the planar Navier-Stokes (NS) equations either in vorticity or in stream function formulation. The fluid is assumed to occupy a bounded possibly multiply connected domain. The velocity field satisfies either homogeneous (no-slip boundary conditions) or prescribed Dirichlet boundary conditions. We prove that the analysis of the 2D Navier-Stokes equations can be carried out in terms of the so-called nonprimitive variables only (vorticity field and stream function) without resorting to the classical NS theory (stated in primitive variables, i.e. velocity and pressure fields). Both approaches (in primitive and nonprimitive variables) are shown to be equivalent for weak (Leray) and strong (Kato) solutions. Explicit, Bernoulli-like formulas are derived and allow recovering the pressure field from the vorticity fields or the stream function. In the last section, the functional framework described earlier leads to a simplified rephrasing of the vorticity dynamics, as introduced by Maekawa in [52]. At this level of regularity, the vorticity equation splits into a coupling between a parabolic and an elliptic equation corresponding respectively to the non-harmonic and harmonic parts of the vorticity equation. By exploiting this structure it is possible to prove new existence and uniqueness results, as well as the exponential decay of the palinstrophy (that is, loosely speaking, the H^1 norm of the vorticity) for large time, an estimate which was not known so far.

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1. Introduction

The NS equations stated in primitive variables (velocity and pressure) have been received much attention since the pioneering work of Leray [44, 46, 45]. Strong solutions were shown to exist in 2D by Lions and Prodi [50] and Lions [47]. Henceforth, we will refer for instance to the books of Lions [48, Chap. 1, Section 6], Ladyzhenskaya [42] and Temam [63] for the main results that we shall need on this topic.

In 2D, the vorticity equation provides an attractive alternative model to the classical NS equations for describing the dynamics of a viscous, incompressible fluid. Thus it exhibits many advantages: It is a nice advection-diffusion scalar equation while the classical NS system, although parabolic as well, is a coupling between an unknown vector field (the velocity) and an unknown scalar field (the pressure). However the lack of natural and simple boundary conditions for the vorticity field makes the analysis of the vortex dynamics troublesome and explains why the problem has been addressed mainly so far in the case where the fluid occupies the whole space. In this configuration, a proof of existence and uniqueness for the corresponding Cauchy problem assuming the initial data to be integrable and twice continuously differentiable was first provided by McGrath [54]. Existence results were extended independently by Cottet [12] and Giga *et al.* [25] to the case where the initial data is a finite measure. These authors proved that uniqueness also holds when the atomic part of the initial vorticity is sufficiently small; see also [34]. For initial data in $L^1(\mathbb{R}^2)$, the Cauchy problem was proved to be well posed by Ben-Artzi [5], and Brézis [9]. Then, Gallay and Wayne [22] and Gallagher *et al.* [17] proved the uniqueness of the solution for an initial vorticity that is a large Dirac mass. Finally, Gallagher and Gallay [18] succeeded in removing the smallness assumption on the atomic part of the initial measure and shown that the Cauchy problem is globally well-posed for any initial data in $\mathcal{M}(\mathbb{R}^2)$.

As explained in [26, Chap. 11, §2.7], the vorticity equation (still set in the whole space) provides an interesting line of attack to study the large time behavior of the NS equations. This idea was exploited for instance by Giga and Kambe [24], Carpio [10], Gallay *et al.* [21, 20, 22, 19] and Kukavica and Reis [40].

Among the quoted authors above, some of them, such as McGrath [54] and Ben-Artzi [5] were actually interested in studying the convergence of solutions to the NS equations towards solutions of the Euler equations when the viscosity vanishes. This is a very challenging problem, well understood in the absence of solid walls (that is, when the fluid fills the whole space) and for which the vorticity equation plays a role of paramount importance. In the introduction of the chapter "Boundary Layer Theory" in the book [59], Lighthill argues that *To explain convincingly the existence of boundary layers, and, also to show what consequences of flow separation (including matters of such practical importance as the effect of trailing vortex wakes) may be expected, arguments concerning vorticity are needed.* More recently, Chemin in [11] claims *The key quantity for understanding 2D incompressible fluids is the vorticity.* There exists a burgeoning literature treating the problem of vanishing viscosity limit and we refer to the recently-released book [26, Chap. 15] for a comprehensive list of references. When the fluid is partially of totally confined, the analysis of the vanishing viscosity limits turns into a more involved problem due to the formation of a boundary layer. In this case, the vorticity equation still plays a crucial role: In [33], Kato gives a necessary and sufficient condition for the vanishing viscosity limit to hold and this condition is shown by Kelliher [35, 36, 38] to be equivalent to the formation of a vortex sheet on the boundary of the fluid domain.

In the presence of walls, the derivation of suitable boundary conditions for the vorticity was also of prime importance for the design of numerical schemes. A review of these conditions (and more generally on stream-vorticity based numeral schemes), can be found in [23], [29], [14] and [56]. However, it has been actually well known since the work of Quatarpelle and co-workers [57, 31, 32, 16, 3, 8], that the vorticity does not satisfy pointwise conditions on the boundary but rather a *non local* or integral condition which reads:

$$\text{for all } h\in\mathfrak{H},\qquad \int_{\mathcal F}\omega h\,\mathrm{d}x=0, \tag{1.1}$$

where $\mathcal F$ is the domain of the fluid and $\mathfrak{H}$ the closed subspace of the harmonic functions in $L^2(\mathcal F)$ (see also [6, Lemma 1.2]). Anderson [1] and more recently Maekawa [52] propose nonlinear boundary conditions that will be shown to be equivalent (see Section 8) to:

$$\text{for all } h\in\mathfrak{H},\qquad \int_{\mathcal F}\big(-\nu\Delta\omega+u\cdot\nabla\omega\big)h\,\mathrm{d}x=0, \tag{1.2}$$

where $\nu>0$ is the kinematic viscosity and u the velocity field deduced from ω via the Biot-Savart law. Providing that ω is a solution to the classical vorticity equation, Equality (1.2) is nothing but the time derivative of (1.1).

Starting from (1.1), the aim of this paper is to provide a Hilbertian functional framework allowing the analysis of the 2D vorticity equation in a bounded multiply connected domain. The analysis is wished to be self-contained, without recourse to classical results on the NS equations in primitive variables. We shall prove that the analysis can equivalently be carried out at the level of the stream function. Homogeneous and nonhomogeneous boundary conditions for the velocity field will be considered and explicit formulas for the pressure will be derived. In the last section, new estimates (in particular for the palinstrophy) will be established.

2. General settings

The planar domain $\mathcal F$ occupied by the fluid is assumed to be open, bounded and path-connected. We assume furthermore that its boundary Σ can be decomposed into a disjoint union of $\mathcal C^{1,1}$ Jordan curves:

$$\Sigma=\Big(\bigcup_{k=1}^N\Sigma_k^-\Big)\cup\Sigma^+. \tag{2.1}$$

The curves Σ_k^- for $k\in\{1,\dots,N\}$ are the inner boundaries of $\mathcal F$ while Σ^+ is the outer boundary. On Σ we denote by n the unit normal vector directed toward the exterior of the fluid and by τ the unit tangent vector oriented in such a way that $\tau^\perp=n$ (see Fig. 1). Here and subsequently in the paper, for every $x=(x_1,x_2)\in\mathbb R^2$, the notation $x^\perp$ is used to represent the vector $(-x_2,x_1)$.

Let now T be a positive real number and define the space-time cylinder $\mathcal F_T=(0,T)\times\mathcal F$, whose lateral boundary is $\Sigma_T=(0,T)\times\Sigma$. The velocity of the fluid is supposed to be prescribed, equal on Σ_T to some vector field b satisfying the compatibility condition:

$$\int_\Sigma b\cdot n\,\mathrm{d}s=0\qquad\text{on } (0,T). \tag{2.2}$$

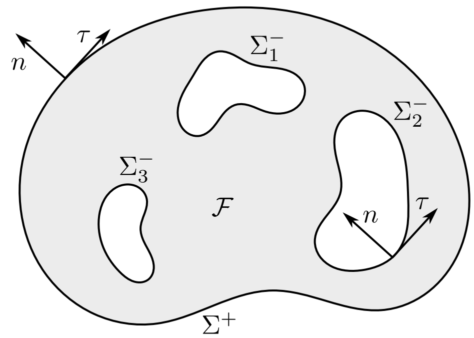

FIGURE 1. The domain of the fluid: an open, $\mathcal{C}^{1,1}$ and $N-$connected open set.

The density and the dynamic viscosity of the fluid, denoted respectively by ϱ and μ, are both positive constants. The flow is governed by the Navier-Stokes equations. Introducing u the Eulerian velocity field and π the (static) pressure field, the equations read:

$$\partial_t u + \omega u^\perp - \nu\Delta u + \nabla\left(p + \frac{1}{2}|u|^2\right) = f \quad \text{in } \mathcal{F}_T \tag{2.3a}$$

$$\nabla\cdot u = 0 \quad \text{in } \mathcal{F}_T \tag{2.3b}$$

$$u = b \quad \text{on } \Sigma_T \tag{2.3c}$$

$$u(0) = u^{\mathrm{i}} \quad \text{in } \mathcal{F}. \tag{2.3d}$$

In this system $\nu = \mu/\varrho$ is the kinematic viscosity, $\frac{1}{2}\varrho|u|^2$ is the dynamic pressure, $p = \pi/\varrho$, f is a body force, u^{i} is a given initial condition and ω the vorticity field defined as the curl of u, namely:

$$\omega = \nabla^\perp\cdot u \qquad \text{in } \mathcal{F}_T. \tag{2.4}$$

2.1. **The NS system in nonprimitive variables.** The Helmholtz-Weyl decomposition of the velocity field (see [27, Theorem 3.2]) leads to the existence, at every moment t, of a potential function $\varphi(t,\cdot)$ and a stream function $\psi(t,\cdot)$ such that:

$$u(t,\cdot) = \nabla\varphi(t,\cdot) + \nabla^\perp\psi(t,\cdot) \qquad \text{in } \mathcal{F}. \tag{2.5}$$

The potential function (also referred to as Kirchhoff potential) depends only on the boundary conditions satisfied by the velocity field of the fluid. It is defined at every moment t as the solution (unique up to an additive constant) of the Neumann problem:

$$\Delta\varphi(t,\cdot) = 0 \quad \text{in } \mathcal{F} \qquad \text{and} \qquad \frac{\partial\varphi}{\partial n}(t,\cdot) = b(t,\cdot)\cdot n \quad \text{on } \Sigma. \tag{2.6}$$

The stream function ψ in (2.5) vanishes on Σ^+ and is constant on every connected component Σ_j^- $(j = 1,\dots,N)$ of the inner boundary Σ^-. Moreover, it satisfies:

$$\Delta\psi(t,\cdot) = \omega(t,\cdot) \quad \text{in } \mathcal{F} \qquad \text{and} \qquad \frac{\partial\psi}{\partial n}(t,\cdot) = -\big[b(t,\cdot) - \nabla\varphi(t,\cdot)\big]\cdot\tau \quad \text{on } \Sigma \qquad \text{for all } t>0. \tag{2.7}$$

Forming, at any moment, the scalar product in $\mathbf{L}^2(\mathcal{F})$ (the bold font notation $\mathbf{L}^2(\mathcal{F})$ stands for $L^2(\mathcal{F};\mathbb{R}^2)$) of (2.3a) with $\nabla^\perp\theta$ where θ is a test function that vanishes on Σ^+ and is constant on every Σ_j^-, we obtain (up to an integration by parts):

$$\left(\int_{\mathcal{F}} \nabla\partial_t\psi\cdot\nabla\theta\,\mathrm{d}x + \int_{\mathcal{F}}\omega u\cdot\nabla\theta\mathrm{d}x\right) - \nu\int_{\mathcal{F}}\nabla\omega\cdot\nabla\theta\,\mathrm{d}x = \int_{\mathcal{F}}\nabla\psi_f\cdot\nabla\theta\,\mathrm{d}x \qquad \text{on } (0,T). \tag{2.8}$$

In this equality, the force field $f(t,\cdot)$ has been decomposed according to the Helmholtz-Weyl theorem:

$$f(t,\cdot) = \nabla\varphi_f(t,\cdot) + \nabla^\perp\psi_f(t,\cdot) \qquad \text{for all } t>0.$$

Integrating by parts again the terms in (2.8), we end up with the system:

$$\partial_t\omega + u\cdot\nabla\omega - \nu\Delta\omega = f_V \qquad \text{in } \mathcal{F}_T \tag{2.9a}$$

$$-\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\Sigma_k^-} b\cdot\tau\,\mathrm{d}s\right) + \int_{\Sigma_k^-}\omega(b\cdot n)\,\mathrm{d}s - \nu\int_{\Sigma_k^-}\frac{\partial\omega}{\partial n}\,\mathrm{d}s = \int_{\Sigma_k^-}\frac{\partial\psi_f}{\partial n}\,\mathrm{d}s \quad \text{on } (0,T),\quad k=1,\ldots,N, \tag{2.9b}$$

$$\omega(0) = \omega^{\mathrm{i}} \qquad \text{in } \mathcal{F}, \tag{2.9c}$$

where $f_V = \Delta\psi_f$ and the initial condition ω^{i} is the curl of u^{i} in (2.3d). To be closed, System (2.9) has to be supplemented with the identities (2.5), (2.6) and (2.7).

Remark 2.1. *The N equations* (2.9b) *(that will be termed "Lamb's fluxes conditions" in the sequel) cannot be derived from* (2.9a) *(this is well explained in* [32, Remark 3.2]*). They control the mean amount of vorticity produced on the inner boundaries. Such relations can be traced back to Lamb in* [43, Art. 328a] *(see also* [65] *for more recent references), where in a two-dimensional viscous flow the change of circulation along any curve is given by:*

$$\frac{\mathrm{D}\Gamma}{\mathrm{D}t} = \nu\oint\frac{\partial\omega}{\partial n}\,\mathrm{d}s.$$

At this point, the lack of boundary conditions for ω might indicate that System (2.9) is unlikely to be solved. Indeed, seeking for an *a priori* enstrophy estimate (enstrophy is the square of the $L^2(\mathcal{F})$ norm of the vorticity), we multiply (2.9a) by ω an integrate over $\mathcal{F}$, but shortly get stuck with the term:

$$\int_{\mathcal{F}}\Delta\omega\omega\,\mathrm{d}x, \tag{2.10}$$

that cannot be integrated by parts. The other sticking point is that the boundary value problem (2.7) permitting the reconstruction of the stream function from the vorticity is overdetermined since the stream function ψ has to satisfy both Dirichlet and Neumann boundary conditions on Σ. All these observations are well known.

2.2. **Some leading ideas.** Before going into details, we wish to give some insights on how the aforementioned difficulties can be circumvented. To simplify, we shall focus for the time being on the case of homogeneous boundary conditions (i.e. $b = 0$) and of a simply connected fluid domain (i.e. $\Sigma = \Sigma^+$). The latter assumption leads to the disappearance of the equations (2.9b) in the system.

The first elementary observation, that can be traced back to Quartapelle and Valz-Gris in [57], is that a function ω defined in $\mathcal{F}$ is the Laplacian of some function ψ if and only if the following equality holds for every harmonic function h:

$$\int_{\mathcal{F}}\omega h\,\mathrm{d}x = \int_{\Sigma}\left(\frac{\partial\psi}{\partial n}\Big|_{\Sigma} - \Lambda_{DN}\psi|_{\Sigma}\right)h|_{\Sigma}\,\mathrm{d}s,$$

where the notation Λ_{DN} stands to the Dirichlet-to-Neumann operator. Introducing $\mathfrak{H}$, the closed subspace of the harmonic functions in $L^2(\mathcal{F})$, we deduce from this assertion that:

$$\Delta H^2_0(\mathcal{F}) = \mathfrak{H}^\perp \quad \text{in} \quad L^2(\mathcal{F}). \tag{2.11}$$

We denote by V_0 the closed space $\mathfrak{H}^\perp$ and decompose the space $L^2(\mathcal{F})$ into the orthogonal sum

$$L^2(\mathcal{F}) = V_0 \overset{\perp}{\oplus} \mathfrak{H}. \tag{2.12}$$

This orthogonality condition satisfied by the vorticity plays the role of boundary conditions classically expected when dealing with a parabolic type equation like (2.9a). The authors in [57] and in [31] do not elaborate on this idea and instead of deriving an autonomous functional framework for the analysis of the vorticity equation (2.9a), System (2.9) is supplemented with the identity:

$$\omega(t,\cdot) = \Delta\psi(t,\cdot) \qquad \text{in } \mathcal{F} \qquad \text{for all } t\in(0,T),$$

and some function spaces for the stream function are introduced. However, as it will be explained later on, the dynamics of the flow can be dealt with with any one of the nonprimitive variable alone (vorticity or stream function) by introducing the appropriate functional framework.

Let us go back to the splitting (2.12). The orthogonal projection onto $\mathfrak{H}$ in $L^2(\mathcal{F})$ is usually referred to as the harmonic Bergman projection and has been received much attention so far. The Bergman projection, as well as the orthogonal projection onto V_0, denoted by P in the sequel, enjoys some useful properties (see for instance [4], [62] and references therein). In particular, P maps continuously $H^k(\mathcal{F})$ onto $H^k(\mathcal{F})$ for every nonnegative integer k, providing that Σ is of class $\mathcal{C}^{k+1,1}$. This leads us to define the spaces $V_1 = \mathsf{P}H^1_0(\mathcal{F})$, which is therefore a subspace of $H^1(\mathcal{F})$. We denote by P_1 the restriction to $H^1_0(\mathcal{F})$ of the projection P. A quite surprising result is that $\mathsf{P}_1 : H^1_0(\mathcal{F}) \to V_1$ is invertible and we denote by Q_1 its inverse. The operator Q_1 will be proved to be the orthogonal projector onto $H^1_0(\mathcal{F})$ in $H^1(\mathcal{F})$ for the semi-norm $\|\nabla \cdot\|_{\mathbf{L}^2(\mathcal{F})}$. The space V_1 is next equipped with the scalar product

$$(\omega_1, \omega_2)_{V_1} = (\nabla \mathsf{Q}_1\omega_1, \nabla \mathsf{Q}_1\omega_2)_{\mathbf{L}^2(\mathcal{F})}, \qquad \omega_1, \omega_2 \in V_1,$$

and the corresponding norm is shown to be equivalent to the usual norm of $H^1(\mathcal{F})$. Since the inclusion $H^1_0(\mathcal{F}) \subset L^2(\mathcal{F})$ is continuous, dense and compact, we can draw the same conclusion for the inclusion $V_1 \subset V_0$. Identifying V_0 with its dual space by means of Riesz Theorem and denoting by V_{-1} the dual space of V_1, we end up with a so-called Gelfand triple of Hilbert spaces (see for instance [7, Chap. 14]):

$$V_1 \subset V_0 \subset V_{-1},$$

where V_0 is the pivot space. With these settings, it is classical to introduce first the isometric operator $\mathsf{A}^V_1 : V_1 \to V_{-1}$ defined by the relation:

$$\langle \mathsf{A}^V_1 \omega_1, \omega_2\rangle_{V_{-1},V_1} = (\omega_1, \omega_2)_{V_1} \qquad \text{for all } \omega_1, \omega_2 \in V_1,$$

and next the space V_2 as the preimage of V_0 by A^V_1. The space V_2 is a Hilbert space as well, once equipped with the scalar product

$$(\omega_1, \omega_2)_{V_2} = (\mathsf{A}^V_1\omega_1, \mathsf{A}^V_1\omega_2)_{V_0} \qquad \text{for all } \omega_1, \omega_2 \in V_2,$$

and the inclusion $V_2 \subset V_1$ is continuous dense and compact. We denote by A^V_2 the restriction of A^V_1 to V_2 and classical results on Gelfand triples assert that the operator A^V_2 is an isometry from V_2 onto V_0. The crucial observation for our purpose is that, providing that Σ is of class $\mathcal{C}^{3,1}$:

$$V_2 = \left\{\omega \in H^2(\mathcal{F}) \cap V_1 \,:\, \frac{\partial \omega}{\partial n}\Big|_\Sigma = \Lambda_{DN}\omega|_\Sigma\right\} \qquad \text{and} \qquad \mathsf{A}^V_2\omega = -\Delta\omega \quad \text{for every } \omega \text{ in } V_2.$$

In particular, every vorticity in V_2 has zero mean flux through the boundary Σ. Denoting by V_{k+2} the preimage of V_k by A^V_2 for every integer $k \geqslant 1$, we define by induction a chain of embedded Hilbert spaces V_k whose dual spaces are denoted by V_{-k}. Each one of the following inclusion is continuous dense and compact:

$$\ldots \subset V_{k+1} \subset V_k \subset V_{k-1} \subset \ldots \subset V_1 \subset V_0 \subset V_{-1} \subset \ldots \subset V_{-k+1} \subset V_{-k} \subset V_{-k-1} \subset \ldots$$

We define as well isometries $\mathsf{A}^V_k : V_k \to V_{k-2}$ for all the integers k. This construction is made precise in Appendix A. It supplies a suitable functional framework to deal with the linearized vorticity equation. Thus, we shall prove in the sequel that for every $T > 0$, every integer k, every $f_V \in L^2(0, T; V_{k-1})$ and every ω^{i} in V_k there exists a unique solution

$$\omega \in H^1(0, T; V_{k-1}) \cap \mathcal{C}([0, T]; V_k) \cap L^2(0, T; V_{k+1}), \tag{2.13a}$$

to the Cauchy problem:

$$\partial_t\omega + \nu \mathsf{A}^V_{k+1}\omega = f_V \quad \text{in } \mathcal{F}_T \tag{2.13b}$$

$$\omega(0) = \omega^{\mathrm{i}} \quad \text{in } \mathcal{F}. \tag{2.13c}$$

Let us go back to the problem of enstrophy estimate where we got stuck with the term (2.10). At the level of regularity corresponding to $k = 0$ in (2.13) for instance, we obtain:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|^2_{V_0} + \nu\|\omega\|^2_{V_1} = \langle f_V, \omega\rangle_{V_1,V_{-1}} \qquad \text{on } (0, T). \tag{2.14}$$

By definition $\|\omega\|^2_{V_0} = \|\omega\|^2_{L^2(\mathcal{F})}$, which is the expected quantity but the second term in the left hand side is $\|\omega\|^2_{V_1} = \|\nabla \mathsf{Q}_1\omega\|^2_{\mathbf{L}^2(\mathcal{F})}$, whereas one would naively expect $\|\nabla\omega\|^2_{\mathbf{L}^2(\mathcal{F})}$. We recall that Q_1 is the orthogonal projection onto $H^1_0(\mathcal{F})$. So now, instead of multiplying (2.9a) by ω, let multiply this equation by $\mathsf{Q}_1\omega$, whose trace vanishes on Σ, and integrate over $\mathcal{F}$. The term (2.10) is replaced by a quantity that can now be integrated by parts. Thus:

$$\int_{\mathcal{F}} \Delta\omega \mathsf{Q}_1\omega \,\mathrm{d}x = -(\nabla\omega, \nabla\mathsf{Q}_1\omega)_{\mathbf{L}^2(\mathcal{F})} = -\|\nabla\mathsf{Q}_1\omega\|^2_{\mathbf{L}^2(\mathcal{F})} = -\|\omega\|^2_{V_1}.$$

On the other hand, regarding the first term in (2.9a), we still have (at least formally):

$$\int_{\mathcal{F}} \partial_t\omega \mathsf{Q}_1\omega \,\mathrm{d}x = \int_{\mathcal{F}} \partial_t\omega\omega \,\mathrm{d}x = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{F}} |\omega|^2 \,\mathrm{d}x,$$

because ω is orthogonal in $L^2(\mathcal{F})$ to the harmonic functions and $\mathsf{Q}_1\omega$ and ω differ only up to an harmonic function. To sum up, in the enstrophy estimate, the natural dissipative term is not $\|\nabla\omega\|^2_{\mathbf{L}^2(\mathcal{F})}$ but $\|\nabla\mathsf{Q}_1\omega\|^2_{\mathbf{L}^2(\mathcal{F})}$. Notice that, since Q_1 is the orthogonal projector onto $H^1_0(\mathcal{F})$:

$$\|\nabla\mathsf{Q}_1\omega\|^2_{\mathbf{L}^2(\mathcal{F})} \leqslant \|\nabla\omega\|^2_{\mathbf{L}^2(\mathcal{F})} \qquad \text{for all } \omega \in H^1(\mathcal{F}).$$

Defining the lowest eigenvalue of A^V_1 by means of a Rayleigh quotient:

$$\lambda_{\mathcal{F}} = \min_{\substack{\omega\in V_1\\ \omega\neq 0}} \frac{\|\omega\|^2_{V_1}}{\|\omega\|^2_{V_0}} = \min_{\substack{\omega\in V_1\\ \omega\neq 0}} \frac{\|\nabla\mathsf{Q}_1\omega\|^2_{\mathbf{L}^2(\mathcal{F})}}{\|\omega\|^2_{L^2(\mathcal{F})}}, \tag{2.15}$$

the following Poincaré-type estimate holds true:

$$\lambda_{\mathcal{F}}\|\omega\|^2_{V_0} \leqslant \|\omega\|^2_{V_1} \quad \text{for all } \omega \in V_1,$$

and classically leads with (2.14) (assuming that $f_V = 0$ to simplify) and Grönwall's inequality to the estimate:

$$\|\omega(t)\|_{V_0} \leqslant \|\omega^{\mathrm{i}}\|_{V_0} e^{-\nu\lambda_{\mathcal{F}} t}, \qquad t \geqslant 0,$$

where the constant $\lambda_{\mathcal{F}}$ is optimal. This constant governing the exponential decay of the solution is actually the same at any level of regularity. Thus, the solution to (2.13) (with $\beta = 0$) satisfies for every integer k:

$$\|\omega(t)\|_{V_k} \leqslant \|\omega^{\mathrm{i}}\|_{V_k} e^{-\nu\lambda_{\mathcal{F}} t}, \qquad t \geqslant 0.$$

Remark 2.2. *Kato's criteria for the existence of the vanishing viscosity limit in* [33] *and rephrased in terms of the vorticity by Kelliher in* [36] *will be shown to be equivalent to the convergence of ω^ν toward ω in the space V_{-1} (ω^ν stands for the vorticity of NS equations with vorticity ν and ω is the vorticity of Euler equations). Some care should be taken with the space V_{-1} because it is not a distribution space, what may result in some mistakes or misunderstandings (we refer here to the very instructive paper of Simon* [61]*).*

As mentioned earlier, the analysis of the dynamics of the flow can as well be carried out in terms of the sole stream function. It suffices to introduce the function spaces $S_0 = H^1_0(\mathcal{F})$ and $S_1 = H^2_0(\mathcal{F})$. The inclusion $S_1 \subset S_0$ being continuous dense and compact, the configuration $S_1 \subset S_0 \subset S_{-1}$ (with S_{-1} the dual space of S_1) is a Gelfand triple where S_0 is the pivot space. We proceed as for the vorticity spaces and define a chain of embedded Hilbert spaces S_k and related isometries $\mathsf{A}^S_k : S_k \to S_{k-2}$ for every integer k (we refer again to Appendix A for the details). In particular, providing that Σ is of class $\mathcal{C}^{2,1}$, we will verify that:

$$S_2 = H^3(\mathcal{F}) \cap H^2_0(\mathcal{F}) \qquad \text{and} \qquad \mathsf{A}^S_2\psi = -\mathsf{Q}_1\Delta\psi \qquad \text{for all } \psi \in S_2.$$

The counterpart of the Cauchy problem (2.13), restated in terms of the stream function is:

$$\partial_t\psi + \nu\mathsf{A}^S_{k+1}\psi = f_S \quad \text{in } \mathcal{F}_T \tag{2.16a}$$

$$\psi(0) = \psi^{\mathrm{i}} \quad \text{in } \mathcal{F}. \tag{2.16b}$$

For every $T > 0$, every integer k, every $f_S \in L^2(0,T;S_{k-1})$ and every ψ^{i} in S_k, this problem admits a unique solution:

$$\psi \in H^1(0,T;S_{k-1}) \cap \mathcal{C}([0,T];S_k) \cap L^2(0,T;S_{k+1}), \tag{2.16c}$$

which satisfies in addition the exponential decay estimate (assuming that $f_S = 0$ to simplify):

$$\|\psi(t)\|_{S_k} \leqslant \|\psi^{\mathrm{i}}\|_{S_k} e^{-\nu\lambda_{\mathcal{F}} t} \qquad \text{for all } t \geqslant 0.$$

The constant $\lambda_{\mathcal{F}}$ is defined in (2.15) and is therefore the same as the one governing the exponential decay of the enstrophy.

The solution to problem (2.13) can easily be deduced from the solution to problem (2.16) and vice versa. Indeed, for every integer k, the operator:

$$\Delta_k : \psi \in S_{k+1} \longmapsto \Delta\psi \in V_k,$$

can be shown to be an isometry. Thus, let be given $T > 0$ and consider

- $\omega \in H^1(0,T;V_{k-1}) \cap \mathcal{C}([0,T];V_k) \cap L^2(0,T;V_{k+1})$ the unique solution to Problem (2.13) for some integer k, some initial condition $\omega^{\mathrm{i}} \in V_k$ and some source term $f_V \in L^2(0,T;V_{k-1})$;
- $\psi \in H^1(0,T;S_{k'-1}) \cap \mathcal{C}([0,T];S_{k'}) \cap L^2(0,T;S_{k'+1})$ the unique solution to Problem (2.16) for some integer k', some initial condition $\psi^{\mathrm{i}} \in S_{k'}$ and some source term $f_S \in L^2(0,T;S_{k'-1})$.

Providing that $k' = k+1$, we claim that both following assertions are equivalent:

(1) $\omega = \Delta_k\psi$;
(2) $\omega^{\mathrm{i}} = \Delta_k\psi^{\mathrm{i}}$ and $f_V = \Delta_{k-1}f_S$.

If we take for granted that the operators $\mathsf{P}_k : S_{k-1} \to V_k$ and $\mathsf{Q}_k : V_k \to S_{k-1}$ can be defined at any level of regularity in such a way that P_k extend $\mathsf{P}_{k'}$ if $k \leqslant k'$ and $\mathsf{Q}_k = \mathsf{P}_k^{-1}$, we can show that the diagram in Fig. 2 commutes and all the operators are isometries.

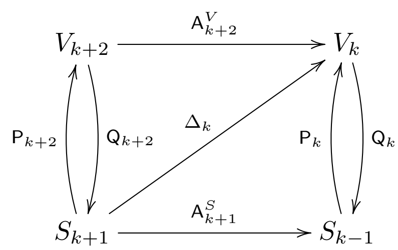

FIGURE 2. The top row contains the spaces V_k for the vorticity fields while the bottom row contains the stream function spaces S_k. The operators A^V_k and A^S_k appears in the Cauchy problems (2.13) and (2.16) respectively. The operators Δ_k link isometrically the stream functions to the corresponding vorticity fields.

To accurately state the equivalence result between Problems (2.13) (Stokes problem in vorticity variable), (2.16) (Stokes problem in stream function variable) and the evolution homogeneous Stokes equations in primitive variables, it is worth recalling the functional framework for the Stokes equations by introducing the spaces:

$$\mathbf{J}_0 = \big\{u \in \mathbf{L}^2(\mathcal{F}) \,:\, \nabla\cdot u = 0 \text{ in } \mathcal{F} \text{ and } u|_\Sigma\cdot n = 0\big\}, \tag{2.17a}$$

$$\mathbf{J}_1 = \big\{u \in \mathbf{H}^1(\mathcal{F}) \,:\, \nabla\cdot u = 0 \text{ in } \mathcal{F} \text{ and } u|_\Sigma = 0\big\}, \tag{2.17b}$$

whose scalar products are respectively:

$$(u_1,u_2)_{\mathbf{J}_0} = \int_{\mathcal{F}} u_1\cdot u_2 \,\mathrm{d}x \qquad \text{for all } u_1,u_2 \in \mathbf{J}_0, \tag{2.17c}$$

$$(u_1,u_2)_{\mathbf{J}_1} = \int_{\mathcal{F}} \nabla u_1 : \nabla u_2 \,\mathrm{d}x \qquad \text{for all } u_1,u_2 \in \mathbf{J}_1. \tag{2.17d}$$

The inclusion $\mathbf{J}_1 \subset \mathbf{J}_0$ being continuous dense and compact, from the Gelfand triple $\mathbf{J}_1 \subset \mathbf{J}_0 \subset \mathbf{J}_{-1}$ we can define a chain of embedded Hilbert spaces $\mathbf{J}_k$ and isometries $\mathsf{A}^{\mathbf{J}}_k : \mathbf{J}_k \to \mathbf{J}_{k-2}$ for every integer k. Providing that Σ is of class $\mathcal{C}^{1,1}$, it can be shown in particular that:

$$\mathbf{J}_2 = \mathbf{J}_1 \cap \mathbf{H}^2(\mathcal{F}) \qquad \text{and} \qquad \mathsf{A}^{\mathbf{J}}_2 = -\Pi_0\Delta,$$

where $\Pi_0 : \mathbf{L}^2(\mathcal{F}) \to \mathbf{J}_0$ is the Leray projector. For every $T > 0$, every integer k, every $f_{\mathbf{J}} \in L^2(0,T;\mathbf{J}_{k-1})$ and every u^{i} in $\mathbf{J}_k$, it is well known that there exists a unique solution

$$u \in H^1(0,T;\mathbf{J}_{k-1}) \cap \mathcal{C}([0,T];\mathbf{J}_k) \cap L^2(0,T;\mathbf{J}_{k+1}),$$

to the Cauchy problem:

$$\partial_t u + \nu \mathsf{A}^{\mathbf{J}}_{k+1} u = f_{\mathbf{J}} \quad \text{in } \mathcal{F}_T \tag{2.18a}$$

$$u(0) = u^{\mathrm{i}} \quad \text{in } \mathcal{F}. \tag{2.18b}$$

The operator:

$$\nabla^{\perp}_k : \psi \in S_k \longmapsto \nabla^{\perp}\psi \in \mathbf{J}_{k-1},$$

will be proved to be an isometry for every integer k. It allows us to link Problem (2.18) to the equivalent problems (2.13) and (2.16). More precisely, let be given $T > 0$ and consider

- $u \in H^1(0,T;\mathbf{J}_{k-1}) \cap \mathcal{C}([0,T];\mathbf{J}_k) \cap L^2(0,T;\mathbf{J}_{k+1})$ the unique solution to Problem (2.13) for some integer k, some initial condition $u^{\mathrm{i}} \in \mathbf{J}_k$ and some source term $f_{\mathbf{J}} \in L^2(0,T;\mathbf{J}_{k-1})$;
- $\psi \in H^1(0,T;S_{k'-1}) \cap \mathcal{C}([0,T];S_{k'}) \cap L^2(0,T;S_{k'+1})$ the unique solution to Problem (2.16) for some integer k', some initial condition $\psi^{\mathrm{i}} \in S_{k'}$ and some source term $f_S \in L^2(0,T;S_{k'-1})$.

Providing that $k' = k$, we claim that both following assertions are equivalent:

(1) $u = \nabla^{\perp}_k \psi$;
(2) $u^{\mathrm{i}} = \nabla^{\perp}_k \psi^{\mathrm{i}}$ and $f_{\mathbf{J}} = \nabla^{\perp}_{k-1} f_S$.

To conclude this short presentation of the main ideas that will be further elaborated in this paper, it is worth noticing that, contrary to what happens with primitive variables, the case where $\mathcal{F}$ is multiply connected is notably more involved than the simply connected case. The same observation could still be came across in the articles of Glowinski and Pironneau [28] and Guermond and Quartapelle [31].

2.3. **Organization of the paper.** The next section is devoted to the study of the Stokes operator in nonprimitive variables (namely the operators A^V_k and A^S_k mentioned in the preceding section). The expression of the Biot-Savart law is also provided. Then, in Section 4, lifting operators (for both the vorticity field and the stream function) are defined. They are required in Section 5 for the analysis of the evolution Stokes problem (in nonprimitive variables) with nonhomogeneous boundary conditions. The NS equations in nonprimitive variables is dealt with in Section 6 where weak and strong solutions are addressed. Explicit formulas to recover the pressure from the vorticity or the stream function are supplied in Section 7. The existence and uniqueness of more regular vorticity solutions is examined in Section 8. In this section we also prove the exponential decay of the palinstrophy (i.e. of the quantity $\|\nabla\omega\|_{\mathbf{L}^2(\mathcal{F})}$) when time growths. In Section 9 we conclude with providing some insights on upcoming generalization results for coupled fluid-structure systems.

3. Stokes operator

3.1. **Function spaces.** Let Σ_0 stands for either Σ^+ or Σ^-_j for some $j \in \{1,\dots,N\}$. Providing that Σ_0 is of class $\mathcal{C}^{k,1}$ (k being a nonnegative integer), it makes sense to consider the boundary Sobolev space $H^{k+\frac12}(\Sigma_0)$ and its dual space $H^{-k-\frac12}(\Sigma_0)$. Using $L^2(\Sigma_0)$ as pivot space, we shall use a boundary integral notation in place of the duality pairing all along this paper. More precisely, we adopt the following convention of notation:

$$\langle g_1, g_2\rangle_{H^{-k-\frac12}(\Sigma_0),H^{k+\frac12}(\Sigma_0)} = \int_{\Sigma_0} g_1 g_2 \,\mathrm{d}s \qquad \text{for all } g_1 \in H^{-k-\frac12}(\Sigma_0) \text{ and } g_2 \in H^{k+\frac12}(\Sigma_0). \tag{3.1}$$

In particular, following this rule:

$$\langle g, 1\rangle_{H^{-k-\frac12}(\Sigma_0),H^{k+\frac12}(\Sigma_0)} = \int_{\Sigma_0} g \,\mathrm{d}s \qquad \text{for all } \in H^{-k-\frac12}(\Sigma_0).$$

Fundamental function spaces. For every nonnegative integer k, we denote by $H^k(\mathcal{F})$ the classical Sobolev spaces of index k and we define the Hilbert spaces:

$$S_0 = \{\psi \in H^1(\mathcal{F}) \,:\, \psi|_{\Sigma^+} = 0 \quad \text{and} \quad \psi|_{\Sigma^-_j} = c_j, \quad c_j \in \mathbb{R}, \quad j = 1,\dots,N\}, \tag{3.2a}$$

$$S_1 = \left\{\psi \in S_0 \cap H^2(\mathcal{F}) \,:\, \frac{\partial\psi}{\partial n}\Big|_{\Sigma} = 0\right\}, \tag{3.2b}$$

provided with the scalar products:

$$(\psi_1,\psi_2)_{S_0} = (\nabla\psi_1,\nabla\psi_2)_{\mathbf{L}^2(\mathcal{F})} \qquad \text{for all } \psi_1,\psi_2\in S_0, \tag{3.2c}$$

$$(\psi_1,\psi_2)_{S_1} = (\Delta\psi_1,\Delta\psi_2)_{L^2(\mathcal{F})} \qquad \text{for all } \psi_1,\psi_2\in S_1. \tag{3.2d}$$

The norm $\|\cdot\|_{S_0}$ is equivalent in S_0 to the usual norm of $H^1(\mathcal{F})$. For every $j=1,\dots,N$, we define the continuous linear form $\mathsf{Tr}_j : \psi\in S_0\mapsto \psi|_{\Sigma_j^-}\in\mathbb{R}$ and the function ξ_j as the unique solution in S_0 to the variational problem:

$$(\xi_j,\theta)_{S_0} + \mathsf{Tr}_j\theta = 0 \qquad \text{for all } \theta\in S_0. \tag{3.3a}$$

The functions ξ_j are harmonic in $\mathcal{F}$ and obey the mean fluxes conditions:

$$\int_{\Sigma_k^-}\frac{\partial\xi_j}{\partial n}\,\mathrm{d}s = -\delta_j^k \qquad \text{for } k=1,\dots,N, \tag{3.3b}$$

where δ_j^k is the Kronecker symbol. We denote by $\mathbb{F}_S$ the N dimensional subspace of S_0 spanned by the functions ξ_j $(j=1,\dots,N)$ that will account for the fluxes of the stream functions through the inner boundaries. Notice that the Gram matrix $\big((\xi_j,\xi_k)_{S_0}\big)_{1\leqslant j,k\leqslant N}$ is invertible and equal to the matrix of the traces $\big(-\mathsf{Tr}_k\xi_j\big)_{1\leqslant j,k\leqslant N}$. Therefore, by means of a Gram-Schmidt process, we can derive from the free family $\{\xi_j,\ j=1,\dots,N\}$, an orthonormal family in S_0, denoted by $\{\hat\xi_j,\ j=1,\dots,N\}$. The space S_0 admits the following orthogonal decomposition:

$$S_0 = H^1_0(\mathcal{F})\overset{\perp}{\oplus}\mathbb{F}_S. \tag{3.4}$$

In S_1, the norm $\|\cdot\|_{S_1}$ is equivalent to the usual norm of $H^2(\mathcal{F})$. For every $j=1,\dots,N$, we denote by χ_j the unique solution in S_1 such that:

$$\Delta^2\chi_j = 0 \quad \text{in } \mathcal{F} \qquad \text{and} \qquad \int_{\Sigma_k^-}\frac{\partial\Delta\chi_j}{\partial n}\,\mathrm{d}s = -\delta_j^k \qquad \text{for } k=1,\dots,N, \tag{3.5}$$

where the normal derivative of $\Delta\chi_j$ is in $H^{-\frac{3}{2}}(\Sigma_k^-)$ (see the convention of notation (3.1)). We denote by $\mathbb{B}_S$ the N dimensional subspace of S_1 spanned by the functions χ_j. The Gram matrix $\big((\chi_j,\chi_k)_{S_1}\big)_{1\leqslant j,k\leqslant N}$ being invertible and equal to the matrix of traces $\big(\mathsf{Tr}_j\chi_k\big)_{1\leqslant j,k\leqslant N}$, we infer that:

$$S_1 = H^2_0(\mathcal{F})\overset{\perp}{\oplus}\mathbb{B}_S. \tag{3.6}$$

In $L^2(\mathcal{F})$, we denote by $\mathfrak{H}$ the closed subspace of the harmonic functions with zero mean flux through every connected part Σ_j^- of the inner boundary $(j=1,\dots,N)$, namely:

$$\mathfrak{H} = \Big\{h\in L^2(\mathcal{F})\,:\,\Delta h = 0 \text{ in } \mathcal{D}(\mathcal{F}) \text{ and } (h,\Delta\chi_j)_{L^2(\mathcal{F})}=0,\quad j=1,\dots,N\Big\}. \tag{3.7}$$

The space $L^2(\mathcal{F})$ admits the orthogonal decomposition:

$$L^2(\mathcal{F}) = \mathfrak{H}\overset{\perp}{\oplus}V_0 \qquad \text{where} \qquad V_0=\mathfrak{H}^\perp. \tag{3.8}$$

It results from the following lemma that the space V_0 is the natural function space for the vorticity field.

