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# THE EDDY CURRENT MODEL AS A LOW-FREQUENCY, HIGH-CONDUCTIVITY ASYMPTOTIC FORM OF THE MAXWELL TRANSMISSION PROBLEM

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ABSTRACT. We study the relationship between the Maxwell and eddy current (EC) models for three-dimensional configurations involving bounded regions with high conductivity $\sigma$ in air and with sources placed remotely from the conducting objects, which typically occur in the numerical simulation of eddy current nondestructive testing (ECT) experiments. The underlying Maxwell transmission problem is formulated using boundary integral formulations of PMCHWT type. In this context, we derive and rigorously justify an asymptotic expansion of the Maxwell integral problem with respect to the non-dimensional parameter $\gamma := \sqrt{\omega\varepsilon_0/\sigma}$. The EC integral problem is shown to constitute the limiting form of the Maxwell integral problem as $\gamma \to 0$, i.e. as its low-frequency and high-conductivity limit. Estimates in $\gamma$ are obtained for the solution remainders (in terms of the surface currents, which are the primary unknowns of the PMCHWT problem, and the electromagnetic fields) and the impedance variation measured at the extremities of the excitating coil. In particular, the leading and remainder orders in $\gamma$ of the surface currents are found to depend on the current component (electric or magnetic, charge-free or not). These theoretical results are demonstrated on three-dimensional illustrative numerical examples, where the mathematically established estimates in $\gamma$ are reproduced by the numerical results.

**1. Introduction.** Eddy currents have diverse industrial applications, e.g. inductive heating, braking of heavy vehicles, or nondestructive testing, the latter being the primary inspiration for this work. In addition, the ongoing advent of mid-frequency testing for complex media (e.g. made of fibers with diverse conductivities) spurs the development of computational formulations allowing transition from the full Maxwell equation system to modified versions of eddy current (EC) models. Insight into such transition can be gained by considering the influence of a non-dimensional physical parameter, chosen such that the eddy current model is the limiting form of the Maxwell model as that parameter becomes arbitrarily small. This work investigates one such asymptotic approach.

We consider situations involving media with piecewise constant electromagnetic coefficients (dielectric permittivity $\varepsilon^{\mathrm{d}}$, electric conductivity $\sigma$, magnetic permeability $\mu$) and assume time-harmonic conditions with angular frequency $\omega$. Maxwell's equations govern the electric field $\boldsymbol{E}$ and the magnetic field $\boldsymbol{H}$, from which one deduces the electric induction $\boldsymbol{D} := \varepsilon^{\mathrm{d}}\boldsymbol{E}$ and magnetic induction $\boldsymbol{B} := \mu\boldsymbol{H}$. Eddy currents are electric currents $\sigma\boldsymbol{E}$ created in a conducting medium by a time-dependent magnetic field. The electromagnetic state is said to pertain to the eddy current (EC) regime when the displacement currents $\mathrm{i}\omega\varepsilon^{\mathrm{d}}\boldsymbol{E}$ are small relative to $\sigma\boldsymbol{E}$, see e.g. [**14, 23**] or [**21,** Chap. 1]. The EC model then results from neglecting the displacement currents in Maxwell's equations. This suggests an asymptotic approach, whereby the EC model is treated as the limiting form of the Maxwell problem for a dielectric permittivity $\varepsilon^{\mathrm{d}}_{\delta} = \delta\varepsilon^{\mathrm{d}}$ with $\delta \to 0$ (see e.g. [**3,** Chap. 8] or [**11**]). In this framework, the electric fields $\boldsymbol{E}_{\delta}$ and $\boldsymbol{E}_{\mathrm{EC}}$, respectively solving the Maxwell and EC problems, have been shown (in [**11**] and later in [**21,** Sec. 2.2]) to verify

$$\big\| \boldsymbol{E}_{\delta} - \boldsymbol{E}_{\mathrm{EC}} \big\| \leq C\delta.$$

In addition, the justification in a transient setting of the eddy current model (again defined as the zero-permittivity limit of the Maxwell model) is addressed in [**18**].

Alternatively, the low-frequency asymptotic viewpoint (see e.g. [**17**]) is also relevant since displacement currents $\mathrm{i}\omega\varepsilon^{\mathrm{d}}\boldsymbol{E}$ may become negligible for $\omega$ small enough. Indeed, low-frequency expansions of both the Maxwell and EC models are compared in [**2**] for bounded conductors in air, resulting (under suitable assumptions on the source that are for example verified by an excitating coil placed in air) in the estimate

$$\big\| \boldsymbol{E}(\omega) - \boldsymbol{E}_{\mathrm{EC}}(\omega) \big\| \leq C\omega^2, \qquad \big\| \boldsymbol{H}(\omega) - \boldsymbol{H}_{\mathrm{EC}}(\omega) \big\| \leq C\omega^2.$$

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(with underlying function-analytic issues however undergoing some critical discussion in [**19,** Sec. 5.4]) This viewpoint is also pursued, for bounded media, in [**1**], [**21,** Sec. 2.3]. and [**23**]. In [**23**], the electric field is decomposed as $\boldsymbol{E} = \boldsymbol{E}_{\text{EC}} + \delta\boldsymbol{E}$, where $\boldsymbol{E}_{\text{EC}}$ solves the EC model while $\delta\boldsymbol{E}$ also solves a EC problem, this time with displacement currents acting as sources; the estimate $\delta\boldsymbol{E} = O(\omega^2)$ is then established for cases where the source has no galvanic connection with the conductor.

In this work, the relationship between the Maxwell and EC models is studied by investigating the asymptotic behavior of the former with respect to the nondimensional parameter $\gamma := \sqrt{\omega\varepsilon_0/\sigma}$, whose choice is well suited to eddy current nondestructive testing (ECT) [**25**] conditions. In ECT experiments, a coil carrying an alternative current is placed in air above a potentially flawed conducting part undergoing inspection, the impedance variation $\Delta Z$ (see eq. (27)) measured at the coil extremities being related to perturbations caused by defects to the eddy currents induced in the part. ECT experiments are modeled as Maxwell (or eddy current) transmission problems, formulated in this work using boundary integral equations (BIEs) [**5, 8, 14, 16, 20, 22, 26, 28**]. Operating ECT conditions can be characterized in terms of the non-dimensional parameters $\gamma$ and $\xi := L\sqrt{\omega\sigma\mu_0} = \sqrt{2}L/d$ (with $L$ denoting the characteristic diameter of the inspected part and $d$ the skin depth). The experimental goal being to set $d$ to a given target value commensurate with the expected depth of subsurface defects to be detected, we assume $\xi = O(1)$. When testing highly conducting parts, this goal is achieved by choosing a low value for $\omega$, in which case we have $\gamma \ll 1$. This leads us to investigate the transition from Maxwell to EC models by seeking and establishing an asymptotic expansion of the Maxwell BIE formulation with respect to $\gamma$ about $\gamma = 0$. In particular, we show that, under the previously mentioned assumptions on the source, the electric and magnetic fields $\boldsymbol{E}_\gamma, \boldsymbol{H}_\gamma$ and their EC counterparts satisfy, away from the surface separating the two media, the pointwise estimates

$$|\boldsymbol{E}_\gamma| = O(\gamma), \quad |\boldsymbol{E}_\gamma - \boldsymbol{E}_{\text{EC}}| = O(\gamma^3), \qquad |\boldsymbol{H}_\gamma| = O(1), \quad |\boldsymbol{H}_\gamma - \boldsymbol{H}_{\text{EC}}| = O(\gamma^2), \tag{1}$$

implying that the impedance variation $\Delta Z$ verifies

$$|\Delta Z_\gamma| = O(\gamma), \quad |\Delta Z_\gamma - \Delta Z_{\text{EC}}| = O(\gamma^3). \tag{2}$$

The present small-$\gamma$ asymptotic approach is distinct from the previously-considered asymptotic justifications of the EC model. In particular, its results serve to highlight the scaling disparities between components of the surface currents (which are the primary unknowns in BEM formulations of PMCHWT type) that can severely affect solution accuracy in the low-frequency, high-conductivity limit. We note in passing that the parameter $\gamma$ is also introduced in [**6**], where conducting bodies coated by a thin dielectric layer are studied in the large-conductivity limit.

