



Secure building-blocks against differential and linear attacks

Anne Canteaut

► **To cite this version:**

| Anne Canteaut. Secure building-blocks against differential and linear attacks. 2018. hal-01955315

HAL Id: hal-01955315

<https://hal.inria.fr/hal-01955315>

Submitted on 14 Dec 2018

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Secure building-blocks against differential and linear attacks

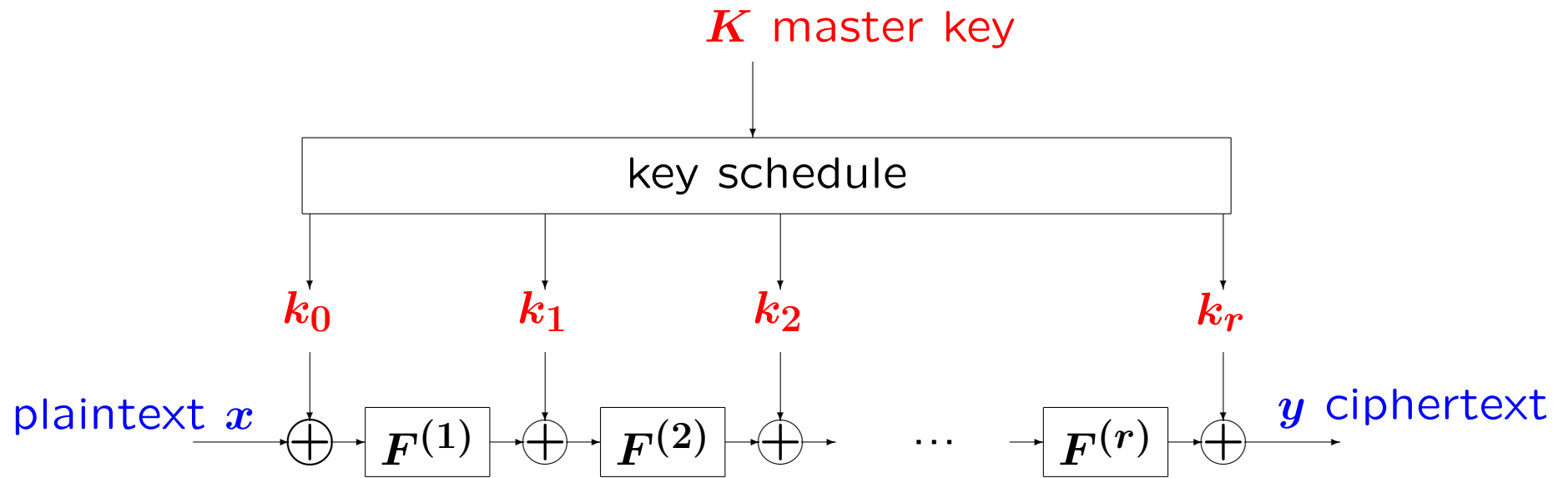
Anne Canteaut

`Anne.Canteaut@inria.fr`

`https://www.paris.inria.fr/secret/Anne.Canteaut/`

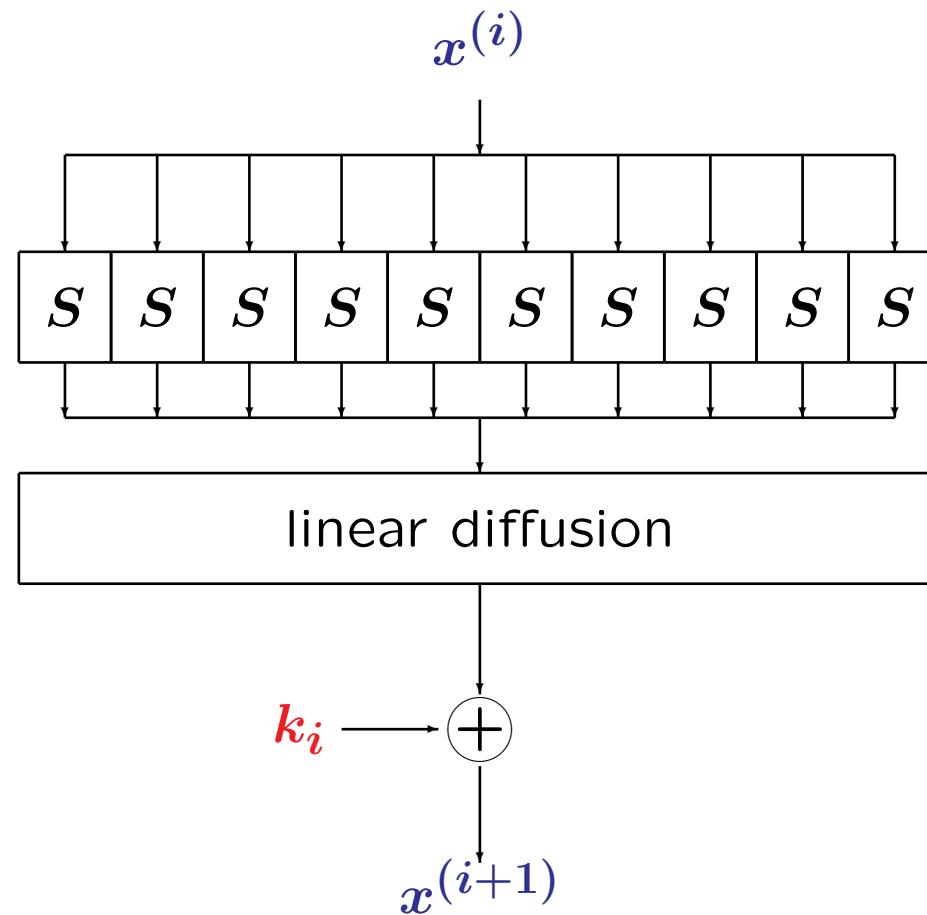
COST Training School, Torremolinos, February 2018

Key-alternating block ciphers



where each $F^{(i)}$ is a permutation of \mathbb{F}_2^n .

