On CCZ-Equivalence, Extended-Affine Equivalence and Function Twisting

Anne Canteaut, Léo Perrin

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Definition (CCZ-Equivalence)

$F : \mathbb{F}_2^n \to \mathbb{F}_2^m$ and $G : \mathbb{F}_2^n \to \mathbb{F}_2^m$ are *C(arlet)-C(harpin)-Z(inoviev) equivalent* if

$$\Gamma_G = \left\{(x, G(x)), \forall x \in \mathbb{F}_2^n\right\} \; = \; L\left(\left\{(x, F(x)), \forall x \in \mathbb{F}_2^n\right\}\right) = L(\Gamma_F)\,,$$

where $L : \mathbb{F}_2^{n+m} \to \mathbb{F}_2^{n+m}$ is an affine permutation.

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Definition (EA-Equivalence; EA-mapping)

F and G are *E(xtented) A(ffine) equivalent* if $G(x) = (B \circ F \circ A)(x) + C(x)$, where A, B, C are affine and A, B are permutations; so that

$$\left\{(x, G(x)), \forall x \in \mathbb{F}_2^n\right\} \; = \left[\begin{array}{cc} A^{-1} & 0 \\ CA^{-1} & B \end{array}\right] \left(\left\{(x, F(x)), \forall x \in \mathbb{F}_2^n\right\}\right)\; .$$

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Affine permutations with such linear part are **EA-mappings**; their transposes are **TEA-mappings**

What is the relation between functions that are CCZ- but not EA-equivalent?

Admissible Mapping

For $F : \mathbb{F}_2^n \to \mathbb{F}_2^m$, the affine permutation L is **admissible for F** if

$$L\Big(\{(x, F(x)) \, , \forall x \in \mathbb{F}_2^n\} \Big) = \{(x, G(x)) \, , \forall x \in \mathbb{F}_2^n\}$$

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Definition (LAT/Walsh Spectrum)

The L(inear) A(pproximation) T(able) of $F : \mathbb{F}_2^n \to \mathbb{F}_2^m$ is

$$\mathcal{W}_F(\alpha, \beta) = \sum_{x \in \mathbb{F}_2^n} (-1)^{\alpha \cdot x + \beta \cdot F(x)} .$$

Structure of this talk

0 - CCZ-Equivalence ; Bijectivity

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1.1 - Vector spaces of zeroes in LAT

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1.1 - Vector spaces of zeroes in LAT

1.2 - Partition CCZ-class into EA-classes

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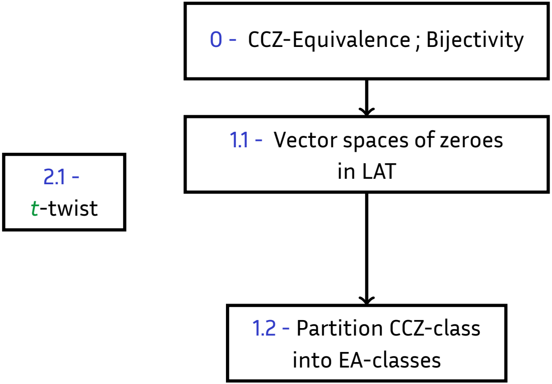

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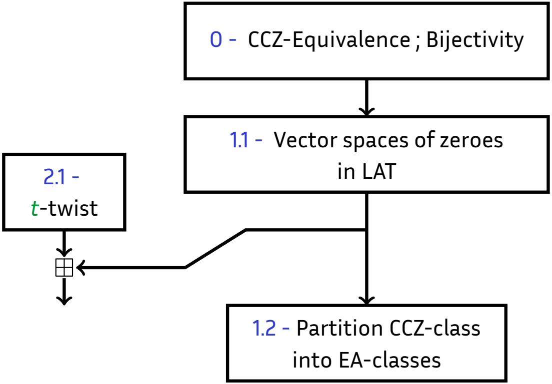

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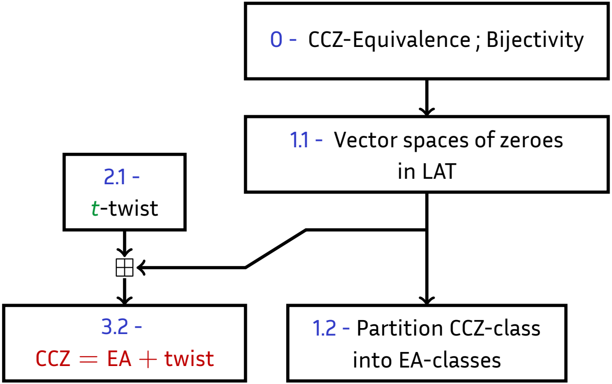