This article is organized as follows. Our starting point is the well-known PMCHWT integral formulation for the electromagnetic transmission problem [**20**] with a Hodge decomposition applied to the unknown surface currents, which is recalled in Sec. 2. Then, in Section 3, we introduce the parameter $\gamma$ into the PMCHWT integral problem, define rescaled surface currents, derive governing problems for the first two coefficients of the solution expansion in $\gamma$ and state the corresponding (main) result on the small-$\gamma$ expansion of the Maxwell problem, whose complete proof is given next in Sec. 4. The integral problem for the leading-order contribution to the surface currents is found to coincide with the EC integral problem established in [**14**], and the proof of Sec. 4 exploits the known coercivity of the latter [**14,** Thm. 12]. Then, estimates (1) and (2) stem from related expansions of integral representations given in Sec. 3.3. Finally, in Sec. 5, the discretized form of the PMCHWT problem incorporating a Hodge decomposition is briefly described and the established asymptotic properties are demonstrated on corroborating numerical results for a simple 3D configuration.

**2. Preliminaries: Maxwell transmission problem and PMCHWT integral formulation.** We assume time-harmonic conditions with given angular frequency $\omega$, the time-harmonic factor $e^{-\mathrm{i}\omega t}$ being implicitly understood wherever relevant. Electromagnetic testing can be mathematically modeled as a transmission problem whereby a three-dimensional bounded conducting object (or a set thereof) with complex permittivity $\varepsilon_1 := \varepsilon_0\varepsilon_{\text{r}} = \varepsilon^{\text{d}} + \mathrm{i}\sigma/\omega$ and permeability $\mu_1 = \mu_0\mu_{\text{r}}$, which occupies the bounded Lipschitz domain $\Omega_1 \subset \mathbb{R}^3$, is surrounded by vacuum filling the unbounded surrounding space $\Omega_0 := \mathbb{R}^3 \setminus \overline{\Omega_1}$ (Fig. 1). The unit normal $\boldsymbol{n}$ on the boundary $\Gamma = \partial\Omega_1$ points from $\Omega_1$ to $\Omega_0$ (i.e. is

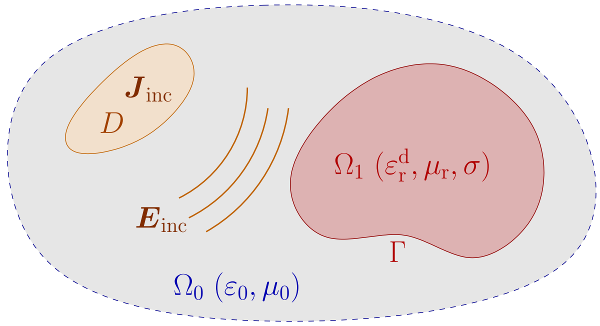



***Figure 1.*** *Scattering by a conducting object: geometry and notation.*

chosen exterior to $\Gamma$). The conducting object is excited by electric and magnetic fields created by a given current density $\boldsymbol{J}_{\text{inc}}$, which is assumed to have a compact support $D \Subset \Omega_0$. As a result of the foregoing assumptions, the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{H}$ solve the linear frequency-domain Maxwell equations

$$\begin{aligned} &\mathbf{rot}\,\boldsymbol{E} = \mathrm{i}\omega\mu_0\boldsymbol{H}, && \mathbf{rot}\,\boldsymbol{H} = -\mathrm{i}\omega\varepsilon_0\boldsymbol{E} + \boldsymbol{J}_{\text{inc}} && \text{in } \Omega_0,\\ &\mathbf{rot}\,\boldsymbol{E} = \mathrm{i}\omega\mu_1\boldsymbol{H}, && \mathbf{rot}\,\boldsymbol{H} = -\mathrm{i}\omega\varepsilon^{\rm d}\boldsymbol{E} - \sigma\boldsymbol{E} && \text{in } \Omega_1, \end{aligned}$$

where $\mathbf{rot}$ denotes the curl operator. In addition, $\boldsymbol{E}$ and $\boldsymbol{H}$ in $\Omega_0$ are assumed to satisfy the Silver–Müller radiation condition at infinity:

$$\Big|\,|\boldsymbol{r}|^{-1}\mathbf{rot}\,\boldsymbol{u}\times\boldsymbol{r} - \mathrm{i}\omega\sqrt{\varepsilon_0\mu_0}\,\boldsymbol{u}\Big| = O(|\boldsymbol{r}|^{-1}), \quad \text{uniformly for } |\boldsymbol{r}|\to\infty \qquad (\boldsymbol{u}=\boldsymbol{E},\boldsymbol{H}). \tag{3}$$

On adopting $\boldsymbol{E}$ as the primary unknown, the above-described problem leads to the following transmission problem for the electric fields $\boldsymbol{E}_0$ in the vacuum $\Omega_0$ and $\boldsymbol{E}_1$ in the conducting part $\Omega_1$:

$$\begin{aligned} &(\mathbf{rot}\,\mathbf{rot} - \kappa_0^2)\boldsymbol{E}_0 = \mathrm{i}\omega\mu_0\boldsymbol{J}_{\text{inc}} && \text{in } \Omega_0, && \boldsymbol{\gamma}_\times^-\boldsymbol{E}_1 - \boldsymbol{\gamma}_\times^+\boldsymbol{E}_0 = \boldsymbol{0} \quad \text{on } \Gamma,\\ &(\mathbf{rot}\,\mathbf{rot} - \kappa_1^2)\boldsymbol{E}_1 = \boldsymbol{0} && \text{in } \Omega_1, && \mu_{\rm r}^{-1}\boldsymbol{\gamma}_N^-\boldsymbol{E}_1 - \boldsymbol{\gamma}_N^+\boldsymbol{E}_0 = \boldsymbol{0} \quad \text{on } \Gamma,\\ &\boldsymbol{E}_0 \text{ satisfies (3)} && \text{at infinity}, \end{aligned} \tag{4}$$

wherein the wavenumbers $\kappa_0$ and $\kappa_1$ associated with the respective media are given by

$$\kappa_0^2 = \varepsilon_0\mu_0\omega^2, \qquad \kappa_1^2 = \kappa_0^2\varepsilon_{\rm r}\mu_{\rm r} \qquad \text{with} \quad \varepsilon_{\rm r} := \varepsilon_{\rm r}^{\rm d} + \mathrm{i}\frac{\sigma}{\omega\varepsilon_0}$$