Round function in a substitution-permutation network



Outline

- Representations of Sboxes
- Linear approximations of a Boolean function and Walsh transform
- Resistance to differential attacks
- Finding good Sboxes
- Security criteria for the linear layer

Representations of Sboxes

Boolean functions

Definition. A **Boolean function of n variables** is a function from \mathbb{F}_2^n into \mathbb{F}_2 .

Truth table of a Boolean function.

x_1	0	1	0	1	0	1	0	1
x_2	0	0	1	1	0	0	1	1
x_3	0	0	0	0	1	1	1	1
$f(x_1, x_2, x_3)$	0	1	0	0	0	1	1	1

Value vector of f : word of 2^n bits corresponding to all $f(x), x \in \mathbb{F}_2^n$.

Vectorial Boolean functions

Definition. A **vectorial Boolean function** with n inputs and m outputs is a function from \mathbb{F}_2^n into \mathbb{F}_2^m :

$$S : \mathbb{F}_2^n \longrightarrow \mathbb{F}_2^m$$

$$(x_1, \dots, x_n) \longmapsto (y_1, \dots, y_m)$$

Each function

$$S_i : (x_1, \dots, x_n) \longmapsto y_i$$

is called a **coordinate** of S .

Example.

x	0	1	2	3	4	5	6	7	8	9	a	b	c	d	e	f
$S(x)$	f	e	b	c	6	d	7	8	0	3	9	a	4	2	1	5
$S_1(x)$	1	0	1	0	0	1	1	0	0	1	1	0	0	0	1	1
$S_2(x)$	1	1	1	0	1	0	1	0	0	1	0	1	0	1	0	0
$S_3(x)$	1	1	0	1	1	1	1	0	0	0	0	0	1	0	0	1
$S_4(x)$	1	1	1	1	0	1	0	1	0	0	1	1	0	0	0	0

Hamming weight of a Boolean function

Hamming weight of a Boolean function.

The Hamming weight of a Boolean function f , $wt(f)$, is the Hamming weight of its value vector.

A function of n variables is **balanced** if and only if $wt(f) = 2^{n-1}$.

Proposition. A vectorial function S with n inputs and n outputs is a permutation if and only if any nonzero linear combination of its coordinates

$$x \mapsto \bigoplus_{i=1}^n \lambda_i S_i(x), \quad \lambda = (\lambda_1, \dots, \lambda_n) \neq 0$$

is a balanced Boolean function.

Algebraic normal form (ANF)

Monomials in $\mathbb{F}_2[x_1, \dots, x_n]/(x_1^2 + x_1, \dots, x_n^2 + x_n)$:

$$\{x^u, u \in \mathbb{F}_2^n\} \text{ where } x^u = \prod_{i=1}^n x_i^{u_i}.$$

Example: $x^{1011} = x_1^1 x_2^0 x_3^1 x_4^1 = x_1 x_3 x_4$.

Proposition.

Any Boolean function of n variables has a **unique polynomial representation** in $\mathbb{F}_2[x_1, \dots, x_n]/(x_1^2 + x_1, \dots, x_n^2 + x_n)$:

$$f(x_1, \dots, x_n) = \sum_{u \in \mathbb{F}_2^n} a_u x^u, \quad a_u \in \mathbb{F}_2.$$

Moreover, the coefficients of the ANF and the values of f satisfy:

$$a_u = \bigoplus_{x \preceq u} f(x) \text{ and } f(u) = \bigoplus_{x \preceq u} a_x,$$

where $x \preceq y$ if and only if $x_i \leq y_i$ for all $1 \leq i \leq n$.

Example

x_1	0	1	0	1	0	1	0	1
x_2	0	0	1	1	0	0	1	1
x_3	0	0	0	0	1	1	1	1
$f(x_1, x_2, x_3)$	0	1	0	0	0	1	1	1

$$a_{000} = f(000) = 0$$

$$a_{100} = f(100) \oplus f(000) = 1$$

$$a_{010} = f(010) \oplus f(000) = 0$$

$$a_{110} = f(110) \oplus f(010) \oplus f(100) \oplus f(000) = 1$$

$$a_{001} = f(001) \oplus f(000) = 0$$

$$a_{101} = f(101) \oplus f(001) \oplus f(100) \oplus f(000) = 0$$

$$a_{011} = f(011) \oplus f(001) \oplus f(010) \oplus f(000) = 1$$

$$a_{111} = \bigoplus_{x \in \mathbb{F}_2^3} f(x) = wt(f) \bmod 2 = 0$$

$$f = x_1 \oplus x_1x_2 \oplus x_2x_3.$$

Computing the ANF

$n = 3$:

0	1	2	3	4	5	6	7
$f(0)$	$f(1)$	$f(2)$	$f(3)$	$f(4)$	$f(5)$	$f(6)$	$f(7)$
$f(0)$	$f(0) \oplus f(1)$	$f(2)$	$f(2) \oplus f(3)$	$f(4)$	$f(4) \oplus f(5)$	$f(6)$	$f(6) \oplus f(7)$
$f(0)$	$f(0) \oplus f(1)$	$f(0) \oplus f(2)$	$f(0) \oplus f(1) \oplus f(2) \oplus f(3)$	$f(4)$	$f(4) \oplus f(5)$	$f(4) \oplus f(6)$	$f(4) \oplus f(5) \oplus f(6) \oplus f(7)$
$f(0)$	$f(0) \oplus f(1)$	$f(0) \oplus f(2)$	$f(0) \oplus f(1) \oplus f(2) \oplus f(3)$	$f(0) \oplus f(4)$	$f(0) \oplus f(1) \oplus f(4) \oplus f(5)$	$f(0) \oplus f(2) \oplus f(4) \oplus f(6)$	$f(0) \oplus f(1) \oplus f(2) \oplus f(3) \oplus f(4) \oplus f(5) \oplus f(6) \oplus f(7)$

first step:

$$f(2i + 1) \leftarrow f(2i + 1) \oplus f(2i)$$

second step:

$$f(4i + j + 2) \leftarrow f(4i + j + 2) \oplus f(4i + j), \quad \forall 0 \leq j < 2$$

third step:

$$f(8i + j + 4) \leftarrow f(8i + j + 4) \oplus f(8i + j), \quad \forall 0 \leq j < 4$$