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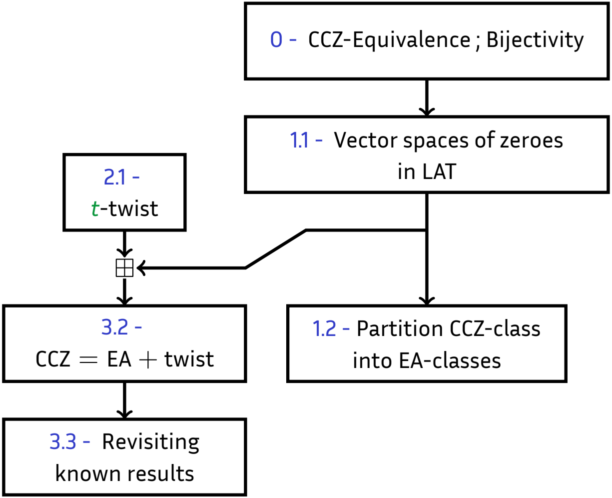

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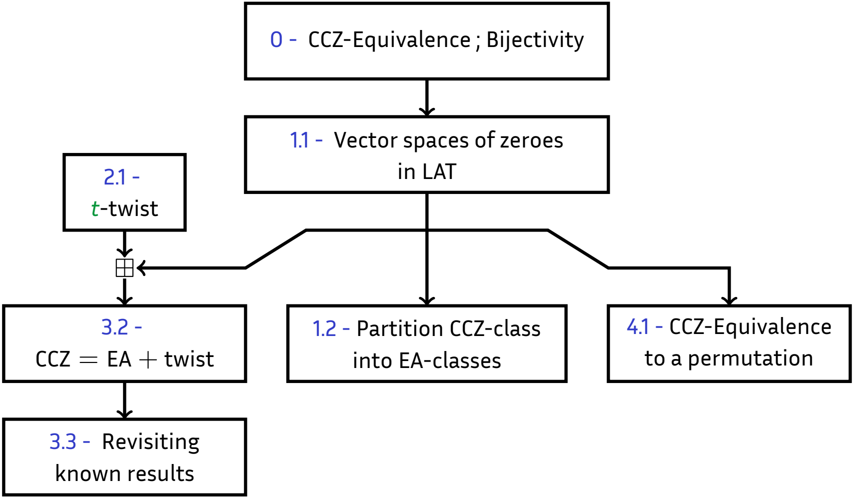

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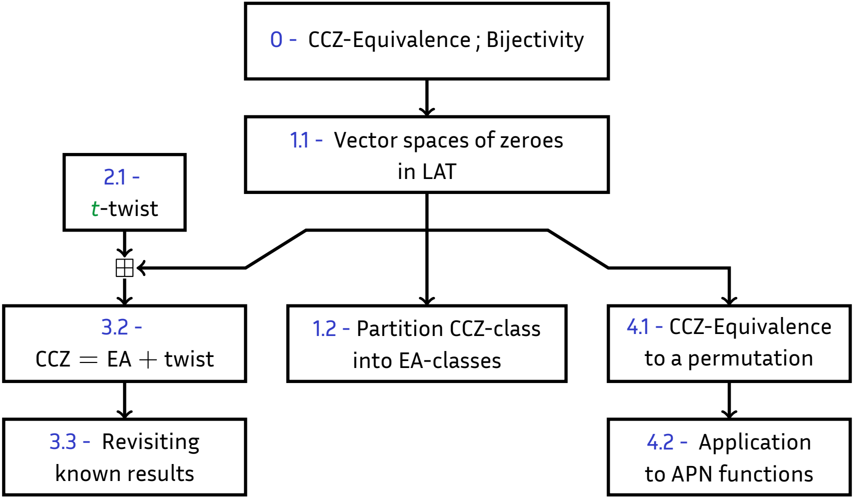

Outline

Plan of this Section

Walsh Zeroes

For all $F : \mathbb{F}_2^n \to \mathbb{F}_2^m$, we have

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The *Walsh zeroes* of $F : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^m$ is the set

$$\mathcal{Z}_F = \{u \in \mathbb{F}_2^n \times \mathbb{F}_2^m, \mathcal{W}_F(u) = 0\} \cup \{0\}.$$

With $\mathcal{V} = \{(x, 0), \forall x \in \mathbb{F}_2^n\} \subset \mathbb{F}_2^{n+m}$, we have $\mathcal{V} \subset \mathcal{Z}_F$.