Lemma 3.1. *The operator $\Delta_0 : \psi\in S_1\mapsto\Delta\psi\in V_0$ is an isometry.*

Proof. Let ω be in V_0 and denote by ψ_0 the unique function in $H^1_0(\mathcal{F})\cap H^2(\mathcal{F})$ satisfying $\Delta\psi_0=\omega$. On the other hand, using the rule of notation (3.1), the function

$$h_0 = h - \sum_{j=1}^N\left(\int_\Sigma\frac{\partial\hat\xi_j}{\partial n}h\,\mathrm{d}s\right)\hat\xi_j,$$

is in the space $\mathfrak{H}^1=\mathfrak{H}\cap H^1(\mathcal{F})$ providing that h is a harmonic function in $H^1(\mathcal{F})$. It follows that:

$$(\omega,h_0)_{L^2(\mathcal{F})} = (\Delta\psi_0,h_0)_{L^2(\mathcal{F})} = \int_\Sigma\frac{\partial\psi_0}{\partial n}h_0\,\mathrm{d}s = \int_\Sigma\left[\frac{\partial\psi_0}{\partial n} - \sum_{j=1}^N\left(\int_\Sigma\frac{\partial\psi_0}{\partial n}\hat\xi_j\,\mathrm{d}s\right)\frac{\partial\hat\xi_j}{\partial n}\right]h\,\mathrm{d}s = 0.$$

Since every element in $H^{\frac{1}{2}}(\Sigma)$ can be achieved as the trace of a harmonic function in $H^1(\mathcal{F})$, the equality above entails that:

$$\frac{\partial\psi_0}{\partial n}=\sum_{j=1}^N\left(\int_\Sigma\frac{\partial\psi_0}{\partial n}\hat{\xi}_j\,\mathrm{d}s\right)\frac{\partial\hat{\xi}_j}{\partial n}\qquad\text{in } H^{-\frac{1}{2}}(\Sigma).$$

We are done by noticing now that the function:

$$\psi=\psi_0-\sum_{j=1}^N\left(\int_\Sigma\frac{\partial\psi_0}{\partial n}\hat{\xi}_j\,\mathrm{d}s\right)\hat{\xi}_j,$$

is in S_1 and solves $\Delta\psi=\omega$. Uniqueness being straightforward, the proof is then complete. □

The Bergman projection and its inverse. Considering the orthogonal splitting (3.8) of $L^2(\mathcal{F})$, we denote by P the orthogonal projection from $L^2(\mathcal{F})$ onto V_0 while the notation $\mathsf{P}^\perp$ will stand for the orthogonal projection onto $\mathfrak{H}$. When the domain $\mathcal{F}$ is simply connected, the operator $\mathsf{P}^\perp$ is referred to as the harmonic Bergman projection and has been extensively studied (see for instance [4], [62] and references therein). The projector P (and also $\mathsf{P}^\perp$ which we are less interested in) enjoys the following property:

Lemma 3.2. *Assume that Σ is of class $\mathcal{C}^{k+1,1}$ for some nonnegative integer k, then P (and $\mathsf{P}^\perp$) maps $H^k(\mathcal{F})$ into $H^k(\mathcal{F})$ and P, seen as an operator from $H^k(\mathcal{F})$ into $H^k(\mathcal{F})$, is bounded.*

Proof. Let u be in $L^2(\mathcal{F})$. The proof consists in verifying that

$$\mathsf{P}u=\Delta w_0+\sum_{k=1}^N(\Delta\chi_k,u)_{L^2(\mathcal{F})}\Delta\chi_k,$$

where the functions w_0 belongs to $H^2_0(\mathcal{F})$ and satisfies the variational formulation:

$$(\Delta w_0,\Delta\theta_0)_{L^2(\mathcal{F})}=(u,\Delta\theta_0)_{L^2(\mathcal{F})},\qquad\text{for all }\theta_0\in H^2_0(\mathcal{F}).\tag{3.9}$$

The conclusion of the Lemma will follow according to elliptic regularity results for the biharmonic operator stated for instance in [27, Theorem 1.11]. By definition:

$$\mathsf{P}u=\operatorname{argmin}\left\{\frac{1}{2}\int_\mathcal{F}|v-u|^2\,\mathrm{d}x\,:\,v\in V_0\right\}.$$

According to Lemma 3.1, there exists a unique $w\in S_1$ such that $\mathsf{P}u=\Delta w$ and:

$$w=\operatorname{argmin}\left\{\frac{1}{2}\int_\mathcal{F}|\Delta\theta-u|^2\,\mathrm{d}x\,:\,\theta\in S_1\right\}.$$

Owning to the orthogonal decomposition (3.6), the function w can be decomposed as:

$$w=w_0+\sum_{k=1}^N\alpha_k\chi_k,$$

where $w_0\in H^2_0(\mathcal{F})$ and $(\alpha_1,\ldots,\alpha_N)\in\mathbb{R}^N$ are such that:

$$(w_0,\alpha_1,\ldots,\alpha_N)=\operatorname{argmin}\left\{\frac{1}{2}\int_\mathcal{F}\left|\Delta\theta_0+\sum_{k=1}^N\beta_k\Delta\chi_k-u\right|^2\mathrm{d}x\,:\,(\theta_0,\beta_1,\ldots,\beta_N)\in H^2_0(\mathcal{F})\times\mathbb{R}^N\right\}.$$

It follows that w_0 solves indeed the variational problem (3.9) and $\alpha_k=(\Delta\chi_k,u)_{L^2(\mathcal{F})}$ for every $k=1,\ldots,N$. □

Remark 3.3. *The following observations are in order:*

1. *The harmonic Bergman projection is quite demanding in terms of boundary regularity, and one may wonder if the assumption on the regularity of Σ is optimal in the statement of Lemma 3.2. Focusing on the case $k=0$, the definition of the space $\mathfrak{H}$ requires defining the flux of harmonic functions through the connected parts of Σ^-. The normal derivative of harmonic functions in $L^2(\mathcal{F})$ can be defined as elements of $H^{-\frac{3}{2}}(\Sigma)$. However, it requires the boundary to be $\mathcal{C}^{1,1}$ (see [30, page 54]), which is the default level of regularity assumed for the domain $\mathcal{F}$ throughout this article.*

(2) *For every $u \in L^2(\mathcal{F})$, the function $\mathsf{P}^\perp u$ belongs to $\mathfrak{H}$ and therefore admits a normal trace on every Σ_j^- ($j = 1, \ldots, N$) in $H^{-\frac{3}{2}}(\Sigma_j^-)$. We deduce that, when u belongs to $H^2(\mathcal{F})$, the fluxes of u across the parts Σ_j^- ($j = 1, \ldots, N$) of the boundary are conserved by the projection P, namely:*

$$\int_{\Sigma_j^-} \frac{\partial \mathsf{P}u}{\partial n}\,\mathrm{d}s = \int_{\Sigma_j^-} \frac{\partial u}{\partial n}\,\mathrm{d}s \qquad \text{for all } j = 1, \ldots, N,$$

where $\partial \mathsf{P}u/\partial n$ belongs to $H^{-\frac{3}{2}+k}(\Sigma_j^-)$ providing that Σ_j^- is of class $\mathcal{C}^{k+1,1}$ for $k = 0, 1, 2$.

Let us define now the operator:

$$\mathsf{Q} : u \in H^1(\mathcal{F}) \longmapsto \mathsf{Q}u = \operatorname{argmin}\left\{ \int_{\mathcal{F}} |\nabla\theta - \nabla u|^2\,\mathrm{d}x \,:\, \theta \in S_0 \right\} \in S_0. \tag{3.10}$$

The variational formulation corresponding to the minimization problem reads:

$$(\nabla \mathsf{Q}u, \nabla\theta)_{\mathbf{L}^2(\mathcal{F})} = (\nabla u, \nabla \theta)_{\mathbf{L}^2(\mathcal{F})} \qquad \text{for all } \theta \in S_0, \tag{3.11}$$

what means that $\mathsf{Q}u$ is the unique function in S_0 that satisfies $\Delta \mathsf{Q}u = \Delta u$ in $\mathcal{F}$. Denoting $\mathsf{Q}^\perp = \mathrm{Id} - \mathsf{Q}$, this entails that $\mathsf{Q}^\perp u$ is harmonic and choosing χ_k ($k = 1, \ldots, N$) as test function in (3.11) and integrating by parts, we obtain:

$$(\mathsf{Q}^\perp u, \Delta \chi_k)_{L^2(\mathcal{F})} = 0, \qquad k = 1, \ldots, N,$$

whence we deduce that $\mathsf{Q}^\perp u$ lies in $\mathfrak{H}^1$. Besides, there exists real coefficients α_j such that the function:

$$u_0 = \mathsf{Q}u - \sum_{j=1}^N \alpha_j \chi_j,$$

belongs to $H^1_0(\mathcal{F})$ because the Gram matrix $\big((\chi_j, \chi_k)_{S_1}\big)_{1 \leqslant j,k \leqslant N}$ is invertible and equal to the matrix of traces $\big(\mathsf{T}_j \chi_k\big)_{1 \leqslant j,k \leqslant N}$. It follows that for every $h \in \mathfrak{H}^1$:

$$(\nabla \mathsf{Q}u, \nabla h)_{\mathbf{L}^2(\mathcal{F})} = (\nabla u_0, \nabla h)_{\mathbf{L}^2(\mathcal{F})} - \sum_{j=1}^N \alpha_j (h, \Delta \chi_j)_{L^2(\mathcal{F})} = 0. \tag{3.12}$$

So the operators P and Q are both orthogonal projections whose kernels are harmonic functions ($\mathfrak{H}$ for P and $\mathfrak{H}^1$ for Q) but for different scalar products. They are tightly related, as expressed in the next lemma, the statement of which requires introducing a new function space. Thus, we define V_1 as the image of S_0 by P and we denote by P_1 the restriction of P to S_0. It is elementary to verify that

$$\mathsf{P}_1 : S_0 \longrightarrow V_1$$

is one-to-one. We denote by Q_1 the inverse of P_1. The space V_1 is then provided with the image topology, namely with the scalar product:

$$(\omega_1, \omega_2)_{V_1} = (\mathsf{Q}_1\omega_1, \mathsf{Q}_1\omega_2)_{S_0} = (\nabla \mathsf{Q}_1\omega_1, \nabla \mathsf{Q}_1 \omega_2)_{\mathbf{L}^2(\mathcal{F})} \qquad \text{for all } \omega_1, \omega_2 \in V_1.$$

Observe that since $H^1(\mathcal{F}) = S_0 \oplus \mathfrak{H}^1$, we have also $V_1 = \mathsf{P}H^1(\mathcal{F})$.

Lemma 3.4. *If Σ is of class $\mathcal{C}^{2,1}$, V_1 is a subspace of $H^1(\mathcal{F})$, Q_1 is the restriction of Q to V_1 and the topology of V_1 is equivalent to the topology of $H^1(\mathcal{F})$.*

Proof. According to Lemma 3.2, if Σ is of class $\mathcal{C}^{2,1}$, the space V_1 is a subspace of $H^1(\mathcal{F})$ and P is bounded from S_0 onto V_1 (seen as subspace of $H^1(\mathcal{F})$). The operator $\mathsf{P}_1 : S_0 \to V_1$ being bounded and invertible, it is an isomorphism according to the bounded inverse theorem. Moreover, for every ψ be in S_0:

$$\mathsf{Q}\mathsf{P}_1\psi = \mathsf{Q}(\psi + (\mathsf{P}_1\psi - \psi)) = \mathsf{Q}\psi + \mathsf{Q}(\mathsf{P}_1\psi - \psi) = \mathsf{Q}\psi = \psi,$$

since the function $\mathsf{P}_1\psi - \psi$ is in $\mathfrak{H}^1$. The proof is now complete. □

Remark 3.5. *When Σ is only of class $\mathcal{C}^{1,1}$, V_1 is a subspace of $S_0 + \mathfrak{H}$. In particular, every element of V_1 has a trace on Σ in the space $H^{-\frac{1}{2}}(\Sigma)$. This trace is in $H^{\frac{1}{2}}(\Sigma)$ when Σ is of class $\mathcal{C}^{2,1}$.*

Further scalar products. For every nonnegative integer k, we define $\mathfrak{H}^k = \mathfrak{H} \cap H^k(\mathcal{F})$. Assuming that Σ is of class $\mathcal{C}^{k-1,1}$, the space $\mathfrak{H}^k$ is provided with the scalar product:

$$(h_1, h_2)_{\mathfrak{H}^k} = (h_1|_\Sigma, h_2|_\Sigma)_{H^{k-\frac{1}{2}}(\Sigma)} \qquad \text{for all } h_1, h_2 \in \mathfrak{H}^k.$$

It would sometimes come in handy to provide $H^1(\mathcal{F})$ with a scalar product that turns Q into an orthogonal projection. To do that, it suffices to define:

$$(u_1, u_2)^S_{H^1} = (\mathsf{Q}u_1, \mathsf{Q}u_2)_{S_0} + (\mathsf{Q}^\perp u_1, \mathsf{Q}^\perp u_2)_{\mathfrak{H}^1} \qquad \text{for all } u_1, u_2 \in H^1(\mathcal{F}). \tag{3.13a}$$

Similarly, the scalar product:

$$(u_1, u_2)^V_{H^1} = (\mathsf{P}u_1, \mathsf{P}u_2)_{V_1} + (\mathsf{P}^\perp u_1, \mathsf{P}^\perp u_2)_{\mathfrak{H}^1} \qquad \text{for all } u_1, u_2 \in H^1(\mathcal{F}), \tag{3.13b}$$

turns the direct sum $H^1(\mathcal{F}) = V_1 \oplus \mathfrak{H}^1$ into an orthogonal sum.

3.2. **Stokes operator in nonprimitive variables.** The inclusion $S_1 \subset S_0$ is clearly continuous dense and compact. Identifying the Hilbert space S_0 with its dual and denoting by S_{-1} the dual space of S_1, we obtain the Gelfand triple:

$$S_1 \subset S_0 \subset S_{-1}. \tag{3.14}$$

Following the lines of Appendix A, we can define (with obvious notation) a family of embedded Hilbert spaces $\{S_k,\, k \in \mathbb{Z}\}$, a family of isometries $\{\mathsf{A}^S_k : S_k \to S_{k-2},\, k \in \mathbb{Z}\}$ and a positive constant:

$$\lambda^S_{\mathcal{F}} = \min_{\substack{\psi \in S_1 \\ \psi \neq 0}} \frac{\|\psi\|^2_{S_1}}{\|\psi\|^2_{S_0}}. \tag{3.15}$$

Lemma 3.6. *The space S_2 is equal to $H^3(\mathcal{F}) \cap S_1$ providing that Σ is of class $\mathcal{C}^{2,1}$. For $k \geqslant 2$, the expressions of the operator A^S_k is:*

$$\mathsf{A}^S_k : \psi \in S_k \longmapsto -\mathsf{Q}_1 \Delta\psi \in S_{k-2}. \tag{3.16}$$

If Σ is of class $\mathcal{C}^{k,1}$ then S_k is a subspace of $H^{k+1}(\mathcal{F})$ and the norm in S_k is equivalent to the classical norm of $H^{k+1}(\mathcal{F})$.

Proof. We recall that A^S_1 is the operator $\psi \in S_1 \longmapsto (\psi, \cdot)_{S_1} \in S_{-1}$. The space S_2 is defined as the preimage of S_0 by A^S_1, namely:

$$S_2 = \{\psi \in S_1 \,:\, (\psi, \cdot)_{S_1} = (f, \cdot)_{S_0} \text{ in } S_{-1} \text{ for some } f \text{ in } S_0\}.$$

Upon an integration by parts and according to Lemma 3.1, one easily obtains that the identity $(\psi, \cdot)_{S_1} = (f, \cdot)_{S_0}$ in S_{-1} is equivalent to the equality $-\mathsf{P}\Delta\psi = \mathsf{P}f$ in V_0. Invoking Lemma 3.1 again, we deduce, on the one hand, that $\mathsf{P}\Delta\psi = \Delta\psi$. Under the assumption on the regularity of the boundary Σ, the equality $-\Delta\psi = \mathsf{P}f$ where f and hence also $\mathsf{P}f$ is in $H^1(\mathcal{F})$ entails that ψ belongs to $H^3(\mathcal{F})$. On the other hand, since f belongs to S_0, $\mathsf{P}f = \mathsf{P}_1 f$. Applying then the operator Q_1 to both sides of the identity $-\Delta\psi = \mathsf{P}_1 f$, we end up with the equality $-\mathsf{Q}_1\Delta\psi = f$ and (3.16) is proven for $k = 2$. The expressions for $k > 2$ follow from the general settings of Appendix A. Then, by induction on k, invoking classical elliptic regularity results, one proves the inclusion $S_k \subset H^{k+1}(\mathcal{F})$ and the equivalence of the norms. □

We straightforwardly deduce that, by definition of the space V_1, the inclusion $V_1 \subset V_0$ enjoys the same properties as the inclusion $S_1 \subset S_0$, namely it is continuous dense and compact. We consider then the Gelfand triple:

$$V_1 \subset V_0 \subset V_{-1}, \tag{3.17}$$

in which V_0 is the pivot space and V_{-1} is the dual space of V_1. As beforehand, we define a family of embedded Hilbert spaces $\{V_k,\, k \in \mathbb{Z}\}$, the corresponding family of isometries $\{\mathsf{A}^V_k : V_k \to V_{k-2},\, k \in \mathbb{Z}\}$ and the positive constant:

$$\lambda^V_{\mathcal{F}} = \min_{\substack{\omega \in V_1 \\ \omega \neq 0}} \frac{\|\omega\|^2_{V_1}}{\|\omega\|^2_{V_0}}. \tag{3.18}$$

Remark 3.7. (1) *As already mentioned earlier, the space V_{-1} is clearly not a distributions space.*

(2) *The guiding principle that the vorticity should be L^2-orthogonal to harmonic functions is somehow still verified in a weak sense in V_{-1}. Indeed, A_1^V being an isometry, every element ω of V_{-1} is equal to some $\mathsf{A}_1^V\omega'$ with $\omega' \in V_1$ and:*

$$\langle \omega, \cdot\rangle_{V_{-1},V_1} = \langle \mathsf{A}_1^V\omega', \cdot\rangle_{V_{-1},V_1} = (\nabla \mathsf{Q}_1\omega', \nabla\mathsf{Q}_1\cdot)_{\mathbf{L}^2(\mathcal{F})}.$$

Identifying the duality pairing with the scalar product in $L^2(\mathcal{F})$ (i.e. the scalar product of the pivot space V_0), we obtain that formally "$(\omega, h)_{L^2(\mathcal{F})} = 0$" for every $h \in \mathfrak{H}^1$.

For the analysis of the spaces V_k and their relations with the spaces S_k, it is worth introducing at this point an additional Gelfand triple, that will come in handy later on. Thus, denote by Z_0 the space $L^2(\mathcal{F})$ (equipped with the usual scalar product) and by Z_1 the space S_0. The configuration:

$$Z_1 \subset Z_0 \subset Z_{-1},$$

is obviously a Gelfand triple in which Z_0 is the pivot space and Z_{-1} the dual space of Z_1. As usual, following Appendix A, we define a family of embedded Hilbert spaces $\{Z_k,\, k \in \mathbb{Z}\}$, and a family of isometries $\{\mathsf{A}_k^Z : Z_k \to Z_{k-2},\, k \in \mathbb{Z}\}$. Focusing on the case $k = 2$, a simple integration by parts leads to:

Lemma 3.8. *The expressions of the space Z_2 and of the operator A_2^Z are respectively:*

$$Z_2 = \left\{ \psi \in H^2(\mathcal{F}) \cap S_0 \,:\, \int_{\Sigma_j^-} \frac{\partial\psi}{\partial n}\,\mathrm{d}s = 0, \quad j = 1,\dots,N \right\} \quad \text{and} \quad \mathsf{A}_2^Z : \psi \in Z_2 \longmapsto -\Delta\psi \in Z_0. \tag{3.19}$$

The space of biharmonic functions in $L^2(\mathcal{F})$ with zero mean flux through the inner boundaries is denoted by $\mathfrak{B}$, namely:

$$\mathfrak{B} = \left\{ \theta \in L^2(\mathcal{F}) \,:\, \Delta\theta \in \mathfrak{H} \text{ and } \int_{\Sigma_j^-} \frac{\partial\theta}{\partial n}\,\mathrm{d}s = 0, \quad j = 1,\dots,N \right\}. \tag{3.20}$$

Since $\Delta\theta$ belongs to $L^2(\mathcal{F})$, the trace of θ on Σ is well defined and belongs to $H^{-\frac{1}{2}}(\Sigma)$ and the trace of the normal derivative is in $H^{-\frac{3}{2}}(\Sigma)$. On the other hand, since $\Delta\theta$ belongs to $\mathfrak{H}$, its trace on Σ is in $H^{-\frac{1}{2}}(\Sigma)$ while its normal trace is in $H^{-\frac{3}{2}}(\Sigma)$.

The space S_1 being a closed subspace of Z_2 it admits an orthogonal complement denoted by $\mathfrak{B}_S$:

$$Z_2 = S_1 \overset{\perp}{\oplus} \mathfrak{B}_S. \tag{3.21}$$

An integration by parts and classical elliptic regularity results allow to deduce that:

$$\mathfrak{B}_S = S_0 \cap \mathfrak{B}, \tag{3.22}$$

and that $\mathfrak{B}_S \subset H^2(\mathcal{F})$.

Lemma 3.9. *The operator A_2^Z is an isometry from S_1 onto V_0 and also an isometry from $\mathfrak{B}_S$ onto $\mathfrak{H}$, i.e. the operator A_2^Z is block-diagonal with respect to the following decompositions of the spaces:*

$$\mathsf{A}_2^Z : S_1 \overset{\perp}{\oplus} \mathfrak{B}_S \longrightarrow V_0 \overset{\perp}{\oplus} \mathfrak{H}.$$

The operators A_1^V and A_1^Z and the operators A_2^V and A_2^Z are connected via the identities:

$$\mathsf{A}_1^V = \mathsf{Q}_1^*\mathsf{A}_1^Z\mathsf{Q}_1 \qquad \text{and} \qquad \mathsf{A}_2^V = \mathsf{A}_2^Z\mathsf{Q}_1 \quad \text{in } V_2, \tag{3.23}$$

where the operator Q_1^* is the adjoint of Q_1.

Proof. The first claim of the lemma is a direct consequence of Lemma 3.1.

By definition, for every $\omega \in V_1$:

$$\mathsf{A}_1^V\omega = (\nabla\mathsf{Q}_1\omega, \nabla\mathsf{Q}_1\cdot)_{\mathbf{L}^2(\mathcal{F})} = \mathsf{Q}_1^*\mathsf{A}_1^Z\mathsf{Q}_1\omega,$$

and the first identity in (3.23) is proved. Addressing the latter, notice that for every $\omega \in V_2$, the function $w = \mathsf{A}_2^V\omega$ is the unique element in V_0 such that:

$$(w, v)_{V_0} = (\omega, v)_{V_1}, \qquad \text{for all } v \in V_1,$$

which can be rewritten as:

$$(w,v)_{L^2(\mathcal{F})} = (\nabla \mathsf{Q}_1\omega, \nabla \mathsf{Q}_1 v)_{\mathbf{L}^2(\mathcal{F})}, \qquad \text{for all } v \in V_1.$$

But the functions v and $\mathsf{Q}_1 v$ differ only up to an element of $\mathfrak{H}$ and since w belongs to $V_0 = \mathfrak{H}^\perp$, it follows that:

$$(w,v)_{L^2(\mathcal{F})} = (w, \mathsf{Q}_1 v)_{L^2(\mathcal{F})}, \qquad \text{for all } v \in V_1.$$

Finally, since $S_0 = Z_1$ and $\mathsf{Q}_1 : V_1 \to S_0$ is an isometry, the function w satisfies:

$$(w,z)_{L^2(\mathcal{F})} = (\nabla \mathsf{Q}_1\omega, \nabla z)_{\mathbf{L}^2(\mathcal{F})}, \qquad \text{for all } z \in Z_1,$$

which means that $w = \mathsf{A}^Z_2 \mathsf{Q}_1 \omega$ and completes the proof. ☐

We can now go back to the study of the vorticity spaces V_k and the related operators A^V_k. Starting with the case $k = 2$, we claim:

Lemma 3.10. *The space V_2 is equal to $\mathsf{P}_1 S_1$, or equivalently:*

$$V_2 = \left\{ \omega \in \mathsf{P}H^2(\mathcal{F}) \,:\, \frac{\partial \mathsf{Q}_1\omega}{\partial n}\Big|_\Sigma = 0 \right\}. \tag{3.24a}$$

Moreover, the expression of the operator A^V_2 is:

$$\mathsf{A}^V_2 : \omega \in V_2 \longmapsto -\Delta\omega \in V_0. \tag{3.24b}$$

Proof. The second formula in (3.23) yields the following identity between function spaces:

$$\mathsf{A}^V_2 V_2 = \mathsf{A}^Z_2 \mathsf{Q}_1 V_2$$

and then, since $\mathsf{A}^V_2 V_2 = V_0$:

$$(\mathsf{A}^Z_2)^{-1} V_0 = \mathsf{Q}_1 V_2.$$

Invoking the first point of Lemma 3.9 we deduce first that $\mathsf{Q}_1 V_2 = S_1$ and then, applying the operator P_1 to both sides of the identity, that $V_2 = \mathsf{P}_1 S_1$. Using again the second formula in (3.23) together with the expression of A^Z_2 given in (3.19), we obtain the expression (3.24b) of the operator A^V_2. ☐

Remark 3.11. *According to Lemma 3.2, if Σ is of class $\mathcal{C}^{3,1}$ then V_2 is a subspace of $H^2(\mathcal{F})$. If Σ is of class $\mathcal{C}^{2,1}$, V_2 is a subspace of $H^1(\mathcal{F})$ and the functions in V_2 can be given a trace in $H^{\frac{1}{2}}(\Sigma)$ and a normal trace in $H^{-\frac{3}{2}}(\Sigma)$. Finally, if Σ is only of class $\mathcal{C}^{1,1}$, the trace still exists in $H^{-\frac{1}{2}}(\Sigma)$ and the normal trace in $H^{-\frac{3}{2}}(\Sigma)$. We use the fact that every function in V_2 is by definition the sum of a function in $H^2(\mathcal{F})$ with a harmonic function in $L^2(\mathcal{F})$.*

For the ease of the reader, we can still state the following lemma which is a straightforward consequence of (3.24b) and the general settings of Appendix A:

Lemma 3.12. *For every positive integer k, the expression of the operators A^V_k are:*

$$\mathsf{A}^V_1 : u \in V_1 \longmapsto (u,\cdot)_{V_1} \in V_{-1} \qquad \text{and} \qquad \mathsf{A}^V_k : u \in V_k \longmapsto (-\Delta)u \in V_{k-2} \qquad \text{for } k \geqslant 2.$$

For nonnegative integers k, V_k is a subspace of $H^k(\mathcal{F})$ providing that Σ is of class $\mathcal{C}^{k+1,1}$ and the norm in V_k is equivalent to the classical norm of $H^k(\mathcal{F})$. For nonpositive indices, the operators are defined by duality as follows:

$$\mathsf{A}^V_{-k} : u \in V_{-k} \longmapsto \langle u, (-\Delta)\cdot\rangle_{V_{-k},V_k} \in V_{-k-2}, \qquad (k \geqslant 0).$$

The next result states that the chain of embedded spaces for the stream function $\{S_k,\, k \in \mathbb{Z}\}$ is globally isometric to the chain of embedded spaces for the vorticity $\{V_k,\, k \in \mathbb{Z}\}$, the isometries being, loosely speaking, the operators P and Q. So far, we have proven that $\mathsf{P}_1 S_1 = V_2$ and $\mathsf{P}_1 S_0 = V_1$. To generalized these relations to every integer k, we need to extend the operators P_1 and Q_1.

Lemma 3.13. *For every positive integer k, the following inclusions hold:*

$$\mathsf{P}_1 S_{k-1} \subset V_k \qquad \text{and} \qquad \mathsf{Q}_1 V_k \subset S_{k-1}.$$

Considering this lemma as granted, it makes sense to define for every positive integer k the operators:

$$\mathsf{P}_k : u \in S_{k-1} \longmapsto \mathsf{P}_1 u \in V_k \qquad \text{and} \qquad \mathsf{Q}_k : u \in V_k \longmapsto \mathsf{Q}_1 u \in S_{k-1}. \tag{3.25}$$

Then, we define also by induction, for every $k \geqslant 0$:

$$\mathsf{P}_{-k} = \mathsf{A}^V_{-k+2}\mathsf{P}_{-k+2}(\mathsf{A}^S_{-k+1})^{-1} : S_{-k-1} \to V_{-k}, \tag{3.26a}$$

and

$$\mathsf{Q}_{-k} = \mathsf{A}^S_{-k+1}\mathsf{Q}_{-k+2}(\mathsf{A}^V_{-k+2})^{-1} : V_{-k} \to S_{-k-1}. \tag{3.26b}$$

Theorem 3.14. *For every integer k, the operators P_k and Q_k defined in* (3.25) *and* (3.26) *are inverse isometries (i.e. $\mathsf{P}_k\mathsf{Q}_k = \mathrm{Id}$ and $\mathsf{Q}_k\mathsf{P}_k = \mathrm{Id}$). Moreover, formulas* (3.26) *can be generalized to every integer k:*

$$\mathsf{A}^V_k\mathsf{P}_k = \mathsf{P}_{k-2}\mathsf{A}^S_{k-1} \qquad \textit{and} \qquad \mathsf{A}^S_{k-1}\mathsf{Q}_k = \mathsf{Q}_{k-2}\mathsf{A}^V_k, \tag{3.27}$$

and for every pair of indices k, k' such that $k' \leqslant k$:

$$\mathsf{P}_{k'} = \mathsf{P}_k \quad \textit{in } S_{k-1} \qquad \textit{and} \qquad \mathsf{Q}_{k'} = \mathsf{Q}_k \quad \textit{in } V_k. \tag{3.28}$$

Remark 3.15. *By definition, P is a projection in $L^2(\mathcal{F})$ and Q a projection in $H^1(\mathcal{F})$. The theorem tells us that formulas* (3.26) *allow extending these projectors to larger spaces.*

The theorem ensures also that to every stream function ψ in some space S_{k-1}, it can be associated a vorticity field $\omega = \mathsf{P}_k\psi$ in V_k. The vorticity ω has the same regularity as ψ and is obviously not the Laplacian of ψ.

The lemma and the theorem are proved at once:

Proof of Lemma 3.13 and Theorem 3.14. Denoting by Q_2 the restriction of Q_1 to V_2, the second formula in (3.23) can be rewritten as $\mathsf{A}^V_2 = \mathsf{A}^Z_2\mathsf{Q}_2$. Since the operators A^V_2 and A^Z_2 are both isometries, this property is also shared by Q_2 and its inverse, which is denoted by P_2. We have now at our disposal two pairs of isometries $(\mathsf{P}_1, \mathsf{Q}_1)$ and $(\mathsf{P}_2, \mathsf{Q}_2)$ corresponding to two successive indices in the chain of embedded spaces. Furthermore, P_2 is the restriction of P_1 to V_2. This fits within the framework of Subsection A.2. We define first P_k and Q_k (for the indices $k \neq 1, 2$) by induction with formulas (3.27) and we apply Lemma A.6. We obtain that the operators P_k and Q_k are indeed isometries from S_{k-1} onto V_k and from V_k onto S_{k-1} respectively. Lemma A.6 also ensures that $\mathsf{P}_k = \mathsf{P}_{k'}$ in V_k and $\mathsf{Q}_k = \mathsf{Q}_{k'}$ in S_{k-1} for indices $k' \leqslant k$, whence we deduce that P_k and Q_k for $k \geqslant 1$ can be equivalently defined by (3.25). Next, since P_1 and Q_1 are reciprocal isometries and P_k and Q_k are just restrictions of P_1 and Q_1, then P_k and Q_k are reciprocal isometries as well. We draw the same conclusion for nonpositive indices using formulas (3.27) and complete the proof. □

Corollary 3.16. *The constant $\lambda^S_{\mathcal{F}}$ defined in* (3.15) *and the constant $\lambda^V_{\mathcal{F}}$ defined in* (3.18) *are equal. We denote simply by $\lambda_{\mathcal{F}}$ their common value.*

Notice also that since $S_1 \subset Z_2$ and $S_0 = Z_1$, we have:

$$\lambda^Z_{\mathcal{F}} = \min_{\substack{\psi \in Z_2 \\ \psi \neq 0}} \frac{\|\psi\|^2_{Z_2}}{\|\psi\|^2_{Z_1}} \leqslant \min_{\substack{\psi \in S_1 \\ \psi \neq 0}} \frac{\|\psi\|^2_{S_1}}{\|\psi\|^2_{S_0}} = \lambda_{\mathcal{F}}. \tag{3.29}$$

Remark 3.15 points out that, for a given stream function ψ in some space S_{k-1}, the function $\omega = \mathsf{P}_k\psi$ is not the (physical) vorticity corresponding to the velocity field $\nabla^\perp\psi$. We shall now define the operators Δ_k that associates the stream function to its corresponding vorticity field.

Definition 3.17. *For every integer k, the operator $\Delta_k : S_{k+1} \to V_k$ is defined equivalently (according to* (3.27)*) by:*

$$\textit{Either} \quad \Delta_k = -\mathsf{A}^V_{k+2}\mathsf{P}_{k+2} \qquad \textit{or} \qquad \Delta_k = -\mathsf{P}_k\mathsf{A}^S_{k+1}. \tag{3.30}$$

The main properties of the operators Δ_k are summarized in the following lemma:

Lemma 3.18. *The following assertions hold:*

(1) *For every integer k, the operator Δ_k is an isometry.*
(2) *For every pair of integers k, k' such that $k' \leqslant k$, $\Delta_k = \Delta_{k'}$ in S_{k+1}.*

(3) *For every nonnegative integer k, the operator Δ_k is the classical Laplacian operator.*

Proof. We recall that the operators P_k and A^V_k are isometries, what yields the first point of the lemma. The second point is a consequence of (3.28) and the similar general property (A.8) satisfied by the operators A^V_k. The third point is a direct consequence of Lemma 3.12. □

For any integer k, the operators Q_k and $-\Delta_{-k}$ can be shown to be somehow adjoint:

Lemma 3.19. *For negative indices, the operators Δ_k and Q_k satisfy the adjointness relations below:*

$$\mathsf{Q}_{-k} = -\Delta_k^* : \omega \in V_{-k} \longmapsto -\langle \omega, \Delta_k \cdot \rangle_{V_{-k},V_k} \in S_{-k-1} \qquad (k \geqslant 0), \tag{3.31a}$$

$$\Delta_{-k} = -\mathsf{Q}_k^* : \psi \in S_{-k+1} \longmapsto -\langle \psi, \mathsf{Q}_k \cdot \rangle_{S_{-k+1},S_{k-1}} \in V_{-k} \quad (k \geqslant 1). \tag{3.31b}$$

Proof. We prove (3.31a) by induction, the proof of (3.31b) being similar. According to (3.27), $\mathsf{Q}_0\mathsf{A}^V_2 = \mathsf{A}^S_1\mathsf{Q}_2$ what means, recalling (A.3) that:

$$\mathsf{Q}_0\mathsf{A}^V_2\omega = (\mathsf{Q}_2\omega, \cdot)_{S_1} = (\omega, \mathsf{P}_2 \cdot)_{V_2} = (\mathsf{A}^V_2\omega, \mathsf{A}^V_2\mathsf{P}_2 \cdot)_{V_0} = (\mathsf{A}^V_2\omega, (-\Delta_0)\cdot)_{V_0},$$

where we have used the fact that the operators P_2 and A^V_2 are isometries. This proves (3.31a) at the step $k = 0$.

According to the definition (3.26b) with $k = 1$, $\mathsf{Q}_{-1} = \mathsf{A}^S_0\mathsf{Q}_1(\mathsf{A}^V_1)^{-1}$. Equivalently stated, for every $\omega \in V_1$:

$$\mathsf{Q}_{-1}\mathsf{A}^V_1\omega = \mathsf{A}^S_0\mathsf{Q}_1\omega = (\mathsf{Q}_1\omega, \mathsf{A}^S_2\cdot)_{S_0} \qquad \text{for all } \omega \in V_1,$$

where we have used the general relation (A.6). The operator P_1 being an isometry:

$$(\mathsf{Q}_1\omega, \mathsf{A}^S_2\cdot)_{S_0} = (\mathsf{P}_1\mathsf{Q}_1\omega, \mathsf{P}_1\mathsf{A}^S_2\cdot)_{V_1} = -(\omega, \Delta_1\cdot)_{V_1} = -\langle \mathsf{A}^V_1\omega, \Delta_1\cdot\rangle_{V_{-1},V_1},$$

and (3.31a) is then proved for $k = 1$.

Let us assume that (3.31a) holds true at the step $k-2$ for some integer $k \geqslant 2$. According to (3.27), $\mathsf{Q}_{-k}\mathsf{A}^V_{-k+2} = \mathsf{A}^S_{-k+1}\mathsf{Q}_{-k+2}$ whence, recalling the definition (A.7):

$$\mathsf{Q}_{-k}\mathsf{A}^V_{-k+2}\omega = \langle \mathsf{Q}_{-k+2}\omega, \mathsf{A}^S_{k+1}\cdot\rangle_{S_{-k+1},S_{k-1}} \qquad \text{for all } \omega \in V_{-k+2}.$$

Using the induction hypothesis for the operator Q_{-k+2}, it comes:

$$\mathsf{Q}_{-k}\mathsf{A}^V_{-k+2}\omega = -\langle \omega, \Delta_{k-2}\mathsf{A}^S_{k+1}\cdot\rangle_{V_{-k+2},V_{k-2}} = -\langle \omega, \mathsf{A}^V_k\Delta_k\cdot\rangle_{V_{-k+2},V_{k-2}} \qquad \text{for all } \omega \in V_{-k+2},$$

where the latter identity results from (3.30). Keeping in mind (A.7), we have indeed proved (3.31a) at the step k and complete the proof. □

3.3. **Biot-Savart operator.** With a slight abuse of terminology, the inverse of the operator Δ_k denoted by N_k will be referred to as the Biot-Savart operator. Quite surprisingly, the expression of this operator is independent from the fluid domain $\mathcal{F}$. We recall that the fundamental solution of the Laplacian is the function:

$$\mathscr{G}(x) = \frac{1}{2\pi}\ln|x|, \qquad x \in \mathbb{R}^2 \setminus \{0\}.$$

In the rest of the paper, we will denote generically by $\mathbf{c}$ the real constants that should arise in the estimates. The value of the constant may change from line to line. The parameters the constant should depend on is indicated in subscript.

Theorem 3.20. *For every nonnegative index k, the Biot-Savart operator $\mathsf{N}_k = (\Delta_k)^{-1}$ is simply the Newtonian potential defined by:*

$$\mathsf{N}_k : \omega \in V_k \longmapsto \mathsf{N}\omega \in S_{k+1}, \tag{3.32}$$

where $\mathsf{N}\omega = \mathscr{G} * \omega$ (ω is extended by 0 outside $\mathcal{F}$).

Proof. Let ω be in V_0 (extended by 0 outside $\mathcal{F}$) and denote by ψ the Newtonian potential $\mathsf{N}\omega$. According to classical properties of the Newtonian potential, $\Delta\psi = \omega$ in $\mathbb{R}^2$ and therefore it suffices to verify that ψ is constant on every connected part of the boundary Σ and that its normal derivative vanishes. Define the constant $\delta = 2\max\{|x-y| \,:\, x\in\Sigma^-,\, y\in\Sigma^+\}$. Let j be in $\{1,\ldots,N\}$ and let q be in $L^2(\Sigma_j^-)$. Notice now that there exists a positive constant $\mathbf{c}_{\Sigma_j^-}$ such that:

$$\int_{\Sigma_j^-}\int_{\mathbb{R}^2}|\mathscr{G}(x-y)\omega(y)q(x)|\mathrm{d}y\,\mathrm{d}s_x = \int_{\Sigma_j^-}\int_{B(0,\delta)}|\mathscr{G}(z)\omega(x-z)q(x)|\mathrm{d}z\,\mathrm{d}s_x$$
$$\leqslant \mathbf{c}_{\Sigma_j^-}\|\omega\|_{V_0}\|\mathscr{G}\|_{L^2(B(0,\delta))}\|q\|_{L^2(\Sigma_j^-)}.$$

We are then allowed to apply Fubini's theorem, which yields:

$$\int_{\mathbb{R}^2}\omega(y)\left(-\int_{\Sigma_j^-}\mathscr{G}(x-y)q(x)\,\mathrm{d}s_x\right)\mathrm{d}y+\int_{\Sigma_j^-}\left(\int_{\mathbb{R}^2}\mathscr{G}(x-y)\omega(y)\mathrm{d}y\right)q(x)\,\mathrm{d}s_x=0,$$

and this identity can be rewritten as:

$$\int_{\mathcal{F}}\Delta\psi(\mathsf{S}_j q)\,\mathrm{d}x+\int_{\Sigma_j^-}\psi\,q\,\mathrm{d}s=0, \tag{3.33}$$

where $\mathsf{S}_j q$ is the single layer potential of density q supported on the boundary Σ_j^-, that is:

$$\mathsf{S}_j q(x) = -\int_{\Sigma_j^-}\mathscr{G}(x-y)q(y)\,\mathrm{d}s_y \qquad \text{for all } x\in\mathbb{R}^2\setminus\Sigma_j^-.$$

The simple layer potential $\mathsf{S}_j q$ is harmonic in $\mathbb{R}^2\setminus\Sigma_j^-$ (we refer to the book of McLean [55] for details about layer potentials) and we denote respectively by $\mathsf{S}_j q^+$ and $\mathsf{S}_j q^-$ the restriction of $\mathsf{S}_j q$ to the unbounded and bounded connected components of $\mathbb{R}^2\setminus\Sigma_j^-$. The functions $\mathsf{S}_j q^+$ and $\mathsf{S}_j q^-$ share the same trace on Σ_j^- and q is the jump of the normal derivative across the boundary Σ_j^-:

$$q=\frac{\partial}{\partial n}\mathsf{S}_j^+ q-\frac{\partial}{\partial n}\mathsf{S}_j^- q \qquad \text{on } \Sigma_j^-. \tag{3.34}$$

From the obvious equality

$$\int_{\Sigma_j^-}\frac{\partial}{\partial n}\mathsf{S}_j^- q\,\mathrm{d}s=0,$$

we deduce that the harmonic function $\mathsf{S}_j^+ q$ has zero mean flux through the boundary Σ_j^- if and only if

$$\int_{\Sigma_j^-} q\,\mathrm{d}s=0.$$

On the other hand, for indices $k\neq j$, we have also:

$$\int_{\Sigma_k^-}\frac{\partial}{\partial n}\mathsf{S}_j^+ q\,\mathrm{d}s=0,$$

because the normal derivative of $\mathsf{S}_j^+ q$ is continuous across Σ_k^- and $\mathsf{S}_j^+ q$ is harmonic inside Σ_k^-. From (3.33), we infer that for every $\omega\in V_0$ and every $q\in L^2(\Sigma_j^-)$ with zero mean value:

$$\int_{\Sigma_j^-}\psi\,q\,\mathrm{d}s=0,$$

and then that ψ is constant on Σ_j^-. For every q in $L^2(\Sigma^+)$, the corresponding single layer potential $\mathsf{S}_0 q$ supported on Σ^+ has zero mean flux through every inner boundary Σ_k^- (for $k\in\{1,\ldots,N\}$) and we deduce from (3.33) again that the trace of ψ is nul on Σ^+. It follows that ψ is in the space S_0.

Let now h be a harmonic function in $\mathcal{F}$ and assume that the normal derivative of h belongs to $L^2(\Sigma)$. Then there exists $q_0\in L^2(\Sigma^+)$ and $q_j\in L^2(\Sigma_j^-)$ $(j=1,\ldots,N)$ such that:

$$h=\mathsf{S}_0 q_0+\sum_{j=1}^N\mathsf{S}_j q_j \qquad \text{in } \mathcal{F}.$$

Using again (3.33) and the fact that ψ is nul on Σ^+ and constant on Σ^-, we deduce that:

$$\int_{\mathcal{F}} \Delta\psi h \,\mathrm{d}x + \int_{\Sigma^-} \psi \frac{\partial h}{\partial n} \,\mathrm{d}s = 0,$$

and therefore, integrating by parts, that:

$$\int_{\Sigma} \frac{\partial \psi}{\partial n} h \,\mathrm{d}s = 0.$$

This last equality being true for every harmonic function h, it follows that the normal derivative of ψ is nul on Σ and therefore that ψ belongs to S_1. The proof is now completed. □

The expression (3.32) when $\mathcal{F} = \mathbb{R}^2$ can be found in the book [53, §2.1]. For the Euler equations in a domain with holes, the Biot-Savart operator (in the sense considered above, that is the operator allowing recovering the stream function) is given by (see [51] for a proof):

$$\mathsf{N}^E\omega(x) = \int_{\mathcal{F}} \mathscr{K}(x,y)\omega(y) \,\mathrm{d}y + \sum_{j=1}^{N} (\mathit{\Gamma}_j - \alpha_j(\omega))\,\xi_j(x) \qquad \text{for all } x \in \mathcal{F}. \tag{3.35}$$

In this identity:

(1) $\mathscr{K} : \mathcal{F} \times \mathcal{F} \to \mathbb{R}$ is the Green's function of the domain $\mathcal{F}$. It is defined by:

$$\mathscr{K}(x,y) = \mathscr{G}(x-y) - \mathscr{H}(x,y) \qquad \text{for all } (x,y) \in \mathcal{F} \times \mathcal{F} \text{ s.t. } x \neq y,$$

where, for every x in $\mathcal{F}$, the function $\mathscr{H}(x,\cdot)$ is harmonic in $\mathcal{F}$ and satisfies

$$\mathscr{H}(x,\cdot) = \mathscr{G}(x-\cdot) \quad \text{ on } \Sigma.$$

(2) The real constants $\alpha_j(\omega)$ are given by:

$$\begin{bmatrix} \alpha_1(\omega) \\ \vdots \\ \alpha_N(\omega) \end{bmatrix} = \begin{bmatrix} (\xi_1,\xi_1)_{S_0} & \dots & (\xi_1,\xi_N)_{S_0} \\ \vdots & & \vdots \\ (\xi_N,\xi_1)_{S_0} & \dots & (\xi_N,\xi_N)_{S_0} \end{bmatrix}^{-1} \begin{bmatrix} \int_{\mathcal{F}} \omega(y)\xi_1(y)\,\mathrm{d}y \\ \vdots \\ \int_{\mathcal{F}} \omega(y)\xi_N(y)\,\mathrm{d}y \end{bmatrix},$$

where we recall the the functions ξ_j $(j = 1, \dots, N)$ are defined in Section 3.1.

(3) For every j, the scalar $\mathit{\Gamma}_j$ is the circulation of the fluid around the inner boundary Σ_j^-.

Notice that for the Euler equations in a multiply connected domain, both the vorticity and the circulation are necessary to recover the stream function.

In case the vorticity is in V_0 and in the absence of circulation, then the Biot-Savart operator for NS equations and the Biot-Savart operator for Euler equations give the same stream function:

Proposition 3.21. *Let ω be in V_0 and assume that the flow is such that $\mathit{\Gamma}_j = 0$ for every $j = 1, \dots, N$. Then $\mathsf{N}_0\omega$ (defined in* (3.32)*) and $\mathsf{N}^E\omega$ (defined in* (3.35)*) are equal.*

The proof relies on the following lemma in which we denote simply by S the simple layer potential supported on the whole boundary Σ.