Computing the ANF

When the value vector is stored as a 32-bit integer x :

```
x ^= (x & 0x55555555) << 1;
```

```
x ^= (x & 0x33333333) << 2;
```

```
x ^= (x & 0x0f0f0f0f) << 4;
```

```
x ^= (x & 0x00ff00ff) << 8;
```

```
x ^= x << 16;
```

Degree of a Boolean function

Definition.

The **degree** of a Boolean function is the degree of the largest monomial in its ANF.

Proposition.

The weight of an n -variable function f is odd if and only if $\deg f = n$.

Definition.

The degree of a vectorial function S with n inputs and m outputs is the maximal degree of its coordinates.

Example

x	0	1	2	3	4	5	6	7	8	9	a	b	c	d	e	f
$S(x)$	f	e	b	c	6	d	7	8	0	3	9	a	4	2	1	5
$S_1(x)$	1	0	1	0	0	1	1	0	0	1	1	0	0	0	1	1
$S_2(x)$	1	1	1	0	1	0	1	0	0	1	0	1	0	1	0	0
$S_3(x)$	1	1	0	1	1	1	1	0	0	0	0	0	1	0	0	1
$S_4(x)$	1	1	1	1	0	1	0	1	0	0	1	1	0	0	0	0

$$S_1 = 1 + x_1 + x_3 + x_2x_3 + x_4 + x_2x_4 + x_3x_4 + x_1x_3x_4 + x_2x_3x_4$$

$$S_2 = 1 + x_1x_2 + x_1x_3 + x_1x_2x_3 + x_4 + x_1x_4 + x_1x_2x_4 + x_1x_3x_4$$

$$S_3 = 1 + x_2 + x_1x_2 + x_2x_3 + x_4 + x_2x_4 + x_1x_2x_4 + x_3x_4 + x_1x_3x_4$$

$$S_4 = 1 + x_3 + x_1x_3 + x_4 + x_2x_4 + x_3x_4 + x_1x_3x_4 + x_2x_3x_4$$

Identifying \mathbb{F}_2^n with a finite field

\mathbb{F}_2^n is identified with the finite field with 2^n elements.

$$\mathbb{F}_{2^n} = \{0\} \cup \{\alpha^i, 0 \leq i \leq 2^n - 2\}$$

where α is a root of a primitive polynomial of degree n .

$$\Rightarrow \text{for any } i, \alpha^i = \sum_{j=0}^{n-1} \lambda_j \alpha^j$$

Example for $n = 4$:

primitive polynomial: $1 + x + x^4$, α a root of this polynomial.

\mathbb{F}_{2^4}	0	1	α	α^2	α^3	α^4	α^5	α^6	α^7
	0	1	α	α^2	α^3	$\alpha + 1$	$\alpha^2 + \alpha$	$\alpha^3 + \alpha^2$	$\alpha^3 + \alpha + 1$
\mathbb{F}_2^4	0000	0001	0010	0100	1000	0011	0110	1100	1011

α^8	α^9	α^{10}	α^{11}	α^{12}	α^{13}	α^{14}
$\alpha^2 + 1$	$\alpha^3 + \alpha$	$\alpha^2 + \alpha + 1$	$\alpha^3 + \alpha^2 + \alpha$	$\alpha^3 + \alpha^2 + \alpha + 1$	$\alpha^3 + \alpha^2 + 1$	$\alpha^3 + 1$
0101	1010	0111	1110	1111	1101	1001

The univariate representation of Sboxes

Any vectorial function with n inputs and n outputs can be seen as

$$S : \mathbb{F}_{2^n} \longrightarrow \mathbb{F}_{2^n}$$

Then,

$$S(X) = \sum_{i=0}^{2^n-1} c_i X^i, c_i \in \mathbb{F}_{2^n}.$$

Example:

x	0	1	2	3	4	5	6	7	8	9	a	b	c	d	e	f
$S(x)$	f	e	b	c	6	d	7	8	0	3	9	a	4	2	1	5

$$S(X) = \alpha^{12} + \alpha^2 X + \alpha^{13} X^2 + \alpha^6 X^3 + \alpha^{10} X^4 + \alpha X^5 + \alpha^{10} X^6 + \alpha^2 X^7 + \alpha^9 X^8 + \alpha^4 X^9 + \alpha^7 X^{10} + \alpha^7 X^{11} + \alpha^5 X^{12} + X^{13} + \alpha^6 X^{14}$$

Linear approximations of a function and Walsh transform

Linear attacks [Matsui 93]

Idea.

Use linear relations between the input and output bits of the cipher which hold with probability significantly greater or significantly less than $1/2$.

$a \in \mathbb{F}_2^n$: input mask

$b \in \mathbb{F}_2^n$: output mask

$$\left| \Pr_x [a \cdot x \oplus b \cdot E_k(x) = 0] - \frac{1}{2} \right|$$

For our 4-bit Sbox.

$$x_1 \oplus x_4 \oplus S_2(x) = 0x9 \cdot x \oplus 0x2 \cdot S(x)$$

equals 0 with probability $\frac{1}{8}$.

Computing the probabilities of all linear relations

Bias of a Boolean function.