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Note that if $\Gamma_G = L(\Gamma_F)$, then $\mathcal{Z}_G = (L^T)^{-1}(\mathcal{Z}_F)$.

Admissibility for F

Lemma

Let $L : \mathbb{F}_2^{n+m} \to \mathbb{F}_2^{n+m}$ be a linear permutation. It is admissible for $F : \mathbb{F}_2^n \to \mathbb{F}_2^m$ if and only if

$$L^T(\mathcal{V}) \subseteq \mathcal{Z}_F$$

Admissibility of EA-mappings

EA-mappings are admissible for all $F : \mathbb{F}_2^n \to \mathbb{F}_2^m$:

$$\begin{bmatrix} A & 0 \\ C & B \end{bmatrix}^T (\mathcal{V}) = \begin{bmatrix} A^T & C^T \\ 0 & B^T \end{bmatrix} \left(\left\{ \begin{bmatrix} x \\ 0 \end{bmatrix}, \forall x \in \mathbb{F}_2^n \right\} \right) = \mathcal{V} .$$

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Theorem (Budaghyan, Carlet (2011))

The CCZ-class of a bent function contains only its EA-class.

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$\Longrightarrow$ no zeroes outside of $\mathcal{V}$

$\Longrightarrow$ no vector spaces of zeroes other than $\mathcal{V}$

$\Longrightarrow$ only 1 EA-class

□

Permutations

We define

$$\mathcal{V}^{\perp} = \{(0, y), \forall y \in \mathbb{F}_2^m\} \subset \mathbb{F}_2^{n+m}.$$

Lemma

$F : \mathbb{F}_2^n \to \mathbb{F}_2^m$ is a permutation if and only if

$$\mathcal{V}^{\perp} \subset \mathcal{Z}_F .$$

EA-classes imply vector spaces

Lemma

*let F, G and G' be such that $\Gamma_G = L(\Gamma_F)$ and $\Gamma_{G'} = L'(\Gamma_F)$.
If $L(\mathcal{V}) = L'(\mathcal{V})$, then G and G' are EA-equivalent.*

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Reality takes it back...

The converse of the lemma is wrong.

Counter-example

Let $F : \mathbb{F}_2^n \to \mathbb{F}_2^n$ be a permutation and let

$$M_n = \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix} .$$

It holds that

$$\begin{aligned} \Gamma_{F^{-1}} &= \left\{ (x, F(x)) , \forall x \in \mathbb{F}_2^n \right\} \\ &= \left\{ \left(F^{-1}(y), (F \circ F^{-1})(y)\right) , \forall y \in \mathbb{F}_2^n \right\} \\ &= \left\{ \left(F^{-1}(y), y\right) , \forall y \in \mathbb{F}_2^n \right\} \\ &= M_n(\Gamma_F) . \end{aligned}$$

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The contradiction

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$\implies$ $M_n(\mathcal{V}) = \mathcal{V}^\perp \neq I_n(\mathcal{V})$

... but M_n and I_n send Γ_F in the same EA-class (namely that of F).

Making the converse work (1/2)

Definition (CCZ-invariants)

The CCZ-invariants of $F : \mathbb{F}_2^n \to \mathbb{F}_2^n$ are the affine permutations L of $\mathbb{F}_2^{n+n}$ such that

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Examples

- For an involution, M_n is a CCZ-invariant.
- For a quadratic function q, there are CCZ-invariants with the following linear parts:

$$\begin{bmatrix} I_n & 0 \\ \Delta_\alpha q & I_n \end{bmatrix} .$$

Making the converse work (2/2)

Theorem (Number of EA-classes)

For $F : \mathbb{F}_2^n \to \mathbb{F}_2^m$, let:

- s_F *be the number of vector spaces of dimension* n *in* $\mathcal{Z}_F$
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Corollary

If $c_F = 1$, then we do have a bijection between EA-classes and vector spaces of 0 of dimension n in $\mathcal{Z}_F$.

Outline

Plan of this Section

EA-equivalence is a simple sub-case of CCZ-Equivalence...