Lemma 3.22. *For every function $h \in \mathbb{F}_S$ (i.e. h harmonic in $\mathcal{F}$ and h belongs to S_0):*

$$\left(\mathsf{S} \frac{\partial h}{\partial n}\Big|_{\Sigma} \right)(x) = h(x) \qquad \text{for all } x \in \mathcal{F}.$$

Proof. Let h be in S_0. Basic results of potential theory ensures that there exists a unique $p \in H^{-\frac{1}{2}}(\Sigma)$ such that $\mathsf{S}p(x) = h(x)$ for every $x \in \mathcal{F}$. The single layer potential $\mathsf{S}p$ is harmonic in $\mathbb{R}^2 \setminus \Sigma$ and belongs to $H^1_{\ell oc}(\mathbb{R}^2)$ (which means that the trace of the function matches on both sides of the boundary Σ). Since the trace of h is equal to 0 on Σ^+, the single layer potential vanishes identically on the unbounded connected component of $\mathbb{R}^2 \setminus \Sigma^+$. For similar reasons, $\mathsf{S}p$ is constant inside Σ_j^- for every $j = 1, \dots, N$. According to the jump formula (3.34), we obtain that $p = \partial h/\partial n$ on Σ and the proof is completed. □

We can move on to the:

Proof of Proposition 3.21. For every $x \in \mathcal{F}$, the function:

$$\mathscr{H}_0(x,\cdot) = \mathscr{H}(x,\cdot) - \sum_{j=1}^N (\nabla \mathscr{H}(x,\cdot), \nabla \hat{\xi}_j)_{\mathbf{L}^2(\mathcal{F})} \hat{\xi}_j,$$

belongs to $\mathfrak{H}$. Integrating by parts the terms in the sum, we obtain for every $j = 1, \dots, N$:

$$(\nabla \mathscr{H}(x,\cdot), \nabla \hat{\xi}_j)_{\mathbf{L}^2(\mathcal{F})} = \int_\Sigma \mathscr{H}(x,y) \frac{\partial \hat{\xi}_j}{\partial n}(y) \,\mathrm{d}s_y = -\left(\mathsf{S}\frac{\partial \hat{\xi}_j}{\partial n}\right)(x) = -\hat{\xi}_j(x),$$

according to Lemma 3.22. Let now ω be in V_0. Then, providing that $\Gamma_j = 0$ for every $j = 1, \dots, N$:

$$\mathsf{N}^E\omega(x) = \mathsf{N}_0\omega(x) - \int_{\mathcal{F}} \mathscr{H}_0(x,y)\omega(y)\,\mathrm{d}y + \sum_{j=1}^N \hat{\alpha}_j(\omega)\hat{\xi}_j(x) - \sum_{j=1}^N \alpha_j(\omega)\xi_j(x) \qquad \text{for all } x \in \mathcal{F}, \tag{3.36}$$

where, for every $j = 1, \dots, N$:

$$\hat{\alpha}_j(\omega) = \int_{\mathcal{F}} \omega(y)\hat{\xi}_j(y)\,\mathrm{d}y.$$

The second term in the right hand side of (3.36) vanishes by definition of V_0 and both last terms cancel out since they stand for the same linear application expressed in two different bases of $\mathbb{F}_S$. □

It remains now to link the spaces S_k for the stream functions to the spaces $\mathbf{J}_k$ for the velocity fields. We recall the definitions (2.17) of the spaces $\mathbf{J}_0$ and $\mathbf{J}_1$. For every other integers k, the spaces $\mathbf{J}_k$ are classically defined from the Gelfand triple $\mathbf{J}_1 \subset \mathbf{J}_0 \subset \mathbf{J}_{-1}$, as well as the isometries $\mathsf{A}^{\mathbf{J}}_k : \mathbf{J}_k \to \mathbf{J}_{k-2}$. The following Lemma can be found in [31]:

Lemma 3.23. *The operators $\nabla^\perp_0 : \psi \in S_0 \mapsto \nabla^\perp\psi \in \mathbf{J}_0$ and $\nabla^\perp_1 : \psi \in S_1 \mapsto \nabla^\perp\psi \in \mathbf{J}_1$ are well defined and are isometries.*

Applying the abstract results of Section A.2, we deduce:

Lemma 3.24. *For every index k, it can be defined an isometry:*

$$\nabla^\perp_k : S_k \longmapsto \mathbf{J}_k,$$

such that, for every pair of indices $k \leqslant k'$, $\nabla^\perp_k = \nabla^\perp_{k'}$ in $S_{k'}$ and Diagram 3 commutes.

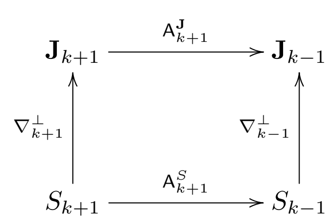

Figure 3. The top row contains the function $\mathbf{J}_k$ for the velocity field and the bottom row contains the spaces S_k for the stream functions. All the operators are isometries.

Remark 3.25. *Let be given a sequence $(\psi_n)_n$ in S_0 and $\bar{\psi} \in S_0$. Define the corresponding velocity fields $u_n = \nabla^\perp_0\psi_n$ and $\bar{u} = \nabla^\perp_0\bar{\psi}$ and the vorticity fields $\omega_n = \Delta_{-1}\psi_n$ and $\bar{\omega} = \Delta_{-1}\bar{\psi}$. Then, the following assertions are equivalent:*

$$\psi_n \rightharpoonup \bar{\psi} \qquad \text{in } S_0, \tag{3.37a}$$

$$u_n \rightharpoonup \bar{u} \qquad \text{in } \mathbf{J}_0, \tag{3.37b}$$

$$\omega_n \rightharpoonup \bar{\omega} \qquad \text{in } V_{-1}. \tag{3.37c}$$

Let a time $T>0$ be given and suppose now that $(\psi_n)_n$ is a sequence in $L^\infty([0,T];S_0)$ and that $\bar\psi$ lies in $L^\infty([0,T];S_0)$. Then the velocity fields u_n and $\bar u$ belongs to $L^\infty([0,T];\mathbf{J}_0)$ and the vorticity fields ω_n and $\bar\omega$ are in $L^\infty([0,T];V_{-1})$. In the context of vanishing viscosity limit, assume that $\bar u$ is a solution to the Euler equations and that u_n is a solution to the NS equations with a viscosity that tends to zero along with n. Following Kelliher [36], the vanishing viscosity limit holds when $u_n \rightharpoonup \bar u$ in $\mathbf{J}_0$, uniformly on $[0,T]$. According to (3.37), this conditions is then equivalent to either $\psi_n \rightharpoonup \bar\psi$ in S_0 , uniformly on $[0,T]$ or to $\omega_n \rightharpoonup \bar\omega$ in V_{-1}, uniformly on $[0,T]$; see also Remark 2.2.

Most of the material elaborated so far in this section is summarized in the commutative diagram of Fig. 4, which contains the main operators and their relations.

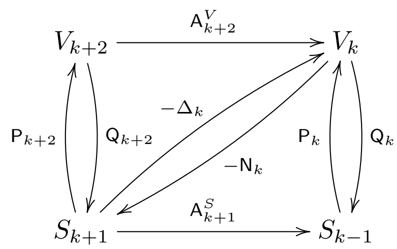

FIGURE 4. The top row contains the function spaces V_k for the vorticity fields while the bottom row contains the spaces S_k for the stream functions. The operators A^V_k and A^S_k are Stokes operators (see the Cauchy problems (5.3) and (5.2) in the next section). The operators Δ_k link the stream functions to the corresponding vorticity fields.

3.4. **A simple example: The unit disk.** In this subsection, we assume that $\mathcal{F}$ is the unit disk and we aim at computing the spectrum of the operator $\mathcal{A}^V_2$ (i.e. the operator A^V_2 seen as an unbounded operator of domain V_2 in V_0; see (A.11)).

All the harmonic functions in $\mathcal{F}$ are equal to the real part of a holomorphic function in $\mathcal{F}$. The holomorphic functions can be expanded as power series with convergence radius equal to 1. It follows that a function $\omega\in L^2(\mathcal{F})$ belongs to $V_0=\mathfrak{H}^\perp$ if and only if, for every nonnegative integer k:

$$\mathrm{Re}\left(\int_{\mathcal{F}}\omega(z)\,z^k\,\mathrm{d}|z|\right)=0\qquad\text{and}\qquad \mathrm{Im}\left(\int_{\mathcal{F}}\omega(z)\,z^k\,\mathrm{d}|z|\right)=0.$$

Using the method of separation of variables in polar coordinates, we find first that a function $\omega(r,\theta)=\rho(r)\Theta(\theta)$ is in V_0 when:

$$\left(\int_0^1\rho(r)r^{k+1}\,\mathrm{d}r\right)\left(\int_0^{2\pi}\Theta(\theta)e^{ik\theta}\,\mathrm{d}\theta\right)=0\qquad\text{for all } k\in\mathbb{N}.$$

Then, providing that $-\Delta\omega=\lambda\omega$ in $\mathcal{F}$ for some positive real number λ, we deduce the expression of the function ω, namely:

$$(r,\theta)\longmapsto\rho_k(r)\cos(k\theta)\qquad\text{or}\qquad(r,\theta)\longmapsto\rho_k(r)\sin(k\theta)$$

for some nonnegative integer k. The function ρ_k solves the differential equation in $(0,1)$:

$$\rho_k''(r)+\frac{1}{r}\rho_k'(r)+\left(\lambda-\frac{k^2}{r^2}\right)\rho_k(r)=0\qquad r\in(0,1),\tag{3.38a}$$

and satisfies:

$$\int_0^1\rho_k(r)r^{k+1}\,\mathrm{d}r=0.\tag{3.38b}$$

The solution of (3.38a) (regular at $r=0$) is $\rho_k(r)=J_k(\sqrt{\lambda}r)$ where J_k is the Bessel function of the first kind. Multiplying the equation (3.38a) by r^{k+1} and integrating over the interval $(0,1)$, we show that (3.38b) is equivalent to:

$$\sqrt{\lambda_k}J_k'(\sqrt{\lambda})-kJ_k(\sqrt{\lambda})=0.$$

Using the identity $J_k'(r) = kJ_k(r)/r - J_{k+1}(r)$, the condition above can be rewritten as:

$$J_{k+1}(\sqrt{\lambda}) = 0. \tag{3.39}$$

We denote by α_k^j (for every integers $j, k \geqslant 1$) the j-th zero of the Bessel function J_k and we set

$$\lambda_k^j = (\alpha_{k+1}^j)^2 \qquad \text{for all } k \geqslant 0 \text{ and } j \geqslant 1.$$

Proposition 3.26. *The eigenvalues of A_2^V (and then also of A_k^V, A_k^S and $\mathsf{A}_k^{\mathbf{J}}$ for every index k, since they all have the same spectrum) are the real positive numbers λ_k^j ($k \geqslant 0$, $j \geqslant 1$). The eigenspaces corresponding to λ_0^j ($j \geqslant 1$) are of dimension 1, spanned by the eigenfunctions:*

$$(r,\theta) \longmapsto J_0\Big(\sqrt{\lambda_0^j}\,r\Big). \tag{3.40a}$$

The eigenspaces of the other eigenvalues λ_k^j (for $k \geqslant 1$) are of dimension 2, spanned by the eigenfunctions:

$$(r,\theta) \longmapsto J_k\Big(\sqrt{\lambda_k^j}\,r\Big)\cos(k\theta) \qquad \text{and} \qquad (r,\theta) \longmapsto J_k\Big(\sqrt{\lambda_k^j}\,r\Big)\sin(k\theta). \tag{3.40b}$$

Proof. By construction, the functions defined in (3.40) are indeed eigenfunctions of A_2^V. To prove that every eigenfunctions of this operator is of the form (3.40), it suffices to follow the lines of the proof of [13, §8.1.1d.] for the Dirichlet operator in the unit disk. ☐

We recover the spectrum of the Stokes operator as computed for instance in [37].

4. Lifting operators of the boundary data

4.1. **Lifting operators for the stream functions.** Considering (2.3c), the velocity field u solution to the NS equations in primitive variables is assumed to satisfy Dirichlet boundary conditions on Σ, the trace of u on Σ being denoted by b. Classically, this constraint is dealt with by means of a lifting operator. We refer to [58] and references therein for a quite comprehensive survey on this topic. In nonprimitive variables, as already mentioned earlier in (2.5), (2.6) and (2.7), the Dirichlet conditions for u translate into Neumann boundary conditions for both the potential and the stream function, namely:

$$\frac{\partial\varphi}{\partial n} = b\cdot n \quad \text{and} \quad \frac{\partial\psi}{\partial n} = \frac{\partial\varphi}{\partial\tau} - b\cdot\tau \quad \text{on } \Sigma. \tag{4.1a}$$

Around every inner boundaries Σ_j^-, the circulation of the fluid is classically defined by:

$$\mathit{\Gamma}_j = \int_{\Sigma_j^-} b\cdot\tau\,\mathrm{d}s = -\int_{\Sigma_j^-}\frac{\partial\psi}{\partial n}\,\mathrm{d}s \qquad (j = 1,\dots,N). \tag{4.1b}$$

This being reminded, identities (4.1a) and (4.1b) suggest that instead of the field b, the prescribed data on the boundary shall rather be given at every moment under the form of a triple $(g_n, g_\tau, \mathit{\Gamma})$ where g_n and g_τ are scalar functions defined on Σ and $\mathit{\Gamma} = (\mathit{\Gamma}_1,\dots,\mathit{\Gamma}_N)$ is a vector in $\mathbb{R}^N$ in such a way that:

$$b = g_n n + \Big(g_\tau - \sum_{j=1}^N \mathit{\Gamma}_j\frac{\partial\xi_j}{\partial n}\Big)\tau \quad \text{with} \int_\Sigma g_n\,\mathrm{d}s = 0 \quad \text{and} \quad \int_{\Sigma_j^-} g_\tau\,\mathrm{d}s = 0 \qquad (j = 1,\dots,N). \tag{4.1c}$$

We recall that n and τ stand respectively for the unit outer normal and unit tangent vectors to Σ. The definition of suitable function spaces for g_n and g_τ requires introducing the following indices used to make precise the regularity of the boundary Σ. Thus, for every integer k, we define:

$$I_1(k) = \Big|k - \frac{1}{2}\Big| - \frac{1}{2}, \qquad J_1(k) = \Big|k - \frac{1}{2}\Big| + \frac{1}{2} = \max\{I_1(k-1), I_1(k+1)\}, \tag{4.2a}$$

$$I_2(k) = |k-1| + 1, \qquad J_2(k) = ||k| - 1| + 2 = \begin{cases}\max\{I_2(k-1), I_2(k+1)\} & \text{if } k \geqslant 0\\ I_2(k+1) & \text{if } k \leqslant -1,\end{cases} \tag{4.2b}$$

and we can now state:

Definition 4.1. *Let k be an integer. Assuming that Σ is of class $\mathcal{C}^{I_1(k),1}$, it makes sense to define:*

$$G^n_k = \left\{ g \in H^{k-\frac{1}{2}}(\Sigma) \,:\, \int_\Sigma g \,\mathrm{d}s = 0 \right\} \text{ if } k \geqslant -1 \quad \text{and} \quad G^n_k = G^n_{-1} \text{ otherwise} \tag{4.3a}$$

$$\text{and} \qquad G^\tau_k = \left\{ g \in H^{k-\frac{1}{2}}(\Sigma) \,:\, \int_{\Sigma^-_j} g \,\mathrm{d}s = 0, \quad j = 1, \dots, N \right\}, \tag{4.3b}$$

where the boundary integrals are understood according to the rule of notation (3.1).

The only purpose of setting $G^n_k = G^n_{-1}$ when $k \leqslant -2$ in (4.3a) is to simplify the statement of the next results.

The problem of lifting the normal component g_n by the harmonic Kirchhoff potential function is addressed in the lemma below, where, for every nonnegative integer k:

$$\mathfrak{H}^k_K = \left\{ \varphi \in H^k(\mathcal{F}) \,:\, \Delta\varphi = 0 \text{ in } \mathcal{D}'(\mathcal{F}), \int_\mathcal{F} \varphi \,\mathrm{d}x = 0 \text{ and } \int_\Sigma \frac{\partial \varphi}{\partial n} = 0 \right\}. \tag{4.4}$$

Lemma 4.2. *Assume that Σ is of class $\mathcal{C}^{|k|,1}$ for some integer $k \geqslant -1$ and that g_n belongs to G^n_k. Then the operator*

$$\mathsf{L}^n_k : g_n \in G^n_k \longmapsto \varphi \in \mathfrak{H}^{k+1}_K \qquad \text{where} \quad \frac{\partial \varphi}{\partial n}\Big|_\Sigma = g_n, \tag{4.5}$$

is well defined and bounded. The operator

$$\mathsf{T}_k : g_n \in G^n_k \longmapsto \frac{\partial \varphi}{\partial \tau}\Big|_\Sigma \in G^\tau_k, \tag{4.6}$$

is bounded as well. Moreover, as for the definition of G^k_n, the definition of T_k is extended to integers $k \leqslant -2$ by setting $\mathsf{T}_k = \mathsf{T}_{-1}$.

Proof. Let us only consider the weakest case, i.e. $k = -1$. We introduce the Hilbert space E and its scalar product whose corresponding norm is equivalent in E to the usual norm of $H^2(\mathcal{F})$:

$$E = \left\{ \theta \in H^2(\mathcal{F}) \,:\, \frac{\partial \theta}{\partial n}\Big|_\Sigma = 0, \int_\Sigma \theta|_\Sigma \,\mathrm{d}s = 0 \right\}, \qquad (\theta_1, \theta_2)_E = \int_\mathcal{F} \Delta\theta_1 \Delta\theta_2 \,\mathrm{d}x.$$

According to Riesz representation Theorem, for every $g_n \in G^n_{-1}$, there exists a unique $\theta_g \in E$ such that:

$$(\theta, \theta_g)_E = -\int_\Sigma g_n \theta|_\Sigma \,\mathrm{d}s \qquad \text{for all } \theta \in E.$$

One easily verifies that the function $\varphi = \Delta\theta_g$ is in $L^2(\mathcal{F})$ and satisfies $\int_\mathcal{F} \varphi \,\mathrm{d}x = 0$ and $(\partial\varphi/\partial n)|_\Sigma = g_n$ in $H^{-\frac{3}{2}}(\Sigma)$. The rest of the lemma being either classical or obvious, the proof is complete. □

The operator T_k is the tangential differential operator composed with the classical Neumann-to-Dirichlet map. Regarding now the second identity in (4.1a), we seek a lifting operator valued in the kernel of the operator $\mathsf{Q}\Delta$, that is the kernel of the Stokes operator for the stream function (see Lemma 3.6). Loosely speaking (disregarding regularity issues), this kernel is $\mathfrak{B}_S$, the space of the biharmonic stream functions defined in (3.22).

Definition 4.3. *Let k be an integer and assume that Σ is of class $\mathcal{C}^{I_2(k),1}$. The space of biharmonic functions $\mathfrak{B}^k_S$ and the lifting operator $\mathsf{L}^\tau_k : G^\tau_k \to \mathfrak{B}^k_S$ are defined differently, depending upon the sign of k:*

(1) *When $k \geqslant 1$, $\mathfrak{B}^k_S = \mathfrak{B}_S \cap H^{k+1}(\mathcal{F})$ (and hence $\mathfrak{B}^1_S$ is simply equal to $\mathfrak{B}_S$ defined in* (3.22)*) and for any $g_\tau \in G^\tau_k$, $\mathsf{L}^\tau_k g_\tau$ is the unique stream function ψ in $\mathfrak{B}^k_S$ satisfying the Neumann boundary condition:*

$$\frac{\partial \psi}{\partial n}\Big|_\Sigma = g_\tau \qquad \text{on } \Sigma.$$

(2) *When $k \leqslant 0$, for any $g_\tau \in G_k^\tau$, $\mathsf{L}_k^\tau g_\tau$ is the element of the dual space S_k given by:*

$$\langle \mathsf{L}_k^\tau g_\tau, \theta \rangle_{S_{-k}, S_k} = \int_\Sigma (\mathsf{P}_{-k+1}\theta) g_\tau \,\mathrm{d}s \qquad \textit{for all } \theta \in S_{-k}, \tag{4.7}$$

and the space $\mathfrak{B}_S^k$ is defined as the image of L_k^τ in S_k.

Remark 4.4. (1) *For $k \leqslant 0$, the operator L_k^τ is well defined according to Lemma 3.2 and Lemma 3.6, under the regularity assumption on the boundary Σ of Definition 4.3.*

(2) *For every integer k, the space G_k^τ is actually well defined as soon as the boundary Σ is of class $\mathcal{C}^{I_1(k),1}$ (see Definition 4.1). However, further regularity is needed to define the lifting operator, namely $\mathcal{C}^{I_2(k),1}$.*

For every pair of integers (k', k), both positive or both nonpositive, the inequality $k' \geqslant k$ entails the inclusion $\mathfrak{B}_S^{k'} \subset \mathfrak{B}_S^k$. We shall prove that the inclusion $\mathfrak{B}_S^1 \subset \mathfrak{B}_S^0$ still holds and that the diagram on Fig. 5 commutes. Notice that L_k^τ is clearly invertible when k is positive. The question of invertibility for nonpositive indices k, or more precisely of injectivity (since surjectivity is obvious) is not clear. This amounts to determine whether the traces of the functions of V_{-k+1} are dense in $H^{-k+\frac{1}{2}}(\Sigma)$.

$$\begin{array}{ccc} G_{k'}^\tau & \subset & G_k^\tau \\ \big\downarrow \mathsf{L}_{k'}^\tau & & \big\downarrow \mathsf{L}_k^\tau \\ \mathfrak{B}_S^{k'} & \subset & \mathfrak{B}_S^k \end{array}$$

FIGURE 5. The diagram commutes for any pair of integers (k, k') such that $k' \geqslant k$.

Lemma 4.5. *The operator L_k^τ is bounded for every integer k and is an isomorphism when k is positive. For every pair of integers (k', k) such that $k' \geqslant k$, the restriction of L_k^τ to $G_{k'}^\tau$ is equal to $\mathsf{L}_{k'}^\tau$ (providing that Σ is of class $\mathcal{C}^{\max\{I_2(k), I_2(k')\},1}$).*

Proof. The boundedness is a consequence of Lemma 3.2, Lemma 3.6 and the continuity of the trace operator.

Let Σ be of class $\mathcal{C}^{2,1}$, g_τ belong to G_1^τ and introduce the stream function $\psi = \mathsf{L}_1^\tau g_\tau$. Considering $\psi \in \mathfrak{B}_S^1$ as an element of S_0 identified with its dual space, we get:

$$(\psi, \theta)_{S_0} = (\nabla\psi, \nabla \mathsf{P}_1\theta)_{\mathbf{L}^2(\mathcal{F})} = \int_\Sigma (\mathsf{P}_1\theta) g_\tau \,\mathrm{d}s - (\Delta\psi, \mathsf{P}_1\theta)_{L^2(\mathcal{F})} \qquad \text{for all } \theta \in S_0,$$

where the last term vanishes because $\Delta\psi$ belongs to $\mathfrak{H}$. This proves that $\mathsf{L}_1^\tau = \mathsf{L}_0^\tau$ in G_1^τ. The other cases derive straightforwardly and the proof is complete. ☐

We can gather Lemma 4.2 and Definition 4.3 in order to define a lifting operator taking into account the circulation of the fluid around the fixed obstacles. In view of (4.1a) and (4.1b), we are led to set:

Definition 4.6. *Let k be any integer and assume that Σ is of class $\mathcal{C}^{I_2(k),1}$ and that the triple (g_n, g_τ, Γ) is in $G_k^n \times G_k^\tau \times \mathbb{R}^N$ with $\Gamma = (\Gamma_1, \ldots, \Gamma_N)$. We define the operator:*

$$\mathsf{L}_k^S(g_n, g_\tau, \Gamma) = \mathsf{L}_k^\tau(\mathsf{T}_k g_n - g_\tau) + \sum_{j=1}^N \Gamma_j \xi_j, \tag{4.8}$$

which is valued in the space

$$S_k^{\mathrm{b}} = \mathfrak{B}_S^k \oplus \mathbb{F}_S. \tag{4.9}$$

We can address the case of time dependent spaces:

Definition 4.7. *Let T be a positive real number, k be an integer and assume that Σ is of class $\mathcal{C}^{J_1(k),1}$ (the expression of $J_1(k)$ is given in (4.2)). We begin by introducing the spaces:*

$$G^n_k(T) = L^2(0,T;G^n_{k+1}) \cap \mathcal{C}([0,T];G^n_k) \cap H^1(0,T;G^n_{k-1})$$
$$G^\tau_k(T) = L^2(0,T;G^\tau_{k+1})) \cap \mathcal{C}([0,T];G^\tau_k) \cap H^1(0,T;G^\tau_{k-1}),$$

and also:

$$G_k(T) = \begin{cases} G^n_k(T) \times G^\tau_k(T) \times H^1(0,T;\mathbb{R}^N) & \text{when } k \geqslant 0,\\ L^2(0,T;G^n_{k+1}) \times L^2(0,T;G^\tau_{k+1}) \times L^2(0,T;\mathbb{R}^N) & \text{when } k \leqslant -1.\end{cases} \tag{4.10}$$

Assuming that Σ is of class $\mathcal{C}^{J_2(k),1}$ (with $J_2(k)$ defined in (4.2)) the operator L^S_{k+1} maps to space $G_k(T)$ into the space:

$$S^{\rm b}_k(T) = \begin{cases} H^1(0,T;S^{\rm b}_{k-1}) \cap \mathcal{C}([0,T];S^{\rm b}_k) \cap L^2(0,T;S^{\rm b}_{k+1}) & \text{if } k \geqslant 0,\\ L^2(0,T;S^{\rm b}_{k+1}) & \text{if } k \leqslant -1.\end{cases} \tag{4.11}$$

As a direct consequence of Lemmas 4.2 and 4.5, we can state:

Lemma 4.8. *Let k be any integer and assume that Σ is of class $\mathcal{C}^{I_2(k),1}$. Then the lifting operator for the stream function:*

$$\mathsf{L}^S_k : G^n_k \times G^\tau_k \times \mathbb{R}^N \longrightarrow S^{\rm b}_k,$$

is well defined and is bounded. Moreover, if k and k' are two integers such that $k' \leqslant k$, then $\mathsf{L}^S_{k'} = \mathsf{L}^S_k$ in $G^n_k \times G^\tau_k \times \mathbb{R}^N$. It follows that for every positive real number T and every integer k, providing that Σ is of class $C^{J_2(k),1}$, the operator:

$$\mathsf{L}^S_{k+1} : G_k(T) \longrightarrow S^{\rm b}_k(T),$$

is well defined and bounded as well, the bound being uniform with respect to T.

4.2. **Additional function spaces.** We aim now at building a lifting operator valued in vorticity spaces (i.e. we aim at giving the counterpart of Definitions 4.6-4.7 and Lemma 4.8 for the vorticity). We recall that, for every positive integer k, the lifting operator L^S_k is valued in $S^{\rm b}_k$. For nonpositive integers k, $S^{\rm b}_k$ is a subspace of S_k and therefore, the corresponding vorticity space is simply $V^{\rm b}_{k-1} = \Delta_{k-1} S^{\rm b}_k$. However, when k is positive, $S^{\rm b}_k \cap S_k = \{0\}$. A somehow naive approach would consist in taking simply the Laplacian of $S^{\rm b}_k$ but one easily verifies that $\Delta S^{\rm b}_k \subset \mathfrak{H}$ and $\mathfrak{H}$ is in no space V_j for any integer j. This difficulty is circumvented by noticing that $S^{\rm b}_k \subset S_0$ (still considering positive integers k). So $V^{\rm b}_k = \Delta_{-1} S^{\rm b}_k$ (with Δ_{-1} defined in (3.31b)) seems to be a good candidate for our purpose, an idea we are now going to elaborate on. More precisely, for every integer k, $S^{\rm b}_k$ is a subspace of $\bar{S}_k$ defined by:

$$\bar{S}_k = S_0 \cap H^{k+1}(\mathcal{F}) \quad \text{if } k \geqslant 1 \qquad \text{and} \qquad \bar{S}_k = S_k \quad \text{if } k \leqslant 0. \tag{4.12}$$

The corresponding vorticity space is therefore in the image of $\bar{S}_k$ (seen as a subspace of S_0) by Δ_{-1} if $k \geqslant 1$ and by Δ_{k-1} if $k \leqslant 0$ (see Fig. 4). Thus we define:

$$\bar{V}_k = \Delta_{-1} \bar{S}_{k+1} \quad \text{if } k \geqslant 0 \qquad \text{and} \qquad \bar{V}_k = \Delta_k \bar{S}_{k+1} = V_k \quad \text{if } k \leqslant -1. \tag{4.13}$$

It is crucial to understand that, no matter how regular the functions are, the spaces $\bar{V}_k$ are always dual spaces (for every integer k). They are subspaces of V_{-1}. We will show that V_{-1} is the space of largest index that contains in some sense the harmonic functions. We shall focus our analysis on the pairs $(\bar{S}_1, \bar{V}_0)$, $(\bar{S}_2, \bar{V}_1)$ and $(\bar{S}_3, \bar{V}_2)$ only, the other cases being of less importance as it will becomes clear in the next section.

The pair $(\bar{S}_1, \bar{V}_0)$. The space $\bar{S}_1$ is provided with the scalar product:

$$(\bar\psi_1, \bar\psi_2)_{\bar{S}_1} = (\Delta\bar\psi_1, \Delta\bar\psi_2)_{L^2(\mathcal{F})} + \Gamma(\bar\psi_1) \cdot \Gamma(\bar\psi_2), \qquad \text{for all } \bar\psi_1, \bar\psi_2 \in \bar{S}_1, \tag{4.14}$$

where, for every $\theta \in H^2(\mathcal{F})$:

$$\Gamma(\theta) = \big(\Gamma_1(\theta), \ldots, \Gamma_N(\theta)\big)^t \in \mathbb{R}^N \qquad \text{with} \qquad \Gamma_j(\theta) = -\int_{\Sigma^-_j} \frac{\partial\theta}{\partial n}\,{\rm d}s, \quad (j = 1, \ldots, N).$$

Lemma 4.9. *The space $\bar{S}_1$ enjoys the following properties:*

(1) *The norm induced by the scalar product* (4.14) *is equivalent in* $\bar{S}_1$ *to the usual norm of* $H^2(\mathcal{F})$.

(2) *The space* $\bar{S}_1$ *admits the following orthogonal decompositions:*

$$\bar{S}_1 = Z_2 \overset{\perp}{\oplus} \mathbb{F}_S = S_1 \overset{\perp}{\oplus} \mathfrak{B}_S \overset{\perp}{\oplus} \mathbb{F}_S, \tag{4.15}$$

where we recall that the expression of the space $\mathfrak{B}_S$ *is given in* (3.22) *and that the finite dimensional space* $\mathbb{F}_S$ *is spanned by the functions* ξ_j $(j = 1, \ldots, N)$.

The orthogonal decomposition (4.15) can be given a physical meaning: The subspace S_1 contains the stream functions with homogeneous boundary conditions while the subspace $\mathfrak{B}_S$ contains the stream functions that solve stationary Stokes problems (with zero circulation though). Finally, the space $\mathbb{F}_S$ contains the harmonic stream functions accounting for the circulation of the fluid around the inner boundaries Σ_j^- $(j = 1, \ldots, N)$.

Proof of Lemma 4.9. The equivalence of the norms derives from classical elliptic regularity results. On the other hand, according to (3.21):

$$Z_2 = S_1 \overset{\perp}{\oplus} \mathfrak{B}_S \subset \bar{S}_1. \tag{4.16}$$

Applying the fundamental homomorphism theorem to the surjective operator $(-\Delta) : \bar{S}_1 \longrightarrow Z_0$ whose kernel is the space $\mathbb{F}_S$, we next obtain that:

$$\bar{S}_1 = Z_2 \oplus \mathbb{F}_S. \tag{4.17}$$

Finally, combining (4.16) and (4.17) yields (4.15) after verifying that the direct sum is orthogonal for the scalar product (4.14). The proof is then completed. □

Let us determine now the corresponding decomposition for the vorticity space $\bar{V}_0 = \Delta_{-1}\bar{S}_1$ which is a subspace of the dual space V_{-1} (see Fig. 4). We shall prove in particular that $\bar{V}_0$ contains V_0 whose expression is (seen as a subspace of V_{-1}):

$$V_0 = \big\{(\omega, \mathsf{Q}_1 \cdot)_{L^2(\mathcal{F})} \,:\, \omega \in \mathfrak{H}^\perp\big\}. \tag{4.18}$$

Notice that in (4.18), one would expect merely the term $(\omega, \cdot)_{L^2(\mathcal{F})}$ in place of $(\omega, \mathsf{Q}_1 \cdot)_{L^2(\mathcal{F})}$, but both linear forms are equal in V_1. We define below two additional subspaces of V_{-1}:

$$\mathfrak{H}_V = \big\{(\omega, \mathsf{Q}_1 \cdot)_{L^2(\mathcal{F})} \,:\, \omega \in \mathfrak{H}\big\} \qquad \text{and} \qquad L^2_V = \big\{(\omega, \mathsf{Q}_1 \cdot)_{L^2(\mathcal{F})} \,:\, \omega \in L^2(\mathcal{F})\big\}. \tag{4.19}$$

The space $\mathfrak{H}_V$ contains in some sense the harmonic vorticity field. Finally, we introduce the finite dimensional subspace of V_{-1}:

$$\mathbb{F}^*_V = \operatorname{span}\{\zeta_j,\, j = 1, \ldots, N\}, \tag{4.20}$$

where, for every $j = 1, \ldots, N$:

$$\langle \zeta_j, \omega \rangle_{V_{-1}, V_1} = -(\nabla \xi_j, \nabla \mathsf{Q}_1 \omega)_{\mathbf{L}^2(\mathcal{F})} = -\int_{\Sigma_j^-} \frac{\partial \xi_j}{\partial n} \mathsf{Q}_1 \omega \, \mathrm{d}s \qquad \text{for all } \omega \in V_1. \tag{4.21}$$

We can now state:

Theorem 4.10. *The space* $\bar{V}_0$ *can be decomposed as follows:*

$$\bar{V}_0 = L^2_V \overset{\perp}{\oplus} \mathbb{F}^*_V = V_0 \overset{\perp}{\oplus} V_0^{\mathrm{b}} \qquad \textit{where} \qquad V_0^{\mathrm{b}} = \mathfrak{H}_V \overset{\perp}{\oplus} \mathbb{F}^*_V. \tag{4.22}$$

The direct sum above is orthogonal for the scalar product defined, for every $\bar{\omega}_1$ *and* $\bar{\omega}_2$ *in* $\bar{V}_0$ *by:*

$$(\bar{\omega}_1, \bar{\omega}_2)_{\bar{V}_0} = (\omega_1, \omega_2)_{L^2(\mathcal{F})} + \sum_{j=1}^N \alpha_{1,j} \alpha_{2,j}, \tag{4.23}$$

where, for $k = 1, 2$, $\bar{\omega}_k = (\omega_k, \mathsf{Q}_1 \cdot)_{L^2(\mathcal{F})} + \zeta^k$ *with* $\omega_k \in L^2(\mathcal{F})$ *and* $\zeta^k = \sum_{j=1}^N \alpha_{k,j}\zeta_j$ *in* $\mathbb{F}^*_V$ $(\alpha_{k,j} \in \mathbb{R}$ *for* $j = 1, \ldots, N)$.

Moreover, the restriction of Δ_{-1} *to* $\bar{S}_1$, *denoted by* $\bar{\Delta}_0$, *is an isometry from* $\bar{S}_1$ *onto* $\bar{V}_0$ *(see Fig. 6).*

Remark 4.11. *Let us emphasize that:*

(1) *In the decomposition* $\bar\omega = (\omega, \mathsf{Q}_1\cdot)_{L^2(\mathcal{F})} + \zeta$ *of every* $\bar\omega$ *of* $\bar V_0$*, the term* ω *which belongs to* $L^2(\mathcal{F})$ *will be referred to as the regular part of* $\bar\omega$ *while* ζ *will stand for the singular part.*

(2) *Loosely speaking, the space* $\bar V_0$ *consists in functions in* $L^2(\mathcal{F})$ *and measures* ζ_j $(j = 1, \ldots, N)$ *supported on the boundaries* Σ_j^- *(notice again that* $\mathbb{F}_V^*$ *is not a distributions space). This can be somehow understood from a physical point of view by observing that* $-\zeta_j$ *is the vorticity corresponding to the harmonic stream function* ξ_j *which accounts for the circulation of the fluid around* Σ_j^-*. Hence the vorticity is a measure supported on the boundary of the obstacle. In connection with this topic, wondering how is vorticity imparted to the fluid when a stream flow past an obstacle, Lighthill answers in* [59] *that the solid boundary is a distributed source of vorticity (just as, in some flows, it may be a distributed source of heat).*

(3) *The fact that* $\bar\Delta_0$ *is an isometry asserts in particular that to any given vorticity in* $\bar V_0$ *corresponds a unique stream function* $\bar\psi$ *in* $\bar S_1$ *that can be uniquely decomposed as* $\psi + \psi_S + \psi_C$ *where* $\nabla^\perp\psi = 0$ *on the boundary* Σ*,* $\nabla^\perp\psi_S$ *solves a stationary Stokes system and* ψ_C *is harmonic in* $\mathcal{F}$ *and accounts for the circulation of the fluid around the boundaries* Σ_j^- $(j = 1, \ldots, N)$*.*

The rest of this subsection is dedicated to the proof of Theorem 4.10. In order to determine the image of $\bar S_1$ by the operator Δ_{-1}, the factorization $\Delta_{-1} = -\mathsf{A}_1^V\mathsf{P}_1$ suggests to determine first the expression of the space $\mathsf{P}_1\bar S_1$. This space is provided with the scalar product:

$$(\omega_1, \omega_2)_{\mathsf{P}_1\bar S_1} = (\Delta\omega_1, \Delta\omega_2)_{L^2(\mathcal{F})} + \Gamma(\omega_1)\cdot\Gamma(\omega_2), \qquad \text{for all } \omega_1, \omega_2 \in \mathsf{P}_1\bar S_1,$$

and we denote by $\bar{\mathsf{P}}_2$ the restriction of P_1 to $\bar S_1$.

Remark 4.12. *According to Lemma 3.2, when* Σ *is of class* $\mathcal{C}^{3,1}$*, the space* $\mathsf{P}_1\bar S_1$ *is simply equal to* $V_1 \cap H^2(\mathcal{F})$*. When* Σ *is less regular Remark 3.11 applies replacing* V_2 *with* $\mathsf{P}_1\bar S_1$*.*

The decomposition (4.15) leads to introducing the spaces:

$$\mathfrak{B}_V = \mathsf{P}_1\mathfrak{B}_S \qquad \text{and} \qquad \mathbb{F}_V = \mathsf{P}_1\mathbb{F}_S. \tag{4.24}$$

Nothing more than $\mathfrak{B}_V = \mathfrak{B} \cap V_0$ (where $\mathfrak{B}$ is defined in (3.20)) can be said on the space $\mathfrak{B}_V$. The space $\mathbb{F}_V$ however can be bound to the space $\mathbb{B}_S$ spanned by the functions χ_j $(j = 1, \ldots, N)$ defined in (3.5) (see the definition below the identity (3.6)).

Lemma 4.13. (1) *The space* $\mathbb{B}_S$ *is a subspace of* S_2 *and* $\mathbb{F}_V = \Delta_1\mathbb{B}_S$*.*

(2) *The space* $\mathsf{P}_1\bar S_1$ *admits the following orthogonal decomposition:*

$$\mathsf{P}_1\bar S_1 = V_2 \overset{\perp}{\oplus} \mathfrak{B}_V \overset{\perp}{\oplus} \mathbb{F}_V. \tag{4.25}$$

Moreover, the operator $\bar{\mathsf{P}}_2$ *is an isometry from* $\bar S_1$ *onto* $\mathsf{P}_1\bar S_1$ *(see Fig. 6).*

Proof. For every $\theta \in \mathcal{D}$, an integration by parts yields:

$$(\chi_j, \theta)_{S_1} = -(\mathsf{Q}_1\Omega_j, \theta)_{S_0} \qquad \text{for all } j = 1, \ldots, N, \tag{4.26a}$$

where $\Omega_j = \Delta\chi_j$, because χ_j is of class $\mathcal{C}^\infty$ in the support of θ. On the other hand, for every pair of indices $j, k \in \{1, \ldots, N\}$, we have also:

$$(\chi_j, \chi_k)_{S_1} = \int_\Sigma \mathsf{Q}_1\Omega_j \frac{\partial\chi_k}{\partial n}\,\mathrm{d}s - (\mathsf{Q}_1\Omega_j, \chi_k)_{S_0} = -(\mathsf{Q}_1\Omega_j, \chi_k)_{S_0}, \tag{4.26b}$$

where we have used the rule of notation (3.1) (as being harmonic in $L^2(\mathcal{F})$, the trace of Ω_j on Σ is well defined in $H^{-\frac12}(\Sigma)$). Since the space $\mathcal{D}(\mathcal{F}) \oplus \mathbb{B}_S$ is dense in S_1 according to the decomposition (3.6), we deduce from the identities (4.26) that $\mathbb{B}_S \subset S_2$ (recall the S_2 is the preimage of S_0 by A_1^S).

Notice now that the functions $\mathsf{P}_1\xi_j$ $(j = 1, \ldots, N)$ belong to V_1, are harmonic in $\mathcal{F}$ and according to the second point of Remark 3.3 they satisfy the same fluxes conditions (3.3b) as the functions ξ_j. All these properties are also shared by the functions Ω_j whence we deduce first that:

$$\mathsf{P}_1\xi_j = \Omega_j \qquad (j = 1, \ldots, N), \tag{4.27}$$

and then that $\mathbb{F}_V = \Delta_1 \mathbb{B}_S$, which is the first point of the lemma. The second point can easily be deduced from (4.15) and the second occurence of Remark 3.3. $\square$

Yet it remains to apply the operator A_1^V to the equality (4.25) in order to get the expression of $\bar{V}_0 = \Delta_{-1}\bar{S}_1$. Since A_1^V is an isometry, the decomposition (4.22) is a direct consequence of the decomposition (4.25) and the following lemma, where the spaces $\mathfrak{H}_V$ and $\mathbb{F}_V^*$ are defined respectively in (4.19) and (4.20).

Lemma 4.14. *The following equalities hold:*

$$\mathsf{A}_1^V \mathfrak{B}_V = \mathfrak{H}_V \qquad \text{and} \qquad \mathsf{A}_1^V \mathbb{F}_V = \mathbb{F}_V^*. \tag{4.28}$$

Proof. By definition, every element of $\mathfrak{B}_V$ can be written $\mathsf{P}_1\psi$ for some $\psi \in \mathfrak{B}_S$. The definitions of the operator A_1^V and of the scalar product in V_1 lead to:

$$\langle \mathsf{A}_1^V \mathsf{P}_1\psi, \theta\rangle_{V_{-1},V_1} = (\mathsf{P}_1\psi,\theta)_{V_1} = (\psi, \mathsf{Q}_1\theta)_{Z_1}, \qquad \text{for all } \theta \in V_1.$$

But $\mathfrak{B}_S$ is a subspace of Z_2 according to (3.21) and $\mathsf{Q}_1 V_1 = Z_1$. It follows that:

$$(\psi, \mathsf{Q}_1\theta)_{Z_1} = (\mathsf{A}_2^Z\psi, \mathsf{Q}_1\theta)_{Z_0} = (\omega, \mathsf{Q}_1\theta)_{L^2(\mathcal{F})}, \qquad \text{for all } \theta \in V_1,$$

where $\omega = \mathsf{A}_2^Z\psi$ belongs to $\mathfrak{H}$ according to Lemma 3.9. The first equality in (4.28) being proven, let us address the latter. For every $j = 1, \ldots, N$, some elementary algebra yields:

$$\langle \mathsf{A}_1^V \mathsf{P}_1\xi_j, \omega\rangle_{V_{-1},V_1} = (\mathsf{P}_1\xi_j,\omega)_{V_1} = (\nabla\xi_j, \nabla\mathsf{Q}_1\omega)_{\mathbf{L}^2(\mathcal{F})} = (\xi_j, \mathsf{Q}_1\omega)_{S_0}, \qquad \text{for all } \omega \in V_1.$$

Comparing with (4.21), we obtain indeed that $\mathsf{A}_1^V\mathsf{P}_1\xi_j = -\zeta_j$ and recalling the definition of $\mathbb{F}_V$ given in (4.24), we are done with both identities in (4.28) and the proof is completed. $\square$

As we did for $\bar{S}_1$ and $\mathsf{P}_1\bar{S}_1$, the space $\bar{V}_0$ can be provided with a norm stronger than the one of the ambiant space V_{-1}, namely the norm which derives from the scalar product (4.23). One easily verifies that the direct sum (4.22) is indeed orthogonal for this scalar product. Furthermore, the operator $\bar{\mathsf{A}}_2^V : \mathsf{P}_1\bar{S}_1 \to \bar{V}_0$ which is the restriction of A_1^V to $\mathsf{P}_1\bar{S}_1$ is an isometry. Since $\bar{\Delta}_0 = -\bar{\mathsf{A}}_2^V\bar{\mathsf{P}}_2$ and the operators $\bar{\mathsf{A}}_2^V$ and $\bar{\mathsf{P}}_2$ are both isometries, we can draw the same conclusion for $\bar{\Delta}_0$. The proof of the theorem is now completed. $\square$

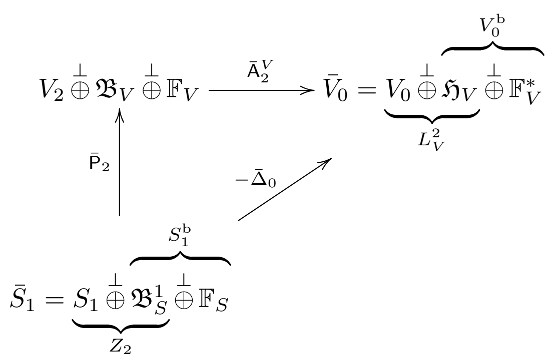

FIGURE 6. Some function spaces and isometric operators appearing in the statement of Theorem 4.10 and its proof. As usual, the top row contains the vorticity spaces while the bottom row contains the spaces for the stream functions.