For any Boolean function f of n variables

$$\mathcal{E}(f) = \sum_{x \in \mathbb{F}_2^n} (-1)^{f(x)} = 2^n - 2wt(f).$$

Equivalently,

$$\Pr[f(x) = 0] = \frac{wt(f)}{2^n} = \frac{1}{2} \left(1 - \frac{\mathcal{E}(f)}{2^n} \right).$$

→ we need to compute the biases of all Boolean functions

$$x \longmapsto b \cdot S(x) + a \cdot x .$$

Linear approximations of an Sbox

$a \setminus b$	1	2	3	4	5	6	7	8	9	a	b	c	d	e	f
1	-4	.	4	.	-4	8	-4	4	8	4	.	-4	.	4	.
2	4	-4	.	-4	.	.	4	4	8	.	4	8	-4	-4	.
3	8	4	4	-4	4	4	-4	-4	-4	.	8
4	.	-4	4	4	-4	.	.	-8	.	4	4	4	4	.	8
5	-4	4	.	4	8	.	4	-4	8	.	-4	.	4	-4	.
6	-4	.	4	.	4	8	4	4	-8	4	.	4	.	-4	.
7	.	.	.	8	.	-8	8	.	8	.	.
8	.	-4	4	-8	.	4	4	-8	.	-4	-4	.	.	4	-4
9	-4	-12	.	.	4	-4	.	4	.	.	-4	-4	.	.	4
a	-4	.	-12	-4	.	4	.	-4	.	4	.	.	-4	.	4
b	.	.	.	4	-4	4	-4	.	.	-8	-8	4	-4	-4	4
c	.	.	.	-4	-4	-4	-4	.	.	8	-8	4	4	-4	-4
d	-4	.	4	4	.	-4	.	-4	.	4	.	.	-12	.	-4
e	4	-4	.	.	4	4	-8	-4	.	.	4	-4	.	-8	-4
f	-8	4	4	-8	.	-4	-4	.	.	-4	4	.	.	-4	4

$$\Pr[a \cdot x + b \cdot S(x) = 0] = \frac{1}{2} \left(1 + \frac{\mathcal{E}(a, b)}{2^n} \right)$$

For instance, for $a = 0x9$ and $b = 0x2$, we have $p = \frac{1}{2} \left(1 - \frac{12}{16} \right) = \frac{1}{8}$.

Walsh transform of a Boolean function

Walsh transform of a Boolean function f of n variables

$$\begin{aligned} \mathbb{F}_2^n &\longrightarrow \mathbb{Z} \\ a &\longmapsto \mathcal{E}(f + \ell_a) = \sum_{x \in \mathbb{F}_2^n} (-1)^{f(x) + a \cdot x} \end{aligned}$$

where $\ell_a : x \longmapsto a \cdot x$

Walsh transform of a vectorial function S :

$$\begin{aligned} \mathbb{F}_2^n \times \mathbb{F}_2^m &\longrightarrow \mathbb{Z} \\ (a, b) &\longmapsto \mathcal{E}(b \cdot S + \ell_a) = \sum_{x \in \mathbb{F}_2^n} (-1)^{b \cdot S(x) + a \cdot x} \end{aligned}$$

Computing the Walsh transform

$f(x)$	0	1	0	0	0	1	1	1
$T(x) = (-1)^{f(x)}$	1	-1	1	1	1	-1	-1	-1
step 1	0	2	2	0	0	2	-2	0
step 2	2	2	-2	2	-2	2	2	2
$\mathcal{E}(f + \ell_a)$	0	4	0	4	4	0	-4	0

first step:

$$T(2i) \leftarrow T(2i) + T(2i + 1)$$

$$T(2i + 1) \leftarrow T(2i) - T(2i + 1)$$

second step:

$$T(4i + j) \leftarrow T(4i + j) + T(4i + j + 2), \quad \forall 0 \leq j < 2$$

$$T(4i + j + 2) \leftarrow T(4i + j) - T(4i + j + 2), \quad \forall 0 \leq j < 2$$

third step:

$$T(8i + j) \leftarrow T(8i + j) + T(8i + j + 4), \quad \forall 0 \leq j < 4$$

$$T(8i + j + 4) \leftarrow T(8i + j) - T(8i + j + 4), \quad \forall 0 \leq j < 4$$

Complexity : $n2^n$ operations.

Some basic properties of the Walsh transform

Lemma:

$$\mathcal{E}(\ell_a) = \sum_{x \in \mathbb{F}_2^n} (-1)^{a \cdot x} = \begin{cases} 2^n & \text{if } a = 0 \\ 0 & \text{otherwise} \end{cases}.$$

Proposition. The Walsh transform is an **involution** (up to a multiplicative constant): for any $x \in \mathbb{F}_2^n$,

$$\begin{aligned} \sum_{a \in \mathbb{F}_2^n} \mathcal{E}(f + \ell_a) (-1)^{a \cdot x} &= \sum_{a \in \mathbb{F}_2^n} \sum_{u \in \mathbb{F}_2^n} (-1)^{f(u) + a \cdot u + a \cdot x} \\ &= \sum_{u \in \mathbb{F}_2^n} (-1)^{f(u)} \sum_{a \in \mathbb{F}_2^n} (-1)^{a \cdot (x+u)} \\ &= 2^n (-1)^{f(x)} \end{aligned}$$

Some basic properties of the Walsh transform

Parseval equality.

$$\sum_{a \in \mathbb{F}_2^n} \mathcal{E}^2(f + \ell_a) = 2^{2n}.$$

Proof.

$$\begin{aligned} \sum_{a \in \mathbb{F}_2^n} \mathcal{E}^2(f + \ell_a) &= \sum_{a \in \mathbb{F}_2^n} \left(\sum_{x \in \mathbb{F}_2^n} (-1)^{f(x) + a \cdot x} \right) \left(\sum_{y \in \mathbb{F}_2^n} (-1)^{f(y) + a \cdot y} \right) \\ &= \sum_{x \in \mathbb{F}_2^n} \sum_{y \in \mathbb{F}_2^n} (-1)^{f(x) + f(y)} \sum_{a \in \mathbb{F}_2^n} (-1)^{a \cdot (x+y)} \\ &= 2^n \sum_{x \in \mathbb{F}_2^n} (-1)^{f(x) + f(x)} \\ &= 2^{2n}. \end{aligned}$$

[Check it on each column of the table on Slide 19]

Linearity of a Boolean function

Definition. For any Boolean function f of n variables,

$$\mathcal{L}(f) = \max_a |\mathcal{E}(f + \ell_a)|$$

is called the **linearity** of f (highest bias for an affine approximation).

$$\mathcal{NL}(f) = 2^{n-1} - \frac{1}{2}\mathcal{L}(f)$$

is called the **nonlinearity** of f (distance of f to the affine functions).