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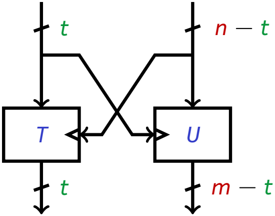
What must we add to EA-equivalence to fully describe CCZ-Equivalence?

Definition of the Twist

Any function $F : \mathbb{F}_2^n \to \mathbb{F}_2^m$ can be projected on $\mathbb{F}_2^t \times \mathbb{F}_2^{m-t}$.



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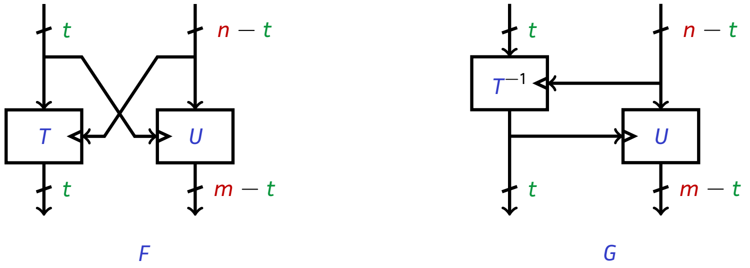

If T is a permutation for all secondary inputs, then we define the t-twist equivalent of F as G, where

$$G(x, y) = \left(T_y^{-1}(x), U_{T_y^{-1}(x)}(y)\right)$$

for all $(x, y) \in \mathbb{F}_2^t \times \mathbb{F}_2^{n-t}$.

Examples of Twisting

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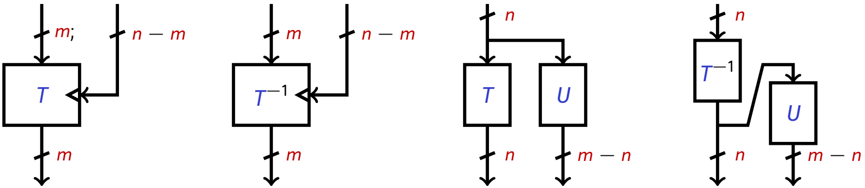

$t = m$ (start) $t = m$ (end) $t = n$ (start) $t = n$ (end)

Swap Matrices

The swap matrix permuting $\mathbb{F}_2^{n+m}$ is defined for $t \leq \min(n, m)$ as

$$M_t = \begin{bmatrix} 0 & 0 & I_t & 0 \\ 0 & I_{n-t} & 0 & 0 \\ I_t & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{m-t} \end{bmatrix} .$$

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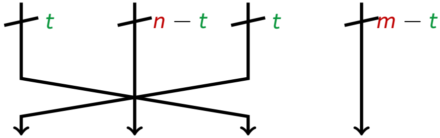

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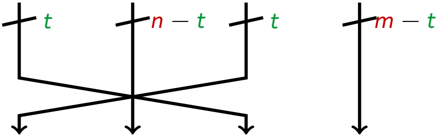
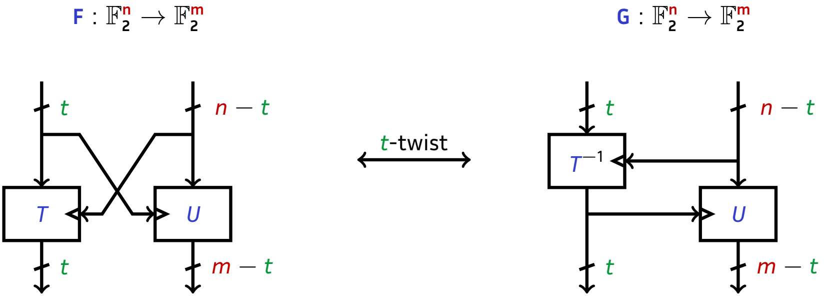


For all $t \leq \min(n, m)$, M_t is an **orthogonal** and **symmetric involution**.