The pair $(\bar{S}_2, \bar{V}_1)$. We assume that Σ is of class $\mathcal{C}^{2,1}$ and we consider the spaces:

$$\bar{S}_2 = S_0 \cap H^3(\mathcal{F}) \qquad \text{and} \qquad \bar{V}_1 = \Delta_{-1}\bar{S}_2. \tag{4.29}$$

The analysis of these spaces being very similar to those of $\bar{S}_1$ and $\bar{V}_0$, we shall skip the details and focus on the main results.

Lemma 4.15. *The spaces $\bar{S}_2$ and $\mathsf{P}_1\bar{S}_2$ admit respectively the following orthogonal decompositions:*

$$\bar{S}_2 = S_2 \overset{\perp}{\oplus} \mathfrak{B}_S^2 \overset{\perp}{\oplus} \mathbb{F}_S \qquad \text{and} \qquad \mathsf{P}_1\bar{S}_2 = V_3 \overset{\perp}{\oplus} \mathfrak{B}_V^3 \overset{\perp}{\oplus} \mathbb{F}_V, \tag{4.30}$$

where $\mathfrak{B}_S^2 = \mathfrak{B}_S \cap H^3(\mathcal{F})$ was introduced in Definition 4.3 and $\mathfrak{B}_V^3 = \mathsf{P}_1\mathfrak{B}_S^2$. The spaces $\bar{S}_2$ and $\mathsf{P}_1\bar{S}_2$ are provided with the same scalar product, namely:

$$(\psi_1,\psi_2)_{\bar{S}_2} = (\Delta\psi_1,\Delta\psi_2)_{H^1}^V + \Gamma(\psi_1)\cdot\Gamma(\psi_2) \qquad \text{for all } \psi_1,\psi_2 \in \bar{S}_2,$$
$$(\omega_1,\omega_2)_{\mathsf{P}_1\bar{S}_2} = (\Delta\omega_1,\Delta\omega_2)_{H^1}^V + \Gamma(\omega_1)\cdot\Gamma(\omega_2) \qquad \text{for all } \omega_1,\omega_2 \in \mathsf{P}_1\bar{S}_2,$$

the scalar product $(\cdot,\cdot)_{H^1}^V$ being defined in (3.13b).

Finally, the operator $\bar{\mathsf{P}}_3$ defined as the restriction of P_1 to $\bar{S}_2$ is an isometry from $\bar{S}_2$ onto $\mathsf{P}_1\bar{S}_2$ (see Fig. 7).

We turn now our attention to $\bar{V}_1 = \Delta_{-1}\bar{S}_2$, which is a subspace of the dual space V_{-1}. As a subspace of V_{-1} the spaces V_1 is identified with $\{(\omega,\mathsf{Q}_1\cdot)_{L^2(\mathcal{F})} : \omega \in V_1\}$ and we define as well:

$$\mathfrak{H}_V^1 = \big\{(\omega,\mathsf{Q}_1\cdot)_{L^2(\mathcal{F})} : \omega \in \mathfrak{H}^1\big\},$$

where we recall that $\mathfrak{H}^1 = \mathfrak{H} \cap H^1(\mathcal{F})$ (defined in Subsection 3.1). Finally, in the same way, we introduce:

$$H_V^1 = \{(\omega,\mathsf{Q}_1\cdot)_{L^2(\mathcal{F})} : \omega \in H^1(\mathcal{F})\},$$

that can be compared with the space L_V^2 defined in (4.19).

Theorem 4.16. *The space $\bar{V}_1$ is a subspace of V_{-1} which can be decomposed as follows:*

$$\bar{V}_1 = H_V^1 \overset{\perp}{\oplus} \mathbb{F}_V^* = V_1 \overset{\perp}{\oplus} V_1^{\rm b} \qquad \text{with} \qquad V_1^{\rm b} = \mathfrak{H}_V^1 \overset{\perp}{\oplus} \mathbb{F}_V^*. \tag{4.31}$$

It is provided with the scalar product, defined for every $\bar{\omega}_1,\bar{\omega}_2 \in \bar{V}_1$ by:

$$(\bar{\omega}_1,\bar{\omega}_2)_{\bar{V}_1} = (\omega_1,\omega_2)_{H^1}^V + \sum_{j=1}^N \alpha_{1,j}\alpha_{2,j},$$

where, for $k=1,2$, $\bar{\omega}_k = (\omega_k,\mathsf{Q}_1\cdot)_{L^2(\mathcal{F})} + \zeta^k$ with $\omega_k \in H^1(\mathcal{F})$ and $\zeta^k = \sum_{j=1}^N \alpha_{k,j}\zeta_j$ in $\mathbb{F}_V^*$ ($\alpha_{k,j} \in \mathbb{R}$ for $j=1,\dots,N$).

Finally, the operator $\bar{\Delta}_1$ which is the restriction of Δ_{-1} to $\bar{S}_2$ is an isometry from $\bar{S}_2$ onto $\bar{V}_1$ (see Fig. 7).

Remark 4.17. *As in Remark 4.11, in the decomposition $\bar{\omega} = (\omega,\mathsf{Q}_1\cdot)_{L^2(\mathcal{F})} + \zeta$ of every vorticity field $\bar{\omega}$ in $\bar{V}_1$, the term ω (belonging to $H^1(\mathcal{F})$) will be called the regular part of $\bar{\omega}$ and ζ, the singular part.*

These results can be summarized in the commutative diagram on Fig. 7 where the operator $\bar{\mathsf{A}}_3^V$ defined as the restriction of A_1^V to the space $\mathsf{P}_1\bar{S}_2$ is an isometry from $\mathsf{P}_1\bar{S}_2$ onto $\bar{V}_1$.

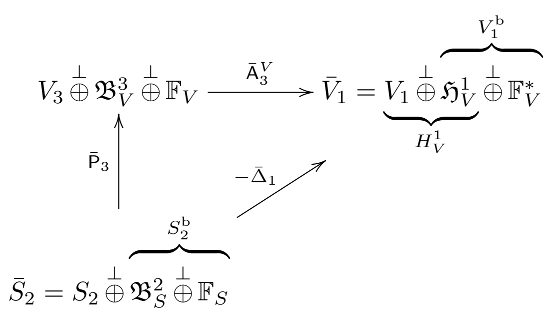

FIGURE 7. Some function spaces and isometric operators appearing in the statement of Lemma 4.15 and Theorem 4.16. This diagram is worth being compared with the diagrams on Fig. 4 and Fig. 6. In particular, the following inclusions hold: $\bar{S}_2 \subset \bar{S}_1 \subset S_0$ and $\bar{V}_1 \subset \bar{V}_0 \subset V_{-1}$.

The pair $(\bar S_3, \bar V_2)$. The decompositions of $\bar V_0$ and $\bar V_1$ rested mainly on the simple equalities $L^2(\mathcal F) = V_0 \oplus \mathfrak H$ and $H^1(\mathcal F) = V_1 \oplus \mathfrak H^1$. However, $H^2(\mathcal F)$ is not equal to $V_2 \oplus \mathfrak H^2$. Indeed, according to Fig. 6, the correct decomposition is more complex, namely:

$$H^2(\mathcal F) = V_2 \oplus \mathfrak B_V \oplus \mathbb F_V \oplus \mathfrak H^2.$$

We are led to define:

$$H^2_V = \{(\omega, \mathsf Q_1\cdot)_{L^2(\mathcal F)} \,:\, \omega \in H^2(\mathcal F)\} \qquad \text{and} \qquad \mathfrak H^2_V = \{(\omega, \mathsf Q_1\cdot)_{L^2(\mathcal F)} \,:\, \omega \in \mathfrak H^2\}.$$

Since V_2, $\mathfrak B_V$ and $\mathbb F_V$ are subspaces of the pivot space V_0, they are identified to subspaces of V_{-1} and the following decompositions hold:

$$\bar V_2 = H^2_V \oplus \mathbb F^*_V = V_2 \oplus \mathfrak B_V \oplus \mathbb F_V \oplus V^{\rm b}_2 \quad \text{with} \quad H^2_V = V_2 \oplus \mathfrak B_V \oplus \mathbb F_V \oplus \mathfrak H^2_V \quad \text{and} \quad V^{\rm b}_2 = \mathfrak H^2_V \oplus \mathbb F^*_V. \tag{4.32}$$

This direct sum is orthogonal once $\bar V_2$ is provided with the scalar product:

$$(\bar\omega_1, \bar\omega_2)_{\bar V_2} = (\Delta\omega_1, \Delta\omega_2)_{L^2(\mathcal F)} + (\mathsf P^\perp\omega_1, \mathsf P^\perp\omega_2)_{\mathfrak H^2} + \Gamma(\omega_1)\cdot\Gamma(\omega_2) + \sum_{j=1}^N \alpha_{1,j}\alpha_{2,j},$$

for every $\bar\omega_k \in \bar V_2$ such that $\bar\omega_k = (\omega_k, \mathsf Q_1\cdot)_{L^2(\mathcal F)} + \sum_{j=1}^N \alpha_{k,j}\zeta_j$ with $\omega_k \in H^2(\mathcal F)$ and $\alpha_{k,j} \in \mathbb R$ for $k = 1, 2$. The decompositions (4.32) will play an important role in Section 8 and in particular the fact that $\bar V_2$ is not equal to $V_2 \oplus V^{\rm b}_2$.

We do not need to enter into the details of the decomposition of $\bar S_3$. Let us just make precise the norm this space is equipped with, namely:

$$(\psi_1, \psi_2)_{\bar S_3} = (\Delta^2\psi_1, \Delta^2\psi_2)_{L^2(\mathcal F)} + (\mathsf P^\perp\Delta\psi_1, \mathsf P^\perp\Delta\psi_2)_{\mathfrak H^2} + \Gamma(\Delta\psi_1)\cdot\Gamma(\Delta\psi_2) + \Gamma(\psi_1)\cdot\Gamma(\psi_2),$$

for every ψ_1, ψ_2 in $\bar S_3$. As usual, we denote by $\bar\Delta_2$ the restriction of Δ_{-1} to $\bar S_3$ and we let is to the reader to verify that:

Lemma 4.18. *The operator $\bar\Delta_2$ is an isometry from $\bar S_3$ onto $\bar V_2$.*

Notice that obviously $\bar S_3$ contains the space $S^{\rm b}_3$.

4.3. **Lifting operators for the vorticity field.** The expressions of the lifting operators for the vorticity derive straightforwardly from Fig. 6 and Fig. 7. Following the lines of Definition 4.6 and recalling that the indices $I_2(k)$ and $J_2(k)$ are defined in (4.2), we can write:

Definition 4.19. *Let k be an integer such that $k \leqslant 2$ and assume that Σ is of class $\mathcal C^{I_2(k+1),1}$. For every triple $(g_n, g_\tau, \mathit\Gamma)$ in $G^n_{k+1} \times G^\tau_{k+1} \times \mathbb R^N$ with $\mathit\Gamma = (\mathit\Gamma_1, \ldots, \mathit\Gamma_N)$ we define:*

$$\mathsf L^V_k(g_n, g_\tau, \mathit\Gamma) = \bar\Delta_k \mathsf L^S_{k+1}(g_n, g_\tau, \mathit\Gamma) = \bar\Delta_k L^\tau_{k+1}(\mathsf T_{k+1}g_n - g_\tau) + \sum_{j=1}^N \mathit\Gamma_j\zeta_j \qquad \text{if } k = 0, 1, 2, \tag{4.33a}$$

$$\mathsf L^V_k(g_n, g_\tau, \mathit\Gamma) = \Delta_k \mathsf L^S_{k+1}(g_n, g_\tau, \mathit\Gamma) \qquad \text{if } k \leqslant -1. \tag{4.33b}$$

The operator $\mathsf L^V_k$ is valued in the space $V^{\rm b}_k$ defined by:

$$V^{\rm b}_k = \bar\Delta_k S^{\rm b}_{k+1} = \mathfrak H^k_V \oplus \mathbb F^*_V \quad \text{if } k = 0, 1, 2 \qquad \text{and} \qquad V^{\rm b}_k = \Delta_k S^{\rm b}_{k+1} \subset V_k \quad \text{if } k \leqslant -1.$$

Let T be a positive real number, k be an integer such that $k \leqslant 1$ and assume that Σ is of class $\mathcal C^{J_2(k+1),1}$. The operator $\mathsf L^V_{k+1}$ maps the space $G_{k+1}(T)$ (defined in (4.10)) into the space:

$$V^{\rm b}_k(T) = \begin{cases} H^1(0, T; V^{\rm b}_{k-1}) \cap \mathcal C([0, T]; V^{\rm b}_k) \cap L^2(0, T; V^{\rm b}_{k+1}) & \text{if } k = -1, 0, 1,\\ L^2(0, T; V^{\rm b}_{k+1}) & \text{if } k \leqslant -2. \end{cases} \tag{4.34}$$

As a direct consequence of Lemmas 4.8, we are allowed to claim:

Lemma 4.20. *Let k be an integer such that $k \leqslant 2$ and assume that Σ is of class $\mathcal{C}^{I_2(k+1),1}$. Then the lifting operator for the vorticity:*

$$\mathsf{L}_k^V : G_{k+1}^n \times G_{k+1}^\tau \times \mathbb{R}^N \longrightarrow V_k^{\mathrm{b}},$$

is well defined and is bounded. Moreover if k and k' are two integers such that $k' \leqslant k \leqslant 2$, then $\mathsf{L}_{k'}^V = \mathsf{L}_k^V$ in $G_{k+1}^n \times G_{k+1}^\tau \times \mathbb{R}^N$. It follows that for every positive real number T and every $k \leqslant 1$, providing that Σ is of class $C^{J_2(k+1),1}$, the operator:

$$L_{k+1}^V : G_{k+1}(T) \longrightarrow V_k^{\mathrm{b}}(T),$$

is well defined and bounded as well, the bound being uniform with respect to T.

This lemma makes precise the expression of the vorticity corresponding to any prescribed boundary Dirichlet conditions for the velocity field on Σ.

Definition 4.19 and Lemma 4.20 justify the lengthy construction of the the spaces $\bar{V}_k$ $(k = 0, 1, 2)$ carried out in Subsection 4.2. As already mentioned, the naive approach consisting in taking the Laplacian of a lifting stream function does not result in the correct result, first because the correct vorticity (in both cases of Fig. 6 and Fig. 7) belongs actually to dual spaces, the expressions of which requires the construction of the spaces V_k and $\bar{V}_k$ and second because the circulation would vanish at the vorticity level.

5. Evolution Stokes problem in nonprimitive variables

The evolution Stokes problem, stated in the original primitive variables (u, p), reads:

$$\partial_t u - \nu \Delta u + \nabla\Big(\frac{p}{\varrho}\Big) = f \quad \text{in } \mathcal{F}_T \tag{5.1a}$$

$$\nabla \cdot u = 0 \quad \text{in } \mathcal{F}_T \tag{5.1b}$$

$$u = b \quad \text{on } \Sigma_T \tag{5.1c}$$

$$u(0) = u^{\mathrm{i}} \quad \text{in } \mathcal{F}, \tag{5.1d}$$

where the source term f, the boundary data b and the initial data u^{i} are prescribed. We recall that the constant $\nu > 0$ is the kinematic viscosity of the fluid.

5.1. Homogeneous boundary conditions.

We have at our disposal all the material allowing to deal with the evolution Stokes problem in terms of both the vorticity field and the stream function.

Definition 5.1. *in terms of the stream function, the evolution Stokes problem (called $\psi-$Stokes problem) can be stated as follows: Let k be any integer, T be a positive real number, ψ^{i} be in S_k and f_S be an element of $L^2(0, T; S_{k-1})$. The Cauchy problem for the stream function with homogeneous boundary conditions, at regularity level k, reads:*

$$\partial_t \psi + \nu \mathsf{A}_{k+1}^S \psi = f_S \quad \textit{in } \mathcal{F}_T, \tag{5.2a}$$

$$\psi(0) = \psi^{\mathrm{i}} \quad \textit{in } \mathcal{F}. \tag{5.2b}$$

Problem (5.2) can be rephrased in terms of the vorticity field: Let k be any integer, T be a positive real number, ω^{i} be in V_k and f_V be an element of $L^2(0, T; V_{k-1})$. The Cauchy problem for the vorticity field, called $\omega-$Stokes problem, at regularity level k reads:

$$\partial_t \omega + \nu \mathsf{A}_{k+1}^V \omega = f_V \quad \textit{in } \mathcal{F}_T, \tag{5.3a}$$

$$\omega(0) = \omega^{\mathrm{i}} \quad \textit{in } \mathcal{F}. \tag{5.3b}$$

For every integer k, we introduce the function spaces:

$$S_k(T) = H^1(0, T; S_{k-1}) \cap \mathcal{C}([0, T]; S_k) \cap L^2(0, T; S_{k+1}), \tag{5.4a}$$

$$V_k(T) = H^1(0, T; V_{k-1}) \cap \mathcal{C}([0, T]; V_k) \cap L^2(0, T; V_{k+1}). \tag{5.4b}$$

Invoking for instance [49, Theorem 4.1] or simply Proposition A.10 (we felt somewhat uncomfortable with quoting general results on semigroups in Banach spaces in such a simple case for which everything can be shown "by hand"; see the short subsection A.3), we claim:

Proposition 5.2. *For every integer k, every $T>0$, every $\psi^{\rm i}\in S_k$ and every $f_S\in L^2(0,T;S_{k-1})$, there exists a unique solution ψ to problem* (5.2) *in the space $S_k(T)$. Moreover, there exists a real positive constant $\mathbf{c}_\nu$ (depending on ν but uniform in $\mathcal{F}$, k and T) such that:*

$$\|\psi\|_{S_k(T)}\leqslant \mathbf{c}_\nu\Big(\|\psi^{\rm i}\|_{S_k}+\|f_S\|_{L^2(0,T;S_{k-1})}\Big). \tag{5.5a}$$

For every integer k, every $T>0$, every $\omega^{\rm i}\in V_k$ and every $f_V\in L^2(0,T;V_{k-1})$, there exists a unique solution ω to problem (5.3) *in the space $V_k(T)$. Moreover, the following estimate holds with the same constant $\mathbf{c}_\nu$ as in* (5.5a)*:*

$$\|\omega\|_{V_k(T)}\leqslant \mathbf{c}_\nu\Big(\|\omega^{\rm i}\|_{V_k}+\|f_V\|_{L^2(0,T;V_{k-1})}\Big). \tag{5.5b}$$

The solutions ψ and ω to problems (5.2) and (5.3) respectively, satisfy the following exponential decay estimates:

Lemma 5.3. *Let ψ be a solution to the Cauchy problem* (5.2) *in the space $S_k(T)$ for some integer k, some source term $f_S\in L^2(0,T;S_{k-1})$ and some initial condition $\psi^{\rm i}\in S_k$. Then, the following estimate holds:*

$$\|\psi(t)\|_{S_k}\leqslant e^{-[\nu(1-\varepsilon)\lambda_{\mathcal{F}}]t}\Big[\|\psi^{\rm i}\|^2_{S_k}+\frac{1}{2\nu\varepsilon}\|f_S\|^2_{L^2(0,T;S_{k-1})}\Big]^{\frac12}\ \text{for all } t\in[0,T] \text{ and } \varepsilon\in(0,1), \tag{5.6a}$$

where $\lambda_{\mathcal{F}}>0$ is the constant defined in Corollary 3.16. If $f_S=0$, we can choose $\varepsilon=0$ in (5.6a).

Let ω be a solution to the Cauchy problem (5.3) *in the space $V_k(T)$ for some integer k, some source term $f_V\in L^2(0,T;V_{k-1})$ and some initial condition $\omega^{\rm i}\in V_k$. Then, the following estimate holds:*

$$\|\omega(t)\|_{V_k}\leqslant e^{-[\nu(1-\varepsilon)\lambda_{\mathcal{F}}]t}\Big[\|\omega^{\rm i}\|^2_{V_k}+\frac{1}{2\nu\varepsilon}\|f_V\|^2_{L^2(0,T;V_{k-1})}\Big]^{\frac12}\ \text{for all } t\in[0,T] \text{ and } \varepsilon\in(0,1). \tag{5.6b}$$

If $f_V=0$, we can choose $\varepsilon=0$ in (5.6b).

We can easily connect problems (5.2) and (5.3) by means of either the operators P_k and Q_k or with the operator Δ_k. The proof is straightforward, resting on the commutative diagrams of Fig. 3 and Fig. 4:

Theorem 5.4. *Let k and k' be two integers and T be a positive real number. Let ψ be the solution in $S_k(T)$ to Problem* (5.2) *with source term $f_S\in L^2(0,T;S_{k-1})$ and initial condition $\psi^{\rm i}\in S_k$. Let ω be the solution in $V_{k'}(T)$ to Problem* (5.3) *with source term $f_V\in L^2(0,T;V_{k'-1})$ and initial condition $\omega^{\rm i}\in V_{k'}$.*
If $k'=k+1$, then the following assertions are equivalent:

(1) *$\omega^{\rm i}=\mathsf{P}_{k+1}\psi^{\rm i}$ and for a.e. $t\in(0,T)$, $f_V(t)=\mathsf{P}_kf_S(t)$;*
(2) *For a.e. $t\in(0,T)$, $\omega(t)=\mathsf{P}_{k+2}\psi(t)$.*

If $k'=k-1$ then the following assertions are equivalent:

(1) *$\omega^{\rm i}=\Delta_{k-1}\psi^{\rm i}$ and for a.e. $t\in(0,T)$, $f_V(t)=\Delta_{k-2}f_S(t)$;*
(2) *For a.e. $t\in(0,T)$, $\omega(t)=\Delta_k\psi(t)$.*

In addition to the data already introduced, let

$$u\in H^1(0,T;\mathbf{J}_{k-1})\cap\mathcal{C}([0,T];\mathbf{J}_k)\cap L^2(0,T;\mathbf{J}_{k+1})$$

be the solution to Problem (2.18) *for some $f_{\mathbf{J}}\in L^2(0,T;\mathbf{J}_{k-1})$ and $u^{\rm i}$ in $\mathbf{J}_k$. The following assertions are equivalent:*

(1) *$u^{\rm i}=\nabla_k^\perp\psi^{\rm i}$ and for a.e. $t\in(0,T)$, $f_{\mathbf{J}}(t)=\nabla^\perp_{k-1}f_S(t)$;*
(2) *For a.e. $t\in(0,T)$, $u(t)=\nabla^\perp_{k+1}\psi(t)$.*

5.2. **Nonhomogeneous boundary conditions.** Following the definition (5.4) of $S_k(T)$ and $V_k(T)$, (4.11) of $S_k^{\rm b}(T)$ and (4.34) of $V_k^{\rm b}(T)$ we introduce for every real positive number T and every integer $k\leqslant 2$:

$$\bar S_k(T)=H^1(0,T;\bar S_{k-1})\cap\mathcal{C}([0,T];\bar S_k)\cap L^2(0,T;\bar S_{k+1}), \tag{5.7a}$$

where we recall that the spaces $\bar S_k$ are defined in (4.12). The counterpart stated in terms of the vorticity is, for every integer $k\leqslant 1$, the space (see Fig. 6 and Fig. 7):

$$\bar V_k(T)=H^1(0,T;\bar V_{k-1})\cap\mathcal{C}([0,T];\bar V_k)\cap L^2(0,T;\bar V_{k+1}), \tag{5.7b}$$

where the spaces $\bar V_k$ are defined in (4.13).

Definition 5.5. *Let a positive real number T, an integer $k\leqslant 1$, a source term $f_S\in L^2(0,T;S_{k-1})$, an initial data $\psi^{\rm i}\in\bar S_k$ and a triple $(g_n,g_\tau,\varGamma)\in G_k(T)$ be given. Define $\psi^{\rm i}_0=\psi^{\rm i}-\mathsf{L}^S_k(g_n(0),g_\tau(0),\varGamma(0))$ when $k=0,1$ and $\psi^{\rm i}_0=\psi^{\rm i}$ when $k\leqslant -1$. Finally, assume that Σ is of class $C^{J_2(k),1}$ and that the following compatibility condition holds:*

$$\psi^{\rm i}_0\in S_k\qquad \text{if } k=1. \tag{5.8}$$

We say that a function $\psi\in\bar S_k(T)$ is solution of the evolution $\psi-$Stokes problem satisfying the Dirichlet boundary conditions on Σ_T as described in (4.1) by the triple $(g_n,g_\tau,\varGamma)$ if:

1. *When $k=0$ or $k=1$: There exists $\psi_0\in S_k(T)$ solution to the homogeneous $\psi-$Stokes Cauchy problem*

$$\partial_t\psi_0+\nu\mathsf{A}^S_{k+1}\psi_0=-\partial_t\mathsf{L}^S_{k+1}(g_n,g_\tau,\varGamma)+f_S\qquad \text{in } \mathcal{F}_T, \tag{5.9a}$$

$$\psi_0(0)=\psi^{\rm i}_0\qquad \text{in } \mathcal{F}, \tag{5.9b}$$

   such that $\psi=\psi_0+\mathsf{L}^S_{k+1}(g_n,g_\tau,\varGamma)$.

2. *When $k\leqslant -1$: The function ψ is the solution to the Cauchy problem:*

$$\partial_t\psi+\nu\mathsf{A}^S_{k+1}\psi=\nu\mathsf{A}^S_{k+1}\mathsf{L}^S_{k+1}(g_n,g_\tau,\varGamma)+f_S\qquad \text{in } \mathcal{F}_T, \tag{5.10a}$$

$$\psi(0)=\psi^{\rm i}_0\qquad \text{in } \mathcal{F}. \tag{5.10b}$$

The case $k=2$ is more involved and will be treated in Section 8. The difference of definition depending on the level of regularity k is worth some additional explanation. Before that, combining Proposition 5.2 and Lemma 4.20, we are allowed to claim:

Proposition 5.6. *Every $\psi-$Stokes problem as stated in Definition 5.5 admits a unique solution. Moreover, there exists a positive constant $\mathbf{c}_{[k,\mathcal{F},\nu]}$ uniform in T such that the solution $\psi\in\bar S_k(T)$ satisfies the estimate:*

$$\|\psi\|_{\bar S_k(T)}\leqslant \mathbf{c}_{[k,\mathcal{F},\nu]}\Big[\|\psi^{\rm i}_0\|^2_{S_k}+\|f_S\|^2_{L^2(0,T;S_{k-1})}+\|(g_n,g_\tau,\varGamma)\|^2_{G_k(T)}\Big]^{\frac12}. \tag{5.11}$$

The consistency of Definition 5.5 is asserted by the following results:

Proposition 5.7. *Let a positive real number T, an integer k, a source term $f_S\in L^2(0,T;S_{k-1})$, an initial data $\psi^{\rm i}\in\bar S_k$ and a triple $(g_n,g_\tau,\varGamma)\in G_k(T)$ be given as in Definition 5.5. Denote by ψ^k the solution whose existence and uniqueness in the space $\bar S_k(T)$ are asserted in Proposition 5.6.*

Let k' be any integer lower than k and, all other data remaining equal, denote by $\psi^{k'}$ the corresponding solution in $\bar S_{k'}(T)$. Then $\psi^k=\psi^{k'}$.

Proof. The proposition is obvious when k and k' are both nonnegative or when k and k' are both negative, so let us focus on the case $k=0$ and $k'=-1$ and compare the solutions ψ^0 and ψ^{-1}.

By definition, the function ψ^0 solves the Cauchy problem:

$$\begin{aligned}\partial_t\big(\psi^0-\mathsf{L}^S_1(g_n,g_\tau,\varGamma)\big)+\nu\mathsf{A}^S_1\big(\psi^0-\mathsf{L}^S_1(g_n,g_\tau,\varGamma)\big)&=-\partial_t\mathsf{L}^S_1(g_n,g_\tau,\varGamma)+f_S&&\text{in } \mathcal{F}_T,\\ \psi^0(0)&=\psi^{\rm i}&&\text{in } \mathcal{F}.\end{aligned}$$

Since the operator A^S_0 extends the operator A^S_1 to S_0, the function ψ^0 belongs to $L^2(0,T;\bar S_1)\subset L^2(0,T;S_0)$ and the lifting operator L^S_1 is valued in $S^{\rm b}_1$, which is a subspace of S_0, we are allowed to write that:

$$\mathsf{A}^S_1\big(\psi^0-\mathsf{L}^S_1(g_n,g_\tau,\varGamma)\big)=\mathsf{A}^S_0\psi^0-\mathsf{A}^S_0\mathsf{L}^S_1(g_n,g_\tau,\varGamma).$$

It follows that ψ^0 solves as well the Cauchy problem:

$$\begin{aligned}\partial_t\psi^0+\nu\mathsf{A}^S_0\psi^0&=\nu\mathsf{A}^S_0\mathsf{L}^S_1(g_n,g_\tau,\varGamma)+f_S&&\text{in } \mathcal{F}_T,\\ \psi^0(0)&=\psi^{\rm i}&&\text{in } \mathcal{F},\end{aligned}$$

a solution of which is ψ^{-1}. The proof is now completed. ∎

The definition of weak solutions (i.e. for negative integers k) with nonhomogeneous boundary conditions given in Definition 5.5 can be rephrased by means of the duality method (or transposition method; see [58] and references therein).

Proposition 5.8. *Let data be given as in Definition 5.5 and assume that k is a negative integer and $f_S = 0$. Denote by ψ the unique solution to the corresponding Cauchy nonhomogeneous $\psi-$Stokes problem* (5.10)*. Then for every $\vartheta \in L^2(0,T;S_{-k-1})$ and $\theta \in S_{-k}(T)$ solution to the backward Cauchy problem:*

$$\begin{aligned} -\partial_t\theta + \nu \mathsf{A}^S_{-k+1}\theta &= \vartheta && \text{in } \mathcal{F}_T,\\ \theta(T) &= 0 && \text{in } \mathcal{F}, \end{aligned}$$

the following identity holds:

$$\int_0^T \langle \psi, \vartheta\rangle_{S_{k+1},S_{-k-1}}\,\mathrm{d}t = \langle \psi^{\mathrm{i}}, \theta(0)\rangle_{S_k,S_{-k}} - \nu \int_0^T \langle \Delta_{-k}\theta\big|_\Sigma, b_\tau\rangle_{H^{-k-\frac12}(\Sigma),H^{k+\frac12}(\Sigma)}\,\mathrm{d}t, \tag{5.12}$$

where (see identities (4.1)*):*

$$b_\tau = \mathsf{T}_{k+1}g_n - g_\tau + \sum_{j=1}^N \mathit{\Gamma}_j \frac{\partial \xi_j}{\partial n}\Big|_\Sigma.$$

Proof. Equation (5.10a) holds in $L^2(0,T;S_{k-1})$, which is the dual space of $L^2(0,T;S_{-k+1})$. Forming the duality pairing of (5.10a) with θ yields:

$$\int_0^T \langle \partial_t\psi, \theta\rangle_{S_{k-1},S_{-k+1}}\,\mathrm{d}t + \nu\int_0^T \langle \mathsf{A}^S_{k+1}\psi,\theta\rangle_{S_{k-1},S_{-k+1}}\,\mathrm{d}t = \nu\int_0^T \langle \mathsf{A}^S_{k+1}\mathsf{L}^S_{k+1}(g_n,g_\tau,\mathit{\Gamma}),\theta\rangle_{S_{k-1},S_{-k+1}}\,\mathrm{d}t.$$

Integrating by parts and using the definition of the operator A^S_{k+1}, we obtain for the term in the left hand side:

$$\begin{aligned} \int_0^T \langle \partial_t\psi, \theta\rangle_{S_{k-1},S_{-k+1}}\,\mathrm{d}t + \nu\int_0^T \langle \mathsf{A}^S_{k+1}\psi,\theta\rangle_{S_{k-1},S_{-k+1}}\,\mathrm{d}t &= \\ - \langle \psi^{\mathrm{i}}, \theta(0)\rangle_{S_k,S_{-k}} + \int_0^T &\langle \psi, -\partial_t\theta + \mathsf{A}^S_{-k+1}\theta\rangle_{S_{k+1},S_{-k-1}}\,\mathrm{d}t. \end{aligned} \tag{5.13}$$

The right hand side term is dealt with as follows:

$$\int_0^T \langle \mathsf{A}^S_{k+1}\mathsf{L}^S_{k+1}(g_n,g_\tau,\mathit{\Gamma}),\theta\rangle_{S_{k-1},S_{-k+1}}\,\mathrm{d}t = \int_0^T \langle \mathsf{L}^S_{k+1}(g_n,g_\tau,\mathit{\Gamma}), \mathsf{A}^S_{-k+1}\theta\rangle_{S_{k+1},S_{-k-1}}\,\mathrm{d}t,$$

where, by definition (see (4.8)):

$$\begin{aligned} \int_0^T &\left\langle \mathsf{L}^S_{k+1}(g_n,g_\tau,\mathit{\Gamma}), \mathsf{A}^S_{-k+1}\theta\right\rangle_{S_{k+1},S_{-k-1}}\mathrm{d}t \\ &= \int_0^T \left\langle \mathsf{L}^\tau_{k+1}(\mathsf{T}_{k+1}g_n - g_\tau) + \sum_{j=1}^N \mathit{\Gamma}_j\xi_j, \mathsf{A}^S_{-k+1}\theta\right\rangle_{S_{k+1},S_{-k-1}}\mathrm{d}t. \end{aligned} \tag{5.14}$$

Since the index k is negative, $\mathsf{A}^S_{-k+1}\theta$ belongs to S_0. It follows that for every $j\in\{1,\ldots,N\}$:

$$\langle \xi_j, \mathsf{A}^S_{-k+1}\theta\rangle_{S_{k+1},S_{-k-1}} = -(\nabla\xi_j,\nabla\Delta_{-k}\theta)_{\mathbf{L}^2(\mathcal{F})} = -\int_\Sigma \frac{\partial\xi_j}{\partial n}\Delta_{-k}\theta\,\mathrm{d}s. \tag{5.15}$$

Recalling the definition (4.7) of the operator L^τ_{k+1} and the factorization (3.30) of Δ_k and then gathering (5.13), (5.14) and (5.15), we obtain indeed (5.12) and complete the proof. ∎

Definition 5.5 and Propositions 5.6, 5.7 and 5.8 can be restated in terms of the vorticity field.

Definition 5.9. *Let a positive real number T, a nonpositive integer k, a source term $f_V \in L^2(0,T;V_{k-1})$, an initial data $\omega^{\rm i} \in \bar{V}_k$ and a triple $(g_n, g_\tau, \varGamma) \in G_{k+1}(T)$ be given. Define $\omega_0^{\rm i} = \omega^{\rm i} - \mathsf{L}_k^V(g_n(0), g_\tau(0), \varGamma(0))$ when $k = -1, 0$ and $\omega_0^{\rm i} = \omega^{\rm i}$ when $k \leqslant -2$. Finally, assume that Σ is of class $C^{J_2(k+1),1}$ and that the following compatibility condition holds:*

$$\omega_0^{\rm i} \in V_k \qquad \text{if } k = 0. \tag{5.16}$$

We say that a function $\omega \in \bar{V}_k(T)$ is solution of the evolution $\omega-$Stokes problem satisfying the Dirichlet boundary conditions on Σ_T as described in (4.1) by the triple $(g_n, g_\tau, \varGamma)$ if:

(1) *When $k = -1$ or $k = 0$: There exists $\omega_0 \in V_k(T)$ solution to the homogeneous $\omega-$Stokes Cauchy problem*

$$\partial_t \omega_0 + \nu \mathsf{A}_{k+1}^V \omega_0 = -\partial_t \mathsf{L}_{k+1}^V(g_n, g_\tau, \varGamma) + f_V \qquad \text{in } \mathcal{F}_T, \tag{5.17a}$$

$$\omega_0(0) = \omega_0^{\rm i} \qquad \text{in } \mathcal{F}, \tag{5.17b}$$

such that $\omega = \omega_0 + \mathsf{L}_{k+1}^V(g_n, g_\tau, \varGamma)$.

(2) *When $k \leqslant -2$: The function ω is the solution to the Cauchy problem:*

$$\partial_t \omega + \nu \mathsf{A}_{k+1}^V \omega = \nu \mathsf{A}_{k+1}^V \mathsf{L}_{k+1}^V(g_n, g_\tau, \varGamma) + f_V \qquad \text{in } \mathcal{F}_T, \tag{5.18a}$$

$$\omega(0) = \omega_0^{\rm i} \qquad \text{in } \mathcal{F}. \tag{5.18b}$$

Proposition 5.10. *Every $\omega-$Stokes problem as stated in Definition 5.9 admits a unique solution. Moreover, there exists a positive constant $\mathbf{c}_{[k,\mathcal{F},\nu]}$ (uniform in T) such that the solution $\omega \in \bar{S}_k(T)$ satisfies the estimate:*

$$\|\omega\|_{\bar{V}_k(T)} \leqslant \mathbf{c}_{[k,\mathcal{F},\nu]} \Big[\|\omega_0^{\rm i}\|_{V_k}^2 + \|f_V\|_{L^2(0,T;V_{k-1})}^2 + \|(g_n, g_\tau, \varGamma)\|_{G_{k+1}(T)}^2 \Big]^{\frac{1}{2}}. \tag{5.19}$$

Proposition 5.11. *Let a positive real number T, an integer k, a source term $f_V \in L^2(0,T;V_{k-1})$, an initial data $\omega^{\rm i} \in \bar{V}_k$ and a triple $(g_n, g_\tau, \varGamma) \in G_{k+1}(T)$ be given as in Definition 5.9. Denote by ω^k the solution whose existence and uniqueness in the space $\bar{V}_k(T)$ are asserted in Proposition 5.10.*

Let k' be any integer lower than k and, all other data remaining equal, denote by $\omega^{k'}$ the corresponding solution in $\bar{V}_{k'}(T)$. Then $\omega^k = \omega^{k'}$.

Proposition 5.12. *Let data be given as in Definition 5.9 with $k \leqslant -2$ and $f_V = 0$. Denote by ω the unique solution to the corresponding Cauchy nonhomogeneous $\omega-$Stokes problem (5.18). Then for every $\vartheta \in L^2(0,T;V_{-k-1})$ and $\theta \in V_{-k}(T)$ solution to the backward Cauchy problem:*

$$-\partial_t \theta + \nu \mathsf{A}_{-k+1}^V \theta = \vartheta \qquad \text{in } \mathcal{F}_T,$$

$$\theta(T) = 0 \qquad \text{in } \mathcal{F},$$

the following identity holds:

$$\int_0^T \langle \omega, \vartheta \rangle_{V_{k+1}, V_{-k-1}} \, {\rm d}t = \langle \omega^{\rm i}, \theta(0) \rangle_{V_k, V_{-k}} - \nu \int_0^T \langle A_{-k+1}^V \theta\big|_\Sigma, b_\tau \rangle_{H^{-k-\frac{3}{2}}(\Sigma), H^{k+\frac{3}{2}}(\Sigma)} \, {\rm d}t, \tag{5.20}$$

where (see identities (4.1)):

$$b_\tau = \mathsf{T}_{k+2} g_n - g_\tau + \sum_{j=1}^N \varGamma_j \frac{\partial \xi_j}{\partial n}\Big|_\Sigma.$$

Proof. We form the duality pairing of (5.18a) in $L^2(0,T;V_{k-1})$ with θ in $L^2(0,T;V_{-k+1})$, then the proof follows mainly the lines of the proof of Proposition 5.8. Let us focus on the right hand side term only, namely:

$$\int_0^T \Big\langle \mathsf{A}_{k+1}^V \mathsf{L}_{k+1}^V(g_n, g_\tau, \varGamma), \theta \Big\rangle_{V_{k-1}, V_{-k+1}} {\rm d}s.$$

According to (A.7), the duality pairing can be turned into:

$$\Big\langle \mathsf{A}_{k+1}^V \mathsf{L}_{k+1}^V(g_n, g_\tau, \varGamma), \theta \Big\rangle_{V_{k-1}, V_{-k+1}} = \Big\langle \mathsf{L}_{k+1}^V(g_n, g_\tau, \varGamma), \mathsf{A}_{-k+1}^V \theta \Big\rangle_{V_{k+1}, V_{-k-1}}.$$

Then, using the definition (4.33b) of L^V_{k+1} and the second formula in Lemma 3.19, we obtain:

$$\left\langle \mathsf{L}^V_{k+1}(g_n, g_\tau, \Gamma), \mathsf{A}^V_{-k+1}\theta \right\rangle_{V_{k+1},V_{-k-1}} = \left\langle \mathsf{L}^S_{k+2}(g_n, g_\tau, \Gamma), \mathsf{Q}_{-k-1}A^V_{-k+1}\theta \right\rangle_{S_{k+2},S_{-k-2}}.$$

Resting on formula (3.27), the last term is proven to be equal to:

$$\left\langle \mathsf{L}^S_{k+2}(g_n, g_\tau, \Gamma), A^S_{-k}\mathsf{Q}_{-k+1}\theta \right\rangle_{S_{k+2},S_{-k-2}},$$

and therefore it is very much alike the left hand side in (5.14). The proof is then completed after noticing that $-\Delta_{-k-1}\mathsf{Q}_{-k+1} = \mathsf{A}^V_{-k+1}$ (see for instance Fig. 4). ☐

6. Navier-Stokes equations in nonprimitive variables

6.1. **Estimates for the nonlinear advection term.** Following our rules of notation, we define for every positive integer k and every positive time T, the time dependent space for the Kirchhoff potential:

$$\mathfrak{H}^k_K(T) = H^1(0,T;\mathfrak{H}^{k-1}_K) \cap \mathcal{C}([0,T];\mathfrak{H}^k_K) \cap L^2(0,T;\mathfrak{H}^{k+1}_K), \tag{6.1}$$

where we recall that the spaces $\mathfrak{H}^k_K$ were defined in (4.4). Then, we aim at establishing some (very classical) estimates for the nonlinear advection term of the Navier-Stokes equations. Denoting by u a smooth velocity field defined in $\mathcal{F}$, an integration par parts yields the equality:

$$(\nabla u u, \nabla^\perp\theta)_{\mathbf{L}^2(\mathcal{F})} = (D^2\theta u, u^\perp)_{\mathbf{L}^2(\mathcal{F})} \qquad \text{for all } \theta \in S_1.$$

The main estimates satisfied by the right hand side term are summarized in the lemma below:

Lemma 6.1. *Let a stream function $\bar\psi$ be in $\bar S_1$ (this space being defined in (4.15)) and a Kirchhoff potential φ be in $\mathfrak{H}^2_K$ (defined in (4.4)). Then the linear form:*

$$\varLambda^S_1(\bar\psi,\varphi) : \theta \in S_1 \longmapsto -(D^2\theta\nabla\bar\psi, \nabla^\perp\bar\psi)_{\mathbf{L}^2(\mathcal{F})} + (D^2\theta\nabla^\perp\varphi, \nabla^\perp\bar\psi)_{\mathbf{L}^2(\mathcal{F})} - (D^2\theta\nabla\varphi, \nabla\bar\psi)_{\mathbf{L}^2(\mathcal{F})} \in \mathbb{R}, \tag{6.2}$$

is well defined and bounded. Moreover, there exists a positive constant $\mathbf{c}_{[\mathcal{F},\nu]}$ such that:

1. *For every $\theta \in S_1$:*

$$|\langle \varLambda^S_1(\bar\psi+\theta,\varphi),\theta\rangle_{S_{-1},S_1}| \leqslant \frac{\nu}{2}\|\theta\|^2_{S_1} + \mathbf{c}_{[\mathcal{F},\nu]}\Big(\|\bar\psi\|^2_{\bar S_1}\|\bar\psi\|^2_{S_0} + \|\varphi\|^2_{\mathfrak{H}^2_K}\|\varphi\|^2_{\mathfrak{H}^1_K}\Big)\|\theta\|^2_{S_0} + \mathbf{c}_{[\mathcal{F},\nu]}\Big(\|\bar\psi\|^2_{\bar S_1}\|\bar\psi\|^2_{S_0} + \|\varphi\|^2_{\mathfrak{H}^2_K}\|\varphi\|^2_{\mathfrak{H}^1_K}\Big). \tag{6.3a}$$

2. *For every pair $(\theta_1,\theta_2) \in S_1 \times S_1$:*

$$|\langle \varLambda^S_1(\bar\psi+\theta_2,\varphi) - \varLambda^S_1(\bar\psi+\theta_1,\varphi),\Theta\rangle_{S_{-1},S_1}| \leqslant \frac{\nu}{2}\|\Theta\|^2_{S_1} + \mathbf{c}_{[\mathcal{F},\nu]}\Big[\|\bar\psi\|^2_{\bar S_1}\|\bar\psi\|^2_{S_0} + \|\varphi\|^2_{\mathfrak{H}^2_K}\|\varphi\|^2_{\mathfrak{H}^1_K} + \|\theta_1\|^2_{S_1}\|\theta_1\|^2_{S_0}\Big]\|\Theta\|^2_{S_0}, \tag{6.3b}$$

   where $\Theta = \theta_2 - \theta_1$.