Can we say something about $\mathcal{L}(f)$?

$$\mathcal{L}(f) = \max_a |\mathcal{E}(f + \ell_a)|$$

Theorem. [Rothaus 76] For any Boolean function with n variables,

$$\mathcal{L}(f) \geq 2^{\frac{n}{2}},$$

with equality for even n only. The functions achieving this bound are called **bent functions**. They are not balanced.

Proof. From Parseval equality:

$$2^{2n} = \sum_{a \in \mathbb{F}_2^n} \mathcal{E}^2(f + \ell_a) \leq \max_{a \in \mathbb{F}_2^n} \mathcal{E}^2(f + \ell_a) \times 2^n = 2^n \mathcal{L}^2(f)$$

with equality if and only if all $\mathcal{E}^2(f + \ell_a)$ are equal.

Then, $\mathcal{L}(f) \geq 2^{\frac{n}{2}}$ with equality if and only if

$$\mathcal{E}(f + \ell_a) = \pm 2^{\frac{n}{2}}, \quad \forall a \in \mathbb{F}_2^n .$$

Can we say something about $\mathcal{L}(f)$?

What is the lowest possible value for $\mathcal{L}(f)$ when n is odd?
When f is balanced?

Functions of degree 2.

For n odd, $n = 2t + 1$

$$x_1x_2 \oplus x_3x_4 \oplus \dots \oplus x_{2t-1}x_{2t} \oplus x_{2t+1}$$

satisfies $\mathcal{L}(f) = 2^{\frac{n+1}{2}}$. Moreover, f is balanced and

$$\forall a \in \mathbb{F}_2^n, \mathcal{E}(f + \ell_a) \in \{0, \pm 2^{\frac{n+1}{2}}\}.$$

Theorem.

$$2^{\frac{n}{2}} \leq \min_{f \in \mathcal{B}ool_n} \mathcal{L}(f) \leq 2^{\frac{n+1}{2}}$$

Boolean functions with a low linearity

n	$\min_{f \in \mathcal{B}ool_n} \mathcal{L}(f)$	
5	8	[Berlekamp-Welch 72]
7	16	[Mykkelveit 80]
9	24, 26, 28, 30	[Kavut-Maitra-Yücel 06]
11	46-60	
13	92-120	
15	182-216	[Paterson-Wiedemann 83]

Open problem. Find the lowest possible linearity for a Boolean function of n variables, where n is odd and $n \geq 9$.

Balanced Boolean functions with a low linearity

n	$\min_{f \in \mathcal{Bal}_n} \mathcal{L}(f)$
4	8
5	8
6	12
7	16
8	20, 24
9	24, 28, 32
10	36, 40

Open problem. Find the highest possible nonlinearity for a balanced Boolean function of n variables, where n is even and $n \geq 8$.

Proposition. [Katz 71] If f is balanced, all values $\mathcal{E}(f + \ell_a)$ are divisible by $2^{\lceil \frac{n-1}{\deg f} \rceil + 1}$, i.e., at least by 4 (and by 8 if $\deg f < n - 1$).

Linearity of an Sbox

Criterion on the Sbox.

All linear approximations of S should have a small bias, *i.e.*,

$$\mathcal{L}(S) = \max_{a \in \mathbb{F}_2^n, b \in \mathbb{F}_2^n, b \neq 0} |\mathcal{E}(b \cdot S + \ell_a)| = \max_{b \neq 0} \mathcal{L}(b \cdot S)$$

must be as small as possible.

$$\mathcal{NL}(S) = 2^{n-1} - \frac{1}{2} \mathcal{L}(S)$$

is called the **nonlinearity** of S .

Sboxes with a low linearity

What is the lowest possible value for $\mathcal{L}(S)$ when S is a vectorial function with n inputs and n outputs?

Theorem. [Chabaud-Vaudenay 94] For any function S with n inputs and n outputs,

$$\mathcal{L}(S) \geq 2^{\frac{n+1}{2}},$$

with equality for odd n only. The functions achieving this bound are called **almost bent functions**.

For n even.

There exist Sboxes with

$$\mathcal{L}(S) = 2^{\frac{n+2}{2}}$$

but we do not know if this value is minimal.