Swap Matrices and Twisting



Swap Matrices and Twisting

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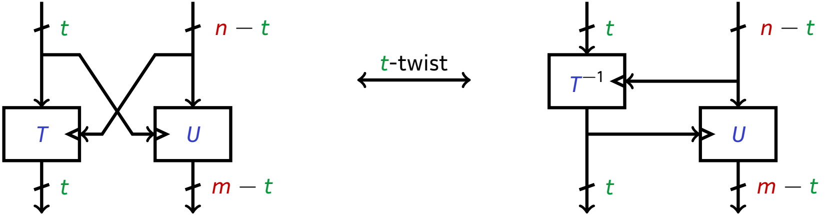

$$\Gamma_F = \{ (x, F(x)) , \forall x \in \mathbb{F}_2^n \} \xleftrightarrow{M_t} \Gamma_G = \{ (x, G(x)) , \forall x \in \mathbb{F}_2^n \}$$

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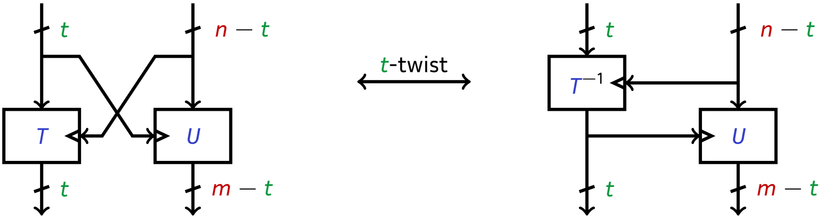

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$$\mathcal{W}_F(u) = \mathcal{W}_G(M_t(u))$$

Twisting and CCZ-Class

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Twisting preserves the CCZ-equivalence class.

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Main Result

Theorem

If $F : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^m$ and $G : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^m$ are CCZ-equivalent, then

$$\Gamma_G = (B \times M_t \times A)(\Gamma_F) ,$$

where A and B are EA-mappings and where

$$t = \dim \left(proj_{\mathcal{V}^\perp} \left((A^T \times M_t \times B^T)(\mathcal{V}) \right) \right) .$$

In other words, EA-equivalence and twists are sufficient to fully describe CCZ-equivalence!

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Corollary

If a function is CCZ-equivalent but not EA-equivalent to another function, then they have to be EA-equivalent to functions for which a t-twist is possible.

Proof sketch

1. As F is CCZ-equivalent to G, there is a linear permutation $L : \mathbb{F}_2^{n+m} \rightarrow \mathbb{F}_2^{n+m}$ such that

$$\Gamma_G = L(\Gamma_F) \text{ and } L^T(\mathcal{V}) \subset \mathcal{Z}_F \ .$$

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1+2+lem. As $L^T(\mathcal{V}) = (A^T \times M_t)(\mathcal{V})$, the functions G and G' such that $\Gamma_G = L(\Gamma_F)$ and $\Gamma_{G'} = (A^T \times M_t)(\Gamma_F)$ are EA-equivalent.
We conclude that
$$\Gamma_G = (B \times M_t \times A)(\Gamma_F) \, .$$

Usage?

What can we do with this knowledge?

Boolean Functions

Theorem (Budaghyan, Carlet (2011))

The CCZ-class of $F : \mathbb{F}_2^n \rightarrow \mathbb{F}_2$ is limited to its EA-class.

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$\implies$ $F(x||y) = T_y(x), \forall (x, y) \in \mathbb{F}_2 \times \mathbb{F}_2^{n-1}$, where T_y is always a permutation of $\mathbb{F}_2$

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$\Longrightarrow$ $F(x||y) = x \oplus f(y), \forall (x, y) \in \mathbb{F}_2 \times \mathbb{F}_2^{n-1}$,

$\Longrightarrow$ 1-twisting F does not change the EA-class

$\Longrightarrow$ it is impossible to leave the EA-class of F

$\square$

Modular Addition (1/2)

Theorem (Schulte-Geers'13)

Addition modulo 2^m is CCZ-equivalent to

$$q(x,y) = \left(0, x_0y_0, x_0y_0 + x_1y_1, ..., x_0y_0 + ... + x_{n2}y_{n2}\right),$$

where $\Gamma_{\boxplus} = L(\Gamma_q)$ *with*

$$L = \begin{bmatrix} I_m & 0 & I_m \\ 0 & I_m & I_m \\ I_m & I_m & I_m \end{bmatrix}.$$

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It holds that

$$L^{-1} = \underbrace{\begin{bmatrix} I_m & 0 & 0 \\ I_m & I_m & 0 \\ I_m & 0 & I_m \end{bmatrix}}_{A_1} \times \underbrace{\begin{bmatrix} 0 & 0 & I_m \\ 0 & I_m & 0 \\ I_m & 0 & 0 \end{bmatrix}}_{M_m} \times \underbrace{\begin{bmatrix} I_m & 0 & 0 \\ I_m & I_m & 0 \\ 0 & I_m & I_m \end{bmatrix}}_{A_2} .$$