If for some positive real number T, $\bar\psi$ belongs to $\mathcal{C}([0,T];S_0)\cap L^2(0,T;\bar S_1)$ and φ to $\mathcal{C}([0,T];\mathfrak{H}^1_K)\cap L^2(0,T;\mathfrak{H}^2_K)$ then $\varLambda^S_1(\bar\psi,\varphi)$ is in $L^2(0,T;S_{-1})$ and there exists a positive constant $\mathbf{c}_\mathcal{F}$ such that:

$$\|\varLambda^S_1(\bar\psi,\varphi)\|_{L^2(0,T;S_{-1})} \leqslant \mathbf{c}_\mathcal{F}\Big[\|\bar\psi\|_{\mathcal{C}([0,T];S_0)}\|\bar\psi\|_{L^2(0,T;\bar S_1)} + \|\varphi\|_{\mathcal{C}([0,T];\mathfrak{H}^1_K)}\|\varphi\|_{L^2(0,T;\mathfrak{H}^2_K)}\Big]. \tag{6.3c}$$

Proof. Considering the first term in the right hand side of (6.2), Hölder's inequality yields:

$$|(D^2\theta\nabla\bar\psi, \nabla^\perp\bar\psi)_{\mathbf{L}^2(\mathcal{F})}| \leqslant \|\nabla\bar\psi\|^2_{\mathbf{L}^4(\mathcal{F})}\|\theta\|_{S_1} \qquad \text{for all } \theta\in S_1. \tag{6.4a}$$

Then Sobolev embedding Theorem followed by an interpolation inequality between the spaces $L^2(\mathcal{F})$ and $H^1(\mathcal{F})$ leads to:

$$\|u\|_{L^4(\mathcal{F})} \leqslant \mathbf{c}_\mathcal{F}\|u\|_{H^{\frac12}(\mathcal{F})} \leqslant \mathbf{c}_\mathcal{F}\|u\|^{\frac12}_{L^2(\mathcal{F})}\|u\|^{\frac12}_{H^1(\mathcal{F})} \qquad \text{for all } u\in H^1(\mathcal{F}).$$

Combining this inequality with Lemma 4.9 (equivalence of the norms in $\bar S_1$ and $H^2(\mathcal{F})$), we obtain first:

$$\|\nabla\bar\psi\|_{\mathbf{L}^4(\mathcal{F})} \leqslant \mathbf{c}_\mathcal{F}\|\bar\psi\|^{\frac12}_{S_0}\|\bar\psi\|^{\frac12}_{\bar S_1}.$$

Once plugged in (6.4a), it gives rise to:

$$|(D^2\theta\nabla\bar\psi,\nabla^\perp\bar\psi)_{\mathbf{L}^2(\mathcal{F})}|\leqslant \mathbf{c}_{\mathcal{F}}\|\bar\psi\|_{S_0}\|\bar\psi\|_{\bar S_1}\|\theta\|_{S_1}\qquad\text{for all }\theta\in S_1. \tag{6.4b}$$

Based on the same arguments, it is then straightforward to prove the existence of a positive constant $\mathbf{c}_{\mathcal{F}}$ such that:

$$|\langle\Lambda_1^S(\bar\psi,\varphi),\theta\rangle_{S_{-1},S_1}|\leqslant \mathbf{c}_{\mathcal{F}}\Big(\|\bar\psi\|_{\bar S_1}\|\bar\psi\|_{S_0}+\|\varphi\|_{\mathfrak{H}^2_K}\|\varphi\|_{\mathfrak{H}^1_K}\Big)\|\theta\|_{S_1}\qquad\text{for all }\theta\in S_1.$$

This shows that the linear form $\Lambda_1^S(\bar\psi,\varphi)$ is indeed bounded and satisfies:

$$\|\Lambda_1^S(\bar\psi,\varphi)\|_{S_{-1}}\leqslant \mathbf{c}_{\mathcal{F}}\Big(\|\bar\psi\|_{\bar S_1}\|\bar\psi\|_{S_0}+\|\varphi\|_{\mathfrak{H}^2_K}\|\varphi\|_{\mathfrak{H}^1_K}\Big). \tag{6.5}$$

Let us move on to the estimate (6.3a). Some of the terms vanishing after an integration by parts, we end up with the following equality:

$$\langle\Lambda_1^S(\bar\psi+\theta,\varphi),\theta\rangle_{S_{-1},S_1}=-(D^2\theta\nabla\bar\psi,\nabla^\perp\bar\psi)_{\mathbf{L}^2(\mathcal{F})}+(D^2\theta\nabla^\perp\varphi,\nabla^\perp\bar\psi)_{\mathbf{L}^2(\mathcal{F})}\\+(D^2\theta\nabla^\perp\varphi,\nabla^\perp\theta)_{\mathbf{L}^2(\mathcal{F})}-(D^2\theta\nabla\varphi,\nabla\bar\psi)_{\mathbf{L}^2(\mathcal{F})}. \tag{6.6}$$

Addressing the first term in the right hand side, let us start over from the inequality (6.4b) to which we apply Young's inequality:

$$|(D^2\theta\nabla\bar\psi,\nabla^\perp\bar\psi)_{\mathbf{L}^2(\mathcal{F})}|\leqslant\frac{\nu}{8}\|\theta\|^2_{S_1}+\mathbf{c}_{[\mathcal{F},\nu]}\|\bar\psi\|^2_{\bar S_1}\|\bar\psi\|^2_{S_0}.$$

The four remaining terms in the right hand side of (6.6) can be handle the same way and summing the resulting estimates yields (6.3a).

With the notation of the occurence (2) of the Lemma, some elementary algebra leads to:

$$\langle\Lambda_1^S(\bar\psi+\theta_2,\varphi)-\Lambda_1^S(\bar\psi+\theta_1,\varphi),\Theta\rangle_{S_{-1},S_1}=-(D^2\Theta\nabla(\bar\psi+\theta_1),\nabla^\perp\Theta)_{\mathbf{L}^2(\mathcal{F})}+(D^2\Theta\nabla^\perp\varphi,\nabla^\perp\Theta)_{\mathbf{L}^2(\mathcal{F})},$$

and proceeding as for (6.3a) we quickly obtain (6.3b).

Finally (6.3c) derives straightforwardly from (6.5) and the proof is completed. □

In case the stream function is more regular, the nonlinear term satisfies better estimates:

Lemma 6.2. *Let a stream function $\bar\psi$ be in $\bar S_2$ (see* (4.30) *for a definition) and a Kirchhoff potential φ be in $\mathfrak{H}^2_K$ (see* (4.4)*). Then the linear form $\Lambda_1^S(\bar\psi,\varphi)$ defined in* (6.2) *extends to a continuous linear form in S_0 whose expression is:*

$$\Lambda_0^S(\bar\psi,\varphi):\theta\in S_0\longmapsto(\Delta\bar\psi(\nabla^\perp\bar\psi+\nabla\varphi),\nabla\theta)_{\mathbf{L}^2(\mathcal{F})}\in\mathbb{R}. \tag{6.7}$$

Moreover, there exists a positive constant $\mathbf{c}_{\mathcal{F}}$ such that:

$$\|\Lambda_0^S(\bar\psi,\varphi)\|_{S_0}\leqslant\mathbf{c}_{\mathcal{F}}\Big(\|\bar\psi\|^2_{\bar S_1}+\|\varphi\|^2_{\mathfrak{H}^2_K}\Big)^{\frac12}\|\bar\psi\|^{\frac15}_{\bar S_1}\|\bar\psi\|^{\frac45}_{\bar S_2}. \tag{6.8a}$$

Let T be a positive real number and let $\bar\psi$ be in $\bar S_1(T)$, φ be in $\mathfrak{H}^2_K(T)$ and θ be in $S_1(T)$ (these spaces being defined respectively in (5.7a), (6.1) *and* (5.4a)*). Then $\Lambda_0^S(\bar\psi+\theta,\varphi)$ belongs to $L^2(0,T;S_0)$ and:*

$$\|\Lambda_0^S(\bar\psi+\theta,\varphi)\|_{L^2(0,T;S_0)}\leqslant\mathbf{c}_{\mathcal{F}}T^{\frac1{10}}\Big(\|\bar\psi\|^2_{\bar S_1(T)}+\|\varphi\|^2_{\mathfrak{H}^2_K(T)}+\|\theta\|^2_{S_1(T)}\Big). \tag{6.8b}$$

Finally, for every pair $(\theta_1,\theta_2)\in S_1(T)\times S_1(T)$:

$$\|\Lambda_0^S(\bar\psi+\theta_2,\varphi)-\Lambda_0^S(\bar\psi+\theta_1,\varphi)\|_{L^2(0,T;S_0)}\\\leqslant\mathbf{c}_{\mathcal{F}}T^{\frac1{10}}\Big(\|\bar\psi\|^2_{\bar S_1(T)}+\|\theta_1\|^2_{S_1(T)}+\|\theta_2\|^2_{S_1(T)}+\|\varphi\|^2_{\mathfrak{H}^2_K(T)}\Big)^{\frac12}\|\theta_2-\theta_1\|_{S_1(T)}. \tag{6.8c}$$

Proof. Assume that θ belongs to S_1. Then, integrating by parts, we obtain:

$$(\Delta\bar\psi(\nabla^\perp\bar\psi+\nabla\varphi),\nabla\theta)_{\mathbf{L}^2(\mathcal{F})}=-(D^2\varphi\nabla\bar\psi,\nabla\theta)_{\mathbf{L}^2(\mathcal{F})}-(D^2\theta\nabla\bar\psi,\nabla^\perp\bar\psi)_{\mathbf{L}^2(\mathcal{F})}-(D^2\theta\nabla\bar\psi,\nabla\varphi)_{\mathbf{L}^2(\mathcal{F})}. \tag{6.9}$$

Integrating by parts again the first term in the right hand side, it comes:

$$(D^2\varphi\nabla\bar\psi,\nabla\theta)_{\mathbf{L}^2(\mathcal{F})}=-(\Delta\theta\nabla\varphi,\nabla\bar\psi)_{\mathbf{L}^2(\mathcal{F})}-(D^2\bar\psi\nabla\varphi,\nabla\theta)_{\mathbf{L}^2(\mathcal{F})}, \tag{6.10a}$$

while the last term can be rewritten as follows:

$$(D^2\bar{\psi}\nabla\varphi,\nabla\theta)_{\mathbf{L}^2(\mathcal{F})}=(\nabla(\nabla\theta\cdot\nabla\bar{\psi}),\nabla\varphi)_{\mathbf{L}^2(\mathcal{F})}-(D^2\theta\nabla\bar{\psi},\nabla\varphi)_{\mathbf{L}^2(\mathcal{F})},\tag{6.10b}$$

and the first term in the right hand side vanishes. Gathering (6.10a) and (6.10b) yields:

$$(D^2\varphi\nabla\bar{\psi},\nabla\theta)_{\mathbf{L}^2(\mathcal{F})}=-(\Delta\theta\nabla\varphi,\nabla\bar{\psi})_{\mathbf{L}^2(\mathcal{F})}+(D^2\theta\nabla\bar{\psi},\nabla\varphi)_{\mathbf{L}^2(\mathcal{F})}=-(D^2\theta\nabla^\perp\bar{\psi},\nabla^\perp\varphi)_{\mathbf{L}^2(\mathcal{F})}.\tag{6.11}$$

Replacing this expression in (6.9), we recover indeed the definition (6.2) of $\Lambda_1^S(\bar{\psi},\varphi)$.

Applying Hölder's inequality yields:

$$\|\Delta\bar{\psi}(\nabla^\perp\bar{\psi}+\nabla\varphi)\|^2_{\mathbf{L}^2(\mathcal{F})}\leqslant\mathbf{c}_{\mathcal{F}}\|\Delta\bar{\psi}\|^{\frac{2}{5}}_{L^2(\mathcal{F})}\|\Delta\bar{\psi}\|^{\frac{8}{5}}_{L^4(\mathcal{F})}\Big(\|\nabla\bar{\psi}\|^2_{\mathbf{L}^5(\mathcal{F})}+\|\nabla\varphi\|^2_{\mathbf{L}^5(\mathcal{F})}\Big),$$

and then, Sobolev embedding Theorem leads straightforwardly to (6.8a).

From (6.8a), we deduce that:

$$\|\Lambda_0^S(\bar{\psi}+\theta,\varphi)\|_{S_0}\leqslant\mathbf{c}_{\mathcal{F}}\Big(\|\bar{\psi}\|^2_{\bar{S}_1}+\|\varphi\|^2_{\mathfrak{H}^2_K}+\|\theta\|^2_{S_1}\Big)^{\frac{3}{5}}\Big(\|\bar{\psi}\|^2_{\bar{S}_2}+\|\theta\|^2_{S_2}\Big)^{\frac{2}{5}},$$

and therefore, in particular:

$$\|\Lambda_0^S(\bar{\psi}+\theta,\varphi)\|_{L^{\frac{5}{2}}(0,T;S_0)}\leqslant\mathbf{c}_{\mathcal{F}}\Big(\|\bar{\psi}\|^2_{\bar{S}_1(T)}+\|\varphi\|^2_{\mathfrak{H}^2_K(T)}+\|\theta\|^2_{S_1(T)}\Big).$$

The estimate (6.8b) follows with Hölder's inequality. The last inequality (6.8c) is proved the same way. □

Remark 6.3. (1) *Denoting $u=\nabla^\perp\bar{\psi}+\nabla\varphi$ in the statement of the lemma, it is classical to verify that:*

$$(\Lambda_0^S(\bar{\psi},\varphi),\theta)_{S_0}=(\nabla uu,\nabla^\perp\theta)_{\mathbf{L}^2(\mathcal{F})}\qquad\text{for all }\theta\in S_0.\tag{6.12}$$

(2) *In this lemma, the assumption $\varphi\in\mathfrak{H}^2_K(T)$ is too strong. Indeed, the Kirchhoff potential is not required to belong to $L^2(0,T;\mathfrak{H}^3_K)$ and one can verify that $\varphi\in\mathcal{C}([0,T];\mathfrak{H}^2_K)$ would be suffisant. However, the hypothesis $\varphi\in\mathfrak{H}^2_K(T)$ will be necessary later on.*

6.2. **Weak solutions.**

Definition 6.4. *Let a positive real number T, a source term $f_S\in L^2(0,T;S_{-1})$, an initial data $\psi^{\rm i}\in S_0$ and a triple $(g_n,g_\tau,\Gamma)\in G_0(T)$ be given. Define $\psi^{\rm i}_0=\psi^{\rm i}-\mathsf{L}_0^S(g_n(0),g_\tau(0),\Gamma(0))$ and assume that Σ is of class $C^{3,1}$.*

We say that a stream function $\psi\in\bar{S}_0(T)$ is a weak (or Leray) solution to the $\psi-$Navier-Stokes equations satisfying the Dirichlet boundary conditions on Σ_T as described in (4.1) by the triple (g_n,g_τ,Γ) if $\psi=\psi_b+\psi_\ell+\psi_\Lambda$ where:

(1) *The function ψ_b accounts for the boundary conditions. It belongs to $S_0^b(T)$ defined in (4.11) and is equal to $\mathsf{L}_1^S(g_n,g_\tau,\Gamma)$;*

(2) *The function ψ_ℓ accounts for the source term and the initial condition. It is defined as the unique solution in $S_0(T)$ of the homogeneous (linear) $\psi-$Stokes Cauchy problem*

$$\partial_t\psi_\ell+\nu\mathsf{A}_1^S\psi_\ell=-\partial_t\psi_b+f_S\qquad\text{in }\mathcal{F}_T,\tag{6.13a}$$

$$\psi_\ell(0)=\psi^{\rm i}_0\qquad\text{in }\mathcal{F}.\tag{6.13b}$$

(3) *The function ψ_Λ accounts for the nonlinear advection term. It belongs to the space $S_0(T)$ and solves the nonlinear Cauchy problem:*

$$\partial_t\psi_\Lambda+\nu\mathsf{A}_1^S\psi_\Lambda=-\Lambda_1^S(\psi_b+\psi_\ell+\psi_\Lambda,\varphi)\qquad\text{in }\mathcal{F}_T,\tag{6.14a}$$

$$\psi_\Lambda(0)=0\qquad\text{in }\mathcal{F},\tag{6.14b}$$

where $\varphi=\mathsf{L}_1^n g_n$ is the Kirchhoff potential that belongs to $\mathfrak{H}^1_K(T)$.

Theorem 6.5. *For any set of data as described in Definition 6.4, there exists a unique (weak) solution in $\bar{S}_0(T)$ to the $\psi-$Navier-Stokes equations.*

Proof. The existence and uniqueness of ψ_ℓ is asserted by Proposition 5.6. Lemma 6.1 being granted, the proof of existence and uniqueness of the function ψ_Λ is quite similar to the proof [48, Chap. 1, Section 6]. Let us focus on the main differences and omit some details.

Denote by $\bar\psi$ the function in $\bar S_0(T)$ equal to the sum $\psi_b + \psi_\ell$ (where the functions ψ_b and ψ_ℓ are given) and notice that, according to Proposition 5.6 and Lemma 4.2:

$$\|\bar\psi\|_{\bar S_0(T)} \leqslant \mathbf{c}_{[\mathcal{F},\nu]}\Big[\|\psi_0^{\mathrm{i}}\|^2_{S_0} + \|f_S\|^2_{L^2(0,T;S_{-1})} + \|(g_n,g_\tau,\varGamma)\|^2_{G_0(T)}\Big]^{\frac12} \tag{6.15a}$$

$$\|\varphi\|_{\mathfrak{H}^1_K(T)} \leqslant \mathbf{c}_{\mathcal{F}}\|g_n\|_{G_0^n(T)} \leqslant \mathbf{c}_{\mathcal{F}}\|(g_n,g_\tau,\varGamma)\|_{G_0(T)}. \tag{6.15b}$$

Then, for every positive integer m, introduce $\mathbb{S}_1^m$, the finite dimensional subspace of S_1 spanned by the $m-$th first eigenvalues of $\mathcal{A}_1^S$ (loosely speaking, this operator is equal to A_1^S seen as an unbounded operator in S_{-1} of domain S_1; see (A.11)). Denote by Π_m the orthogonal projector from S_1 onto $\mathbb{S}_1^m$ and by Π_m^* its adjoint for the duality pairing $S_{-1}\times S_1$. Finally, let ψ_Λ^m be the unique solution in $\mathbb{S}_1^m$ of the Cauchy problem:

$$\partial_t\psi_\Lambda^m + \nu\mathsf{A}_1^S\psi_\Lambda^m = -\Pi_m^*\Lambda_1^S(\bar\psi+\psi_\Lambda^m,\varphi) \qquad \text{in } \mathcal{F}_T, \tag{6.16a}$$

$$\psi_\Lambda^m(0) = 0 \qquad \text{in } \mathcal{F}. \tag{6.16b}$$

The existence and uniqueness of $\psi_\Lambda^m \in \mathcal{C}^1([0,T_m];S_1)$ on a time interval $(0,T_m)$ is guaranteed by Cauchy-Lipschitz Theorem. Forming now for any $s\in(0,T_m)$ the duality pairing of equation (6.14a) set in S_{-1} with $\psi_\Lambda^m(s)$ in S_1 and using the estimate (6.3a) for the nonlinear term, we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\psi_\Lambda^m(s)\|^2_{S_0} + \nu\|\psi_\Lambda^m(s)\|^2_{S_1} \leqslant \Phi(s)\|\psi_\Lambda^m(s)\|^2_{S_0} + \Phi(s), \tag{6.17}$$

where, for every s in $(0,T)$:

$$\Phi(s) = \mathbf{c}_{[\mathcal{F},\nu]}\Big[\|\bar\psi(s)\|^2_{\bar S_1}\|\bar\psi(s)\|^2_{\bar S_0} + \|\varphi(s)\|^2_{\mathfrak{H}^1_K}\|\varphi(s)\|^2_{\mathfrak{H}^2_K}\Big].$$

One easily verifies that Φ belongs to $L^1(0,T)$ and that, according to the estimates (6.15) above:

$$\|\Phi\|_{L^1(0,T)} \leqslant \mathbf{c}_{[\mathcal{F},\nu]}\Big[\|\psi_0^{\mathrm{i}}\|^2_{S_0} + \|f_S\|^2_{L^2(0,T;S_{-1})} + \|(g_n,g_\tau,\varGamma)\|^2_{G_0(T)}\Big]. \tag{6.18}$$

Then, integrating (6.17) over $(0,t)$ for any $t\in(0,T_m)$ and introducing the constant $\lambda_{\mathcal{F}}$ defined in (3.29) yields the estimate:

$$\|\psi_\Lambda^m(t)\|^2_{S_0} + \int_0^t \big(\nu\lambda_{\mathcal{F}} - \Phi(s)\big)\|\psi_\Lambda^m(s)\|^2_{S_0}\,\mathrm{d}s \leqslant \int_0^t \Phi(s)\,\mathrm{d}s \qquad \text{for all } t\in(0,T),$$

which, with Grönwall's inequality, leads to the estimate below, uniform in t according to (6.18):

$$\|\psi_\Lambda^m(t)\|_{S_0} \leqslant \|\Phi\|^{\frac12}_{L^1(0,T)}e^{\frac12\|\Phi\|_{L^1(0,T)}} \leqslant \mathbf{c}_{[\mathcal{F},\nu,\|\psi_0^{\mathrm{i}}\|_{S_0},\|f_S\|_{L^2(0,T;S_{-1})},\|(g_n,g_\tau,\varGamma)\|_{G_0(T)}]}. \tag{6.19a}$$

We deduce that T_m can be chosen equal to T. Going back to inequality (6.17), integrating it again over the time interval $(0,T)$ and using the estimate (6.19a), we get another estimate uniform in t:

$$\|\psi_\Lambda^m\|_{L^2(0,T;S_1)} \leqslant \mathbf{c}_{[\mathcal{F},\nu,\|\psi_0^{\mathrm{i}}\|_{S_0},\|f_S\|_{L^2(0,T;S_{-1})},\|(g_n,g_\tau,\varGamma)\|_{G_0(T)}]}. \tag{6.19b}$$

From identity (6.16a), we deduce now that:

$$\|\partial_t\psi_\Lambda^m\|_{L^2(0,T;S_{-1})} \leqslant \nu\|\psi_\Lambda^m\|_{L^2(0,T;S_1)} + \|\Lambda(\bar\psi+\psi_\Lambda^m,\varphi)\|_{L^2(0,T;S_{-1})}.$$

Combining (6.3c), (6.19a) and (6.19b) allows us to deduce that:

$$\|\partial_t\psi_\Lambda^m\|_{L^2(0,T;S_{-1})} \leqslant \mathbf{c}_{[\mathcal{F},\nu,\|\psi_0^{\mathrm{i}}\|_{S_0},\|f_S\|_{L^2(0,T;S_{-1})},\|(g_n,g_\tau,\varGamma)\|_{G_0(T)}]}. \tag{6.19c}$$

It follows from the estimates (6.19) that the sequence $(\psi_\Lambda^m)_{m\geqslant1}$ remains in a ball of $S_0(T)$, centered at the origin and whose radius depends only on $\mathcal{F}$, ν and the norms of the data $\|\psi_0^{\mathrm{i}}\|_{S_0}$, $\|f_S\|_{L^2(0,T;S_{-1})}$ and $\|(g_n,g_\tau,\varGamma)\|_{G_0(T)}$. The existence of a solution as limit of a subsequence of $(\psi_\Lambda^m)_m$ is next obtained, following exactly the lines of the proof [48, Chap. 1, Section 6].

Let us address now the uniqueness of the solution. We denote by Ψ_Λ the difference $\psi^2_\Lambda - \psi^1_\Lambda$ between two solutions to the Cauchy problem (6.14) and this function satisfies:

$$\partial_t \Psi_\Lambda + \nu \mathsf{A}^S_1 \Psi_\Lambda = -\Lambda^S_1(\bar\psi + \psi^2_\Lambda, \varphi) + \Lambda^S_1(\bar\psi + \psi^1_\Lambda, \varphi) \qquad \text{in } \mathcal{F}_T.$$

Forming, for a.e. $t \in (0,T)$, the duality pairing of this identity set in S_{-1} with $\Psi_\Lambda(t) \in S_1$ and using the inequality (6.3b) results in the estimate:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Psi_\Lambda(t)\|^2_{S_0} + \Big[\nu\lambda_{\mathcal{F}} - \mathbf{c}_{[\mathcal{F},\nu]}\big(\|\bar\psi\|^2_{\bar S_1}\|\bar\psi\|^2_{S_0} + \|\varphi\|^2_{\mathfrak{H}^2_K}\|\varphi\|^2_{\mathfrak{H}^1_K} + \|\psi^1_\Lambda\|^2_{S_1}\|\psi^1_\Lambda\|^2_{S_0}\big)\Big]\|\Psi_\Lambda(t)\|^2_{S_0} \leqslant 0.$$

The conclusion follows with Grönswall's inequality, keeping in mind that inequalities (6.19a) and (6.19b) hold for ψ^1_Λ as well. The proof is now completed. □

Definition 6.4 and Theorem 6.5 can easily be rephrased in terms of the vorticity field. The nonlinear advection term is defined for every $\bar\omega \in \bar V_0$ and $\varphi \in \mathfrak{H}^2_K$ as an element of V_{-2} by:

$$\Lambda^V_1(\bar\omega, \varphi) = \Delta_{-2}\Lambda^S_1(\bar\Delta_0^{-1}\bar\omega, \varphi) = -\langle \Lambda^S_1(\bar\Delta_0^{-1}\bar\omega, \varphi), \mathsf{Q}_2\cdot\rangle_{S_{-1},S_1},$$

the latter identity being deduced from (3.31b).

Definition 6.6. *Let a positive real number T, a source term $f_V \in L^2(0,T;V_{-2})$, an initial data $\omega^{\mathrm{i}} \in V_{-1}$ and a triple $(g_n, g_\tau, \Gamma) \in G_0(T)$ be given. Define $\omega^{\mathrm{i}}_0 = \omega^{\mathrm{i}} - \mathsf{L}^V_{-1}(g_n(0), g_\tau(0), \Gamma(0))$ and assume that Σ is of class $C^{3,1}$.*

We say that a vorticity function $\omega \in \bar V_{-1}(T)$ is a (weak) solution to the $\omega-$Navier-Stokes equations satisfying the Dirichlet boundary conditions on Σ_T as described in (4.1) by the triple (g_n, g_τ, Γ) if $\omega = \omega_b + \omega_\ell + \omega_\Lambda$ where:

(1) *The function ω_b accounts for the boundary conditions. It belongs to $V^b_{-1}(T)$ (defined in(4.34)) and is equal to $\mathsf{L}^V_0(g_n, g_\tau, \Gamma)$;*
(2) *The function ω_ℓ accounts for the source term and the initial condition. It is defined as the unique solution in $V_{-1}(T)$ of the homogeneous (linear) $\omega-$Stokes Cauchy problem*

$$\partial_t\omega_\ell + \nu\mathsf{A}^V_0\omega_\ell = -\partial_t\omega_b + f_V \qquad \text{in } \mathcal{F}_T, \tag{6.20a}$$

$$\omega_\ell(0) = \omega^{\mathrm{i}}_0 \qquad \text{in } \mathcal{F}. \tag{6.20b}$$

(3) *The function ω_Λ accounts for the nonlinear advection term. It belongs to the space $V_{-1}(T)$ and solves the nonlinear Cauchy problem:*

$$\partial_t\omega_\Lambda + \nu\mathsf{A}^V_0\omega_\Lambda = -\Lambda^V_1(\omega_b + \omega_\ell + \omega_\Lambda, \varphi) \qquad \text{in } \mathcal{F}_T, \tag{6.21a}$$

$$\omega_\Lambda(0) = 0 \qquad \text{in } \mathcal{F}, \tag{6.21b}$$

where $\varphi = \mathsf{L}^n_1 g_n$ is the Kirchhoff potential that belongs to $\mathfrak{H}^1_K(T)$.

Theorem 6.7. *With any set of data as described in Definition 6.6, there exists a unique solution ω to the $\omega-$Navier-stokes equations. Moreover, if ψ is the unique solution to the $\psi-$Navier-Stokes equations as defined in Definition 6.4 and $\omega^{\mathrm{i}} = \Delta_{-1}\psi^{\mathrm{i}}$, $f_V = \Delta_{-2}f_S$, all the other data being equal, then $\omega = \bar\Delta_0\psi$.*

Proof. It suffices to apply the operator Δ_{-2} to (6.13) and (6.14) to obtain (6.20) and (6.21), because, according to the commutative diagram of Fig. 4, $\Delta_{-2}\mathsf{A}^S_1 = \mathsf{A}^V_0\Delta_0$. □

In case of homogeneous boundary conditions, we recover the exponential decay estimates as stated in Lemma 5.3 for the $\psi-$Stokes Cauchy problem, namely:

Corollary 6.8. *Assume that ψ is solution to the $\psi-$Navier-Stokes equations with homogeneous boundary conditions (i.e. $\psi_b = 0$ in Definition 6.4). Then the exponential decay* (5.6a) *holds true with $k = 0$.*

Similarly, if ω is solution to the $\omega-$Navier-Stokes equations with homogeneous boundary conditions (i.e. $\omega_b = 0$ in Definition 6.6), then the exponential decay (5.6b) *holds true with $k = -1$.*

Proof. In case of homogeneous boundary conditions, ψ solves the Cauchy problem:

$$\begin{aligned} \partial_t\psi + \nu \mathsf{A}_1^S\psi &= -\Lambda_1^S(\psi,0) + f_S && \text{in } \mathcal{F}_T,\\ \psi(0) &= \psi^{\mathrm{i}} && \text{in } \mathcal{F}. \end{aligned}$$

It suffices to form the duality pairing with $\psi \in S_1$, notice that $\langle \Lambda_1^S(\psi,0),\psi\rangle_{S_{-1},S_1} = 0$ and apply Grönwall's inequality to complete the proof. ☐

Remark 6.9. *In Definition 6.6, the initial condition ω^{i} can be taken in the dual space V_{-1}. This space contains, for every $1 < p < 2$:*

$$L^p_V = \{(\omega, \mathsf{Q}_1\cdot)_{L^2(\mathcal{F})} \,:\, \omega \in L^p(\Omega)\},$$

which can be identified with $L^p(\mathcal{F})$. The space V_{-1} contains also, for every Lipschitz curve $\mathscr{C}$ included in $\mathcal{F}$ and for every $q \in H^{-\frac12}(\mathscr{C})$ what can be identified as a vorticity filament:

$$\omega_q : \theta \in V_1 \mapsto \int_{\mathscr{C}} q\,\mathsf{Q}_1\theta\,\mathrm{d}s.$$

6.3. **Strong solutions.**

Definition 6.10. *Let a positive real number T, a source term $f_S \in L^2(0,T;S_0)$, an initial data $\psi^{\mathrm{i}} \in \bar{S}_1$ and a triple $(g_n, g_\tau, \Gamma) \in G_1(T)$ be given. Define $\psi^{\mathrm{i}}_0 = \psi^{\mathrm{i}} - \mathsf{L}_1^S(g_n(0), g_\tau(0), \Gamma(0))$ and assume that Σ is of class $C^{2,1}$ and that the compatibility condition:*

$$\psi_0^{\mathrm{i}} \in S_1 \quad \text{or equivalently that} \quad \frac{\partial \psi_0^{\mathrm{i}}}{\partial n}\bigg|_\Sigma = 0, \tag{6.22}$$

is satisfied.

We say that a stream function $\psi \in \bar{S}_1(T)$ is a strong (or Kato) solution to the $\psi-$Navier-Stokes equations satisfying the Dirichlet boundary conditions on Σ_T as described in (4.1) by the triple (g_n, g_τ, Γ) if $\psi = \psi_b + \psi_\ell + \psi_\Lambda$ where:

1. *The function ψ_b accounts for the boundary conditions. It belongs to $S_1^b(T)$ and is equal to $\mathsf{L}_2^S(g_n, g_\tau, \Gamma)$;*
2. *The function ψ_ℓ accounts for the source term and the initial condition. It is defined as the unique solution in $S_1(T)$ of the homogeneous (linear) $\psi-$Stokes Cauchy problem*

$$\partial_t\psi_\ell + \nu\mathsf{A}_2^S\psi_\ell = -\partial_t\psi_b + f_S \qquad \text{in } \mathcal{F}_T, \tag{6.23a}$$

$$\psi_\ell(0) = \psi_0^{\mathrm{i}} \qquad \text{in } \mathcal{F}. \tag{6.23b}$$

3. *The function ψ_Λ accounts for the nonlinear advection term. It belongs to the space $S_1(T)$ and solves the nonlinear Cauchy problem:*

$$\partial_t\psi_\Lambda + \nu\mathsf{A}_2^S\psi_\Lambda = -\Lambda_0^S(\psi_b+\psi_\ell+\psi_\Lambda, \varphi) \qquad \text{in } \mathcal{F}_T, \tag{6.24a}$$

$$\psi_\Lambda(0) = 0 \qquad \text{in } \mathcal{F}, \tag{6.24b}$$

where $\varphi = \mathsf{L}_2^n g_n$ is the Kirchhoff potential that belongs to $\mathfrak{H}^2_K(T)$.

Theorem 6.11. *With any set of data as described in Definition 6.10, there exists a unique (strong) solution in $\bar{S}_1(T)$ to the $\psi-$Navier-Stokes equations.*

Proof of Theorem 6.11. The proof is based on a fixed point argument. The existence and uniqueness of ψ_b and ψ_ℓ being granted, denote by $\bar\psi$ the sum $\psi_b + \psi_\ell$ that belongs to $\bar{S}_1(T)$ and introduce the constant:

$$R_0 = \Big[\|\psi_0^{\mathrm{i}}\|^2_{S_1} + \|f_S\|^2_{L^2(0,T;S_0)} + \|(g_n,g_\tau,\Gamma)\|^2_{G_1(T)}\Big]^{\frac12}.$$

Then, define three maps:

1. $\mathsf{X}_T : L^2(0,T;S_0) \longrightarrow S_1(T)$ where, for every $f \in L^2(0,T;S_0)$, $\theta = \mathsf{X}_T f$ is the unique solution in $S_1(T)$ to the Cauchy problem:

$$\begin{aligned} \partial_t\theta + \nu\mathsf{A}_2^S\theta &= f && \text{in } \mathcal{F}_T,\\ \theta(0) &= 0 && \text{in } \mathcal{F}. \end{aligned}$$

(2) $\mathsf{Y}_T : S_1(T) \longrightarrow L^2(0,T;S_0)$ where, for every $\theta \in S_1(T)$, $\mathsf{Y}_T(\theta) = -\Lambda_0^S(\bar\psi + \theta, \varphi)$ (remind that the Kirchhoff potential $\varphi = \mathsf{L}_2^n g_n$ is given).

(3) $\mathsf{Z}_T = \mathsf{Y}_T \circ \mathsf{X}_T : L^2(0,T;S_0) \longrightarrow L^2(0,T;S_0)$.

Combining the estimates (6.8b), (6.8c) and (5.11), we deduce that for every $f, f_1, f_2 \in L^2(0,T;S_0)$:

$$\|\mathsf{Z}_T f\|_{L^2(0,T,S_0)} \leqslant \mathbf{c}_{[\mathcal{F},\nu]} T^{\frac{1}{10}} \Big(R_0^2 + \|f\|^2_{L^2(0,T;S_0)}\Big),$$

$$\|\mathsf{Z}_T f_2 - \mathsf{Z}_T f_1\|_{L^2(0,T,S_0)} \leqslant \mathbf{c}_{[\mathcal{F},\nu]} T^{\frac{1}{10}} \Big(R_0 + \|f_1\|_{L^2(0,T;S_0)} + \|f_2\|_{L^2(0,T;S_0)}\Big)\|f_2 - f_1\|_{L^2(0,T,S_0)}.$$

Let now R be equal to $2R_0$. Then, for every f, f_1, f_2 in B_R, the ball of center 0 and radius R in $L^2(0,T;S_0)$:

$$\|\mathsf{Z}_T f\|_{L^2(0,T,S_0)} \leqslant \mathbf{c}_{[\mathcal{F},\nu]} T^{\frac{1}{10}} R^2,$$

$$\|\mathsf{Z}_T f_2 - \mathsf{Z}_T f_1\|_{L^2(0,T,S_0)} \leqslant \mathbf{c}_{[\mathcal{F},\nu]} T^{\frac{1}{10}} R\|f_2 - f_1\|_{L^2(0,T,S_0)}.$$

Thus, for $T_0 = 1/(4\mathbf{c}_{[\mathcal{F},\nu]}R_0)^{10}$, the mapping Z_{T_0} is a contraction from B_R into itself. Banach fixed point Theorem asserts that Z_{T_0} admits a unique fixed point whose image by X_{T_0} yields a solution ψ_Λ to the Cauchy problem (6.24) on $(0,T_0)$. Let $(0,T^*)$ be the larger time interval to which the solution $\psi = \psi_\Lambda + \psi_\ell + \psi_b$ can be extended. The time of existence T_0 depending only on R_0, standard arguments ensure that the following alternative holds:

$$\text{Either} \quad T^* = T \qquad \text{or} \qquad \lim_{t \to T^*} \|\psi(t)\|_{\bar S_1} = +\infty. \tag{6.25}$$

As being a weak solution, estimates (6.19a) and (6.19b) hold for ψ_Λ, namely $\|\psi_\Lambda\|_{\mathcal{C}(0,T;S_0)}$ and $\|\psi_\Lambda\|_{L^2(0,T;S_1)}$ are bounded. On the other hand, forming the scalar product of equation (6.24a) with $\mathsf{A}_2^S\psi_\Lambda$ in S_0, we obtain:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|\psi_\Lambda\|^2_{S_1} + \nu\|\psi_\Lambda\|^2_{S_2} = -(\Lambda_0(\psi,\varphi), \mathsf{A}_2^S\psi_\Lambda)_{S_0} \qquad \text{on } (0,T^*).$$

Considering the nonlinear term in the right hand side, Hölder's inequality yields:

$$|(\Lambda_0(\psi,\varphi), \mathsf{A}_2^S\psi_\Lambda)_{S_0}| \leqslant \|\Delta\psi\|_{L^4(\mathcal{F})}\|\nabla^\perp\psi + \nabla\varphi\|_{\mathbf{L}^4(\mathcal{F})}\|\psi_\Lambda\|_{S_2}. \tag{6.26}$$

whence we deduce, proceeding as in the proof of Lemma 6.1, that:

$$|(\Lambda_0\psi, \mathsf{A}_2^S\psi_\Lambda)_{S_0}| \leqslant \Big[\frac{\nu}{2} + \mathbf{c}_{[\mathcal{F},\nu]}\Theta_1\Big]\|\psi_\Lambda\|^2_{S_2} + \mathbf{c}_{[\mathcal{F},\nu]}\Theta_2 \qquad \text{on } (0,T^*),$$

with $\Theta_1 = \big[\|\psi_\Lambda\|^2_{S_1} + \|\psi_b\|^2_{\bar S_1} + \|\psi_\ell\|^2_{\bar S_1} + \|\varphi\|^2_{\mathfrak{H}^2_K}\big]$ and $\Theta_2 = \big[\|\psi_\Lambda\|^2_{S_0} + \|\psi_b\|^2_{\bar S_0} + \|\psi_\ell\|^2_{\bar S_0} + \|\varphi\|^2_{\mathfrak{H}^1_K}\big] + \big[\|\psi_b\|^2_{\bar S_2} + \|\psi_\ell\|^2_{\bar S_2}\big]$. The functions Θ_1 and Θ_2 both belong to $L^1(0,T)$. It follows that:

$$\frac{\mathrm{d}}{\mathrm{dt}}\|\psi_\Lambda\|^2_{S_1} + \big(\nu - \mathbf{c}_{[\mathcal{F},\nu]}\Theta_1\big)\|\psi_\Lambda\|^2_{S_2} \leqslant \mathbf{c}_{[\mathcal{F},\nu]}\Theta_2 \qquad \text{on } (0,T^*),$$

and by Grönwall's inequality, we conclude that $\|\psi_\Lambda\|_{S_1}$ is bounded on $[0,T^*)$. The latter occurence in (6.25) may not happen and therefore $T^* = T$. ☐

Definition 6.10 and Theorem 6.11 can easily be rephrased in terms of the vorticity field. The nonlinear advection term is defined for every $\bar\omega \in \bar V_1$ and $\varphi \in \mathfrak{H}^2_K$ as an element of V_{-1} by:

$$\Lambda_0^V(\bar\omega,\varphi) = \Delta_{-1}\Lambda_0^S(\bar\Delta_1^{-1}\bar\omega,\varphi) = -(\omega(\nabla^\perp\bar\psi + \nabla\varphi), \nabla\mathsf{Q}_1\cdot)_{\mathbf{L}^2(\mathcal{F})}, \tag{6.27}$$

the latter identity being deduced from (3.31b). In (6.27), ω stands for the *regular* part of $\bar\omega$ (see Remark 4.17) and $\bar\psi = \bar\Delta_1^{-1}\bar\omega$.

Remark 6.12. *Notice that even at this level of regularity, the nonlinear term of the vorticity equation cannot be written in the most common form, namely as the advection term $u\cdot\nabla\omega$ (see Section 8).*

Definition 6.13. *Let a positive real number T, a source term $f_V \in L^2(0,T;V_{-1})$, an initial data $\omega^{\mathrm{i}} \in \bar V_0$ and a triple $(g_n, g_\tau, \Gamma) \in G_1(T)$ be given. Define $\omega_0^{\mathrm{i}} = \omega^{\mathrm{i}} - \mathsf{L}_0^V(g_n(0), g_\tau(0), \Gamma(0))$ and assume that Σ is of class $C^{2,1}$ and that the compatibility condition $\omega_0^{\mathrm{i}} \in V_0$ is satisfied.*

We say that a vorticity function $\omega \in \bar V_0(T)$ is a strong (or Kato) solution to the $\omega-$Navier-Stokes equations satisfying the Dirichlet boundary conditions on Σ_T as described in (4.1) by the triple (g_n, g_τ, Γ) if $\omega = \omega_b + \omega_\ell + \omega_\Lambda$ where:

(1) *The function ω_b accounts for the boundary conditions. It lies in $V_0^b(T)$ and is equal to $\mathsf{L}_1^V(g_n, g_\tau, \Gamma)$;*
(2) *The function ω_ℓ accounts for the source term and the initial condition. It is defined as the unique solution in $V_0(T)$ of the homogeneous (linear) $\omega-$Stokes Cauchy problem*

$$\partial_t \omega_\ell + \nu \mathsf{A}_1^V \omega_\ell = -\partial_t \omega_b + f_V \qquad \text{in } \mathcal{F}_T, \tag{6.28a}$$

$$\omega_\ell(0) = \omega_0^{\mathrm{i}} \qquad \text{in } \mathcal{F}. \tag{6.28b}$$

(3) *The function ω_Λ accounts for the nonlinear advection term. It belongs to the space $V_0(T)$ and solves the nonlinear Cauchy problem:*

$$\partial_t \omega_\Lambda + \nu \mathsf{A}_1^V \omega_\Lambda = -\Lambda_0^V(\omega_b + \omega_\ell + \omega_\Lambda, \varphi) \qquad \text{in } \mathcal{F}_T, \tag{6.29a}$$

$$\omega_\Lambda(0) = 0 \qquad \text{in } \mathcal{F}, \tag{6.29b}$$

where $\varphi = \mathsf{L}_2^n g_n$ is the Kirchhoff potential that belongs to $\mathfrak{H}_K^2(T)$.