and the boundary trace operators $\boldsymbol{\gamma}_\times^\pm$, $\boldsymbol{\gamma}_N^\pm$ acting on vector fields are defined by

$$\boldsymbol{\gamma}_\times^\ell\boldsymbol{u} := \boldsymbol{\gamma}_0^\ell\boldsymbol{u}\times\boldsymbol{n}, \quad \boldsymbol{\gamma}_N^\ell\boldsymbol{u} := \boldsymbol{\gamma}_\times^\ell(\mathbf{rot}\,\boldsymbol{u}) \quad (\ell=\pm), \tag{5}$$

where $\boldsymbol{\gamma}_0^+\boldsymbol{u}$ and $\boldsymbol{\gamma}_0^-\boldsymbol{u}$ are the exterior and interior Dirichlet traces of $\boldsymbol{u}$, i.e. (for sufficiently regular fields $\boldsymbol{u}$) the restrictions to $\Gamma$ of $\boldsymbol{u}|_{\overline{\Omega_0}}$ and $\boldsymbol{u}|_{\overline{\Omega_1}}$.

***Eddy current model.*** The eddy current model is formally obtained by removing the displacement current terms from Maxwell's equations, i.e. assuming that $\boldsymbol{E},\boldsymbol{H}$ now solve

$$\begin{aligned} &\mathbf{rot}\,\boldsymbol{E}_0 = \mathrm{i}\omega\mu_0\boldsymbol{H}_0, && \mathbf{rot}\,\boldsymbol{H}_0 = \boldsymbol{J}_{\text{inc}} && \text{in } \Omega_0, && \boldsymbol{\gamma}_\times^-\boldsymbol{E}_1 - \boldsymbol{\gamma}_\times^+\boldsymbol{E}_0 = \boldsymbol{0} && \text{on } \Gamma,\\ &\mathbf{rot}\,\boldsymbol{E}_1 = \mathrm{i}\omega\mu_1\boldsymbol{H}_1, && \mathbf{rot}\,\boldsymbol{H}_1 = -\sigma\boldsymbol{E}_1 && \text{in } \Omega_1, && \mu_{\rm r}^{-1}\boldsymbol{\gamma}_N^-\boldsymbol{E}_1 - \boldsymbol{\gamma}_N^+\boldsymbol{E}_0 = \boldsymbol{0} && \text{on } \Gamma; \end{aligned}$$

moreover the Silver–Müller condition (3) is replaced by the decay conditions $\boldsymbol{E}_0(\boldsymbol{x}) = O(|\boldsymbol{x}|^{-1})$ and $\boldsymbol{H}_0(\boldsymbol{x}) = O(|\boldsymbol{x}|^{-1})$ (uniformly for $|\boldsymbol{x}|\to\infty$).

***Source term.*** Let the volume potentials $\boldsymbol{\Phi}_\ell$ be defined by

$$\boldsymbol{\Phi}_\ell[\boldsymbol{J}](\boldsymbol{x}) := \int_{\mathbb{R}^3} G(\boldsymbol{x}-\boldsymbol{x}';\kappa_\ell)\,\boldsymbol{J}(\boldsymbol{x}')\,\mathrm{d}\boldsymbol{x}' \qquad (\ell=0,1), \tag{6}$$

where $\boldsymbol{J}$ is a (scalar- or vector-valued) density and $G(\boldsymbol{z};\kappa)$ is the well-known fundamental solution of $-(\Delta+\kappa^2)G=\delta$ given by

$$G(\boldsymbol{z};\kappa) = \frac{e^{\mathrm{i}\kappa|\boldsymbol{z}|}}{4\pi|\boldsymbol{z}|}, \qquad \boldsymbol{z}\in\mathbb{R}^3\setminus\{\boldsymbol{0}\}.$$

The field $\boldsymbol{w}:=\boldsymbol{\Phi}_\ell[\boldsymbol{J}]$ then solves $-(\Delta+\kappa_\ell^2)\boldsymbol{w}=\boldsymbol{J}$ in $\mathbb{R}^3$ and satisfies the Sommerfeld radiation condition.

In problem (4), the given current density $\boldsymbol{J}_{\text{inc}}$ is assumed to satisfy $\operatorname{div}\boldsymbol{J}_{\text{inc}}=0$ in $D$ and $\boldsymbol{J}_{\text{inc}}\cdot\boldsymbol{n}=0$ on $\partial D$. The incident electric field $\boldsymbol{E}_{\text{inc}}$ created by $\boldsymbol{J}_{\text{inc}}$ in $\mathbb{R}^3$ filled by the vacuum medium, given by

$$\boldsymbol{E}_{\text{inc}}(\boldsymbol{x}) := \mathrm{i}\omega\mu_0\boldsymbol{\Phi}_0[\boldsymbol{J}_{\text{inc}}](\boldsymbol{x}) \tag{7}$$

(i.e. the Biot-Savart law), satisfies $(\mathbf{rot}\,\mathbf{rot}-\kappa_0^2)\boldsymbol{E}_{\text{inc}}=\mathrm{i}\omega\mu_0\boldsymbol{J}_{\text{inc}}$ in $\mathbb{R}^3$ and the radiation condition (3).

***Function space setting.*** Boldface symbols $\boldsymbol{H}^m,\boldsymbol{L}^2,\ldots$ indicate classical Sobolev spaces of vector-valued fields, e.g. $\boldsymbol{H}^1(X)=H^1(X;\mathbb{C}^3)$ for some domain $X\subset\mathbb{R}^3$.

We first note that the volume potentials (6) can be defined for densities in Sobolev spaces, so that for example the linear mapping $\boldsymbol{\Phi}_\ell:\boldsymbol{H}^{-1}_{\text{comp}}(\mathbb{R}^3)\to\boldsymbol{H}^1_{\text{loc}}(\mathbb{R}^3)$ is well-defined and continuous [**15, 22**]. Moreover, the following spaces of vector fields in $\Omega_0$ or $\Omega_1$ are suitable for Maxwell solutions, see e.g. [**5**]:

$$\begin{aligned}\boldsymbol{H}_{\text{loc}}(\boldsymbol{\mathcal{D}},\Omega_0) &= \big\{\,\boldsymbol{u}\in\boldsymbol{L}^2_{\text{loc}}(\Omega_0):\ \boldsymbol{\mathcal{D}}\boldsymbol{u}\in\boldsymbol{L}^2_{\text{loc}}(\Omega_0)\,\big\}\\ \boldsymbol{H}(\boldsymbol{\mathcal{D}},\Omega_1) &= \big\{\,\boldsymbol{u}\in\boldsymbol{L}^2(\Omega_1):\ \boldsymbol{\mathcal{D}}\boldsymbol{u}\in\boldsymbol{L}^2(\Omega_1)\,\big\}\end{aligned}$$

where $\boldsymbol{\mathcal{D}}$ is the (distributional) partial differential operator $\boldsymbol{\mathcal{D}}=\mathbf{rot}$ or $\boldsymbol{\mathcal{D}}=\mathbf{rot}\,\mathbf{rot}$. In addition, let the space of tangential vector fields $\boldsymbol{\mathcal{V}}$ be defined by

$$\boldsymbol{\mathcal{V}} := \big\{\,\boldsymbol{v}\in\boldsymbol{H}^{-1/2}_{\parallel}(\Gamma):\ \operatorname{div}_S\boldsymbol{v}\in H^{-1/2}(\Gamma)\,\big\}$$

where $\boldsymbol{H}^{-1/2}_{\parallel}(\Gamma)$ is the $\boldsymbol{L}^2(\Gamma)$ dual of $\boldsymbol{H}^{1/2}_{\parallel}(\Gamma):=\boldsymbol{n}\times\big(\gamma_0^-\boldsymbol{H}^1(\Omega_1)\times\boldsymbol{n}\big)$ and $\operatorname{div}_S$ denotes the surface divergence operator, see e.g. [**16,** Sec. 2.5.6]. Then, we know (see [**4**]) that:

**Lemma 1.** *The trace operators $\boldsymbol{\gamma}^+_\times$, $\boldsymbol{\gamma}^-_\times$, $\boldsymbol{\gamma}^+_N$, $\boldsymbol{\gamma}^-_N$ introduced in* (5) *can be extended to linear continuous operators from $\boldsymbol{H}_{loc}(\mathbf{rot}\,,\Omega_0)$, $\boldsymbol{H}(\mathbf{rot}\,,\Omega_1)$, $\boldsymbol{H}_{loc}(\mathbf{rot}\,\mathbf{rot}\,,\Omega_0)$, $\boldsymbol{H}(\mathbf{rot}\,\mathbf{rot}\,,\Omega_1)$, respectively, to $\boldsymbol{\mathcal{V}}$.*

***Stratton–Chu integral representation formulas.*** Introduce the single-layer Helmholtz (vector or scalar) potentials

$$\boldsymbol{\Psi}_\ell[\boldsymbol{u}](\boldsymbol{x}) := \int_\Gamma G(\boldsymbol{x}-\boldsymbol{x}';\kappa_\ell)\boldsymbol{u}(\boldsymbol{x}')\,\mathrm{d}S(\boldsymbol{x}') \qquad (\ell=0,1), \tag{8}$$

which solve the (vector or scalar) Helmholtz equation $-(\Delta+\kappa_\ell^2)\boldsymbol{\Psi}_\ell=\boldsymbol{0}$ in $\mathbb{R}^3\backslash\Gamma$ (note in particular that $\boldsymbol{\Psi}_0[\boldsymbol{u}]$ is defined in $\Omega_1$ and vice versa). Then, the single- and double-layer Maxwell potentials $\boldsymbol{\Psi}^\ell_{\text{S}},\boldsymbol{\Psi}^\ell_{\text{D}}$, defined in terms of $\boldsymbol{\Psi}_\ell$ by

$$\begin{aligned}&\boldsymbol{\Psi}^\ell_{\text{S}}[\boldsymbol{u}] = \boldsymbol{\Psi}_\ell[\boldsymbol{u}] + \kappa_\ell^{-2}\boldsymbol{\nabla}\boldsymbol{\Psi}_\ell[\operatorname{div}_S\boldsymbol{u}] \quad (\kappa_\ell\neq 0),\\ &\boldsymbol{\Psi}^\ell_{\text{D}}[\boldsymbol{u}] = \mathbf{rot}\,\boldsymbol{\Psi}_\ell[\boldsymbol{u}],\end{aligned} \tag{9}$$

are Maxwell solutions in $\mathbb{R}^3\backslash\Gamma$ and define continuous $\boldsymbol{\mathcal{V}}\to\boldsymbol{H}_{\text{loc}}(\mathbf{rot}\,\mathbf{rot}\,,\Omega_0)$ and $\boldsymbol{\mathcal{V}}\to\boldsymbol{H}(\mathbf{rot}\,\mathbf{rot}\,,\Omega_1)$ linear operators [**5**]. Moreover, they verify the interrelations

$$\mathbf{rot}\,\boldsymbol{\Psi}^\ell_{\text{S}} = \boldsymbol{\Psi}^\ell_{\text{D}}, \qquad \mathbf{rot}\,\boldsymbol{\Psi}^\ell_{\text{D}} = \kappa_\ell^2\boldsymbol{\Psi}^\ell_{\text{S}}. \tag{10}$$

Using these definitions, the well-known Stratton–Chu integral representation formula for the electric field (see e.g. [**9,** Thm. 6.2]) reads

$$\begin{aligned}\boldsymbol{E}_0 &= -\boldsymbol{\Psi}^0_{\text{S}}[\boldsymbol{\gamma}^+_N\boldsymbol{E}] - \boldsymbol{\Psi}^0_{\text{D}}[\boldsymbol{\gamma}^+_\times\boldsymbol{E}] + \boldsymbol{E}_{\text{inc}} && \text{in } \Omega_0,\\ \boldsymbol{E}_1 &= \phantom{-}\boldsymbol{\Psi}^1_{\text{S}}[\boldsymbol{\gamma}^-_N\boldsymbol{E}] + \boldsymbol{\Psi}^1_{\text{D}}[\boldsymbol{\gamma}^-_\times\boldsymbol{E}] && \text{in } \Omega_1.\end{aligned}$$

Then, introducing the surface current densities $\boldsymbol{J},\boldsymbol{M}\in\boldsymbol{\mathcal{V}}$ defined by

$$\boldsymbol{\gamma}^+_N\boldsymbol{E} = \mathrm{i}\omega\mu_0\boldsymbol{J}, \qquad \boldsymbol{\gamma}^-_N\boldsymbol{E} = \mathrm{i}\omega\mu_0\mu_{\text{r}}\boldsymbol{J}, \qquad \boldsymbol{\gamma}^+_\times\boldsymbol{E} = \boldsymbol{\gamma}^-_\times\boldsymbol{E} = \mathrm{i}\omega\mu_0\boldsymbol{M}$$

in order to satisfy the transmission conditions of problem (4), the above Stratton–Chu formulas become

$$\begin{aligned} \boldsymbol{E}_0 &= \mathrm{i}\mu_0\omega\big( -\boldsymbol{\Psi}^0_{\mathrm{S}}[\boldsymbol{J}] - \boldsymbol{\Psi}^0_{\mathrm{D}}[\boldsymbol{M}] + \boldsymbol{\Phi}_0[\boldsymbol{J}_{\mathrm{inc}}]\big) && \text{in } \Omega_0, \\ \boldsymbol{E}_1 &= \mathrm{i}\mu_0\omega\big( \mu_{\mathrm{r}}\boldsymbol{\Psi}^1_{\mathrm{S}}[\boldsymbol{J}] + \boldsymbol{\Psi}^1_{\mathrm{D}}[\boldsymbol{M}]\big) && \text{in } \Omega_1. \end{aligned} \tag{11}$$