Modular Addition (2/2)

Lemma

Let $T_z^{\boxplus} : \mathbb{F}_2^m \to \mathbb{F}_2^m$ be defined by

$$T_z^{\boxplus}(x) = \left(x \boxplus (x \oplus z)\right) \oplus (x \oplus z) .$$

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Let $v = T_z^{\boxplus}(x)$. Then:

$$\begin{cases} v_0 & = x_0 \\ v_{i+1} & = x_i + x_{i+1} + v_i z_i \end{cases} \text{ and, convertly, } \begin{cases} x_0 & = v_0 \\ x_{i+1} & = x_i + v_{i+1} + v_i z_i\,. \end{cases}$$

Outline

Plan of this Section

Another Problem

How do we know if a function is CCZ-equivalent to a permutation?

Remainder

Recall that F is a permutation if and only if $\mathcal{V} \subset \mathcal{Z}_F$ and $\mathcal{V}^\perp \subset \mathcal{Z}_F$.

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Lemma

G is CCZ-equivalent to a permutation if and only if

$$V = L(\mathcal{V}) \subset \mathcal{Z}_G \text{ and } V' = L(\mathcal{V}^{\perp}) \subset \mathcal{Z}_G$$

for some linear permutation L. Note that

$$span\left(V \cup V'\right) = \mathbb{F}_2^n \times \mathbb{F}_2^m .$$

3-Spaces Criteria

3-space criteria

Let $F : \mathbb{F}_2^n \to \mathbb{F}_2^m$, not be a permutation. If it is CCZ-equivalent to a permutation then $\mathcal{Z}_F$ must contain at least 3 vector spaces of zeroes of dimension n.

Projected Spaces Criteria

Key observation

The projections

$$p : (x, y) \mapsto x \text{ and } p' : (x, y) \mapsto y$$

mapping $\mathbb{F}_2^n \times \mathbb{F}_2^m$ to $\mathbb{F}_2^n$ and $\mathbb{F}_2^m$ respectively are **linear**.

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Thus, If G is CCZ-equivalent to a permutation then $p(V)$ and $p(V')$ are subspaces of $\mathbb{F}_2^n$ whose span is $\mathbb{F}_2^n$.

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Projected Spaces Criteria

If $F : \mathbb{F}_2^n \to \mathbb{F}_2^m$ is CCZ-equivalent to a permutation, then there are at least two subspaces of dimension $n/2$ in $p(\mathcal{Z}_F)$ and in $p'(\mathcal{Z}_F)$.

QAM

Yu et al. (DCC'14) generated 8180 8-APN quadratic functions from *"QAM"* (matrices).

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None of them are CCZ-equivalent to a permutation

Göloğlu's Candidates (1/2)

Göloğlu's introduced APN functions

$$f_k : x \mapsto x^{2^k+1} + (x + x^{2^{n/2}})^{2^k+1}$$

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> *Unfortunately, f_k are not equivalent to permutations on $n = 4, 8$ and does not **seem** to be equivalent to one on $n = 12$ (we say "it does not seem to be equivalent to a permutation" since checking the existence of CCZ-equivalent permutations **requires huge amount of computing** and is infeasible on $n = 12$; our program was still running at the time of writing).*

Göloğlu's Candidates (2/2)

n	cardinal proj.	time proj. (s)	time `BasesExtraction` (s)
12	1365	0.066	0.0012
16	21845	16.79	0.084
20	349525	10096.00	37.48

Time needed to show that f_k is **not** CCZ-equivalent to a permutation.

Outline

Plan of this Section

Conclusion

- CCZ $=$ EA $+$ Twist, both of which have a simple interpretation.

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- CCZ $=$ EA $+$ Twist, both of which have a simple interpretation.
- Efficient criteria to know if a function is CCZ-equivalent to a permutation...
- ... implemented using a very efficient vector space extraction algorithm (not presented)

The Fourier transform solves everything!

Open Problems

EA-equivalence

How can we efficiently check the EA-equivalence of two functions?

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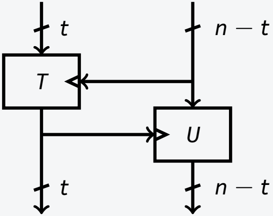
Conjecture

If the CCZ-class of a permutation P is not reduced to the EA-classes of P and P^{-1}, then P has the following decomposition



where both T and U are keyed permutations.