The counterpart of Theorem 6.11 reads:

Theorem 6.14. *For any set of data as described in Definition 6.13, there exists a unique (strong) solution in $\bar{V}_0(T)$ to the $\omega-$Navier-Stokes equations. Moreover, if ψ is the unique solution to the $\psi-$Navier-Stokes equations as defined in Definition 6.10 and $\omega^{\mathrm{i}} = \bar{\Delta}_0 \psi^{\mathrm{i}}$, $f_V = \Delta_{-1} f_S$, all the other data being equal, then $\omega = \bar{\Delta}_1 \psi$.*

Once again, we point out that Equations (6.28a) and (6.29a) are set in $L^2(0,T;V_{-1})$ where V_{-1} is not a distribution space. As very well explained in [61], this may be the cause of numerus mistakes and misunderstandings. Inspired by Guermond and Quartapelle in [32], let us elaborate a "distribution-based" reformulation of Systems (6.28)-(6.29). Any solution ω to the $\omega-$NS equations can be decomposed into:

$$\omega = \omega_\Lambda + \omega_\ell + \omega_b \qquad \text{where} \qquad \omega_b = \omega_b^{\mathfrak{H}} + \zeta_b \quad \text{with} \quad \zeta_b = \sum_{j=1}^N \Gamma_j \zeta_j. \tag{6.30a}$$

In these sums, ω_Λ and ω_ℓ belong to $V_0(T)$, $\omega_b^{\mathfrak{H}}$ is in $H^1(0,T;V_{-1}) \cap \mathcal{C}([0,T], L_V^2) \cap L^2(0,T;H_V^1)$ and $\Gamma_j \in H^1(0,T)$ for every $j = 1,\dots,N$. In the splitting (6.30a) ζ_b is identified as the singular part of ω while the "regular part" is:

$$\omega_{\mathsf{r}} = \omega_\Lambda + \omega_\ell + \omega_b^{\mathfrak{H}}. \tag{6.30b}$$

Recalling the decomposition (3.4) of the space S_0, namely:

$$S_0 = H_0^1(\mathcal{F}) \overset{\perp}{\oplus} \mathbb{F}_S,$$

where the finite dimensional space $\mathbb{F}_S$ is spanned by the functions ξ_j ($j = 1,\dots,N$), we deduce that:

$$V_1 = \mathsf{P}_1 H_0^1(\mathcal{F}) \overset{\perp}{\oplus} \mathbb{F}_V.$$

From any source term $f_V \in L^2(0,T;V_{-1})$, we define $f_V^{\mathsf{r}} \in L^2(0,T;H^{-1}(\mathcal{F}))$ by setting:

$$f_V^{\mathsf{r}} = \langle f_V, \mathsf{P}_1 \cdot \rangle_{V_{-1},V_1}. \tag{6.30c}$$

Theorem 6.15. *Let ω be a solution to the $\omega-$NS equations as described in Definition 6.13 and introduce ω_{r}, ζ_b and f_V^{r} as explained in the relations (6.30). Then ω_{r} obeys the equation:*

$$\partial_t \omega_{\mathsf{r}} - \nu \Delta \omega_{\mathsf{r}} + \nabla \cdot \left[\omega_{\mathsf{r}}(\nabla^\perp \psi + \nabla \varphi)\right] = f_V^{\mathsf{r}} \qquad \text{in} \quad L^2(0,T;H^{-1}(\mathcal{F})), \tag{6.31a}$$

and for every $j = 1,\dots,N$:

$$\Gamma_j' + \nu(\nabla \omega_{\mathsf{r}}, \nabla \xi_j)_{\mathbf{L}^2(\mathcal{F})} - (\omega_{\mathsf{r}}(\nabla^\perp \psi + \nabla \varphi), \nabla \xi_j)_{\mathbf{L}^2(\mathcal{F})} = \langle f_V, \mathsf{P}_1 \xi_j \rangle_{V_{-1},V_1} \quad \text{in} \quad L^2(0,T). \tag{6.31b}$$

Proof. By definition of a strong solution to the $\omega-$NS equations, the following equality holds for every $\theta \in V_1$:

$$\frac{\mathrm{d}}{\mathrm{dt}} \langle \omega_{\mathsf{r}}, \theta \rangle_{V_{-1},V_1} + \sum_{j=1}^N \Gamma_j' \langle \zeta_j, \theta \rangle_{V_{-1},V_1} + \nu(\omega_{\mathsf{r}}, \theta)_{V_1} - (\omega_{\mathsf{r}}(\nabla^\perp \psi + \nabla \varphi), \nabla \mathsf{Q}_1 \theta)_{\mathbf{L}^2(\mathcal{F})} = \langle f_V, \theta \rangle_{V_{-1},V_1} \quad \text{on } (0,T).$$

Notice now that

$$\langle \omega_{\mathsf{r}}, \theta\rangle_{V_{-1},V_1} = (\omega_\Lambda + \omega_\ell, \theta)_{V_0} + (\omega_b^{\mathfrak{H}}, \mathsf{Q}_1\theta)_{L^2(\mathcal{F})} = (\omega_{\mathsf{r}}, \mathsf{Q}_1\theta)_{L^2(\mathcal{F})}.$$

Choosing the test function θ in $\mathsf{P}_1 H^1_0(\mathcal{F})$, we obtain (6.31a) and choosing θ in $\mathbb{F}_V$ leads to (6.31b). □

Appart from the nonlinear advection term, formulation (6.31) is quite similar to System (2.9) displayed at the beginning of this paper. In Section 8, we shall seek more regular solutions to the $\omega-$NS system in order to obtain Identity (6.31a) satisfied in $L^2(0,T;L^2(\mathcal{F}))$.

7. The pressure

The purpose of this section is to explain how the pressure can be recovered from the stream function or the vorticity field, i.e. to derive Bernoulli-like formulas for the $\psi-$NS equations. In the literature, the existence of the pressure field is usually deduced from the Helmholtz-Weyl decomposition and no expression is supplied.

7.1. **Hilbertian framework for the velocity field.** The following Lebesgue spaces shall enter the definition of the pressure:

$$L^2_{\rm m} = \Big\{ f \in L^2(\mathcal{F}) \,:\, \int_{\mathcal{F}} f\,{\rm d}x = 0\Big\} \qquad \text{and} \qquad \mathfrak{H}_{\rm m} = \mathfrak{H} \cap L^2_{\rm m}, \tag{7.1}$$

as well as the Sobolev spaces below:

$$H^1_{\rm m} = H^1(\mathcal{F}) \cap L^2_{\rm m} \qquad \text{and} \qquad H^2_{\rm m} = \Big\{ f \in H^2(\mathcal{F}) \cap L^2_{\rm m} \,:\, \frac{\partial f}{\partial n}\Big|_\Sigma = 0 \Big\}. \tag{7.2}$$

The last two spaces are provided with the norms:

$$(f_1, f_2)_{H^1_{\rm m}} = (\nabla f_1, \nabla f_2)_{\mathbf{L}^2(\mathcal{F})} \qquad \text{for all } f_1, f_2 \in H^1_{\rm m},$$

and

$$(f_1, f_2)_{H^2_{\rm m}} = (\Delta f_1, \Delta f_2)_{L^2(\mathcal{F})} \qquad \text{for all } f_1, f_2 \in H^2_{\rm m}.$$

We recall that the lifting operators L^τ_k (for every integer k) were introduced in Definition 4.3.

Definition 7.1. *For every $f \in L^2_{\rm m}$ we denote by Θ_f the unique function in $H^2_{\rm m}$ satisfying:*

$$\Delta \Theta_f = f \quad \text{in } \mathcal{F},$$

and we denote by Ψ_f the unique preimage of f in Z_2 by the operator A^Z_2 (see Lemma 3.8).

Then, we define the operator $\mathsf{H} : L^2_{\rm m} \longrightarrow L^2_{\rm m}$ by:

$$\mathsf{H} f = \Delta \mathsf{L}^\tau_1 \frac{\partial \Theta_f}{\partial \tau}\Big|_\Sigma \qquad \text{for all } f \in L^2_{\rm m}.$$

It is worth noticing the obvious equality:

$$\Psi_{\mathsf{H}f} = \mathsf{L}^\tau_1 \frac{\partial \Theta_f}{\partial \tau}\Big|_\Sigma \qquad \text{for all } f \in L^2_{\rm m}. \tag{7.3}$$

The operator H will come in handy for defining the pressure from the stream function. The main properties of H are summarized in the following lemma:

Lemma 7.2. *The operator H is bounded,* $\operatorname{Im}\mathsf{H} = \mathfrak{H}_{\rm m}$ *and* $\ker \mathsf{H} = V_0$*, what entails that H is an isomorphism from $\mathfrak{H}_{\rm m}$ onto $\mathfrak{H}_{\rm m}$.*

Denoting classically by H^* the adjoint of H, we deduce that $\operatorname{Im}\mathsf{H}^* = \mathfrak{H}_{\rm m}$, $\ker\mathsf{H}^* = V_0$ *and H^* is an isomorphism from $\mathfrak{H}_{\rm m}$ onto itself. Furthermore, for every $f \in \mathfrak{H}_{\rm m}$, the function $\mathsf{H}^* f$ is the harmonic conjugate of f i.e. the unique function in $\mathfrak{H}_{\rm m}$ such that the complex function*

$$z = (x_1 + ix_2) \longmapsto f(x_1,x_2) + i\,(\mathsf{H}^* f)(x_1,x_2)$$

is holomorphic in $\mathcal{F}$.

Proof. The boundedness results from elliptic regularity results for Θ_f and from the boundedness of the operator L_1^τ (we recall that by default Σ is assumed to be at least of class $\mathcal{C}^{1,1}$). By construction, H is valued in $\mathfrak{H}_{\rm m}$ so let a function h be given in $\mathfrak{H}_{\rm m}$. According to Lemma 3.9, Ψ_h belongs to $\mathfrak{B}_S$ and:

$$\int_{\Sigma^+}\frac{\partial\Psi_h}{\partial n}\,{\rm d}s=\int_{\mathcal{F}}h\,{\rm d}x-\sum_{j=1}^N\int_{\Sigma_j^-}\frac{\partial\Psi_h}{\partial n}\,{\rm d}s=0.$$

We can then define $g\in H^{\frac{3}{2}}(\Sigma)$ such that $\partial g/\partial\tau=\partial\Psi_h/\partial n$ on Σ. We denote now by θ_h the biharmonic function in $H^2(\mathcal{F})$ such that $\partial\theta_h/\partial n=0$ and $\theta_h=g$ on Σ. One easily verifies that $\Delta\theta_h$ belongs to $L^2_{\rm m}$ and $\mathsf{H}\Delta\theta_h=h$. This proves that $\operatorname{Im}\mathsf{H}=\mathfrak{H}_{\rm m}$.

According to Lemma 4.5, the operator L_1^τ is an isomorphism from G_1^τ onto $\mathfrak{B}_S$. Therefore, if $\mathsf{H}f=0$ for some f, then Θ_f is in S_1 and hence $f=\Delta\Theta_f$ is in V_0, which means that indeed $\ker\mathsf{H}=V_0$.

Let now h be in $\mathfrak{H}_{\rm m}$ and f be in $L^2_{\rm m}$. Then:

$$(h,\mathsf{H}f)_{L^2(\mathcal{F})}=(h,\Delta\Psi_{\mathsf{H}f})_{L^2(\mathcal{F})}=\int_\Sigma h\frac{\partial\Psi_{\mathsf{H}f}}{\partial n}\,{\rm d}s=\int_\Sigma h\frac{\partial\Theta_f}{\partial\tau}\,{\rm d}s=-\int_\Sigma\frac{\partial h}{\partial\tau}\Theta_f\,{\rm d}s.$$

Introducing $\bar h$ the harmonic conjugate of h, we deduce that:

$$(h,\mathsf{H}f)_{L^2(\mathcal{F})}=-\int_\Sigma\frac{\partial\bar h}{\partial n}\Theta_f\,{\rm d}s=(\bar h,\Delta\Theta_f)_{L^2(\mathcal{F})}=(\bar h,f)_{L^2(\mathcal{F})},$$

and the proof is completed. ☐

We turn now our attention to the Gelfand triple:

$$\mathbf{H}_1\subset\mathbf{H}_0\subset\mathbf{H}_{-1},\tag{7.4}$$

where $\mathbf{H}_1=\mathbf{H}^1_0(\mathcal{F})$, $\mathbf{H}_0=\mathbf{L}^2(\mathcal{F})$ is the pivot space and $\mathbf{H}_{-1}=\mathbf{H}^{-1}(\mathcal{F})$ is the dual space of $\mathbf{H}_1$. The space $\mathbf{H}_1$ is provided with its usual scalar product, namely:

$$(u,v)_{\mathbf{H}_1}=\int_{\mathcal{F}}\nabla u:\nabla v\,{\rm d}x\qquad\text{for all }u,v\in\mathbf{H}_1.\tag{7.5}$$

Theorem 7.3. *For every $u\in\mathbf{H}_1$ there exists a unique triple $(\psi,\phi,h)\in S_1\times S_1\times\mathfrak{H}_{\rm m}$ such that h is the harmonic Bergman projection of the divergence of u and*

$$u=\nabla^\perp\psi+\nabla^\perp\Psi_{\mathsf{H}h}+\nabla\Theta_h+\nabla\phi\qquad\text{in }\mathcal{F}.\tag{7.6}$$

It follows that the divergence and the curl of u are given respectively by:

$$\nabla\cdot u=\Delta\phi+h\qquad\text{and}\qquad\nabla^\perp\cdot u=\Delta\psi+\mathsf{H}h\qquad\text{in }\mathcal{F},\tag{7.7}$$

and these decompositions in $L^2(\mathcal{F})$ of $\nabla\cdot u$ and $\nabla^\perp\cdot u$ agrees with the orthogonal decomposition $V_0\overset{\perp}{\oplus}\mathfrak{H}$ of the space $L^2(\mathcal{F})$.

Finally, let u_1,u_2 be in $\mathbf{H}_1$ and denote $\psi_1,\psi_2,\phi_1,\phi_2$ the functions in S_1 and h_1,h_2 the functions in $\mathfrak{H}_{\rm m}$ such that:

$$u_k=\nabla^\perp\psi_k+\nabla^\perp\Psi_{\mathsf{H}h_k}+\nabla\Theta_{h_k}+\nabla\phi_k\quad\text{in }\mathcal{F}\qquad(k=1,2).\tag{7.8}$$

Then, the scalar product (7.5) *can be expanded as follows:*

$$\begin{aligned}(u_1,u_2)_{\mathbf{H}_1}&=(\nabla\cdot u_1,\nabla\cdot u_2)_{L^2(\mathcal{F})}+(\nabla^\perp\cdot u_1,\nabla^\perp\cdot u_2)_{L^2(\mathcal{F})}\\&=(\Delta\psi_1,\Delta\psi_2)_{L^2(\mathcal{F})}+(\mathsf{H}h_1,\mathsf{H}h_2)_{L^2(\mathcal{F})}+(h_1,h_2)_{L^2(\mathcal{F})}+(\Delta\phi_1,\Delta\phi_2)_{L^2(\mathcal{F})}.\end{aligned}\tag{7.9}$$

Proof. Let u be given and decompose the L^2 functions $\nabla^\perp\cdot u$ and $\nabla\cdot u$ respectively into the sums $\omega+\omega_h$ and $\delta+h$ with $\omega,\delta\in V_0$ and $\omega_h,h\in\mathfrak{H}$. Since:

$$\int_{\mathcal{F}}h\,{\rm d}x=\int_{\mathcal{F}}(\delta+h)\,{\rm d}x=\int_\Sigma u\cdot n\,{\rm d}s=0,$$

the harmonic function h is actually in $\mathfrak{H}_{\rm m}$. In the same way:

$$\int_{\mathcal{F}}\omega_h\,{\rm d}x=\int_{\mathcal{F}}(\omega+\omega_h)\,{\rm d}x=-\int_\Sigma u\cdot\tau\,{\rm d}s=0,$$

and ω_h is in $\mathfrak{H}_{\rm m}$ as well. Define now ϕ and ψ in S_1 such that $\Delta\phi = \delta$ and $\Delta\psi = \omega$. One easily verifies that the vector field:

$$v = u - \left[\nabla^\perp\psi + \nabla^\perp\Psi_{\mathsf{H}h} + \nabla\Theta_h + \nabla\phi\right] \qquad \text{in } \mathcal{F},$$

is in $\mathbf{H}_1$ and that $\nabla\cdot v = 0$. On the other hand $\nabla^\perp\cdot v = \omega_h - \mathsf{H}h$, which means in particular that $\nabla^\perp\cdot v \in \mathfrak{H}_{\rm m}$. This entails that $v = 0$. Indeed, according to Helmholtz-Weyl decomposition (see [27, Theorem 3.2]), there exists $\Phi \in H^1_{\rm m}$ and $\Psi \in S_0$ such that:

$$v = \nabla^\perp\Psi + \nabla\Phi \qquad \text{in } \mathcal{F}.$$

But $\Phi = 0$ since $\nabla\cdot v = 0$ and Ψ belongs to S_1 according to the boundary conditions and the regularity of v. It follows that $\Delta\Psi \in V_0$ but as observed earlier, $\Delta\Psi = \nabla^\perp\cdot v \in \mathfrak{H}_{\rm m}$, what implies that $\Psi = 0$. This proves the existence and uniqueness of the decomposition (7.6).

Assume now that u_1 and u_2 are in $\boldsymbol{\mathcal{D}}(\mathcal{F}) = \mathcal{D}(\mathcal{F};\mathbb{R}^2)$ and introduce their decompositions as in (7.8). Integrating by parts, we obtain:

$$\begin{aligned}(u_1,u_2)_{\mathbf{H}_1} &= -(\Delta u_1, u_2)_{\mathbf{L}^2(\mathcal{F})}\\ &= -\left\langle \nabla^\perp\Delta\psi_1 + \nabla^\perp\mathsf{H}h_1, u_2\right\rangle_{\boldsymbol{\mathcal{D}}'(\mathcal{F}),\boldsymbol{\mathcal{D}}(\mathcal{F})} - \left\langle \nabla h_1 + \nabla\Delta\phi_1, u_2\right\rangle_{\boldsymbol{\mathcal{D}}'(\mathcal{F}),\boldsymbol{\mathcal{D}}(\mathcal{F})}.\end{aligned} \tag{7.10}$$

We switch to the duality pairing in the second equality because although u_1 is smooth, this does not guaranty that every term in the decomposition (7.6) is also smooth (notice that invoking elliptic regularity results would require the boundary Σ to be smoother than $\mathcal{C}^{1,1}$). The former term in the right hand side of (7.10) yields:

$$\begin{aligned}\left\langle \nabla^\perp\Delta\psi_1 + \nabla^\perp\mathsf{H}h_1, u_2\right\rangle_{\boldsymbol{\mathcal{D}}'(\mathcal{F}),\boldsymbol{\mathcal{D}}(\mathcal{F})} &= (\Delta\psi_1 + \mathsf{H}h_1, \nabla^\perp\cdot u_2)_{L^2(\mathcal{F})}\\ &= -(\Delta\psi_1,\Delta\psi_2)_{L^2(\mathcal{F})} - (\mathsf{H}h_1,\mathsf{H}h_2)_{L^2(\mathcal{F})},\end{aligned}$$

while the latter leads to:

$$\left\langle \nabla h_1 + \nabla\Delta\phi_1, u_2\right\rangle_{\boldsymbol{\mathcal{D}}'(\mathcal{F}),\boldsymbol{\mathcal{D}}(\mathcal{F})} = -(h_1 + \Delta\phi_1, \nabla\cdot u_2)_{\mathbf{L}^2(\mathcal{F})} = -(h_1,h_2)_{L^2(\mathcal{F})} - (\Delta\phi_1,\Delta\phi_2)_{L^2(\mathcal{F})}.$$

The equality (7.9) follows by density of $\boldsymbol{\mathcal{D}}(\mathcal{F})$ into $\mathbf{H}_1$ and the proof is complete. ☐

Remark 7.4. (1) *The decomposition* (7.6) *differs from the one in* [27, Theorem 3.3] *where* $u \in \mathbf{H}_1$ *is decomposed into*

$$u = \nabla^\perp\psi + (-\Delta_D)^{-1}\nabla p \qquad \textit{in } \mathcal{F}, \tag{7.11}$$

with $\psi \in S_1$ *and a potential* p *in* $L^2_{\rm m}$. *The operator* $(-\Delta_D)^{-1}$ *obviously stands for the inverse of the Laplacian operator with homogeneous boundary conditions. The stream function* ψ *is the same in* (7.6) *and* (7.11).

(2) *In* [2, Theorem 3] *or* [39, Theorem 2.1], *every vector field* $u \in \mathbf{L}^2(\mathcal{F})$ *is shown to admit the decomposition:*

$$u = \nabla^\perp\psi + \nabla^\perp h + \nabla p \qquad \textit{in } \mathcal{F}, \tag{7.12}$$

with $\psi \in H^1_0(\mathcal{F})$, $p \in H^1_{\rm m}$ *and* $h \in \mathbb{F}_S$. *This expression is used by Maekawa in* [52] *to derive necessary and sufficient conditions for* u *to be in* $\mathbf{J}_1$.

(3) *Identity* (7.9) *is a trivial version of Friedrich's second inequalities; see* [27, Lemma 2.5 and Remark 2.7] *and also for instance* [41]. *However, it can also be readily deduced from* (7.9) *that there exists a constant* $\mathbf{c}_\mathcal{F}$ *such that for every* $u \in \mathbf{H}_1$*:*

$$\|u\|_{\mathbf{H}_1} \leqslant \mathbf{c}_\mathcal{F}\|\mathsf{P}^\perp\omega\|_{L^2(\mathcal{F})} + \|\delta\|_{L^2(\mathcal{F})} \qquad \textit{or} \qquad \|u\|_{\mathbf{H}_1} \leqslant \mathbf{c}_\mathcal{F}\|\omega\|^2_{L^2(\mathcal{F})} + \|\mathsf{P}^\perp\delta\|_{L^2(\mathcal{F})},$$

where $\omega = \nabla^\perp\cdot u$ *and* $\delta = \nabla\cdot u$. *It means that the* $\mathbf{H}_1$*-norm of a vector field is controlled by:*

(a) *Either the harmonic Bergman projection of the curl and the divergence of this vector field in* L^2*;*

(b) *or by the curl and the harmonic Bergman projection of the divergence of this vector field in* L^2.

We were not able to find this result in the literature.

(4) *The decomposition* (7.6) *of Theorem 7.3 allows to deduce a necessary and suffisant condition for the following overdetermined div-curl problem to be well-posed: There exists a unique u in $\mathbf{H}_1$ such that:*

$$\nabla\cdot u=\delta \qquad \text{and} \qquad \nabla^\perp\cdot u=\omega \qquad \text{in } \mathcal{F},$$

with δ and ω in $L^2(\mathcal{F})$ if and only if $\mathsf{P}^\perp\omega$ is the harmonic conjugate of $\mathsf{P}^\perp\delta$. This result seems to be new as well.

As shown in [60], the definition of the pressure for the Navier-Stokes equations (in classical velocity-pressure formulation) is not possible for a source term $f_{\mathbf{J}}$ in $L^2(0,T;\mathbf{J}_{-1})$, what means in nonprimitive variables, for $f_S\in L^2(0,T;S_{-1})$ (see Fig. 3). The definition of the pressure requires the source term to be in $L^2(0,T;\mathbf{H}_{-1})$. To be more specific, we need to elaborate on the structure of the dual space $\mathbf{H}_{-1}$.

Proposition 7.5. *For every linear form $f_{\mathbf{H}}$ in $\mathbf{H}_{-1}$, there exists a unique pair $(\delta_{\mathbf{H}},\omega_{\mathbf{H}})\in L^2_{\rm m}\times V_0$ such that:*

$$\langle f_{\mathbf{H}},u\rangle_{\mathbf{H}_{-1},\mathbf{H}_1}=(\delta_{\mathbf{H}},\nabla\cdot u)_{L^2(\mathcal{F})}+(\omega_{\mathbf{H}},\nabla^\perp\cdot u)_{L^2(\mathcal{F})} \qquad \text{for all } u\in\mathbf{H}_1.$$

If $f_{\mathbf{H}}$ belongs to $\mathbf{H}_0$, then:

$$\delta_{\mathbf{H}}=-\phi_{\mathbf{H}}-\mathsf{H}^*\psi_{\mathbf{H}} \qquad \text{and} \qquad \omega_{\mathbf{H}}=-\mathsf{P}_1\psi_{\mathbf{H}}, \tag{7.13a}$$

where $f_{\mathbf{H}}=\nabla\phi_{\mathbf{H}}+\nabla^\perp\psi_{\mathbf{H}}$ is the Helmholtz-Weyl decomposition of $f_{\mathbf{H}}$. The identities (7.13a) *can easily be inverted:*

$$\phi_{\mathbf{H}}=-\delta_{\mathbf{H}}+\mathsf{H}^*\omega_{\mathbf{H}} \qquad \text{and} \qquad \psi_{\mathbf{H}}=-\mathsf{Q}_1\omega_{\mathbf{H}}. \tag{7.13b}$$

Proof. Let $f_{\mathbf{H}}$ be in $\mathbf{H}_{-1}$. According to Riesz representation Theorem and Theorem 7.3, there exists $\phi,\psi\in S_1$ and $h\in\mathfrak{H}_{\rm m}$ such that:

$$\langle f_{\mathbf{H}},u\rangle_{\mathbf{H}_{-1},\mathbf{H}_1}=(\Delta\phi+(\mathrm{Id}+\mathsf{H}^*\mathsf{H})h,\nabla\cdot u)_{L^2(\mathcal{F})}+(\Delta\psi,\nabla^\perp\cdot u)_{L^2(\mathcal{F})} \qquad \text{for all } u\in\mathbf{H}_1.$$

It suffices to set $\delta_{\mathbf{H}}=\Delta\phi+(\mathrm{Id}+\mathsf{H}^*\mathsf{H})h$ and $\omega_{\mathbf{H}}=\Delta\psi$.

Assume now that $f_{\mathbf{H}}$ lies in $\mathbf{H}_0$ and denote by $\phi_{\mathbf{H}}\in H^1_{\rm m}$ and $\psi_{\mathbf{H}}\in S_0$ the functions entering the Helmholtz-Weyl decomposition of $f_{\mathbf{H}}$, i.e.

$$f_{\mathbf{H}}=\nabla\phi_{\mathbf{H}}+\nabla^\perp\psi_{\mathbf{H}} \qquad \text{in } \mathcal{F}.$$

By definition of $\mathbf{H}_0$ as pivot space:

$$\langle f_{\mathbf{H}},u\rangle_{\mathbf{H}_{-1},\mathbf{H}_1}=(f_{\mathbf{H}},u)_{\mathbf{L}^2(\mathcal{F})}=-(\phi_{\mathbf{H}},\nabla\cdot u)_{L^2(\mathcal{F})}-(\psi_{\mathbf{H}},\nabla^\perp\cdot u)_{L^2(\mathcal{F})} \qquad \text{for all } u\in\mathbf{H}_1.$$

The orthogonal decomposition $L^2_{\rm m}(\mathcal{F})=V_0\overset{\perp}{\oplus}\mathfrak{H}_{\rm m}$ leads to $\nabla^\perp\cdot u=\mathsf{P}_0^\perp\nabla^\perp\cdot u+\mathsf{P}_0\nabla^\perp\cdot u$. But according to Theorem 7.3, $\mathsf{P}_0^\perp\nabla^\perp\cdot u=\mathsf{H}\nabla\cdot u$, whence we deduce that:

$$\langle f_{\mathbf{H}},u\rangle_{\mathbf{H}_{-1},\mathbf{H}_1}=-(\phi_{\mathbf{H}}+\mathsf{H}^*\psi_{\mathbf{H}},\nabla\cdot u)_{L^2(\mathcal{F})}-(\mathsf{P}_1\psi_{\mathbf{H}},\nabla^\perp\cdot u)_{L^2(\mathcal{F})},$$

and the proof is complete. ☐

Remark 7.6. *It is now easy to verify that if $f_{\mathbf{H}}\in\mathbf{H}_{-1}$ and $f_S\in S_{-1}$ are two linear forms such that:*

$$\langle f_{\mathbf{H}},\nabla^\perp\psi\rangle_{\mathbf{H}_{-1},\mathbf{H}_1}=\langle f_S,\psi\rangle_{S_{-1},S_1} \qquad \text{for all } \psi\in S_1,$$

then $\mathsf{P}f_S=\omega_{\mathbf{H}}$ where $\omega_{\mathbf{H}}\in V_0$ and $\delta_{\mathbf{H}}\in L^2_{\rm m}$ are defined from $f_{\mathbf{H}}$ in Proposition 7.5. We shall prove that the pressure depends only upon $\delta_{\mathbf{H}}$ and therefore is actually independent of the source term f_S.

7.2. **Weak solutions.** When the equation is nonlinear, the operator H is not suffisant to define the pressure. Thus, for every $u\in\mathbf{L}^4(\mathcal{F})$, define $\pi_\Theta[u],\pi_\Psi[u]\in L^2_{\rm m}$ by means of Riesz representation Theorem as:

$$(\pi_\Theta[u],f)_{L^2_{\rm m}}=-(D^2\Theta_f u,u)_{\mathbf{L}^2(\mathcal{F})} \tag{7.14a}$$

$$(\pi_\Psi[u],f)_{L^2_{\rm m}}=(D^2\Psi_f u,u^\perp)_{\mathbf{L}^2(\mathcal{F})} \qquad \text{for all } f\in L^2_{\rm m}. \tag{7.14b}$$

Definition 7.7. *Let T be a positive real number, ψ be a function in $\bar{S}_0(T)$, φ be in $\mathfrak{H}^1_K(T)$ (this space is defined in (6.1)) and $\delta_{\mathbf{H}}$ be in $L^2(0,T;L^2_{\mathrm{m}})$. Then introduce the velocity field $u = \nabla^\perp\psi + \nabla\varphi$ and for a.e. $t \in (0,T)$ define $p_{\mathsf{r}}(t)$ by:*

$$p_{\mathsf{r}}(t) = -\partial_t\varphi(t) + \pi_\Theta[u(t)] + \mathsf{H}^*\Big[\nu\omega(t) - \pi_\Psi[u(t)]\Big] - \delta_{\mathbf{H}}(t), \tag{7.15a}$$

where $\omega(t) = \Delta\psi(t)$ (i.e. $\omega(t)$ is the regular part of $\bar{\Delta}_0\psi(t)$, see Remark 4.11). The pressure p corresponding to these data is obtained by summing p_{r}, called the regular part of the pressure, and a singular part p_{s}:

$$p = p_{\mathsf{r}} + p_{\mathsf{s}} \qquad \text{with} \qquad p_{\mathsf{s}} = -\partial_t \mathsf{H}^*\psi. \tag{7.15b}$$

The proof of the lemma below is obvious:

Lemma 7.8. *The function p_{r} belongs to $L^2(0,T;L^2_{\mathrm{m}})$ and the mapping*

$$(\psi,\varphi,\delta_{\mathbf{H}}) \in \bar{S}_0(T) \times \mathfrak{H}^1_k(T) \times L^2(0,T;L^2_{\mathrm{m}}) \mapsto p_{\mathsf{r}} \in L^2(0,T;L^2_{\mathrm{m}}),$$

is continuous. The function p_{s} lies in $W^{-1,\infty}(0,T;L^2_{\mathrm{m}})$ and the mapping $\psi \in \bar{S}_0(T) \mapsto p_{\mathsf{s}} \in W^{-1,\infty}(0,T;L^2_{\mathrm{m}})$ is continuous.

We can now state the main result of this subsection:

Theorem 7.9. *Let T be a positive real number and let $\psi \in \bar{S}_0(T)$ be a weak solution to the $\psi-$Navier-Stokes equations as defined in Definition 6.4, and whose source term is recalled to be denoted by f_S. Let $\varphi \in \mathfrak{H}^1_K(T)$ be the Kirchhoff potential also introduced in Definition 6.4. Finally, let $f_{\mathbf{H}}$ be in $L^2(0,T;\mathbf{H}_{-1})$ such that (see Remark 7.6):*

$$\langle f_{\mathbf{H}}, \nabla^\perp\theta\rangle_{\mathbf{H}_{-1},\mathbf{H}_1} = \langle f_S,\theta\rangle_{S_{-1},S_1} \qquad \text{for all } \theta \in S_1.$$

According to Proposition 7.5, to the linear form $f_{\mathbf{H}}$ can be associated a pair $(\delta_{\mathbf{H}},\omega_{\mathbf{H}}) \in L^2(0,T;L^2_{\mathrm{m}}) \times L^2(0,T;V_0)$.

Denote now by u the vector field $\nabla\varphi + \nabla^\perp\psi$ and by p the pressure defined from ψ, φ and $\delta_{\mathbf{H}}$ as explained in Definition 7.7. Then the pair (u,p) is a weak (Leray) solution to the Navier-Stokes equations, namely, for every w in $\mathbf{H}_1$:

$$\frac{\mathrm{d}}{\mathrm{d}t}(u,w)_{\mathbf{L}^2(\mathcal{F})} - (\nabla w u, u)_{\mathbf{L}^2(\mathcal{F})} + \nu\int_{\mathcal{F}} \nabla u : \nabla w \,\mathrm{d}x - (p, \nabla\cdot w)_{L^2(\mathcal{F})} = \langle f_{\mathbf{H}}, w\rangle_{\mathbf{H}_{-1},\mathbf{H}_1} \quad \text{on } (0,T). \tag{7.16}$$

Proof. Remind that ψ satisfies, for every $\theta \in S_1$:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\nabla\psi,\nabla\theta)_{\mathbf{L}^2(\mathcal{F})} + \nu(\Delta\psi,\Delta\theta)_{L^2(\mathcal{F})} + (D^2\theta u, u^\perp)_{\mathbf{L}^2(\mathcal{F})} = \langle f_{\mathbf{H}}, \nabla^\perp\theta\rangle_{\mathbf{H}_{-1},\mathbf{H}_1} \quad \text{on } (0,T). \tag{7.17}$$

Let $u_b = \nabla^\perp\psi_b + \nabla\varphi$, $u_0 = u - u_b = \nabla^\perp\psi_\ell + \nabla^\perp\psi_\Lambda$ (see Definition 6.10) and $\omega_b = \Delta\psi_b$. Then, for every $w \in \boldsymbol{\mathcal{D}}(\mathcal{F})$:

$$\int_{\mathcal{F}} \nabla u_b : \nabla w \,\mathrm{d}x = -\langle \Delta u_b, w\rangle_{\boldsymbol{\mathcal{D}}'(\mathcal{F}),\boldsymbol{\mathcal{D}}(\mathcal{F})} = -\langle \nabla^\perp\omega_b, w\rangle_{\boldsymbol{\mathcal{D}}'(\mathcal{F}),\boldsymbol{\mathcal{D}}(\mathcal{F})} = (\omega_b, \nabla^\perp\cdot w)_{L^2(\mathcal{F})}, \tag{7.18a}$$

and this result extends by density to every $w \in \mathbf{H}_1$. According to Theorem 7.3, we can decompose w into

$$w = \nabla^\perp\theta + \nabla^\perp\Psi_{\mathsf{H}h} + \nabla\Theta_h + \nabla\phi \qquad \text{in } \mathcal{F}, \tag{7.18b}$$

with $(\theta,\phi,h) \in S_1 \times S_1 \times \mathfrak{H}_{\mathrm{m}}$ and it follows that $\nabla^\perp\cdot w = \Delta\theta + \mathsf{H}h$. Since $\omega_b \in \mathfrak{H}_V$, we infer that:

$$(\omega_b, \nabla^\perp\cdot w)_{L^2(\mathcal{F})} = (\omega_b, \mathsf{H}h)_{L^2(\mathcal{F})} = (\mathsf{H}^*\omega_b, h)_{L^2(\mathcal{F})} = (\mathsf{H}^*\omega, \nabla\cdot w)_{L^2(\mathcal{F})}, \tag{7.18c}$$

because $\mathsf{H}^*\omega = \mathsf{H}^*\omega_b$. On the other hand, since u_0 belongs to $\mathbf{H}_1$, according to (7.9), it follows that:

$$\int_{\mathcal{F}} \nabla u_0 : \nabla w \,\mathrm{d}x = (\Delta(\psi_\Lambda + \psi_\ell), \Delta\theta)_{L^2(\mathcal{F})} = (\Delta\psi,\Delta\theta)_{L^2(\mathcal{F})}, \tag{7.18d}$$

the latter equality resulting from the orthogonality property $(\Delta\psi_b,\Delta\theta)_{L^2(\mathcal{F})} = 0$. Gathering now the identities (7.18), we obtain:

$$\nu\int_{\mathcal{F}} \nabla u : \nabla w \,\mathrm{d}x = \nu(\Delta\psi,\Delta\theta)_{L^2(\mathcal{F})} + \nu(\mathsf{H}^*\omega, \nabla\cdot w)_{L^2(\mathcal{F})}. \tag{7.19}$$

Invoking again the decomposition (7.18b), we get:

$$(u,w)_{\mathbf{L}^2(\mathcal{F})}=(\nabla\psi,\nabla\theta)_{\mathbf{L}^2(\mathcal{F})}-(\mathsf{H}^*\psi+\varphi,\nabla\cdot w)_{L^2(\mathcal{F})},\tag{7.20}$$

and also:

$$(\nabla w u,u)_{\mathbf{L}^2(\mathcal{F})}=(D^2\theta u,u^\perp)_{\mathbf{L}^2(\mathcal{F})}+(D^2\Psi_{\mathsf{H}h}u,u^\perp)_{\mathbf{L}^2(\mathcal{F})}+(D^2(\Theta_h+\phi)u,u)_{\mathbf{L}^2(\mathcal{F})}.$$

But notice that $\mathsf{H}h=\mathsf{H}(\nabla\cdot w)$ and $\Theta_h+\phi=\Theta_{\nabla\cdot w}$ (both functions share the same boundary conditions and the same Laplacian). Using the notation (7.14), we are then allowed to rewrite the above equality as:

$$(\nabla w u,u)_{\mathbf{L}^2(\mathcal{F})}=(D^2\theta u,u^\perp)_{\mathbf{L}^2(\mathcal{F})}+(\mathsf{H}^*\pi_\Psi[u],\nabla\cdot w)_{L^2(\mathcal{F})}-(\pi_\Theta[u],\nabla\cdot w)_{L^2(\mathcal{F})}.\tag{7.21}$$

Finally, considering the source term:

$$\langle f_{\mathbf{H}},w\rangle_{\mathbf{H}_{-1},\mathbf{H}_1}=\langle f_{\mathbf{H}},\nabla^\perp\theta\rangle_{\mathbf{H}_{-1},\mathbf{H}_1}+\langle f_{\mathbf{H}},w-\nabla^\perp\theta\rangle_{\mathbf{H}_{-1},\mathbf{H}_1}=\langle f_S,\theta\rangle_{S_{-1},S_1}+(\delta_{\mathbf{H}},\nabla\cdot w)_{L^2(\mathcal{F})}.\tag{7.22}$$

Summing now the time derivative of (7.20) with (7.19) and subtracting (7.21) and the term $(p,\nabla\cdot w)_{L^2(\mathcal{F})}$, we obtain (7.22), taking into account (7.17). The resulting equality is therefore (7.16) and the proof is completed. ☐

7.3. **Strong solutions.** For every $u\in\mathbf{H}^2(\mathcal{F})$, define $\Phi[u]$ as the unique element in H^1_{m} such that:

$$(\Phi[u],\theta)_{H^1_{\mathrm{m}}}=-(\nabla u u,\nabla\theta)_{\mathbf{L}^2(\mathcal{F})}\qquad\text{for all }\theta\in H^1_{\mathrm{m}}.\tag{7.23}$$

Definition 7.10. *Let T be a positive time, ψ be a function in $\bar S_1(T)$, φ be in $\mathfrak{H}^2_K(T)$ and $\phi_{\mathbf{H}}$ (accounting for the source term) be in $L^2(0,T;H^1_{\mathrm{m}})$. Then introduce the velocity field $u=\nabla^\perp\psi+\nabla\varphi$. For a.e. $t\in(0,T)$ define $p(t)$ by:*

$$p(t)=-\partial_t\varphi(t)+\Phi[u(t)]+\nu\mathsf{H}^*\mathsf{Q}_1^\perp\omega(t)+\phi_{\mathbf{H}}(t),\tag{7.24}$$

where $\omega(t)=\Delta\psi(t)$ (i.e. $\omega(t)$ is the regular part of $\bar\Delta_1\psi(t)$, see Remark 4.17).

Proposition 7.11. *The function p belongs to $L^2(0,T;H^1_{\mathrm{m}})$ and the mapping*

$$(\psi,\varphi,\phi_{\mathbf{H}})\in\bar S_1(T)\times\mathfrak{H}^2_k(T)\times L^2(0,T;S_0)\mapsto p\in L^2(0,T;H^1_{\mathrm{m}}),$$

is continuous.

Moreover, if ψ is a solution to the $\psi-$NS equations as described in Definition 6.10 with Kirchhoff potential φ and source term $f_S\in L^2(0,T;S_0)$, then p defined in(7.24) from the triple $(\psi,\varphi,\phi_{\mathbf{H}})$ is equal to the pressure of Definition 7.7 computed from the triple $(\psi,\varphi,\delta_{\mathbf{H}})$ with $\delta_{\mathbf{H}}=-\phi_{\mathbf{H}}-\mathsf{H}^*f_S$.

Proof. The continuity of the mapping being obvious, let us verify the claim that p in (7.24) matches the expression given in (7.15b). For a.e. $t\in(0,T)$, $\partial_t\psi$ is in S_0 and since ψ is a strong solution to the $\psi-$NS equations it follows that:

$$\partial_t\mathsf{H}^*\psi=\mathsf{H}^*\partial_t\psi=\mathsf{H}^*\big(-\nu\mathsf{A}^S_2(\psi_\Lambda+\psi_\ell)-\Lambda^S_0(\psi,\varphi)+f_S\big).$$

On the one hand, according to the expression (3.16) of A^S_2:

$$\mathsf{H}^*\mathsf{A}^S_2(\psi_\Lambda+\psi_\ell)=-\mathsf{H}^*\mathsf{Q}_1(\omega_\Lambda+\omega_\ell)=-\mathsf{H}^*\mathsf{P}_1^\perp\mathsf{Q}_1(\omega_\Lambda+\omega_\ell)=\mathsf{H}^*\mathsf{Q}_1^\perp(\omega_\Lambda+\omega_\ell),$$

because $\mathsf{H}^*=\mathsf{H}^*\mathsf{P}^\perp$ and $\mathsf{P}_1^\perp\mathsf{Q}_1=(\mathrm{Id}-\mathsf{P}_1)\mathsf{Q}_1=-\mathsf{Q}_1^\perp$. On the other hand, for every f in L^2_{m}:

$$(\mathsf{H}^*\Lambda^S_0(\psi,\varphi),f)_{L^2(\mathcal{F})}=(\Lambda^S_0(\psi,\varphi),\Delta\Psi_{\mathsf{H}f})_{L^2(\mathcal{F})}=-(\Lambda^S_0(\psi,\varphi),\Psi_{\mathsf{H}f})_{S_0}=-(\nabla u u,\nabla^\perp\Psi_{\mathsf{H}f})_{\mathbf{L}^2(\mathcal{F})},$$

the latter equality resulting from (6.12). Summing up, we obtain that for every $f\in L^2_{\mathrm{m}}$ and a.e. $t\in(0,T)$:

$$\big(\pi_\Theta[u(t)]+\mathsf{H}^*\big[\nu\omega(t)-\pi_\Psi[u(t)]\big]-\partial_t\mathsf{H}^*\psi(t),f\big)_{L^2_{\mathrm{m}}}=-(D^2\Theta_f(t)u(t),u(t))_{\mathbf{L}^2(\mathcal{F})}\tag{7.25}$$

$$-(D^2\Psi_{\mathsf{H}f}(t)u(t),u^\perp(t))_{\mathbf{L}^2(\mathcal{F})}-(\nabla u u,\nabla^\perp\Psi_{\mathsf{H}f})_{\mathbf{L}^2(\mathcal{F})}+\nu(\mathsf{H}^*\omega(t),f)_{L^2_{\mathrm{m}}}+\nu(\mathsf{H}^*\mathsf{Q}_1^\perp(\omega_\Lambda+\omega_\ell)(t),f)_{L^2_{\mathrm{m}}}-(\mathsf{H}^*f_S,f)_{L^2_{\mathrm{m}}}.$$

The two first terms in the right hand side can be rewritten as:

$$(D^2\Theta_f(t)u(t),u(t))_{\mathbf{L}^2(\mathcal{F})}+(D^2\Psi_{\mathsf{H}f}(t)u(t),u^\perp(t))_{\mathbf{L}^2(\mathcal{F})}=(\nabla(\nabla\Theta_f+\nabla^\perp\Psi_{\mathsf{H}f})(t)u(t),u(t))_{\mathbf{L}^2(\mathcal{F})},$$

and the resulting quantity can now be integrated by parts:

$$(\nabla(\nabla\Theta_f+\nabla^\perp\Psi_{\mathsf{H}f})(t)u(t),u(t))_{\mathbf{L}^2(\mathcal{F})}=-(\nabla u(t)u(t),\nabla\Theta_f(t)+\nabla^\perp\Psi_{\mathsf{H}f}(t))_{\mathbf{L}^2(\mathcal{F})}.$$

Turning our attention to the two last terms in the right hand side of (7.25), we observe that $\mathsf{H}^*\omega(t) = \mathsf{H}^*\omega_b(t) = \mathsf{H}^*\mathsf{Q}_1^\perp\omega_b(t)$ according to the properties of H^* stated in Lemma 7.2 and the fact that ω_b is the harmonic part of $\omega = \omega_\Lambda + \omega_\ell + \omega_b$. We have now proved that:

$$\big(\pi_\Theta[u(t)]+\mathsf{H}^*\big[\nu\omega(t)-\pi_\Psi[u(t)]\big]-\partial_t\mathsf{H}^*\psi(t),f\big)_{L^2_{\rm m}} = (\nabla u(t)u(t),\nabla\Theta_f(t))_{\mathbf{L}^2(\mathcal{F})}+\nu(\mathsf{H}^*\mathsf{Q}_1^\perp\omega(t),f)_{L^2_{\rm m}}-(\mathsf{H}^*f_S,f)_{L^2_{\rm m}}.$$

Recalling the definition (7.23) of $\Phi[u(t)]$, we can integrate by parts the first term in the right hand side:

$$(\nabla u(t)u(t),\nabla\Theta_f(t))_{\mathbf{L}^2(\mathcal{F})} = -(\nabla\Phi[u(t)],\nabla\Theta_f(t))_{\mathbf{L}^2(\mathcal{F})} = (\Phi[u(t)],f)_{L^2_{\rm m}},$$

and thus complete the proof. □

Theorem 7.12. *Let T be a positive real number and let $\psi \in \bar{S}_1(T)$ be a strong solution to the $\psi-$Navier-Stokes equations as defined in Definition 6.10. Let $\varphi \in \mathfrak{H}^2_K(T)$ be the Kirchhoff potential also introduced in Definition 6.10.*

Denote now by u the vector field $\nabla\varphi + \nabla^\perp\psi$ and by p the pressure defined from ψ, φ and $\phi_{\mathbf{H}} = -\mathsf{H}^*f_S$ as explained in Definition 7.10. Then the pair (u,p) is a strong (Kato) solution to the Navier-Stokes equations.