***PMCHWT integral formulation.*** By Lemma 1, the boundary traces of $\boldsymbol{\Psi}^\ell_{\mathrm{S}}, \boldsymbol{\Psi}^\ell_{\mathrm{D}}$ under $\boldsymbol{\gamma}^\pm_\times$ and $\boldsymbol{\gamma}^\pm_N$ are well-defined. Let $\{\boldsymbol{\gamma}_\times\} := \frac{1}{2}(\boldsymbol{\gamma}^+_\times + \boldsymbol{\gamma}^-_\times)$ and $\{\boldsymbol{\gamma}_N\} := \frac{1}{2}(\boldsymbol{\gamma}^+_N + \boldsymbol{\gamma}^-_N)$ denote the symmetric parts of $\boldsymbol{\gamma}_\times, \boldsymbol{\gamma}_N$. Then, the boundary traces of $\boldsymbol{\Psi}^\ell_{\mathrm{S}}, \boldsymbol{\Psi}^\ell_{\mathrm{D}}$ under $\boldsymbol{\gamma}^\pm_\times$ are found, using their known jump properties **[5]**, to verify

$$\begin{aligned} \boldsymbol{\gamma}^+_\times\boldsymbol{\Psi}^\ell_{\mathrm{S}} &= \{\boldsymbol{\gamma}_\times\}\boldsymbol{\Psi}^\ell_{\mathrm{S}}, & \boldsymbol{\gamma}^-_\times\boldsymbol{\Psi}^\ell_{\mathrm{S}} &= \{\boldsymbol{\gamma}_\times\}\boldsymbol{\Psi}^\ell_{\mathrm{S}}, \\ \boldsymbol{\gamma}^+_\times\boldsymbol{\Psi}^\ell_{\mathrm{D}} &= \{\boldsymbol{\gamma}_\times\}\boldsymbol{\Psi}^\ell_{\mathrm{D}} - \tfrac{1}{2}\mathcal{I}, & \boldsymbol{\gamma}^-_\times\boldsymbol{\Psi}^\ell_{\mathrm{D}} &= \{\boldsymbol{\gamma}_\times\}\boldsymbol{\Psi}^\ell_{\mathrm{D}} + \tfrac{1}{2}\mathcal{I}. \end{aligned}$$

Moreover, the interrelations (10) imply $\boldsymbol{\gamma}_N\boldsymbol{\Psi}^\ell_{\mathrm{S}} = \boldsymbol{\gamma}_\times\boldsymbol{\Psi}^\ell_{\mathrm{D}}$ and $\boldsymbol{\gamma}_N\boldsymbol{\Psi}^\ell_{\mathrm{D}} = \kappa^2_\ell\boldsymbol{\gamma}_\times\boldsymbol{\Psi}^\ell_{\mathrm{S}}$, so that the boundary traces of $\boldsymbol{\Psi}^\ell_{\mathrm{S}}, \boldsymbol{\Psi}^\ell_{\mathrm{D}}$ under $\boldsymbol{\gamma}^\pm_N$ verify

$$\begin{aligned} \boldsymbol{\gamma}^+_N\boldsymbol{\Psi}^\ell_{\mathrm{S}} &= \{\boldsymbol{\gamma}_\times\}\boldsymbol{\Psi}^\ell_{\mathrm{D}} - \tfrac{1}{2}\mathcal{I}, & \boldsymbol{\gamma}^-_N\boldsymbol{\Psi}^\ell_{\mathrm{S}} &= \{\boldsymbol{\gamma}_\times\}\boldsymbol{\Psi}^\ell_{\mathrm{D}} + \tfrac{1}{2}\mathcal{I}, \\ \boldsymbol{\gamma}^+_N\boldsymbol{\Psi}^\ell_{\mathrm{D}} &= \kappa^2_\ell\{\boldsymbol{\gamma}_\times\}\boldsymbol{\Psi}^\ell_{\mathrm{S}}, & \boldsymbol{\gamma}^-_N\boldsymbol{\Psi}^\ell_{\mathrm{D}} &= \kappa^2_\ell\{\boldsymbol{\gamma}_\times\}\boldsymbol{\Psi}^\ell_{\mathrm{S}}. \end{aligned}$$

We next write the boundary traces of the Stratton–Chu formulas (11), expressing $\boldsymbol{\gamma}^+_\times\boldsymbol{E}_0, \boldsymbol{\gamma}^-_\times\boldsymbol{E}_1, \boldsymbol{\gamma}^+_N\boldsymbol{E}_0, \boldsymbol{\gamma}^-_N\boldsymbol{E}_1$ (in that order) in terms of $\boldsymbol{J}, \boldsymbol{M}$ and rearranging terms, to obtain

$$\begin{aligned} &\text{(a)} && \{\boldsymbol{\gamma}_\times\}\boldsymbol{\Psi}^0_{\mathrm{S}}[\boldsymbol{J}] + \{\boldsymbol{\gamma}_\times\}\boldsymbol{\Psi}^0_{\mathrm{D}}[\boldsymbol{M}] + \tfrac{1}{2}\boldsymbol{M} = \boldsymbol{\gamma}^+_\times\boldsymbol{\Phi}_0[\boldsymbol{J}_{\mathrm{inc}}], \\ &\text{(b)} && \mu_{\mathrm{r}}\{\boldsymbol{\gamma}_\times\}\boldsymbol{\Psi}^1_{\mathrm{S}}[\boldsymbol{J}] + \{\boldsymbol{\gamma}_\times\}\boldsymbol{\Psi}^1_{\mathrm{D}}[\boldsymbol{M}] - \tfrac{1}{2}\boldsymbol{M} = \boldsymbol{0}, \\ &\text{(c)} && \{\boldsymbol{\gamma}_\times\}\boldsymbol{\Psi}^0_{\mathrm{D}}[\boldsymbol{J}] + \tfrac{1}{2}\boldsymbol{J} + \kappa^2_0\{\boldsymbol{\gamma}_\times\}\boldsymbol{\Psi}^0_{\mathrm{S}}[\boldsymbol{M}] = \boldsymbol{\gamma}^+_N\boldsymbol{\Phi}_0[\boldsymbol{J}_{\mathrm{inc}}], \\ &\text{(d)} && \{\boldsymbol{\gamma}_\times\}\boldsymbol{\Psi}^1_{\mathrm{D}}[\boldsymbol{J}] - \tfrac{1}{2}\boldsymbol{J} + \mu_{\mathrm{r}}^{-1}\kappa^2_1\{\boldsymbol{\gamma}_\times\}\boldsymbol{\Psi}^1_{\mathrm{S}}[\boldsymbol{M}] = \boldsymbol{0}. \end{aligned}$$

The combinations (a)+(b) and (c)+(d) of the above equations finally yield the governing system of integral equations

$$\begin{aligned} \mathcal{Z}_J[\boldsymbol{J}] + \mathcal{B}[\boldsymbol{M}] &= \boldsymbol{\gamma}^+_\times\boldsymbol{\Phi}_0[\boldsymbol{J}_{\mathrm{inc}}], \\ \mathcal{B}[\boldsymbol{J}] + \mathcal{Z}_M[\boldsymbol{M}] &= \boldsymbol{\gamma}^+_N\boldsymbol{\Phi}_0[\boldsymbol{J}_{\mathrm{inc}}] \end{aligned} \tag{12}$$

for $\boldsymbol{J}, \boldsymbol{M}$, known as the Poggio-Miller-Chang-Harrington-Wu-Tsai (PMCHWT) integral formulation **[20]** for the scattering problem (4), with the integral operators $\mathcal{Z}_J$, $\mathcal{Z}_M$, $\mathcal{B}$ given by