Resistance to differential attacks

Difference distribution table of an Sbox

$a \setminus b$	1	2	3	4	5	6	7	8	9	a	b	c	d	e	f
1	2	0	4	2	0	2	2	0	0	0	2	0	0	0	2
2	2	2	0	2	4	0	2	0	4	0	0	0	0	0	0
3	2	0	4	0	2	0	0	0	0	6	0	0	0	2	0
4	2	0	2	4	0	0	0	2	2	0	0	2	0	0	2
5	0	4	2	0	0	0	2	2	0	0	4	2	0	0	0
6	4	0	0	0	0	4	0	4	0	0	0	0	4	0	0
7	0	2	0	0	2	2	2	0	2	2	2	0	0	2	0
8	0	4	0	0	0	4	0	0	0	0	0	0	4	0	4
9	2	2	0	2	2	0	0	0	4	0	0	2	0	2	0
a	0	0	2	2	0	2	2	2	0	2	2	0	0	0	2
b	0	0	2	0	4	0	2	2	0	0	0	6	0	0	0
c	0	2	0	0	0	2	0	0	2	2	2	2	0	4	0
d	2	0	0	0	2	0	0	0	0	2	0	0	8	2	0
e	0	0	0	0	0	0	4	0	0	0	4	0	0	4	4
f	0	0	0	4	0	0	0	4	2	2	0	2	0	0	2

$$\delta_S(a, b) = \#\{X \in \mathbb{F}_2^n, S(X \oplus a) \oplus S(X) = b\}$$

Resistance to differential attacks

Criterion on the Sbox. [Nyberg-Knudsen 92]

All entries in the difference table of S should be small.

$$\delta(S) = \max_{a,b \neq 0} \#\{X \in \mathbb{F}_2^n, S(X \oplus a) \oplus S(X) = b\}$$

must be as small as possible.

$\delta(S)$ is called the **differential uniformity** of S (always even).

Theorem. For any Sbox S with n inputs and n outputs,

$$\delta(S) \geq 2.$$

The functions achieving this bound are called **almost perfect nonlinear functions (APN)**.

Link between the difference and square correlation tables

Theorem. [Chabaud Vaudenay 94][Blondeau Nyberg 13]

There is a one-to-one correspondence between the DDT

$$\delta(a, b), a, b \in \mathbb{F}_2^n$$

and the squared LAT

$$\mathcal{E}^2(a, b), a, b \in \mathbb{F}_2^n$$

$$\mathcal{E}^2(u, v) = \sum_{a, b \in \mathbb{F}_2^n} (-1)^{a \cdot u + b \cdot v} \delta(a, b)$$

$$\delta(a, b) = 2^{-2n} \sum_{u, v \in \mathbb{F}_2^n} (-1)^{a \cdot u + b \cdot v} \mathcal{E}^2(u, v)$$

There is a one-to-one correspondence between the Sbox and the LAT.

But several Sboxes may have the same **squared LAT**.

Finding good Sboxes

Affine equivalence between Sboxes

S_1 and S_2 are **affinely equivalent** if there exist two affine permutations of \mathbb{F}_2^n , A_1 and A_2 , such that

$$S_2 = A_2 \circ S_1 \circ A_1$$

Then,

$$\delta(S_2) = \delta(S_1) \text{ and } \mathcal{L}(S_2) = \mathcal{L}(S_1)$$

Permutations of \mathbb{F}_2^n , n odd

$$\mathcal{L}(S) \geq 2^{\frac{n+1}{2}} \text{ and } \delta(S) \geq 2$$

- Any AB Sbox (i.e., with $\mathcal{L}(S) = 2^{\frac{n+1}{2}}$) is APN [Chabaud-Vaudenay 94].
- The converse holds for some specific cases only, e.g for quadratic APN Sboxes [Carlet-Charpin-Zinoviev 98]
- AB functions over \mathbb{F}_2^n have degree at most $\frac{n+1}{2}$.

Known AB permutations of F_2^n , n odd

Monomials permutations $S(x) = x^s$ over F_{2^n} .

quadratic	$2^i + 1$ with $\gcd(i, n) = 1$, $1 \leq i \leq t$	[Gold 68],[Nyberg 93]
Kasami	$2^{2i} - 2^i + 1$ with $\gcd(i, n) = 1$ $2 \leq i \leq t$	[Kasami 71]
Welch	$2^t + 3$	[Dobbertin 98] [C.-Charpin-Dobbertin 00]
Niho	$2^t + 2^{\frac{t}{2}} - 1$ if t is even $2^t + 2^{\frac{3t+1}{2}} - 1$ if t is odd	[Dobbertin 98] [Xiang-Hollmann 01]

Non-monomial permutations. [Budaghyan-Carlet-Leander08]

For n odd, divisible by 3 and not by 9.

$$S(x) = x^{2^i+1} + ux^{2^j\frac{n}{3}+2^{(3-j)\frac{n}{3}+i}} \text{ with } \gcd(i, n) = 1 \text{ and } j = i\frac{n}{3} \pmod{3}$$

Permutations of F_2^4

$$\delta(S) \geq 4 \text{ and } \mathcal{L}(S) \geq 8$$

16 classes of optimal Sboxes [Leander-Poschmann 07]

	0	1	2	3	4	5	6	7	8	9	a	b	c	d	e	f
G_0	0	1	2	13	4	7	15	6	8	11	12	9	3	14	10	5
G_1	0	1	2	13	4	7	15	6	8	11	14	3	5	9	10	12
G_2	0	1	2	13	4	7	15	6	8	11	14	3	10	12	5	9
G_3	0	1	2	13	4	7	15	6	8	12	5	3	10	14	11	9
G_4	0	1	2	13	4	7	15	6	8	12	9	11	10	14	5	3
G_5	0	1	2	13	4	7	15	6	8	12	11	9	10	14	3	5
G_6	0	1	2	13	4	7	15	6	8	12	11	9	10	14	5	3
G_7	0	1	2	13	4	7	15	6	8	12	14	11	10	9	3	5
G_8	0	1	2	13	4	7	15	6	8	14	9	5	10	11	3	12
G_9	0	1	2	13	4	7	15	6	8	14	11	3	5	9	10	12
G_{10}	0	1	2	13	4	7	15	6	8	14	11	5	10	9	3	12
G_{11}	0	1	2	13	4	7	15	6	8	14	11	10	5	9	12	3
G_{12}	0	1	2	13	4	7	15	6	8	14	11	10	9	3	12	5
G_{13}	0	1	2	13	4	7	15	6	8	14	12	9	5	11	10	3
G_{14}	0	1	2	13	4	7	15	6	8	14	12	11	3	9	5	10
G_{15}	0	1	2	13	4	7	15	6	8	14	12	11	9	3	10	5