Proof. The stream function ψ is also a weak solution to the $\psi-$NS equations in the sense of Definition 6.4. According to Proposition 7.11, the pressure p is a "weak" pressure in the sense of Definition 7.7 as well. It follows from Theorem 7.9 that (u,p) is a weak solution to the NS equations in primitive variables. From the regularity of u and p we are allowed to deduce that (u,p) is indeed a strong solution to the NS equations in primitive variables. □

8. More regular vorticity solutions

So far and even for strong solutions as described in the preceding subsection, the regularity of the functions does not allow writing the vorticity equation in the most common form (2.9), that is, loosely speaking, as an advection-diffusion equation set in $L^2(\mathcal{F})$ (see Remark 6.12). To achieve this level of regularity, a first guess would be to seek solutions in $V_1(T)$, in which case, the operator $(-\nu\Delta)$ should be $\nu\mathsf{A}^V_2$. However, since the nonlinear advection term $u\cdot\nabla\omega$ does not belong to V_0 in general, we are inclined to conclude that this approach leads to a dead end. We shall prove that the solution should rather be looked for in the space $\bar{V}_1(T)$. We recall that the spaces $\bar{V}_2, \bar{V}_1$ and $\bar{V}_0$ are all of them subspaces of V_{-1}. They are defined in Subsection 4.2.

Before addressing the $\omega-$NS equations, we begin as usual with Stokes problems. All along this Section, we assume that the boundary Σ is of class $\mathcal{C}^{3,1}$.

8.1. **Regular Stokes vorticity solutions.** For every positive real number T, we aim to define solutions to Stokes problems belonging to:

$$\bar{V}_1(T) = H^1(0,T;\bar{V}_0)\cap\mathcal{C}([0,T];\bar{V}_1)\cap L^2(0,T;\bar{V}_2).$$

We recall that $\bar{V}_2 \subset \bar{V}_1 \subset \bar{V}_0 \subset V_{-1}$ (see Subsection 4.2). So, let a triple (g_n, g_τ, Γ) be given in $G_2(T)$ and define ω_b in $V^{\rm b}_1(T)$ by:

$$\omega_b(t) = \mathsf{L}^V_2(g_n(t),g_\tau(t),\Gamma(t)) = \omega^{\mathfrak{H}}_b(t) + \sum_{j=1}^N \Gamma_j(t)\zeta_j \qquad \text{for a.e. } t\in(0,T), \tag{8.1}$$

where $\omega^{\mathfrak{H}}_b(t)$ belongs to $\mathfrak{H}^2_V$ (the operator L^V_2 is defined in (4.33) and maps continuously $G_2(T)$ into the space $V^{\rm b}_1(T)$ defined in (4.34)). The source term f_V is expected to belong to $L^2(0,T;\bar{V}_0)$ and the decomposition (4.22) of the space $\bar{V}_0$ leads to the splitting:

$$f_V(t) = f^0_V(t) + f^{\mathfrak{H}}_V(t) + \sum_{j=1}^N \alpha^j_V(t)\zeta_j \qquad \text{for a.e. } t\in(0,T), \tag{8.2}$$

with $f_0(t)$ is in V_0, $f^{\mathfrak{H}}_V(t)$ in $\mathfrak{H}_V$ and $\alpha^j_V(t)$ in $\mathbb{R}$ for every $j=1,\dots,N$. Similarly, taking now into account the decomposition (4.32) of $\bar{V}_2$, we seek the total vorticity in the form:

$$\omega(t) = \omega_1(t) + \omega_b(t) \quad \text{with} \quad \omega_1(t) = \omega_0(t) + \omega_{\mathfrak{B}}(t) + \sum_{j=1}^N \beta_j(t)\Omega_j \qquad \text{for a.e. } t\in(0,T), \tag{8.3}$$

where the function $\omega_0(t)$ belongs to V_2, $\omega_{\mathfrak{B}}(t)$ is in $\mathfrak{B}^2_V$ and $\beta_j(t)$ in $\mathbb{R}$ for every $j=1,\dots,N$. They are the unknowns of the problem. The function ω_1 is supposed to satisfy in particular Equation (5.17a) for $k=0$, namely:

$$\partial_t\omega_1 + \nu \mathsf{A}^V_1\omega_1 = f_V - \partial_t\omega_b \qquad \text{in } \mathcal{F}_T, \tag{8.4}$$

this equality being set in $L^2(0,T;V_{-1})$. We want this equation to be satisfy in the slightly more regular space $L^2(0,T;\bar V_0)$. Thus, the operator A^V_1 turns into the operator $\bar{\mathsf{A}}^V_2$ (defined right above Proposition 4.18). Keeping in mind the decomposition (4.22) of $\bar V_0$ we apply successively to Equation (8.4) the orthogonal projections onto the spaces V_0, $\mathfrak{H}_V$ and $\mathbb{F}^*_V$ respectively to obtain the system:

$$\partial_t\omega_0 + \nu\mathsf{A}^V_2\omega_0 = f^0_V - \partial_t\omega_{\mathfrak{B}} - \sum_{j=1}^N \beta_j'\Omega_j \qquad \text{in } \mathcal{F}_T \tag{8.5a}$$

$$\nu\bar{\mathsf{A}}^V_2\omega_{\mathfrak{B}} = f^{\mathfrak{H}}_V - \partial_t\omega^{\mathfrak{H}}_b \qquad \text{in } \mathcal{F}_T \tag{8.5b}$$

$$\nu\beta_j = \Gamma_j' - \alpha^j_V \qquad \text{in } (0,T) \quad \text{for every } j=1,\dots,N. \tag{8.5c}$$

A solution can be worked out by taking the time derivatives of the equations (8.5b) and (8.5c). Thus, one gets the expressions of $\partial_t\omega_{\mathfrak{B}}$ and $\partial_t\beta_j$ that can be used in (8.5a). This leads us to the following statement:

Proposition 8.1. *Let T be a positive real number, (g_n,g_τ,Γ) be a triple in*

$$G_{\mathsf{r}}(T) = G_2(T)\cap\mathcal{C}^1([0,T];G^n_0\times G^\tau_0\times\mathbb{R}^N)\cap H^2(0,T;G^n_{-1}\times G^\tau_{-1}\times\mathbb{R}^N) \tag{8.6a}$$

$$= \big\{(g_n,g_\tau,\Gamma)\in G_2(T)\,:\,(\partial_t g_n,\partial_t g_\tau,\Gamma')\in G_0(T)\big\}, \tag{8.6b}$$

and f_V be a source term in

$$F_{\mathsf{r}}(T) = \Big\{f_V\in L^2(0,T;\bar V_0)\,:\,f_V = f^0_V + f^{\mathfrak{H}}_V + \sum_{j=1}^N \alpha^j_V\zeta_j \text{ with } f^0_V\in L^2(0,T;V_0),$$
$$f^{\mathfrak{H}}_V\in L^2(0,T;\mathfrak{H}_V)\cap\mathcal{C}([0,T];V_{-1})\cap H^1(0,T;V_{-2}) \text{ and } \alpha^j_V\in H^1(0,T) \text{ for every } j=1,\dots,N\Big\}. \tag{8.6c}$$

Let ω_b be defined from the boundary data as in (8.1)*, let ω^{i} be an initial data in $\bar V_1$ satisfying the compatibility condition $\omega^{\mathrm{i}}-\omega_b(0)\in V_1$ and let:*

$$\omega^{\mathrm{i}}_0 = \omega^{\mathrm{i}} - \omega_b(0) - \frac{1}{\nu}\big(\mathsf{A}^V_1\big)^{-1}\big(f^{\mathfrak{H}}_V - \partial_t\omega^{\mathfrak{H}}_b\big)(0) - \frac{1}{\nu}\sum_{j=1}^N\big(\Gamma_j'(0)-\alpha^j_V(0)\big)\Omega_j. \tag{8.7}$$

Then ω^{i}_0 belongs to V_1 and there exists a unique solution $\omega_0\in V_1(T)$ to the Cauchy problem:

$$\partial_t\omega_0 + \nu\mathsf{A}^V_2\omega_0 = f^0_V - \frac{1}{\nu}\big(\mathsf{A}^V_0\big)^{-1}\big(\partial_t f^{\mathfrak{H}}_V - \partial^2_t\omega^{\mathfrak{H}}_b\big) - \frac{1}{\nu}\sum_{j=1}^N\big(\Gamma_j'' - (\alpha^j_V)'\big)\Omega_j \qquad \text{in } \mathcal{F}_T, \tag{8.8a}$$

$$\omega_0(0) = \omega^{\mathrm{i}}_0 \qquad \text{in } \mathcal{F}. \tag{8.8b}$$

The vorticity function:

$$\omega = \omega_0 + \frac{1}{\nu}\big(\bar{\mathsf{A}}^V_2\big)^{-1}\big(f^{\mathfrak{H}}_V - \partial_t\omega^{\mathfrak{H}}_b\big) + \frac{1}{\nu}\sum_{j=1}^N(\Gamma_j' - \alpha^j_V)\Omega_j + \omega_b, \tag{8.9}$$

belongs to $\bar V_1(T)$ and solves System (8.5)*. It will be called a regular vorticity solution to the $\omega-$Stokes equations.*

It is worth noticing that:

(1) System (8.5) is no longer a simple parabolic system but rather a coupled parabolic-elliptic system.
(2) The regularity assumptions (8.6) entail that the functions $f^{\mathfrak{H}}_V$ and $\partial_t\omega^{\mathfrak{H}}_b$ both belong to $\mathcal{C}([0,T];V_{-1})$ and therefore that the equality (8.7) at the initial time makes sense.
(3) Under the hypotheses of the Proposition, the function ω_1 can be defined as in (8.3). One easily verifies that ω_1 solves (8.4).
(4) In the definition (8.6c) of the space $F_{\mathsf{r}}(T)$, the regularities of the harmonic and nonharmonic parts of the source term are different.

Proof. The proof is straightforward: The right-hand side of equation (8.8a) clearly belongs to $L^2(0,T;V_0)$ and hence it suffices to apply Proposition 5.10. □

From Proposition 5.10 and Equality (8.9), we deduce:

Corollary 8.2. *The spaces $G_{\mathsf{r}}(T)$ and $F_{\mathsf{r}}(T)$ being equipped with their natural topologies, there exists a positive constant $\mathbf{c}_{[\mathcal{F},\nu]}$ such that, for every regular vorticity solution to a $\omega-$Stokes problem as defined in Proposition 8.1, the estimate below holds true:*

$$\|\omega\|_{\bar{V}_1(T)} \leqslant \mathbf{c}_{[\mathcal{F},\nu]}\big[\|\omega^{\mathrm{i}}\|^2_{V_1} + \|(g_n,g_\tau,\varGamma)\|^2_{G_{\mathsf{r}}(T)} + \|f_V\|^2_{F_{\mathsf{r}}(T)}\big]^{\frac{1}{2}}. \tag{8.10}$$

8.2. **Regular Navier-Stokes vorticity solutions.**

Setting up the system of equations. To begin with, let us recall the expression (6.27) of the advection term in the NS equation (strong vorticity version). For every $\omega \in \bar{V}_1$ and every Kirchhoff potential $\varphi \in \mathfrak{H}^2_K$:

$$\varLambda^V_0(\omega,\varphi) = -(\omega_{\mathsf{r}}(\nabla^\perp\psi + \nabla\varphi), \nabla \mathsf{Q}_1\cdot)_{\mathbf{L}^2(\mathcal{F})},$$

where ω_{r} is the orthogonal projection of ω on H^1_V (i.e. ω_{r} is the regular part of ω; see Remark 4.17) and the stream function $\psi = (-\bar{\Delta}_1)^{-1}\omega$ belongs to $\bar{S}_2$ (see Fig. 7). Assuming now more regularity, namely that ω is in $\bar{V}_2$ and φ in $\mathfrak{H}^3_k$, an integration by parts yields:

$$\langle \varLambda^V_0(\omega,\varphi),\theta\rangle_{V_{-1},V_1} = -\int_\Sigma \omega_{\mathsf{r}}\frac{\partial\varphi}{\partial n}\mathsf{Q}_1\theta\,\mathrm{d}s + \big((\nabla^\perp\psi+\nabla\varphi)\cdot\nabla\omega_{\mathsf{r}}, \mathsf{Q}_1\theta\big)_{L^2(\mathcal{F})} \qquad \text{for all } \theta\in V_1,$$

which leads us to define:

Definition 8.3. *For every vorticity ω in $\bar{V}_2$, we denote by $\psi = \bar{\Delta}_2^{-1}\omega$ the corresponding stream function and by $(\omega_{\mathsf{r}}, \mathsf{Q}_1\cdot)_{L^2(\mathcal{F})}$ the orthogonal projection of ω on H^2_V (i.e. ω_{r} is the regular part of ω). For every Kirchhoff potential φ in $\mathfrak{H}^2_K$, we define:*

$$\gamma_j(\omega,\varphi) = -\int_{\Sigma_j^-}\omega_{\mathsf{r}}\frac{\partial\varphi}{\partial n}\,\mathrm{d}s \quad (j=1,\ldots,N) \quad \textit{and} \quad \varLambda^V_{\mathsf{r}}(\omega,\varphi) = (\nabla^\perp\psi+\nabla\varphi)\cdot\nabla\omega_{\mathsf{r}} \in L^2(\mathcal{F}), \tag{8.11a}$$

and the linear form on $\bar{V}_0$:

$$\bar{\varLambda}^V_0(\omega,\varphi) = \sum_{j=1}^N \gamma_j(\omega,\varphi)\zeta_j + (\varLambda^V_{\mathsf{r}}(\omega,\varphi), \mathsf{Q}_1\cdot)_{L^2(\mathcal{F})}. \tag{8.11b}$$

Let boundary data $(g_n, g_\tau, \varGamma)$ and a source term f_V be given as in the preceding subsection. Taking now into account the nonlinear advection term, System (8.5) can be rewritten as follows:

$$\partial_t\omega_0 + \nu\mathsf{A}^V_2\omega_0 = f^0_V - \mathsf{P}\varLambda^V_{\mathsf{r}}(\omega,\varphi) - \partial_t\omega_{\mathfrak{B}} - \sum_{j=1}^N\beta_j'\varOmega_j \qquad \text{in } \mathcal{F}_T \tag{8.12a}$$

$$\nu\bar{\mathsf{A}}^V_2\omega_{\mathfrak{B}} = f^{\mathfrak{H}}_V - \mathsf{P}^\perp\varLambda^V_{\mathsf{r}}(\omega,\varphi) - \partial_t\omega^{\mathfrak{H}}_b \qquad \text{in } \mathcal{F}_T \tag{8.12b}$$

$$\nu\beta_j = \varGamma_j' - \alpha^j_V + \gamma_j(\omega,\varphi) \qquad \text{in } (0,T) \text{ for every } j=1,\ldots,N. \tag{8.12c}$$

This formulation allows recovering the formulation (2.9) given at the beginning of the paper and that can be rewritten with the notation of this Section:

$$\partial_t\omega_{\mathsf{r}} + u\cdot\nabla\omega_{\mathsf{r}} - \nu\Delta\omega_{\mathsf{r}} = f_V \qquad \text{in } \mathcal{F}_T \tag{8.13a}$$

$$-\varGamma_j' + \int_{\Sigma_k^-}\omega_{\mathsf{r}}\,g_n\,\mathrm{d}s - \nu\int_{\Sigma_k^-}\frac{\partial\omega_{\mathsf{r}}}{\partial n}\,\mathrm{d}s = -\alpha^j_V \quad \text{on } (0,T), \quad j=1,\ldots,N, \tag{8.13b}$$

with $u = \nabla^\perp\psi + \nabla\varphi$ and $\psi = \bar{\Delta}_2^{-1}\omega$. Thus, decomposing ω in $\bar{V}_2$ as in (8.3), the regular part of the vorticity ω_{r} is given by:

$$\omega_{\mathsf{r}} = \omega_0 + \omega_{\mathfrak{B}} + \sum_{j=1}^N\beta_j\varOmega_j + \omega^{\mathfrak{H}}_b \quad \text{and} \quad \omega = \omega_{\mathsf{r}} + \sum_{j=1}^N\varGamma_j\zeta_j.$$

Summing (8.12a) and (8.12b) gives (8.13a) and (8.13b) is a rephrasing of (8.12c). Indeed, since ω_0, $\omega_{\mathfrak{B}}$ and $\omega^{\mathfrak{H}}_b$ have zero mean flux through the inner boundaries, we have for every $k=1,\ldots,N$:

$$\int_{\Sigma_k^-}\frac{\partial\omega_{\mathsf{r}}}{\partial n}\,\mathrm{d}s = \sum_{j=1}^N\beta_j\int_{\Sigma_k^-}\frac{\partial\varOmega_j}{\partial n}\,\mathrm{d}s = -\beta_k,$$

according to (4.27) and the second point of Remark 3.3.

It is worthwhile comparing also the formulations (8.12) (or equivalently (8.13)) with the results of Maekawa in [52]. Therein, focusing on Section 2, only homogeneous boundary condition are considered and no function space is specified for the vorticity. The author claims that the vorticity has to satisfy the *integral condition*:

$$\frac{\partial}{\partial n}(-\Delta_D)^{-1}\omega+\sum_{j=1}^N\big((-\Delta_D)^{-1}\nabla^\perp\omega,\nabla^\perp\tilde q_j\big)_{L^2}\frac{\partial\tilde q_j}{\partial n}=0\qquad\text{on }\Sigma,\tag{8.14}$$

where the functions $\tilde q_j$ are a free family in $\mathbb{F}_S$ chosen in such a way that $\tilde q_j=c_j\delta_i^j$ on Σ_i^- with c_j a normalizing real constant ensuring that $\|\nabla\tilde q_j\|_{L^2(\mathcal F)}=1$. In equality (8.14), $(-\Delta_D)^{-1}$ obviously stands for the inverse of the Laplacian operator with homogeneous Dirichlet boundary conditions on Σ. It seems however that the family $\{\tilde q_j,\,j=1,\dots,N\}$ should be replaced by an orthonormal family (such as the one we have denoted by $\{\hat\xi_j,\,j=1,\dots,N\}$). This remark holds earlier as well, in [52, Theorem 2.1], the proof of which amounts to quote [39, Theorem 3.20] where the family $\{\tilde q_j,\,j=1,\dots,N\}$ (with different notation though) is indeed an orthonormal family. Besides this observation, the condition (8.14) can be rephrased in a simpler way:

Proposition 8.4. *Assuming that ω belongs to $H^1(\mathcal F)$, condition* (8.14) *(replacing the functions $\tilde q_j$ by the functions $\hat\xi_j$) is equivalent to the condition:*

$$\omega\in V_0.\tag{8.15}$$

Proof. Let ω be smooth in $\mathcal F$ and define ψ the stream function such that:

$$\psi=(-\Delta_D)^{-1}\omega+\sum_{j=1}^N\big((-\Delta_D)^{-1}\nabla^\perp\omega,\nabla^\perp\hat\xi_j\big)_{L^2}\hat\xi_j\qquad\text{in }\mathcal F.\tag{8.16}$$

Then ψ belongs in particular to $\bar S_1$ and $-\Delta\psi=\omega$ in $\mathcal F$. Condition (8.14) means that ψ is in S_1 and hence that ω is in V_0. Reciprocally, let ω be in $V_0\cap H^1(\mathcal F)$ and denote by $\tilde\psi$ the stream function in S_1 such that $-\Delta\tilde\psi=\omega$. In that case, $\nabla^\perp\tilde\psi=(-\Delta_D)^{-1}\nabla^\perp\omega$ in $\mathcal F$. Decomposing $\tilde\psi$ according to the orthogonal decomposition of the space $S_0=H^1_0(\mathcal F)\overset{\perp}{\oplus}\mathbb{F}_S$, we obtain exactly the right hand side of (8.16) (and indeed the family $\{\hat\xi_j,\,j=1,\dots,N\}$ has to be an orthonormal family at this stage). Therefore $\tilde\psi=\psi$ and (8.14) holds. ☐

Further in [52], the dynamics for the vorticity is claimed to be governed by the system of equations:

$$\partial_t\omega-\nu\Delta\omega+u\cdot\nabla\omega=0\qquad\text{in }\mathcal F_T,\tag{8.17a}$$

with $u=\nabla^\perp\psi$, the stream function ψ being given by the Biot-Savart law (8.16). The classical evolution equation (8.17a) is supplemented with an initial condition:

$$\omega(0)=\omega_0\in V_0\tag{8.17b}$$

and a *boundary condition* on Σ_T (once again it seems that the functions $\tilde q_j$ in Maekawa's paper have to be replaced by the functions $\hat\xi_j$):

$$\nu\Big\{\frac{\partial\omega}{\partial n}-\Lambda_{DN}\omega+\sum_{j=1}^N(\nabla\omega,\nabla\hat\xi_j)_{\mathbf{L}^2(\mathcal F)}\frac{\partial\hat\xi_j}{\partial n}\Big\}=-\frac{\partial}{\partial n}\big(-\Delta_D\big)^{-1}(u\cdot\nabla\omega)+\sum_{j=1}^N(\omega u,\nabla\hat\xi_j)_{\mathbf{L}^2(\mathcal F)}\frac{\partial\hat\xi_j}{\partial n}.\tag{8.17c}$$

Notice that in the case of a simply connected domain, this condition was already mentioned by Weinan and Jian-Guo in [14], borrowed from an earlier article of Anderson [1].

Proposition 8.5. *Condition* (8.17c) *is equivalent for every ω solving* (8.17a) *to:*

$$\partial_t\omega(t)\in V_0\qquad\textit{for a.e. } t\in(0,T).$$

Proof. As already mentioned in the proof of Lemma 3.1, for every function h harmonic in $\mathcal F$, the function

$$h_0=h-\sum_{j=1}^N\left(\int_\Sigma\frac{\partial\hat\xi_j}{\partial n}h\,\mathrm{d}s\right)\hat\xi_j,$$

is in $\mathfrak{H}$. For a.e. $t \in (0,T)$, let us form the scalar product in $L^2(\mathcal{F})$ of (8.17a) with h_0. Integrating by parts, we obtain on the one hand:

$$\nu(\Delta\omega, h_0)_{L^2(\mathcal{F})} = \nu \int_\Sigma \left\{ \frac{\partial \omega}{\partial n} - \Lambda_{DN}\omega + \sum_{j=1}^N (\nabla\omega, \nabla\hat{\xi}_j)_{\mathbf{L}^2(\mathcal{F})} \frac{\partial \hat{\xi}_j}{\partial n} \right\} h \,\mathrm{d}s,$$

and on the other hand, considering the advection term:

$$(u\cdot\nabla\omega, h_0)_{L^2(\mathcal{F})} = \int_\Sigma \left\{ -\frac{\partial}{\partial n}\big(-\Delta_D\big)^{-1}(u\cdot\nabla\omega) + \sum_{j=1}^N (\omega u, \nabla\hat{\xi}_j)_{\mathbf{L}^2(\mathcal{F})} \frac{\partial \hat{\xi}_j}{\partial n} \right\} h \,\mathrm{d}s.$$

This shows that (8.17c) is indeed equivalent to $(\partial_t\omega(t), h_0)_{L^2(\mathcal{F})} = 0$ for a.e. $t \in (0,T)$ and completes the proof. □

Summarizing Propositions 8.4 and 8.5, Maekawa's System (8.17) turns out to be equivalent to:

$$\partial_t\omega - \nu\Delta\omega + u\cdot\nabla\omega = 0 \quad \text{in } \mathcal{F}_T, \tag{8.18a}$$

$$\omega(0) = \omega_0 \in V_0 \quad \text{in } \mathcal{F}, \tag{8.18b}$$

$$\partial_t\omega \in V_0 \quad \text{on } (0,T). \tag{8.18c}$$

This seems to contradict the claim of [52, Theorem 2.3] (namely, the equivalence of System (8.17) with the classical NS equations in primitive variables) in a multiply connected domain because Lamb's fluxes conditions (8.12c) (see [43, Art. 328a]) on the inner boundaries:

$$\int_{\Sigma_j^-} \frac{\partial\omega}{\partial n}\,\mathrm{d}s = 0 \qquad \text{for every } j = 1,\ldots,N,$$

are missing and cannot be figured out from System (8.18) (this is explained in [32, Remark 3.2]). Notice however that the equivalence holds in the particular case of a simply connected fluid domain.

Existence and uniqueness of a global solution. We shall now study the existence of solutions to System (8.12) (or equivalently (8.13)). For simplicity purpose, we restrict our analysis to the case where there is no source term and to homogeneous boundary conditions for the velocity field. The system we consider reads therefore as follows:

$$\partial_t\omega_0 + \nu\mathsf{A}_2^V\omega_0 = -\mathsf{P}\Lambda_{\mathsf{r}}^V(\omega) - \partial_t\omega_{\mathfrak{B}} \quad \text{in } \mathcal{F}_T \tag{8.19a}$$

$$\nu\bar{\mathsf{A}}_2^V\omega_{\mathfrak{B}} = -\mathsf{P}^\perp\Lambda_{\mathsf{r}}^V(\omega) \quad \text{in } \mathcal{F}_T, \tag{8.19b}$$

$$\omega(0) = \omega^{\mathrm{i}} \quad \text{in } \mathcal{F}, \tag{8.19c}$$

with $\omega^{\mathrm{i}} \in V_1$, $\omega = \omega_0 + \omega_{\mathfrak{B}} \in V_2 \oplus \mathfrak{B}_V^2$, $\Lambda_{\mathsf{r}}^V(\omega) = \nabla^\perp\psi\cdot\nabla\omega$ and $\psi = \bar{\Delta}_2^{-1}\omega$. System (8.19) can be rephrased as a more standard Cauchy problem whose unknown is ω_0 (the coupling condition (8.19b) cannot be got rid of though since ω still appears in the nonlinear advection term):

$$\partial_t\omega_0 + \nu\mathsf{A}_2^V\omega_0 = -\mathsf{P}\Lambda_{\mathsf{r}}^V(\omega) + \frac{1}{\nu}\big(\mathsf{A}_0^V\big)^{-1}\partial_t\big(\mathsf{P}^\perp\Lambda_{\mathsf{r}}^V(\omega)\big) \quad \text{in } \mathcal{F}_T \tag{8.20a}$$

$$\omega_0(0) = \omega^{\mathrm{i}} + \frac{1}{\nu}\big(\mathsf{A}_1^V\big)^{-1}\mathsf{P}^\perp\Lambda_{\mathsf{r}}^V(\omega^{\mathrm{i}}) \quad \text{in } \mathcal{F}. \tag{8.20b}$$

The solution ω to System (8.19) will be looked for in the space:

$$\Omega(T) = \big[L^2(0,T;H_V^2) \cap H^1(0,T;V_0)\big] \cap \big[\mathcal{C}([0,T];V_1) \cap \mathcal{C}^1([0,T];V_{-1})\big].$$

Theorem 8.6. *For every positive time T and every initial data $\omega^{\mathrm{i}} \in V_1$, System (8.19) admits a unique solution ω in $\Omega(T)$. Moreover this solution satisfies the exponential decay estimate:*

$$\|\omega(t)\|_{V_1} \leqslant \mathbf{c}_{[\mathcal{F},\nu,\omega^{\mathrm{i}}]}e^{-\frac{1}{2}\nu\lambda_{\mathcal{F}}t} \qquad \textit{for all } t \in (0,T), \tag{8.21}$$

where we emphasis that the constant $\mathbf{c}_{[\mathcal{F},\nu,\omega^{\mathrm{i}}]}$ does not depend on T.

Remark 8.7. *It is worth noticing that:*

(1) *In [52], the author shows local in time existence for the same system, considering the particular case where the fluid domain $\mathcal{F}$ is a half-plane.*
(2) *The quantity $\|\nabla\omega\|^2_{\mathbf{L}^2(\mathcal{F})}$ is sometimes called the palinstrophy. The palinstrophy being controlled by $\|\omega\|^2_{V_1}$, estimates (8.21) asserts that the palinstrophy is exponentially decreasing as time growths. This result was not known so far and may play an important role in turbulence theory.*

We begin with establishing the *a priori* estimate (8.21).

Lemma 8.8. *For every initial condition $\omega^{\rm i}\in V_1$, there exists a positive constant $\mathbf{c}_{[\mathcal{F},\nu,\omega^{\rm i}]}$ such that for every positive time T, any solution ω to System (8.19) in $\Omega(T)$ satisfies estimate (8.21).*

Proof. The proof is divided in several steps:

First step: As being a weak solution to the $\omega-$NS equations, we can apply Corollary 6.8 which provides us with the following estimates, satisfied for every t in $(0,T)$:

$$\|\omega(t)\|_{V_{-1}}\leqslant\|\omega^{\rm i}\|_{V_{-1}}e^{-\nu\lambda_{\mathcal{F}}t}\qquad\text{and}\qquad\int_0^t\|\omega(s)\|^2_{V_0}\,{\rm d}s\leqslant\frac{1}{2\nu}\|\omega^{\rm i}\|^2_{V_{-1}}.\tag{8.22a}$$

Arguing that ω is also a strong solution to the $\omega-$NS equation, we obtain that:

$$\frac12\frac{\rm d}{{\rm d}t}\|\omega(t)\|^2_{V_0}+\nu\|\omega\|^2_{V_1}\leqslant\mathbf{c}_{\mathcal{F}}\|\omega\|^{\frac12}_{V_{-1}}\|\omega\|_{V_0}\|\omega\|^{\frac32}_{V_1}\leqslant\frac{\nu}{2}\|\omega\|^2_{V_1}+\frac{\mathbf{c}_{\mathcal{F}}}{\nu^3}\|\omega\|^2_{V_{-1}}\|\omega\|^4_{V_0},$$

that is

$$\frac{\rm d}{{\rm d}t}\|\omega(t)\|^2_{V_0}+\nu\|\omega\|^2_{V_1}\leqslant\frac{\mathbf{c}_{\mathcal{F}}}{\nu^3}\|\omega\|^2_{V_{-1}}\|\omega\|^4_{V_0},$$

which leads us to the estimates:

$$\|\omega(t)\|_{V_0}\leqslant\|\omega^{\rm i}\|_{V_0}\mathbf{E}_{[\mathcal{F},\nu,\omega^{\rm i}]}e^{-\frac12\nu\lambda t}\qquad\text{with}\qquad\mathbf{E}_{[\mathcal{F},\nu,\omega^{\rm i}]}=\exp\left(\frac{\mathbf{c}_{\mathcal{F}}}{\nu^4}\|\omega^{\rm i}\|^4_{V_{-1}}\right),\tag{8.22b}$$

and also:

$$\int_0^t\|\omega(s)\|^2_{V_1}\,{\rm d}s\leqslant\frac1\nu\|\omega^{\rm i}\|^2_{V_0}\left[1+\frac{\mathbf{c}_{\mathcal{F}}}{\nu^4}\|\omega^{\rm i}\|^4_{V_{-1}}\mathbf{E}_{[\mathcal{F},\nu,\omega^{\rm i}]}\right].\tag{8.22c}$$

Second step: We need now to estimate $\|\omega(t)\|_{V_1}$ in term of $\|\omega_0(t)\|_{V_1}$ and $\|\omega(t)\|_{\bar V_2}$ in term of $\|\omega_0(t)\|_{V_2}$ (and possibly some lower order terms).

Starting from the expression (8.19b) and forming for a.e. t in $(0,T)$ the duality pairing with $\omega_{\mathfrak{B}}(t)$, we obtain:

$$\nu\|\omega_{\mathfrak{B}}(t)\|^2_{V_1}=-\langle\mathsf{P}^\perp\varLambda^V_{\mathsf{r}}(\omega(t)),\omega_{\mathfrak{B}}(t)\rangle_{V_{-1},V_1}\leqslant\mathbf{c}_{\mathcal{F}}\|\mathsf{P}^\perp\varLambda^V_{\mathsf{r}}(\omega(t))\|_{V_{-1}}\|\omega_{\mathfrak{B}}(t)\|_{V_1}.\tag{8.23}$$

Introducing the stream function $\psi=\Delta_1^{-1}\omega$, we have for every $\theta\in V_1$:

$$\langle\mathsf{P}^\perp\varLambda^V_{\mathsf{r}}(\omega),\theta\rangle_{V_{-1},V_1}=(\mathsf{P}^\perp(\nabla^\perp\psi\cdot\nabla\omega),\mathsf{Q}_1\theta)_{L^2(\mathcal{F})}=-(\nabla^\perp\psi\cdot\nabla\omega,\mathsf{Q}_1^\perp\theta)_{L^2(\mathcal{F})}=(\omega\nabla^\perp\psi,\nabla\mathsf{Q}_1^\perp\theta)_{\mathbf{L}^2(\mathcal{F})},$$

the latter expression resting on the equalities $\mathsf{P}^\perp\mathsf{Q}_1=(\mathrm{Id}-\mathsf{P})\mathsf{Q}_1=\mathsf{Q}_1-\mathrm{Id}=-\mathsf{Q}_1^\perp$. We can deduce first that:

$$\|\mathsf{P}^\perp\varLambda^V_{\mathsf{r}}(\omega(t))\|_{V_{-1}}\leqslant\mathbf{c}_{\mathcal{F}}\|\omega(t)\|_{L^4(\mathcal{F})}\|\nabla\psi(t)\|_{\mathbf{L}^4(\mathcal{F})}\leqslant\mathbf{c}_{\mathcal{F}}\|\omega(t)\|^{\frac12}_{V_{-1}}\|\omega(t)\|_{V_0}\|\omega(t)\|^{\frac12}_{V_1},\tag{8.24}$$

and next, combining the inequality above with (8.23), that:

$$\|\omega_{\mathfrak{B}}(t)\|_{V_1}\leqslant\frac{\mathbf{c}_{\mathcal{F}}}{\nu}\|\omega(t)\|^{\frac12}_{V_{-1}}\|\omega(t)\|_{V_0}\|\omega(t)\|^{\frac12}_{V_1}.$$

Since $\|\omega(t)\|_{V_1}\leqslant\|\omega_{\mathfrak{B}}(t)\|_{V_1}+\|\omega_0(t)\|_{V_1}$, we obtain, using Young's inequality:

$$\|\omega(t)\|_{V_1}\leqslant\mathbf{c}\|\omega_0(t)\|_{V_1}+\frac{\mathbf{c}_{\mathcal{F}}}{\nu^2}\|\omega(t)\|_{V_{-1}}\|\omega(t)\|^2_{V_0}.\tag{8.25a}$$

We are donne with the term $\|\omega(t)\|_{V_1}$ so let us turn our attention to $\|\omega(t)\|_{\bar V_2}$. Forming, for a.e. t in $(0,T)$, the scalar product of (8.36b) with $\bar{\mathsf{A}}^V_2\omega_{\mathfrak{B}}(t)$ in V_0, we obtain (using Hölder's inequality followed by interpolation inequalities):

$$\nu\|\omega_{\mathfrak{B}}(t)\|^2_{\bar V_2}=\left|(\nabla^\perp\psi\cdot\nabla\omega(t),\Delta\omega_{\mathfrak{B}}(t))_{L^2(\mathcal{F})}\right|\leqslant\mathbf{c}_{\mathcal{F}}\|\omega(t)\|^{\frac12}_{V_{-1}}\|\omega(t)\|^{\frac12}_{V_0}\|\omega(t)\|^{\frac12}_{V_1}\|\omega(t)\|^{\frac12}_{\bar V_2}\|\omega_{\mathfrak{B}}(t)\|_{\bar V_2},$$

that is to say:

$$\|\omega_{\mathfrak{B}}(t)\|_{\bar V_2}\leqslant \frac{\mathbf{c}_{\mathcal F}}{\nu}\|\omega(t)\|_{V_{-1}}^{\frac12}\|\omega(t)\|_{V_0}^{\frac12}\|\omega(t)\|_{V_1}^{\frac12}\|\omega(t)\|_{\bar V_2}^{\frac12}.$$

But $\|\omega(t)\|_{\bar V_2}^2=\|\omega_0(t)\|_{V_2}^2+\|\omega_{\mathfrak B}(t)\|_{\bar V_2}^2$ which, by Young's inequality yields:

$$\|\omega(t)\|_{\bar V_2}\leqslant \mathbf{c}\|\omega_0(t)\|_{V_2}+\frac{\mathbf{c}_{\mathcal F}}{\nu^2}\|\omega(t)\|_{V_{-1}}\|\omega(t)\|_{V_0}\|\omega(t)\|_{V_1}. \tag{8.25b}$$

Our goal for this step is now achieved.

Third step: We form for a.e. $t\in(0,T)$, the scalar product of (8.20a) with $\mathsf{A}_2^V\omega_0(t)$ in V_0 to obtain:

$$\frac12\frac{\mathrm d}{\mathrm dt}\|\omega_0(t)\|_{V_1}^2+\nu\|\omega_0(t)\|_{V_2}^2=-\big(\mathsf{P}\Lambda_{\mathsf r}^V(\omega)(t),\mathsf{A}_2^V\omega_0(t)\big)_{V_0}+\frac1\nu\Big(\big(\mathsf{A}_0^V\big)^{-1}\big[\partial_t\mathsf{P}^\perp\Lambda_{\mathsf r}^V(\omega)(t)\big],\mathsf{A}_2^V\omega_0(t)\Big)_{V_0}. \tag{8.26}$$

Both terms in the right hand side have to be estimated, this task being easier for the first one than for the second one. Indeed, the first term can be rewritten as:

$$\big(\mathsf{P}\Lambda_{\mathsf r}^V(\omega)(t),\mathsf{A}_2^V\omega_0(t)\big)_{V_0}=\big(\nabla^\perp\psi\cdot\nabla\omega(t),\Delta\omega_0(t)\big)_{L^2(\mathcal F)},$$

whence we deduce that:

$$\big|\big(\mathsf{P}\Lambda_{\mathsf r}^V(\omega)(t),\mathsf{A}_2^V\omega_0(t)\big)_{V_0}\big|\leqslant \mathbf{c}_{\mathcal F}\|\omega(t)\|_{V_{-1}}^{\frac12}\|\omega(t)\|_{V_0}^{\frac12}\|\omega(t)\|_{V_1}^{\frac12}\|\omega(t)\|_{\bar V_2}^{\frac12}\|\omega_0(t)\|_{V_2}.$$

Once combined with (8.25b) to get rid of the term $\|\omega(t)\|_{\bar V_2}$, we end up with:

$$\begin{aligned}\big|\big(\mathsf{P}\Lambda_{\mathsf r}^V(\omega)(t),\mathsf{A}_2^V\omega_0(t)\big)_{V_0}\big|\leqslant \frac{\mathbf{c}_{\mathcal F}}{\nu}\|\omega(t)\|_{V_{-1}}\|\omega(t)\|_{V_0}\|\omega(t)\|_{V_1}\|\omega_0(t)\|_{V_2}+\\ \mathbf{c}_{\mathcal F}\|\omega(t)\|_{V_{-1}}^{\frac12}\|\omega(t)\|_{V_0}^{\frac12}\|\omega(t)\|_{V_1}^{\frac12}\|\omega_0(t)\|_{V_2}^{\frac32},\end{aligned} \tag{8.27}$$

and we are now done with the first nonlinear term.

Fourth step: The second term in the right hand side of (8.26) can be turned into:

$$\Big(\big(\mathsf{A}_0^V\big)^{-1}\big[\partial_t\mathsf{P}^\perp\Lambda_{\mathsf r}^V(\omega)(t)\big],\mathsf{A}_2^V\omega_0(t)\Big)_{V_0}=\big(\partial_t\mathsf{P}^\perp\Lambda_{\mathsf r}^V(\omega)(t),\mathsf{A}_0^V\mathsf{A}_2^V\omega_0(t)\big)_{V_{-2}}=\big\langle\partial_t\mathsf{P}^\perp\Lambda_{\mathsf r}^V(\omega)(t),\omega_0(t)\big\rangle_{V_{-2},V_2},$$

and therefore:

$$\Big|\Big(\big(\mathsf{A}_0^V\big)^{-1}\big[\partial_t\mathsf{P}^\perp\Lambda_{\mathsf r}^V(\omega)(t)\big],\mathsf{A}_2^V\omega_0(t)\Big)_{V_0}\Big|\leqslant\|\partial_t\mathsf{P}^\perp\Lambda_{\mathsf r}^V(\omega)(t)\|_{V_{-2}}\|\omega_0(t)\|_{V_2}. \tag{8.28}$$

From the expression (8.38c) we deduce that:

$$\|\partial_t\mathsf{P}^\perp\Lambda_{\mathsf r}^V(\omega)(t)\|_{V_{-2}}\leqslant\mathbf{c}\|\nabla\psi\|_{\mathbf{L}^4(\mathcal F)}\|\nabla\partial_t\psi\|_{\mathbf{L}^4(\mathcal F)}\leqslant\mathbf{c}_{\mathcal F}\|\omega(t)\|_{V_{-1}}^{\frac12}\|\omega(t)\|_{V_0}^{\frac12}\|\partial_t\omega(t)\|_{V_{-1}}^{\frac12}\|\partial_t\omega(t)\|_{V_0}^{\frac12}, \tag{8.29}$$

and we need now to estimate both terms involving a time derivative. As being a strong solution to the $\omega-$NS equation, $\omega(t)$ satisfies for a.e. t in $(0,T)$ the identity below, set in V_{-1}:

$$\partial_t\omega(t)=-\nu\mathsf{A}_1^V\omega(t)-\Lambda_0^V(\omega(t),0),$$

where we recall that the definition of Λ_0^V is given in (6.27). This equality provides us with the inequality:

$$\|\partial_t\omega(t)\|_{V_{-1}}\leqslant\nu\|\omega(t)\|_{V_1}+\|\Lambda_0^V(\omega(t),0)\|_{V_{-1}}.$$

Resting on the definition (6.27), we next easily obtain that:

$$\|\Lambda_0^V(\omega(t),0)\|_{V_{-1}}\leqslant\|\nabla\psi(t)\|_{\mathbf{L}^4(\mathcal F)}\|\omega(t)\|_{L^4(\mathcal F)}\leqslant\mathbf{c}_{\mathcal F}\|\omega(t)\|_{V_{-1}}^{\frac12}\|\omega(t)\|_{V_0}\|\omega(t)\|_{V_1}^{\frac12},$$

and therefore:

$$\|\partial_t\omega(t)\|_{V_{-1}}\leqslant\nu\|\omega(t)\|_{V_1}+\mathbf{c}_{\mathcal F}\|\omega(t)\|_{V_{-1}}^{\frac12}\|\omega(t)\|_{V_0}\|\omega(t)\|_{V_1}^{\frac12}. \tag{8.30a}$$

On the other hand, since by hypothesis ω is a solution on $(0,T)$ to System (8.19), it satisfies for a.e. t in $(0,T)$:

$$\partial_t\omega(t)=\nu\Delta\omega(t)-\nabla^\perp\psi(t)\cdot\nabla\omega(t)\qquad\text{in }V_0,$$

whence we deduce that:

$$\|\partial_t\omega(t)\|_{V_0}\leqslant\nu\|\omega\|_{\bar V_2}+\mathbf{c}_{\mathcal F}\|\omega(t)\|_{V_{-1}}^{\frac12}\|\omega(t)\|_{V_0}^{\frac12}\|\omega(t)\|_{V_1}^{\frac12}\|\omega(t)\|_{\bar V_2}^{\frac12}.$$

Once combined with (8.25b), this estimate becomes:

$$\|\partial_t\omega(t)\|_{V_0} \leqslant \mathbf{c}\nu\|\omega_0\|_{V_2} + \frac{\mathbf{c}_{\mathcal{F}}}{\nu}\|\omega(t)\|_{V_{-1}}\|\omega(t)\|_{V_0}\|\omega(t)\|_{V_1} + \mathbf{c}_{\mathcal{F}}\|\omega(t)\|_{V_{-1}}^{\frac12}\|\omega(t)\|_{V_0}^{\frac12}\|\omega(t)\|_{V_1}^{\frac12}\|\omega_0(t)\|_{V_2}^{\frac12}. \tag{8.30b}$$

Gathering now (8.28) and identities (8.30) we finally obtain the following estimate for the second nonlinear term in (8.26):

$$\begin{aligned}\Big|\Big(\big(\mathsf{A}_0^V\big)^{-1}\big[\partial_t\mathsf{P}^\perp\Lambda_{\mathsf{r}}^V(\omega)(t)\big],\mathsf{A}_2^V\omega_0(t)\Big)_{V_0}\Big| &\leqslant \mathbf{c}_{\mathcal{F}}\|\omega(t)\|_{V_{-1}}^{\frac12}\|\omega(t)\|_{V_0}^{\frac12}\Big(\nu\|\omega(t)\|_{V_1}^{\frac12}\|\omega_0(t)\|_{V_2}^{\frac32}\\ &+\|\omega(t)\|_{V_{-1}}^{\frac12}\|\omega(t)\|_{V_0}^{\frac12}\|\omega(t)\|_{V_1}\|\omega_0(t)\|_{V_2}+\sqrt{\nu}\|\omega(t)\|_{V_{-1}}^{\frac14}\|\omega(t)\|_{V_0}^{\frac14}\|\omega(t)\|_{V_1}^{\frac34}\|\omega_0(t)\|_{V_2}^{\frac54}\\ &+\sqrt{\nu}\|\omega(t)\|_{V_{-1}}^{\frac14}\|\omega(t)\|_{V_0}^{\frac12}\|\omega(t)\|_{V_1}^{\frac14}\|\omega_0(t)\|_{V_2}^{\frac32}+\frac{1}{\sqrt{\nu}}\|\omega(t)\|_{V_{-1}}^{\frac34}\|\omega(t)\|_{V_0}\|\omega(t)\|_{V_1}^{\frac34}\|\omega_0(t)\|_{V_2}\\ &+\|\omega(t)\|_{V_{-1}}^{\frac12}\|\omega(t)\|_{V_0}^{\frac34}\|\omega(t)\|_{V_1}^{\frac12}\|\omega_0(t)\|_{V_2}^{\frac54}\Big).\end{aligned} \tag{8.31}$$

Both terms in the right hand side of (8.26) have now be estimated. Let us collect all the estimates obtained so far and move on to the next step consisting in applying Grönwall's inequality.