$$\begin{aligned} \mathcal{Z}_J &= \{\boldsymbol{\gamma}_\times\}\big(\boldsymbol{\Psi}^0_{\mathrm{S}} + \mu_{\mathrm{r}}\boldsymbol{\Psi}^1_{\mathrm{S}}\big) &&= \{\boldsymbol{\gamma}_\times\}\big(\boldsymbol{\Psi}_0 + \mu_{\mathrm{r}}\boldsymbol{\Psi}_1\big) + \{\boldsymbol{\gamma}_\times\}\big(\kappa_0^{-2}\boldsymbol{\nabla}\boldsymbol{\Psi}_0 + \kappa_1^{-2}\mu_{\mathrm{r}}\boldsymbol{\nabla}\boldsymbol{\Psi}_1\big) \circ \mathrm{div}_S, \\ \mathcal{Z}_M &= \{\boldsymbol{\gamma}_\times\}\big(\kappa^2_0\boldsymbol{\Psi}^0_{\mathrm{S}} + \mu_{\mathrm{r}}^{-1}\kappa^2_1\boldsymbol{\Psi}^1_{\mathrm{S}}\big) &&= \{\boldsymbol{\gamma}_\times\}\big(\kappa^2_0\boldsymbol{\Psi}_0 + \mu_{\mathrm{r}}^{-1}\kappa^2_1\boldsymbol{\Psi}_1\big) + \{\boldsymbol{\gamma}_\times\}\big(\boldsymbol{\nabla}\boldsymbol{\Psi}_0 + \mu_{\mathrm{r}}^{-1}\boldsymbol{\nabla}\boldsymbol{\Psi}_1\big) \circ \mathrm{div}_S, \\ \mathcal{B} &= \{\boldsymbol{\gamma}_\times\}\big(\boldsymbol{\Psi}^0_{\mathrm{D}} + \boldsymbol{\Psi}^1_{\mathrm{D}}\big) &&= \{\boldsymbol{\gamma}_\times\}\big(\mathbf{rot}\,\boldsymbol{\Psi}_0 + \mathbf{rot}\,\boldsymbol{\Psi}_1\big). \end{aligned} \tag{13}$$

We introduce the Cartesian product space $\mathbb{V} := \boldsymbol{\mathcal{V}} \times \boldsymbol{\mathcal{V}}$, define the twisted inner product $\big\langle \cdot\,, \cdot \big\rangle_\times$ by

$$\big\langle \boldsymbol{u}\,, \boldsymbol{v} \big\rangle_\times := -\int_\Gamma \boldsymbol{u}\cdot(\boldsymbol{v}\times\boldsymbol{n})\,\mathrm{d}S,$$

and set $\big\langle \widetilde{\boldsymbol{X}}\,, \boldsymbol{X} \big\rangle_\times := \big\langle \widetilde{\boldsymbol{J}}\,, \boldsymbol{J} \big\rangle_\times + \big\langle \widetilde{\boldsymbol{M}}\,, \boldsymbol{M} \big\rangle_\times$ and $\|\boldsymbol{X}\|^2_{\mathbb{V}} := \|\boldsymbol{J}\|^2_{\boldsymbol{\mathcal{V}}} + \|\boldsymbol{M}\|^2_{\boldsymbol{\mathcal{V}}}$ for any $\boldsymbol{X} = (\boldsymbol{J}, \boldsymbol{M})$, $\widetilde{\boldsymbol{X}} = (\widetilde{\boldsymbol{J}}, \widetilde{\boldsymbol{M}})$. With these definitions, the weak formulation of the PMCHWT integral problem (12) reads

$$\text{Find } \boldsymbol{X} = (\boldsymbol{J}, \boldsymbol{M}) \in \mathbb{V}, \qquad \big\langle \widetilde{\boldsymbol{X}}\,, \boldsymbol{\mathcal{Z}}\boldsymbol{X} \big\rangle_\times = \big\langle \widetilde{\boldsymbol{X}}\,, \boldsymbol{Y} \big\rangle_\times \quad \text{for all } \widetilde{\boldsymbol{X}} = (\widetilde{\boldsymbol{J}}, \widetilde{\boldsymbol{M}}) \in \mathbb{V}$$
$$\text{with} \qquad \boldsymbol{\mathcal{Z}} = \begin{bmatrix} \mathcal{Z}_J & \mathcal{B} \\ \mathcal{B} & \mathcal{Z}_M \end{bmatrix}, \quad \boldsymbol{Y} = \begin{Bmatrix} \boldsymbol{\gamma}^+_\times\boldsymbol{\Phi}_0[\boldsymbol{J}_{\mathrm{inc}}] \\ \boldsymbol{\gamma}^+_N\boldsymbol{\Phi}_0[\boldsymbol{J}_{\mathrm{inc}}] \end{Bmatrix}.$$

***Helmholtz–Hodge decomposition.*** We introduce a Helmholtz–Hodge decomposition $\boldsymbol{\mathcal{V}} = \boldsymbol{\mathcal{V}}_{\mathrm{L}} \oplus \boldsymbol{\mathcal{V}}_{\mathrm{T}}$ of $\boldsymbol{\mathcal{V}}$, where $\boldsymbol{\mathcal{V}}_{\mathrm{L}} := \big\{\boldsymbol{u} \in \boldsymbol{\mathcal{V}} \,:\, \mathrm{div}_S\boldsymbol{u} = 0\big\}$, see **[4]**, and the corresponding additive decompositions

$\boldsymbol{J} = \boldsymbol{J}_{\rm L} + \boldsymbol{J}_{\rm T}$ and $\boldsymbol{M} = \boldsymbol{M}_{\rm L} + \boldsymbol{M}_{\rm T}$ of the unknown surface currents, with $\boldsymbol{J}_{\rm L}, \boldsymbol{M}_{\rm L} \in \boldsymbol{\mathcal{V}}_{\rm L}$ and $\boldsymbol{J}_{\rm T}, \boldsymbol{M}_{\rm T} \in \boldsymbol{\mathcal{V}}_{\rm T}$.

$$\big\langle \widetilde{\boldsymbol{X}} \,,\, \boldsymbol{\mathcal{Z}} \boldsymbol{X} \big\rangle_{\times} = \big\langle \widetilde{\boldsymbol{X}} \,,\, \boldsymbol{Y} \big\rangle_{\times} \quad \text{for all } \widetilde{\boldsymbol{X}} \in \mathbb{V}$$

$$\boldsymbol{\mathcal{Z}} = \begin{bmatrix} \mathcal{A}_J & \mathcal{A}_J & \mathcal{B} & \mathcal{B} \\ \mathcal{A}_J & \mathcal{Z}_J & \mathcal{B} & \mathcal{B} \\ \mathcal{B} & \mathcal{B} & \mathcal{A}_M & \mathcal{A}_M \\ \mathcal{B} & \mathcal{B} & \mathcal{A}_M & \mathcal{Z}_M \end{bmatrix}, \quad \boldsymbol{X} = \begin{Bmatrix} \boldsymbol{J}_{\rm L} \\ \boldsymbol{J}_{\rm T} \\ \boldsymbol{M}_{\rm L} \\ \boldsymbol{M}_{\rm T} \end{Bmatrix}, \quad \widetilde{\boldsymbol{X}} = \begin{Bmatrix} \tilde{\boldsymbol{J}}_{\rm L} \\ \tilde{\boldsymbol{J}}_{\rm T} \\ \tilde{\boldsymbol{M}}_{\rm L} \\ \tilde{\boldsymbol{M}}_{\rm T} \end{Bmatrix}, \quad \boldsymbol{Y} = \begin{Bmatrix} \boldsymbol{\gamma}^+_{\times} \boldsymbol{\Phi}_0[\boldsymbol{J}_{\rm inc}] \\ \boldsymbol{\gamma}^+_{\times} \boldsymbol{\Phi}_0[\boldsymbol{J}_{\rm inc}] \\ \boldsymbol{\gamma}^+_{N} \boldsymbol{\Phi}_0[\boldsymbol{J}_{\rm inc}] \\ \boldsymbol{\gamma}^+_{N} \boldsymbol{\Phi}_0[\boldsymbol{J}_{\rm inc}] \end{Bmatrix}, \tag{14}$$