Permutations of F_2^6

$$\delta(S) \geq 2 \text{ and } \mathcal{L}(S) \geq 12$$

$S = \{0, 54, 48, 13, 15, 18, 53, 35, 25, 63, 45, 52, 3, 20, 41, 33, 59, 36, 2, 34, 10, 8, 57, 37, 60, 19, 42, 14, 50, 26, 58, 24, 39, 27, 21, 17, 16, 29, 1, 62, 47, 40, 51, 56, 7, 43, 44, 38, 31, 11, 4, 28, 61, 46, 5, 49, 9, 6, 23, 32, 30, 12, 55, 22\}$;

satisfies

$$\delta(S) = 2, \text{ deg } S = 4 \text{ and } \mathcal{L}(S) = 16 \text{ [Dillon 09]}$$

The corresponding univariate polynomial over F_{2^6} contains 52 nonzero monomials (out of 56 possible monomials of degree at most 4).

This is the only known APN permutation with an even number of variables.

Good permutations of F_2^n , n even

Usually, we search for permutations S with

$$\delta(S) = 4 \text{ and } \mathcal{L}(S) = 2^{\frac{n+2}{2}}.$$

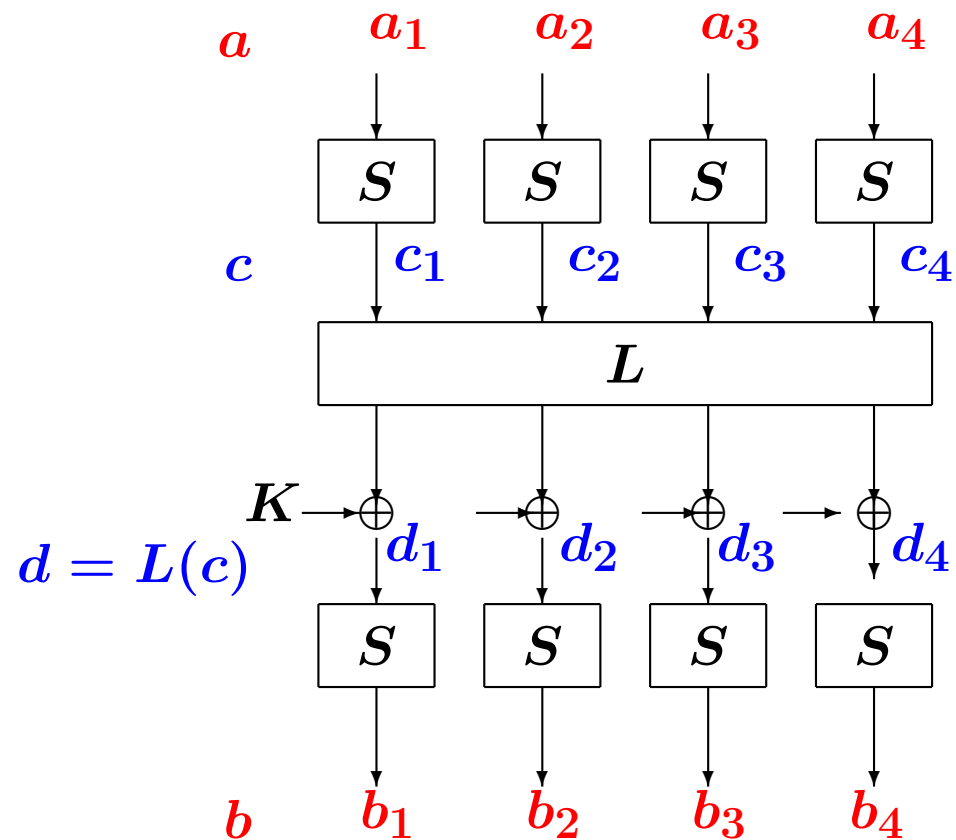
Monomials permutations $S(x) = x^s$ over F_{2^n} .

$2^i + 1, \gcd(i, n) = 2$	$n \equiv 2 \pmod{4}$	[Gold 68]
$2^{2i} - 2^i + 1, \gcd(i, n) = 2$	$n \equiv 2 \pmod{4}$	[Kasami 71]
$2^{\frac{n}{2}} + 2^{\frac{n}{4}} + 1$	$n \equiv 4 \pmod{8}$	[Bracken-Leander 10]
$2^n - 2$		[Lachaud-Wolfmann 90]

The last one is affinely equivalent to the AES Sbox.