Fifth step: Combining (8.27) and (8.31) with (8.26) and using craftily Young's inequality several times, we deduce that for a.e. t in $(0,T)$:

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|\omega_0(t)\|_{V_1}^2+\nu\|\omega_0(t)\|_{V_2}^2 \leqslant \frac{\nu}{2}\|\omega_0(t)\|_{V_2}^2+\frac{\mathbf{c}_{\mathcal{F}}}{\nu^3}\|\omega(t)\|_{V_{-1}}^2\|\omega(t)\|_{V_0}^2\|\omega(t)\|_{V_1}^2+\frac{\mathbf{c}_{\mathcal{F}}}{\nu^7}\|\omega(t)\|_{V_{-1}}^4\|\omega(t)\|_{V_0}^6. \tag{8.32}$$

Applying Grönwall's inequality to (8.32), we obtain:

$$e^{\nu\lambda_{\mathcal{F}}t}\|\omega_0(t)\|_{V_1}^2(t) \le \|\omega_0^{\mathrm{i}}\|_{V_1}^2+\int_0^t\Big[\frac{\mathbf{c}_{\mathcal{F}}}{\nu^3}\|\omega(s)\|_{V_{-1}}^2\|\omega(s)\|_{V_0}^2\|\omega(s)\|_{V_1}^2+\frac{\mathbf{c}_{\mathcal{F}}}{\nu^7}\|\omega(s)\|_{V_{-1}}^4\|\omega(s)\|_{V_0}^6\Big]e^{\nu\lambda_{\mathcal{F}}s}\,\mathrm{d}s, \tag{8.33}$$

where, according to (8.20b), the initial data ω_0^{i} is defined by:

$$\omega_0^{\mathrm{i}}=\omega^{\mathrm{i}}+\frac{1}{\nu}\big(\mathsf{A}_1^V\big)^{-1}\mathsf{P}^\perp\Lambda_{\mathsf{r}}^V(\omega^{\mathrm{i}}).$$

With (8.24), we deduce that:

$$\|\omega_0^{\mathrm{i}}\|_{V_1}\leqslant\|\omega^{\mathrm{i}}\|_{V_1}+\frac{\mathbf{c}_{\mathcal{F}}}{\nu}\|\omega^{\mathrm{i}}\|_{V_{-1}}^{\frac12}\|\omega^{\mathrm{i}}\|_{V_0}\|\omega^{\mathrm{i}}\|_{V_1}^{\frac12}\leqslant\mathbf{c}\|\omega^{\mathrm{i}}\|_{V_1}+\frac{\mathbf{c}_{\mathcal{F}}}{\nu^2}\|\omega^{\mathrm{i}}\|_{V_{-1}}\|\omega^{\mathrm{i}}\|_{V_0}^2. \tag{8.34a}$$

The second term in the right-hand side of (8.33) can be estimated using the estimates (8.22) as follows:

$$\begin{aligned}\int_0^t\frac{\mathbf{c}_{\mathcal{F}}}{\nu^3}\|\omega(s)\|_{V_{-1}}^2\|\omega(s)\|_{V_0}^2\|\omega(s)\|_{V_1}^2e^{\nu\lambda_{\mathcal{F}}s}\,\mathrm{d}s &\le\frac{\mathbf{c}_{\mathcal{F}}}{\nu^3}\|\omega^{\mathrm{i}}\|_{V_{-1}}^2\|\omega^{\mathrm{i}}\|_{V_0}^2\mathbf{E}_{[\mathcal{F},\nu,\omega^{\mathrm{i}}]}\int_0^te^{-2\nu\lambda_{\mathcal{F}}s}\|\omega(s)\|_{V_1}^2\,\mathrm{d}s\\ &\le\frac{\mathbf{c}_{\mathcal{F}}}{\nu^4}\|\omega^{\mathrm{i}}\|_{V_{-1}}^2\|\omega^{\mathrm{i}}\|_{V_0}^4\mathbf{E}_{[\mathcal{F},\nu,\omega^{\mathrm{i}}]}\Big[1+\frac{\mathbf{c}_{\mathcal{F}}}{\nu^4}\|\omega^{\mathrm{i}}\|_{V_{-1}}^4\mathbf{E}_{[\mathcal{F},\nu,\omega^{\mathrm{i}}]}\Big],\end{aligned} \tag{8.34b}$$

and

$$\begin{aligned}\int_0^t\frac{\mathbf{c}_{\mathcal{F}}}{\nu^7}\|\omega(s)\|_{V_{-1}}^4\|\omega(s)\|_{V_0}^6e^{\nu\lambda_{\mathcal{F}}s}\,\mathrm{d}s &\le\frac{\mathbf{c}_{\mathcal{F}}}{\nu^7}\|\omega^{\mathrm{i}}\|_{V_{-1}}^4\|\omega^{\mathrm{i}}\|_{V_0}^4\mathbf{E}_{[\mathcal{F},\nu,\omega^{\mathrm{i}}]}\int_0^te^{-5\nu\lambda_{\mathcal{F}}s}\|\omega(s)\|_{V_0}^2\,\mathrm{d}s\\ &\le\frac{\mathbf{c}_{\mathcal{F}}}{\nu^8}\|\omega^{\mathrm{i}}\|_{V_{-1}}^6\|\omega^{\mathrm{i}}\|_{V_0}^4\mathbf{E}_{[\mathcal{F},\nu,\omega^{\mathrm{i}}]}.\end{aligned} \tag{8.34c}$$

We finally obtain, gathering (8.33) and inequalities (8.34) (using again Young's inequality):

$$\|\omega_0(t)\|_{V_1}^2\leqslant\mathbf{c}_{\mathcal{F}}\Big[\|\omega^{\mathrm{i}}\|_{V_1}^2+\frac{1}{\nu^4}\|\omega^{\mathrm{i}}\|_{V_{-1}}^2\|\omega^{\mathrm{i}}\|_{V_0}^4\mathbf{E}_{[\mathcal{F},\nu,\omega^{\mathrm{i}}]}+\frac{1}{\nu^8}\|\omega^{\mathrm{i}}\|_{V_{-1}}^6\|\omega^{\mathrm{i}}\|_{V_0}^4\mathbf{E}_{[\mathcal{F},\nu,\omega^{\mathrm{i}}]}\Big]e^{-\nu\lambda_{\mathcal{F}}t}, \tag{8.35}$$

what, with (8.25a) and estimates (8.22), completes the proof of the lemma. □

The rest of the section is devoted to the proof of the theorem, which is classical and based again on a fixed point argument. Let us fix $T > 0$ and $\omega^{\mathrm{i}} \in V_1$ and introduce the spaces:

$$\Omega(T,\omega^{\mathrm{i}}) = \big\{\omega \in \Omega(T)\,:\,\omega(0) = \omega^{\mathrm{i}}\big\},$$
$$\Psi(T) = \big[L^2(0,T;\bar S_3)\cap H^1(0,T;S_1)\big]\cap\big[\mathcal{C}([0,T];S_2)\cap\mathcal{C}^1([0,T];S_0)\big],$$

and

$$F(T) = \{f_V \in L^2(0,T;L^2_V)\cap\mathcal{C}([0,T];V_{-1})\,:\,\mathsf{P}^\perp f_V \in H^1(0,T;V_{-2})\}.$$

Then define the mapping $\mathsf{X}_T : f_V \in F(T) \mapsto \omega \in \Omega(T,\omega^{\mathrm{i}})$ where $\omega = \omega_0 + \omega_{\mathfrak{B}}$ is the solution to the $\omega-$Stokes problem:

$$\partial_t\omega_0 + \nu\mathsf{A}^V_2\omega_0 = \mathsf{P}f_V - \partial_t\omega_{\mathfrak{B}} \qquad \text{in } \mathcal{F}_T \tag{8.36a}$$

$$\nu\bar{\mathsf{A}}^V_2\omega_{\mathfrak{B}} = \mathsf{P}^\perp f_V \qquad \text{in } \mathcal{F}_T, \tag{8.36b}$$

$$\omega(0) = \omega^{\mathrm{i}} \qquad \text{in } \mathcal{F}, \tag{8.36c}$$

and $\mathsf{Y}_T : \omega \in \Omega(T,\omega^{\mathrm{i}}) \mapsto \Lambda^V_{\mathsf{r}}(\omega) \in F(T)$ where $\Lambda^V_{\mathsf{r}}(\omega)$ is defined in (8.11a) (with $\varphi = 0$ since, as already mentioned, we consider only homogeneous boundary conditions).

Lemma 8.9. *The mapping X_T is well-defined and there exists a positive constant $\mathbf{c}_{[\mathcal{F},\nu]}$ such that:*

$$\|\mathsf{X}_T(f_V)\|_{\Omega(T)} \leqslant \mathbf{c}_{[\mathcal{F},\nu]}\big[\|\omega^{\mathrm{i}}\|^2_{V_1} + \|f_V\|^2_{F(T)}\big]^{\frac12} \qquad \textit{for all } f_V \in F(T). \tag{8.37a}$$

The mapping Y_T is also well-defined and there exists a positive constant $\mathbf{c}_{\mathcal{F}}$ such that, for all ω_1 and ω_2 in $\Omega(T,\omega^{\mathrm{i}})$:

$$\|\mathsf{Y}_T(\omega_2) - \mathsf{Y}_T(\omega_1)\|_{F(T)} \leqslant \mathbf{c}_{\mathcal{F}}T^{\frac{1}{10}}\big[\|\omega_1\|^2_{\Omega(T)} + \|\omega_2\|^2_{\Omega(T)}\big]^{\frac12}\|\omega_2-\omega_1\|_{\Omega(T)}, \tag{8.37b}$$

providing that $T < 1$.

Proof. The mapping X_T is well-defined from the space $F_{\mathsf{r}}(T)$ (defined in (8.6c)) into $\bar V_1(T)$ according to Proposition 8.1 and following Proposition 8.1, there exists a positive constant $\mathbf{c}_{[\mathcal{F},\nu]}$ such that:

$$\|\omega\|_{\bar V_1(T)} \leqslant \mathbf{c}_{[\mathcal{F},\nu]}\big[\|\omega^{\mathrm{i}}\|^2_{V_1} + \|f_V\|^2_{L^2(0,T;L^2_V)} + \|\partial_t(\mathsf{P}^\perp f_V)\|^2_{L^2(0,T;V_{-2})}\big]^{\frac12}.$$

However, comparing with Proposition 8.1, the source term f_V is assumed herein to satisfy the extra hypothesis $f_V \in \mathcal{C}([0,T];V_{-1})$ (and not only $\mathsf{P}^\perp f_V \in \mathcal{C}([0,T];V_{-1})$). We recall that every solution to the $\omega-$Stokes problem (8.36) satisfies also:

$$\partial_t\omega = -\mathsf{A}^V_1\omega + f_V \qquad \text{in } \mathcal{F}_T.$$

Since ω belongs in particular to $\mathcal{C}([0,T],V_1)$, we infer that $\partial_t\omega$ is in $\mathcal{C}([0,T];V_{-1})$ and finally that there exists a constant $\mathbf{c}_{[\mathcal{F},\nu]}$ such that (8.37a) holds.

For every $\theta \in V_1$, we have by definition:

$$\langle\Lambda^V_{\mathsf{r}}(\omega),\theta\rangle_{V_{-1},V_1} = (\nabla^\perp\psi\cdot\nabla\omega,\mathsf{Q}_1\theta)_{L^2(\mathcal{F})} = -(\omega\nabla^\perp\psi,\nabla(\mathsf{Q}_1\theta))_{\mathbf{L}^2(\mathcal{F})}, \tag{8.38a}$$

$$\langle\mathsf{P}^\perp\Lambda^V_{\mathsf{r}}(\omega),\theta\rangle_{V_{-1},V_1} = (\mathsf{P}^\perp\big(\nabla^\perp\psi\cdot\nabla\omega\big),\mathsf{Q}_1\theta)_{L^2(\mathcal{F})} = -(\nabla^\perp\psi\cdot\nabla\omega,\mathsf{Q}_1^\perp\theta)_{L^2(\mathcal{F})}, \tag{8.38b}$$

the latter expression resting on the equalities $\mathsf{P}^\perp\mathsf{Q}_1 = (\mathrm{Id}-\mathsf{P})\mathsf{Q}_1 = \mathsf{Q}_1 - \mathrm{Id} = -\mathsf{Q}_1^\perp$. Assuming now that θ belongs to V_2, the right hand side in (8.38b) can be integrated by parts twice to obtain:

$$(\nabla^\perp\psi\cdot\nabla\omega,\mathsf{Q}_2^\perp\theta)_{L^2(\mathcal{F})} = (D^2(\mathsf{Q}_2^\perp\theta)\nabla\psi,\nabla^\perp\psi)_{\mathbf{L}^2(\mathcal{F})},$$

whence it can be deduced in particular that:

$$\langle\partial_t\big(\mathsf{P}^\perp\Lambda^V_{\mathsf{r}}(\omega)\big),\theta\rangle_{V_{-2},V_2} = (D^2(\mathsf{Q}_2^\perp\theta)\nabla\partial_t\psi,\nabla^\perp\psi)_{\mathbf{L}^2(\mathcal{F})} + (D^2(\mathsf{Q}_2^\perp\theta)\nabla\psi,\nabla^\perp\partial_t\psi)_{\mathbf{L}^2(\mathcal{F})}. \tag{8.38c}$$

The same arguments as those used in the proof of Equality (6.8b) yield:

$$\|\nabla^\perp\psi\cdot\nabla\omega\|^2_{L^2(0,T;L^2(\mathcal{F}))} \leqslant \mathbf{c}_{\mathcal{F}}T^{\frac15}\|\omega\|^{\frac25}_{\mathcal{C}([0,T];V_1)}\|\psi\|^2_{\mathcal{C}([0,T];S_1)}\|\omega\|^{\frac85}_{L^2(0,T;H^2_V)},$$

which entails that:

$$\|\Lambda^V_{\mathsf{r}}(\omega_2) - \Lambda^V_{\mathsf{r}}(\omega_1)\|_{L^2(0,T;L^2_V)} \leqslant \mathbf{c}_{\mathcal{F}}T^{\frac{1}{10}}\big[\|\omega_1\|^2_{\Omega(T)} + \|\omega_2\|^2_{\Omega(T)}\big]^{\frac12}\|\omega_2-\omega_1\|_{\Omega(T)}. \tag{8.39}$$

Considering now the expression (8.38a), we first easily obtain:

$$\|\omega_2\nabla^\perp\psi_2-\omega_1\nabla^\perp\psi_1\|_{\mathbf{L}^2(\mathcal{F})}\leqslant \mathbf{c}_{\mathcal{F}}\|\omega_2-\omega_1\|_{V_0}^{\frac15}\|\omega_2-\omega_1\|_{L^4(\mathcal{F})}^{\frac45}\|\nabla\psi_2\|_{\mathbf{L}^5(\mathcal{F})}+\mathbf{c}_{\mathcal{F}}\|\omega_1\|_{L^4(\mathcal{F})}\|\nabla(\psi_2-\psi_1)\|_{\mathbf{L}^4(\mathcal{F})}.\tag{8.40a}$$

On the one hand, since ω_1 and ω_2 share the same initial value, we are allowed to write that:

$$\|\omega_2-\omega_1\|_{\mathcal{C}([0,T];V_0)}\leqslant T^{\frac12}\|\partial_t\omega_2-\partial_t\omega_1\|_{L^2(0,T;V_0)}.\tag{8.40b}$$

On the other hand, Sobolev embedding theorem ensures that:

$$\|\omega_2-\omega_1\|_{\mathcal{C}([0,T];L^4(\mathcal{F}))}\leqslant \mathbf{c}_{\mathcal{F}}\|\omega_2-\omega_1\|_{\mathcal{C}([0,T];V_1)}\tag{8.40c}$$

$$\|\nabla\psi_2\|_{\mathcal{C}([0,T];\mathbf{L}^5(\mathcal{F}))}\leqslant \mathbf{c}_{\mathcal{F}}\|\omega_2\|_{\mathcal{C}([0,T];V_0)}.\tag{8.40d}$$

The second term in the right hand side of (8.40a) is estimated in a similar manner, thus:

$$\|\nabla(\psi_2-\psi_1)\|_{\mathcal{C}([0,T];\mathbf{L}^4(\mathcal{F}))}\leqslant \mathbf{c}_{\mathcal{F}}\|\omega_2-\omega_1\|_{\mathcal{C}([0,T];V_0)}\leqslant \mathbf{c}_{\mathcal{F}}T^{\frac12}\|\partial_t\omega_2-\partial_t\omega_1\|_{L^2(0,T;V_0)}.\tag{8.40e}$$

Assuming that $T<1$, the estimates (8.40) give rise to:

$$\|\varLambda^V_{\mathsf{r}}(\omega_2)-\varLambda^V_{\mathsf{r}}(\omega_1)\|_{\mathcal{C}([0,T];V_{-1})}\leqslant \mathbf{c}_{\mathcal{F}}T^{\frac{1}{10}}\left[\|\omega_1\|^2_{\Omega(T)}+\|\omega_2\|^2_{\Omega(T)}\right]^{\frac12}\|\omega_2-\omega_1\|_{\Omega(T)}.\tag{8.41}$$

We turn now our attention to the right hand side of (8.38c). On the one hand, we obtain that:

$$\||\nabla(\partial_t\psi_2-\partial_t\psi_1)||\nabla\psi_2|\|^2_{L^2(0,T;L^2(\mathcal{F}))}\leqslant \mathbf{c}_{\mathcal{F}}T^{\frac15}\|\partial_t\omega_2-\partial_t\omega_1\|^{\frac25}_{\mathcal{C}([0,T];V_{-1})}\|\omega_2\|^2_{\mathcal{C}([0,T];V_0)}\|\partial_t\omega_2-\partial_t\omega_1\|^{\frac85}_{L^2(0,T;V_0)}.\tag{8.42a}$$

On the other hand, using again (8.40b):

$$\||\nabla\partial_t\psi_1||\nabla(\psi_2-\psi_1)|\|_{L^2(0,T;L^2(\mathcal{F}))}\leqslant \mathbf{c}_{\mathcal{F}}T^{\frac12}\|\partial_t\omega_1\|_{L^2(0,T;V_0)}\|\partial_t\omega_2-\partial_t\omega_1\|_{L^2(0,T;V_0)}.\tag{8.42b}$$

Providing again that $T<1$, both estimates (8.42) yield:

$$\|\partial_t(\varLambda^V_{\mathsf{r}}(\omega_2))-\partial_t(\varLambda^V_{\mathsf{r}}(\omega_1))\|_{L^2(0,T;V_{-2})}\leqslant \mathbf{c}_{\mathcal{F}}T^{\frac{1}{10}}\left[\|\omega_1\|^2_{\Omega(T)}+\|\omega_2\|^2_{\Omega(T)}\right]^{\frac12}\|\omega_2-\omega_1\|_{\Omega(T)}.\tag{8.43}$$

Estimate (8.37b) derives now straightforwardly from (8.39), (8.41) and (8.43). This completes the proof. ☐

Proof of Theorem 8.6. Define the mapping $\mathsf{Z}_T:f_V\in F(T)\mapsto \mathsf{Y}_T\circ\mathsf{X}_T(f_V)\in F(T)$ et let $f_V^{\rm i}=\mathsf{Z}_T(0)$. Then, according to the estimates of Lemma 8.9:

$$\|\mathsf{Z}_T(f_V)-f_V^{\rm i}\|_{F(T)}\leqslant \mathbf{c}_{[\mathcal{F},\nu]}T^{\frac{1}{10}}\left(\|f_V\|^2_{F(T)}+\|\omega_0^{\rm i}\|^2_{V_1}\right)\qquad\text{for all } f_V\in F(T),$$

$$\|\mathsf{Z}_T(f^1_V)-\mathsf{Z}_T(f^2_V)\|_{F(T)}\leqslant \mathbf{c}_{[\mathcal{F},\nu]}T^{\frac{1}{10}}\left(\|f^1_V\|^2_{F(T)}+\|f^2_V\|^2_{F(T)}+\|\omega^{\rm i}\|^2_{V_1}\right)^{\frac12}\|f^1_V-f^2_V\|_{F(T)},$$

for every f^1_V,f^2_V in $F(T)$ and $T<1$. For every $R>0$, there exists a time $T^*<1$ (depending only on $\mathcal{F}$, ν, $\|\omega_0^{\rm i}\|_{V_1}$ and R) such that Z_{T^*} is a contraction from $B(f^{\rm i}_V,R)\subset F(T^*)$ into $B(f^{\rm i}_V,R)$. From Banach fixed point theorem, the mapping Z_{T^*} admits a unique fixed point in $B(f^{\rm i}_V,R)$, the image of which by the mapping X_{T^*} is a solution to System (8.19) on $[0,T^*)$. We conclude that T^* can be chosen arbitrarily large following the lines of the proof of Theorem 6.11, using the estimate of Lemma 8.8. Finally, every solution is also a strong solution in the sense of Definition 6.13, which was proved to be unique. ☐

9. Concluding remarks

By introducing a suitable functional framework, the 2D vorticity equation has been shown to be not a classical parabolic equation but rather a parabolic-elliptic coupling. Indeed, applying the harmonic Bergman projection to the equation $\partial_t\omega-\nu\Delta\omega+u\cdot\nabla\omega=0$ leads to its splitting into, on the one hand, an evolution diffusion-advection equation for the non-harmonic part of ω (equation (8.36a)) and on the other hand a (steady) elliptic equation for the remaining harmonic part (equation (8.36b)). By exploiting this structure of the equation, we were able to prove the exponential decay of the palinstrophy for large time, a result which was not known so far. In this work, it is worth noticing the surprising role played by the circulation in this context, circulation being well known for entering the analysis of perfect fluids but usually less came across in the context of viscous fluids. The other point that deserves to be highlighted is the simple form taken by the Biot-Savart operator, described in Theorem 3.20.

In a forthcoming work, we shall apply our method to fluid-structure problems by considering a set of disks, pinned at their centers but free to rotate, immersed in a viscous fluid. The equations governing the coupled fluid-rotating disks system can be stated in terms of the vorticity of the fluid and the angular velocities of the disks only. The analysis of these equations will obviously be carried out in nonprimitive variables.

Appendix A. Gelfand triple

A.1. **General settings.** Let H_1 and H_0 be two Hilbert spaces. Their scalar products are denoted respectively by $(\cdot,\cdot)_1$ and $(\cdot,\cdot)_0$ and their norms by $\|\cdot\|_1$ and $\|\cdot\|_0$. We assume that:

$$H_1 \subset H_0, \tag{A.1}$$

where the inclusion is continuous and dense. Applying Riesz representation theorem, the space H_0 is identified with its dual H_0'. It means that, for every $u \in H_0$, the linear form $(\cdot, u)_0$ is identified with u. The space H_0 is usually referred to as the pivot space. Therefore, the space H_1 cannot be identified with its dual H_{-1} but with a subspace of H_0. Thus, the configuration

$$H_1 \subset H_0 \subset H_{-1}, \tag{A.2}$$

is called (with a slight abuse of terminology) Gelfand triple. The inclusions are both continuous and dense.

We define the operator

$$\mathsf{A}_1 : H_1 \longrightarrow H_{-1}, \qquad \mathsf{A}_1 u = (u,\cdot)_1, \quad \text{for all } u \in H_1, \tag{A.3}$$

and it can be readily verified that A_1 is an isometry. Then, we define the space $H_2 = \mathsf{A}_1^{-1} H_0$ and the operator $\mathsf{A}_2 : H_2 \to H_0$ by setting, for every $u \in H_2$:

$$(\mathsf{A}_2 u, \cdot)_0 = \mathsf{A}_1 u \quad \text{in } H_{-1}. \tag{A.4}$$

We equip the space H_2 with the scalar product:

$$(u, v)_2 = (\mathsf{A}_2 u, \mathsf{A}_2 v)_0, \qquad \text{for all } u, v \in H_2,$$

and the corresponding norm $\|\cdot\|_2$.

Lemma A.1. *The space H_2 is a Hilbert space, the operator A_2 is an isometry and the inclusion $H_2 \subset H_1$ is continuous and dense. It entails that the inclusion $H_{-1} \subset H_{-2}$, where H_{-2} stands for the dual space of H_2, is continuous and dense as well.*

Proof. The estimate below is satisfied by every $u \in H_2$:

$$\|u\|_0 \leqslant \mathbf{c}\|\mathsf{A}_2 u\|_0. \tag{A.5}$$

Indeed, the continuity of the inclusion (A.1) yields $\|u\|_0^2 \leqslant \mathbf{c}\|u\|_1^2$. Then, the definitions of both the operator A_2 and the space H_2 lead to the identity $\|u\|_1^2 = (\mathsf{A}_2 u, u)_0$. Applying Cauchy-Schwarz inequality, we obtain (A.5).

Let assume that $(u_n)_{n\geqslant 0}$ is Cauchy sequence in H_2, or equivalently that $(\mathsf{A}_2 u_n)_{n\geqslant 0}$ is a Cauchy sequence in H_0, and denote by v^* the limit of $(\mathsf{A}_2 u_n)_{n\geqslant 0}$ in H_0. According to (A.5), we deduce that $(u_n)_{n\geqslant 0}$ is a Cauchy sequence in H_0 as well. The equality:

$$(\mathsf{A}_2 u_n - \mathsf{A}_2 u_m, u_n - u_m)_0 = \|u_n - u_m\|_1^2,$$

available for every pair of indices n and m, entails that the sequence $(u_n)_{n\geqslant 0}$ is also a Cauchy sequence in H_1. We denote by u^* its limit in this space. Letting n goes to ∞ in the identity:

$$(\mathsf{A}_2 u_n, \cdot)_0 = \mathsf{A}_1 u_n \quad \text{in } H_{-1},$$

we obtain:

$$(v^*, \cdot)_0 = \mathsf{A}_1 u^*,$$

and therefore u^* belongs to H_2. This proves that H_2 is complete and hence is a Hilbert space.

Let now v be in $H_2^\perp$ in H_1. There exists $u \in H_2$ such that $\mathsf{A}_2 u = v$ and:

$$\|v\|_0^2 = (\mathsf{A}_2 u, v)_0 = (u, v)_1 = 0.$$

It follows that $H_2^\perp = 0$ in H_1 and therefore H_2 is dense in H_1.

The continuity of the inclusion $H_2 \subset H_1$ results from the identity:

$$\|u\|_1^2 = (\mathsf{A}_2 u, u)_0, \quad \text{for all } u \in H_2,$$

combined with Cauchy-Schwarz inequality:

$$(\mathsf{A}_2 u, u)_0 \leqslant \|\mathsf{A}_2 u\|_0 \|u\|_0, , \quad \text{for all } u \in H_2,$$

and estimate (A.5).

Finally, the operator A_2 is onto by definition and it is also injective because the identity $\mathsf{A}_2 u = 0$ for some $u \in H_2$ leads to $(\mathsf{A}_2 u, u)_0 = \|u\|_1^2 = 0$. The proof of the lemma is now completed. □

Let us define the operator $\mathsf{A}_0 : H_0 \to H_{-2}$ by:

$$\mathsf{A}_0 : u \in H_0 \mapsto (\mathsf{A}_2 \cdot, u)_0 \in H_{-2}. \tag{A.6}$$

Lemma A.2. *The operator* A_0 *is an isometry.*

Proof. The operator A_0 is injective. Indeed, the identity $\mathsf{A}_0 u = 0$ for some $u \in H_0$ entails that $(\mathsf{A}_2 \mathsf{A}_2^{-1} u, u)_0 = \|u\|_0^2 = 0$. The operator A_0 is also onto: Any element of H_{-2} can be written, according to Riesz theorem, as $(\cdot, v)_2 = (\mathsf{A}_2 \cdot, \mathsf{A}_2 v)_0$ for some $v \in H_2$, and hence it is equal to $\mathsf{A}_0 u$ with $u = \mathsf{A}_1 v \in H_0$.

Finally, the operator A_0 is also an isometry since we have, for every $u \in H_0$:

$$\|\mathsf{A}_0 u\|_{-2} = \sup_{\substack{v \in H_2 \\ v \neq 0}} \frac{|(\mathsf{A}_2 v, u)_0|}{\|v\|_2} = \sup_{\substack{v \in H_2 \\ v \neq 0}} \frac{|(\mathsf{A}_2 v, u)_0|}{\|\mathsf{A}_2 v\|_0} = \sup_{\substack{w \in H_0 \\ w \neq 0}} \frac{|(w, u)_0|}{\|w\|_0} = \|u\|_0,$$

and the proof is completed. □

So far, we have proved that in the chain of inclusions:

$$H_2 \subset H_1 \subset H_0 \subset H_{-1} \subset H_{-2},$$

every inclusion is continuous and dense and that the operators $\mathsf{A}_k : H_k \to H_{k-2}$ for $k = 0, 1, 2$ are isometries.

By induction, we can next define $H_{k+2} = \mathsf{A}_{k+1}^{-1} H_k$ for every positive integer k. The operator

$$\mathsf{A}_{k+2} : H_{k+2} \longrightarrow H_k$$

is defined from the operator A_{k+1} by setting $\mathsf{A}_{k+2} u = \mathsf{A}_{k+1} u$ for every $u \in H_{k+2}$. The spaces H_{k+2} are Hilbert spaces once equipped with the scalar products:

$$(u, v)_{k+2} = (\mathsf{A}_{k+2} u, \mathsf{A}_{k+2} v)_k, \qquad \text{for all } u, v \in H_{k+2}.$$

For every $k \geqslant 1$, the dual space of H_k is denoted by H_{-k} and we introduce the operator

$$\mathsf{A}_{-k} : H_{-k} \longrightarrow H_{-k-2},$$

defined by duality as follows:

$$\mathsf{A}_{-k} u = \langle u, \mathsf{A}_{k+2} \cdot \rangle_{-k,k} \in H_{-k-2}, \qquad \text{for all } u \in H_{-k}. \tag{A.7}$$

It can be readily verified that the Hilbert spaces H_k $(k \in \mathbb{Z})$ satisfy:

$$\ldots \subset H_{k+1} \subset H_k \subset H_{k-1} \subset \ldots \subset H_1 \subset H_0 \subset H_{-1} \subset \ldots \subset H_{-k+1} \subset H_{-k} \subset H_{-k-1} \subset \ldots$$

each inclusion being continuous and dense. Furthermore, for every integer k, the operator:

$$\mathsf{A}_k : H_k \longrightarrow H_{k-2},$$

is an isometry.

Lemma A.3. *For every integers* n, n' *such that* $n' \leqslant n$ *and for every* $u \in H_n$*, the following equality holds:*

$$\mathsf{A}_n u = \mathsf{A}_{n'} u. \tag{A.8}$$

Proof. For $n' \geqslant 0$, the property (A.8) is obvious.

On the other hand, let $0 \leqslant k' \leqslant k$ be given and assume that $u \in H_{-k'} \subset H_{-k}$. The definition of $\mathsf{A}_{-k}u$ leads to:

$$\mathsf{A}_{-k}u = \langle u, \mathsf{A}_{k+2}\cdot\rangle_{-k,k} \in H_{-k-2}.$$

But $\mathsf{A}_{k+2} = \mathsf{A}_{k'+2}$ in H_{k+2} and therefore:

$$\langle u, \mathsf{A}_{k+2}\cdot\rangle_{-k,k} = \langle u, \mathsf{A}_{k'+2}\cdot\rangle_{-k',k'} = \mathsf{A}_{-k'}u.$$

The proof is now complete. □

Lemma A.4. *For every integer k, the following identity hold:*

$$(\mathsf{A}_{k+1}u, v)_{k-1} = (u, v)_k, \quad \textit{for all } u \in H_{k+1}, \quad \textit{for all } v \in H_k. \tag{A.9}$$

Proof. The proof is by induction on k. The equality (A.9) is true for $k = 1$ according to the definition (A.4) of A_2. Let us assume that (A.9) is true for some integer k. By definition, if z belongs to H_{k+2}, then $\mathsf{A}_{k+1}z$ belongs to H_k. Replacing v by $\mathsf{A}_{k+1}z$ in (A.9), we obtain:

$$(\mathsf{A}_{k+1}u, \mathsf{A}_{k+1}z)_{k-1} = (u, \mathsf{A}_{k+1}z)_k, \quad \text{for all } u \in H_{k+1}, \quad \text{for all } z \in H_{k+2},$$

that is to say, reorganizing the terms:

$$(\mathsf{A}_{k+2}z, u)_k = (z, u)_{k+1}, \quad \text{for all } u \in H_{k+1}, \quad \text{for all } z \in H_{k+2},$$

and therefore, formula (A.9) is true replacing k by $k + 1$. Let us verify that it is also true for $k - 1$. Thus, we have:

$$(\mathsf{A}_{k+1}u, v)_{k-1} = (u, v)_k = (\mathsf{A}_k u, \mathsf{A}_k v)_{k-2}, \quad \text{for all } u \in H_{k+1}, \quad \text{for all } v \in H_k.$$

But A_{k+1} is an isometry from H_{k+1} onto H_{k-1} and $\mathsf{A}_k = \mathsf{A}_{k+1}$ in H_{k+1}, then:

$$(w, v)_{k-1} = (u, v)_k = (w, \mathsf{A}_k v)_{k-2}, \quad \text{for all } w \in H_{k-1}, \quad \text{for all } v \in H_k.$$

The proof is now complete. □

Lemma A.5. *Let k be an integer, w be in H_{k-1} and u be in H_k such that:*

$$(w, v)_{k-1} = (u, v)_k, \quad \textit{for all } v \in H_k.$$

Then $u \in H_{k+1}$ and $w = \mathsf{A}_{k+1}u$.

Proof. Let $\tilde{u} = \mathsf{A}_{k+1}^{-1}w$. Then $(\tilde{u} - u, v)_k = 0$ for every $v \in H_k$ and therefore $u = \tilde{u}$. □

A.2. **Isometric chain of embedded Hilbert spaces.** Let $\{H_k,\, k \in \mathbb{Z}\}$ and $\{\hat{H}_k,\, k \in \mathbb{Z}\}$ be two families of embedded Hilbert spaces build from Gelfand triples. We assume that H_0 and $\hat{H}_0$ are not necessary the pivot spaces. As usual, for every integer k, there exist isometries $\mathsf{A}_k : H_k \to H_{k-2}$ and $\hat{\mathsf{A}}_k : \hat{H}_k \to \hat{H}_{k-2}$ such that $\mathsf{A}_k = \mathsf{A}_{k-1}$ in H_k and $\hat{\mathsf{A}}_k = \hat{\mathsf{A}}_{k-1}$ in $\hat{H}_k$.

We assume furthermore that there exist isometries $p_0 : H_0 \to \hat{H}_0$ and $p_1 : H_1 \to \hat{H}_1$ such that $p_1 = p_0$ in H_1. For every integer $k \geqslant 2$, we define by induction $p_k = \hat{\mathsf{A}}_k^{-1} p_{k-2} \mathsf{A}_k$ and for every $k \geqslant 1$, we set $p_{-k-2} = \hat{\mathsf{A}}_{-k} p_{-k} \mathsf{A}_{-k}^{-1}$.

Lemma A.6. *For every pair of integers k and k' such that $k' \leqslant k$:*

$$p_{k'} = p_k \quad \textit{in } H_k. \tag{A.10}$$

Moreover, for every integer k, the operator $p_k : H_k \to \hat{H}_k$ is an isometry.

Proof. Since A_k and $\hat{\mathsf{A}}_k$ are isometries for every integer k, we can draw the same conclusion for p_k providing that p_{k-2} is an isometry as well. The conclusion follows by induction for every $k \geqslant 0$. The same reasoning allows proving that p_{-k} is also an isometry for every $k \geqslant 1$.

It remains to verify that the equalities (A.10) are true. Assume that for some index $k \geqslant 0$, $p_k = p_{k+1}$ in H_{k+1}. So, from the identity:

$$(\mathsf{A}_{k+2}u, v)_{H_k} = (u, v)_{H_{k+1}}, \qquad \text{for all } u \in H_{k+2}, \quad \text{for all } v \in H_{k+1},$$

we deduce that:

$$(p_k\mathsf{A}_{k+2}u, p_k v)_{\hat{H}_k} = (p_{k+1}u, p_{k+1}v)_{\hat{H}_{k+1}}, \qquad \text{for all } u \in H_{k+2}, \quad \text{for all } v \in H_{k+1}.$$

From the definition of p_{k+2}, we deduce that $p_k\mathsf{A}_{k+2} = \hat{\mathsf{A}}_{k+2}p_{k+2}$, whence, denoting $\mathbf{v} = p_k v = p_{k+1}v$:

$$(\hat{\mathsf{A}}_{k+2}p_{k+2}u, \mathbf{v})_{\hat{H}_k} = (p_{k+1}u, \mathbf{v})_{\hat{H}_{k+1}}, \qquad \text{for all } u \in H_{k+2}, \quad \text{for all } \mathbf{v} \in \hat{H}_{k+1}.$$

This equality entails first that $\hat{\mathsf{A}}_{k+2}p_{k+2}u = \hat{\mathsf{A}}_{k+2}p_{k+1}u$ and next, since $\hat{\mathsf{A}}_{k+2}$ is invertible, that $p_{k+2}u = p_{k+1}u$ for every $u \in H_{k+2}$. The conclusion follows by induction and the cases $k \leqslant 0$ are treated similarly. ☐

A.3. **Semigroup.** We assume that the inclusion (A.1) is in addition compact. In that case, we claim:

Lemma A.7. *For every integer k, the inclusion $H_{k+1} \subset H_k$ is compact.*

Proof. We address the case $k = 1$. Assume that the sequence $(u_n)_{n\geqslant 0}$ is weakly convergent toward 0 in H_2. On the one hand, it means that:

$$(u_n, v)_2 = (\mathsf{A}_2 u_n, \mathsf{A}_2 v)_0 \longrightarrow 0 \text{ as } n \to +\infty \qquad \text{for all } v \in H_2,$$

and therefore, that:

$$(\mathsf{A}_2 u_n, w)_0 \longrightarrow 0 \text{ as } n \to +\infty \qquad \text{for all } w \in H_0.$$

That is, $(\mathsf{A}_2 u_n)_{n\geqslant 0}$ is weakly convergent toward 0 in H_0. On the other hand, since H_2 is continuously included into H_1, the sequence $(u_n)_{n\geqslant 0}$ is also weakly convergent toward 0 in H_1 and hence strongly convergent in H_0. It follows that:

$$\|u_n\|_1^2 = (\mathsf{A}_2 u_n, u_n)_0 \longrightarrow 0 \text{ as } n \to +\infty.$$

Since the operators A_k were proved to be isometries for every k, the other cases follows and the proof is completed. ☐

For every integer k, we define the unbounded operators $\mathcal{A}_k$ of domain $D(\mathcal{A}_k) = H_{k+2}$ in H_k by:

$$\mathcal{A}_k x = \mathsf{A}_{k+2}x \qquad \text{for all } x \in D(\mathcal{A}_k). \tag{A.11}$$

Proposition A.8. *For every integer k, the operator $\mathcal{A}_k$ is self-adjoint with compact inverse. All the operators $\mathcal{A}_k$ share the same spectrum that consists in a sequence $(\lambda_n)_{n\geqslant 1}$ of positive eigenvalues that tends to $+\infty$. All the operators $\mathcal{A}_k$ share also the same eigenfunctions, denoted by e_n ($n \geqslant 1$) and for every nonnegative integer n:*

$$e_n \in H_\infty \qquad \text{with} \qquad H_\infty = \bigcap_{p\geqslant 0} H_p.$$

The eigenfunctions are chosen to form an orthogonal Riesz basis in every H_k and they are scaled to be of unit norm in H_0.

The spaces H_k are isometric to the spaces:

$$\ell_k = \Big\{u = (u_n)_{n\geqslant 1} \in \mathbb{R}^{\mathbb{N}} \,:\, \sum_{n\geqslant 1} \lambda_n^k u_n^2 < +\infty\Big\},$$

provided with the scalar product:

$$(u, v)_{\ell_k} = \sum_{n\geqslant 1} \lambda_n^k u_n v_n \qquad \text{for all } u = (u_n)_{n\geqslant 1} \quad \text{and} \quad v = (v_n)_{n\geqslant 1} \text{ in } \ell_k,$$

the isometries being obviously:

$$\mathcal{I}_k : u \in H_k \mapsto ((u, e_n)_k)_{n\geqslant 1} \in \ell_k \quad \text{with inverse} \quad \mathcal{I}_k^{-1} : u = (u_n)_{n\geqslant 1} \in \ell_k \longmapsto \sum_{n=1}^{+\infty} u_n e_n \in H_k.$$

In ℓ_k we define the strongly continuous semigroup of contraction $(\mathbb{T}_k(t))_{t\geqslant 0}$ by:

$$\mathbb{T}_k(t)u = \Big(e^{-\lambda_n t}u_n\Big)_{n\geqslant 0} \qquad \text{for all } t > 0 \text{ and } u = (u_n)_{n\geqslant 0} \in \ell_k.$$

This semigroup admits the operator $\mathcal{B}_k = \mathcal{I}_k \mathcal{A}_k \mathcal{I}_k^{-1}$ as infinitesimal generator. We deduce that the semigroup $(\mathbb{S}_k(t))_{t\geqslant 0}$ defined by:

$$\mathbb{S}_k(t) = \mathcal{I}_k^{-1}\mathbb{T}(t)\mathcal{I}_k,$$

is a strongly continuous semigroup of contraction in H_k with infinitesimal generator $\mathcal{A}_k$. It is a simple exercice to verify that:

Lemma A.9. (1) *For every $u \in \ell_k$ and for every positive real number T:*

$$\mathbb{T}_k(\cdot)u \in H^1(0,T;\ell_{k-1}) \cap \mathcal{C}([0,T];\ell_k) \cap L^2(0,T;\ell_{k+1}).$$

(2) *Let v be in $L^2(0,T;\ell_{k-1})$ and define $w(s) = \int_0^t \mathbb{T}_k(t-s)v(s)\,\mathrm{d}s$ for every $t \in (0,T)$. Then:*

$$w \in H^1(0,T;\ell_{k-1}) \cap \mathcal{C}([0,T];\ell_k) \cap L^2(0,T;\ell_{k+1}).$$

Considering, for any integer k, any time $T > 0$ and any initial data $u^{\mathrm{i}} \in H_k$ the Cauchy problem:

$$\partial_t u + \mathsf{A}_{k+1} u = f \quad \text{on } (0,T) \tag{A.12a}$$

$$u(0) = u^{\mathrm{i}} \quad \text{in } H_k, \tag{A.12b}$$

where the source term f is given in $L^2(0,T;H_{k-1})$, we deduce:

Proposition A.10. *The Cauchy problem* (A.12) *admits a unique solution in the space:*

$$H^1(0,T;H_{k-1}) \cap \mathcal{C}([0,T];H_k) \cap L^2(0,T;H_{k+1}),$$

and this solution is given by:

$$u(t) = \mathbb{S}_k(t)u^{\mathrm{i}} + \int_0^t \mathbb{S}_k(t-s)f(s)\,\mathrm{d}s \qquad \text{for all } t \in [0,T].$$

Remark A.11. (1) *The chain of embedded spaces H_k and semigroup $(\mathbb{S}_k(t))_{t\geqslant 0}$ fit with the notion of Sobolev towers as described in* [15, §II.2.C].

(2) *The semigroups $(\mathbb{S}_k(t))_{t\geqslant 0}$ are called diagonalizable semigroups; see* [64, §2.6].

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