with integral operators $\mathcal{Z}_J$, $\mathcal{Z}_M$, $\mathcal{B}$ as defined in (13) and

$$\mathcal{A}_J := \{\boldsymbol{\gamma}_{\times}\}\big(\boldsymbol{\Psi}_0 + \mu_{\rm r} \boldsymbol{\Psi}_1\big), \quad \mathcal{A}_M := \{\boldsymbol{\gamma}_{\times}\}\big(\kappa_0^2 \boldsymbol{\Psi}_0 + \mu_{\rm r}^{-1} \kappa_1^2 \boldsymbol{\Psi}_1\big). \tag{15}$$

Discrete Helmholtz–Hodge decompositions of finite-dimensional (BE approximation) subspaces of $\boldsymbol{\mathcal{V}}$ exist in several forms, such as the well-known *loop–tree* decomposition (to which the L,T subscripts introduced above refer) used in Sec. 5. They are in particular used for circumventing low-frequency breakdown in integral equation methods for electromagnetic scattering, see e.g. [**7**]. In this work, the Helmholtz–Hodge decomposition will play an essential role in the asymptotic analysis to follow.

**3. Eddy current problem as asymptotic expansion of PMCHWT problem.** The eddy current (EC) model arises from a quasi-static approximation that is suitable at low frequencies. It is formally obtained by dropping the displacement current term in the Maxwell model, which yields the EC field equations

$$\begin{aligned} &\textbf{rot}\, \boldsymbol{E} = \mathrm{i}\omega\mu_0 \boldsymbol{H}, \qquad \textbf{rot}\, \boldsymbol{H} = \boldsymbol{J}_{\rm inc} \qquad && \text{in } \Omega_0, \\ &\textbf{rot}\, \boldsymbol{E} = \mathrm{i}\omega\mu_1 \boldsymbol{H}, \qquad \textbf{rot}\, \boldsymbol{H} = \sigma \boldsymbol{E} \qquad && \text{in } \Omega_1. \end{aligned}$$

We introduce the non-dimensional parameters

$$\gamma := \sqrt{\omega\varepsilon_0/\sigma}, \qquad \xi := L\sqrt{\omega\mu_0\sigma} \qquad \big(\text{with} \quad L := \text{diam}(\Omega_1)\big),$$

noting for later reference that $\kappa_0 L = \xi\gamma$. Our aim is to investigate the limiting case of the Maxwell (integral equation) model (14) when $\gamma \ll 1$ and $\xi = O(1)$, i.e. for regimes simultaneously involving low frequencies and highly conducting bodies. We observe that $\kappa_0 L = \xi\gamma \to 0$ but $\kappa_1 L = \xi\sqrt{\mathrm{i}\mu_{\rm r}} + O(\gamma^2) \not\to 0$ as $\gamma \to 0$, meaning that usual low-frequency approaches [**7**] (where $\kappa_0 L, \kappa_1 L = O(\omega)$) do not directly apply to ECs. The EC regime is here studied by seeking an expansion in powers of $\gamma$ of the surface currents solving problem (14). For this purpose, we follow the two-step approach commonly used in asymptotic analysis whereby expansions are (i) formally derived, then (ii) precisely formulated and justified. Those two steps are carried out in Secs. 3.1 and 3.2, respectively, with the proof of the main result deferred to Sec. 4. Then, expansions of related quantities are given in Sec. 3.3.

*Remark* 1. The small parameter $\gamma$ is directly related to the notion of displacement currents being small relative to eddy currents since the former take, in either medium, the form $\mathrm{i}\omega\varepsilon^{\rm d}\boldsymbol{E} = \mathrm{i}\gamma^2\varepsilon_{\rm r}^{\rm d}(\sigma\boldsymbol{E})$.

**3.1. Expansion derivation.** To derive the sought expansion of the surface currents, we use a formal expansion of the right-hand side and integral operator matrix of (14). The latter depend on $\gamma$ through the wavenumbers $\kappa_\ell$, for which we have

$$\kappa_0 L = \gamma\xi = O(\gamma), \qquad \kappa_1 L = \xi\sqrt{\mu_{\rm r}(\mathrm{i} + \varepsilon_{\rm r}^{\rm d}\gamma^2)} = \xi\sqrt{\mathrm{i}\mu_{\rm r}} + O(\gamma^2). \tag{16}$$

Then, the (wavenumber-dependent) Helmholtz fundamental solutions admit as a result the expansions

$$\begin{aligned} &\text{(a)} \quad G(\boldsymbol{z};\kappa_0) = G_0^{(0)}(\boldsymbol{z}) + G_0^{(1)}(\boldsymbol{z})\gamma + O(\gamma^2), \qquad &&\text{(b)} \quad G(\boldsymbol{z};\kappa_1) = G_1^{(0)}(\boldsymbol{z}) + O(\gamma^2), \\ &\text{(c)} \ \boldsymbol{\nabla} G(\boldsymbol{z};\kappa_0) = \boldsymbol{\nabla} G_0^{(0)}(\boldsymbol{z}) + O(\gamma^2), &&\text{(d)} \ \boldsymbol{\nabla} G(\boldsymbol{z};\kappa_1) = \boldsymbol{\nabla} G_1^{(0)}(\boldsymbol{z}) + O(\gamma^2), \end{aligned} \tag{17}$$

with

$$G_0^{(0)}(\boldsymbol{z}) := \frac{1}{4\pi|\boldsymbol{z}|}, \qquad G_0^{(1)}(\boldsymbol{z}) := \frac{\mathrm{i}\xi}{4\pi L}, \qquad G_1^{(0)}(\boldsymbol{z}) := \frac{e^{\mathrm{i}\xi\sqrt{\mathrm{i}\mu_{\rm r}}|\boldsymbol{z}|/L}}{4\pi|\boldsymbol{z}|}.$$

The sought expansion of problem (14) can now be set up by using expansions (16) and (17).