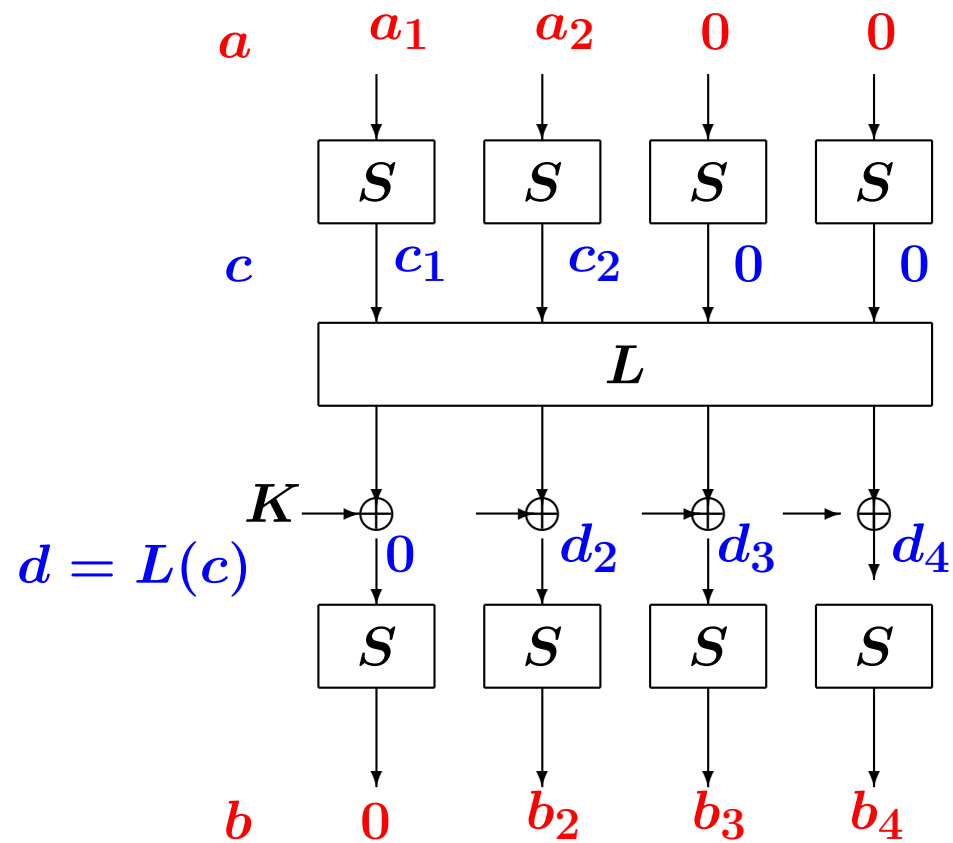
Security criteria for the linear layer

A two-round characteristic



$$\text{EDP}((a, c, L(c), b)) = \prod_{i=1}^t \frac{\delta_S(a_i, c_i)}{2^m} \prod_{j=1}^t \frac{\delta_S(L(c)_j, b_j)}{2^m}$$

A two-round characteristic



$$\text{EDP}((a, c, L(c), b)) \leq \left(2^{-m} \delta(S)\right)^{\text{wt}(c) + \text{wt}(L(c))} \leq \left(2^{-m} \delta(S)\right)^d$$

Differential branch number of L over \mathbb{F}_2^m

minimal number of active Sboxes within a 2-round characteristic

$$d = d_{\min}(\mathcal{C}_L) \text{ where } \mathcal{C}_L = \{(x, L(x)), x \in (\mathbb{F}_2^m)^t\}$$

\mathcal{C}_L is a code over \mathbb{F}_2^m of length $2t$ and size $(2^m)^t$.

Maximizing the differential branch number.

From Singleton's bound,

$$d_{\min}(\mathcal{C}_L) \leq t + 1$$

with equality for **MDS codes**.

MEDP of a two-round differential

A differential may aggregate many differential characteristics.

Bound on the 2-round MEDP [Hong et al00][Daemen-Rijmen02]:

$$\text{MEDP}_2 \leq \left(2^{-m} \delta(S)\right)^{d-1}$$

where d is the differential branch number of L over \mathbf{F}_2^m .

AES [Daemen-Rijmen 98][FIPS PUB 197]

In the AES:

- $S(x) = A(x^{2^{54}})$ over \mathbb{F}_{2^8} where A is an affine permutation of \mathbb{F}_2^8 .
Then, $\delta(S) = 4$.
- $L = \text{MixColumns}$ is such that \mathcal{C}_L is an $[8, 4, 5]$ MDS code over \mathbb{F}_{2^8}

For any 2-round characteristic Ω ,

$$\text{EDP}(\Omega) \leq \left(\frac{\delta(S)}{2^m} \right)^d = 2^{-30}$$

For any 2-round differential (a, b) ,

$$\text{EDP}(a, b) \leq \left(\frac{\delta(S)}{2^m} \right)^{d-1} = 2^{-24}$$

For linear cryptanalysis

Expected squared correlation (linear potential) of a mask (u, v) :

$$\text{ELP}(u, v) = 2^{-2n-\kappa} \sum_{k \in \mathbb{F}_2^\kappa} \left(\sum_{x \in \mathbb{F}_2^n} (-1)^{u \cdot x + v \cdot E_k(x)} \right)^2$$

Expected squared correlation of a 2-round linear trail $(a, L^T(c), c, b)$:

$$\text{ELP}((a, L^T(c), c, b)) \leq \left(2^{-2m} \mathcal{L}(S)^2 \right)^{\text{wt}(L^T(c)) + \text{wt}(c)} \leq \left(2^{-m} \mathcal{L}(S) \right)^{2d'}$$

Linear branch number of L over \mathbb{F}_2^m

$$d' = d_{\min}(\mathcal{C}'_L) \text{ where } \mathcal{C}'_L = \{(L^T(x), x), x \in (\mathbb{F}_2^m)^t\}$$

\mathcal{C}'_L is the dual (orthogonal) of \mathcal{C}_L :

$$\forall x, y : (L^T(x), x) \cdot (y, L(y)) = L^T(x) \cdot y \oplus x \cdot L(y) = 0$$

Then, \mathcal{C}'_L is MDS if and only if \mathcal{C}_L is MDS.

AES [Daemen-Rijmen 98][FIPS PUB 197]

In the AES:

- $S(x) = A(x^{2^{54}})$ over \mathbb{F}_{2^8} where A is an affine permutation of \mathbb{F}_2^8 .
Then, $\mathcal{L}(S) = 2^5$.
- $L = \text{MixColumns}$ is such that \mathcal{C}_L is an $[8, 4, 5]$ MDS code over \mathbb{F}_{2^8}

For any 2-round linear trail Ω ,

$$\text{ELP}(\Omega) \leq \left(\frac{\mathcal{L}(S)}{2^m} \right)^{2d'} = 2^{-30}$$

For any 2-round linear approximation (a, b) ,

$$\text{ELP}(a, b) \leq \left(\frac{\mathcal{L}(S)}{2^m} \right)^{2(d'-1)} = 2^{-24}$$

More detailed lecture notes

<https://www.paris.inria.fr/secret/Anne.Canteaut/poly.pdf>