***Expansion of the right-hand side.*** We first examine the right-hand side of (14). Use of expansion (17a) in $\boldsymbol{\Phi}_0[\boldsymbol{J}_{\text{inc}}]$ yields

$$\boldsymbol{\Phi}_0^{\gamma}[\boldsymbol{J}_{\text{inc}}] = \boldsymbol{\Phi}_0^{(0)}[\boldsymbol{J}_{\text{inc}}] + \gamma\boldsymbol{\Phi}_0^{(1)}[\boldsymbol{J}_{\text{inc}}] + O(\gamma^2) = \boldsymbol{\Phi}_0^{(0)}[\boldsymbol{J}_{\text{inc}}] + O(\gamma^2) \tag{18}$$

where the notation $\boldsymbol{\Phi}_0^{\gamma}$ serves to emphasize the dependence in $\gamma$ of the volume potential $\boldsymbol{\Phi}_0$ defined by (6), and $\boldsymbol{\Phi}_0^{(m)}$ is the volume potential obtained by replacing $G(\cdot;\kappa_0)$ with $G_0^{(m)}$ in (6), and the second equality results from:

**Lemma 2.** *Let $\boldsymbol{J}_{inc}$ be such that $\operatorname{div}\boldsymbol{J}_{inc}=0$ in $D$ and $\boldsymbol{J}_{inc}\cdot\boldsymbol{n}=0$ on $\partial D$. Then $\boldsymbol{\Phi}_0^{(1)}[\boldsymbol{J}_{inc}] = \boldsymbol{0}$.*

*Proof.* Using that $\operatorname{div}(\boldsymbol{x}'\otimes\boldsymbol{J}_{\text{inc}}(\boldsymbol{x}')) = \boldsymbol{x}'\operatorname{div}\boldsymbol{J}_{\text{inc}} + \boldsymbol{J}_{\text{inc}}$ while $G_0^{(1)}$ is a constant function, we have

$$\boldsymbol{\Phi}_0^{(1)}[\boldsymbol{J}_{\text{inc}}](\boldsymbol{x}) = G_0^{(1)}\int_D \boldsymbol{J}_{\text{inc}}(\boldsymbol{x}')\,\mathrm{d}\boldsymbol{x}' = G_0^{(1)}\int_D \big(\operatorname{div}(\boldsymbol{x}'\otimes\boldsymbol{J}_{\text{inc}}(\boldsymbol{x}')) - \boldsymbol{x}'\operatorname{div}\boldsymbol{J}_{\text{inc}}\big)\,\mathrm{d}\boldsymbol{x}' = 0,$$

where the last equality results from the assumptions on $\boldsymbol{J}_{\text{inc}}$ and Green's identity. □

***Expansion of the integral operators.*** Using expansions (16) and (17) in the relevant integral operators given by (13) and (15), we find

$$\begin{aligned} &\mathcal{A}_J = \mathcal{A}_J^{(0)} + \gamma\mathcal{A}_J^{(1)} + O(\gamma^2), && \gamma^2\mathcal{Z}_J = \mathcal{Z}_J^{(0)} + O(\gamma^2), && \mathcal{B} = \mathcal{B}^{(0)} + O(\gamma^2),\\ &\mathcal{A}_M = \mathcal{A}_M^{(0)} \qquad\quad\; + O(\gamma^2), && \mathcal{Z}_M = \mathcal{Z}_M^{(0)} + O(\gamma^2), \end{aligned} \tag{19}$$

with

$$\begin{aligned} &\mathcal{A}_J^{(0)}\boldsymbol{u} = \{\boldsymbol{\gamma}_\times\}\big(\boldsymbol{\Psi}_0^{(0)}\boldsymbol{u} + \mu_{\mathrm{r}}\boldsymbol{\Psi}_1^{(0)}\boldsymbol{u}\big), && \mathcal{B}^{(0)}\boldsymbol{u} = \{\boldsymbol{\gamma}_N\}\big(\boldsymbol{\Psi}_0^{(0)}\boldsymbol{u} + \boldsymbol{\Psi}_1^{(0)}\boldsymbol{u}\big),\\ &\mathcal{A}_J^{(1)}\boldsymbol{u} = \{\boldsymbol{\gamma}_\times\}\boldsymbol{\Psi}_0^{(1)}\boldsymbol{u}, && \mathcal{Z}_J^{(0)}\boldsymbol{u} = \xi^{-2}L^2\{\boldsymbol{\gamma}_\times\}\boldsymbol{\nabla}\boldsymbol{\Psi}_0^{(0)}[\operatorname{div}_S\boldsymbol{u}],\\ &\mathcal{A}_M^{(0)}\boldsymbol{u} = \mathrm{i}\mu_{\mathrm{r}}\xi^2L^{-2}\{\boldsymbol{\gamma}_\times\}\boldsymbol{\Psi}_1^{(0)}\boldsymbol{u}, && \mathcal{Z}_M^{(0)}\boldsymbol{u} = \mathcal{A}_M^{(0)}\boldsymbol{u} + \{\boldsymbol{\gamma}_\times\}\big(\boldsymbol{\nabla}\boldsymbol{\Psi}_0^{(0)} + \mu_{\mathrm{r}}^{-1}\boldsymbol{\nabla}\boldsymbol{\Psi}_1^{(0)}\big)[\operatorname{div}_S\boldsymbol{u}], \end{aligned}$$

and where $\boldsymbol{\Psi}_\ell^{(m)}$ is the potential defined by (8) with $G(\cdot;\kappa_\ell)$ replaced by $G_\ell^{(m)}$.

***Expansion of surface currents.*** Expansions (19) of the integral operators reveal that $\mathcal{Z}_J = O(\gamma^{-2})$. This suggests to recast the PMCHWT problem (14) for $\gamma\neq 0$ in the rescaled form

$$\text{find } \widehat{\boldsymbol{X}}_\gamma \in \mathbb{V} \text{ such that:}\quad \big\langle \widetilde{\boldsymbol{X}}\,,\, \widehat{\boldsymbol{\mathcal{Z}}}_\gamma\widehat{\boldsymbol{X}}_\gamma\big\rangle_\times = \big\langle \widetilde{\boldsymbol{X}}\,,\, \boldsymbol{Y}_\gamma\big\rangle_\times \quad \text{for all } \widetilde{\boldsymbol{X}}\in\mathbb{V}, \tag{20}$$

wherein the notation now emphasizes the dependence on $\gamma$ of the problem and its solution, and having set

$$\widehat{\boldsymbol{\mathcal{Z}}}_\gamma := \begin{bmatrix} \mathcal{A}_J & \gamma^2\mathcal{A}_J & \mathcal{B} & \mathcal{B}\\ \mathcal{A}_J & \gamma^2\mathcal{Z}_J & \mathcal{B} & \mathcal{B}\\ \mathcal{B} & \gamma^2\mathcal{B} & \mathcal{A}_M & \mathcal{A}_M\\ \mathcal{B} & \gamma^2\mathcal{B} & \mathcal{A}_M & \mathcal{Z}_M \end{bmatrix},\quad \widehat{\boldsymbol{X}}_\gamma := \begin{Bmatrix} \boldsymbol{J}_{\mathrm{L}}\\ \gamma^{-2}\boldsymbol{J}_{\mathrm{T}}\\ \boldsymbol{M}_{\mathrm{L}}\\ \boldsymbol{M}_{\mathrm{T}} \end{Bmatrix},\quad \boldsymbol{Y}_\gamma = \begin{Bmatrix} \boldsymbol{\gamma}_\times^+\boldsymbol{\Phi}_0^\gamma[\boldsymbol{J}_{\text{inc}}]\\ \boldsymbol{\gamma}_\times^+\boldsymbol{\Phi}_0^\gamma[\boldsymbol{J}_{\text{inc}}]\\ \boldsymbol{\gamma}_N^+\boldsymbol{\Phi}_0^\gamma[\boldsymbol{J}_{\text{inc}}]\\ \boldsymbol{\gamma}_N^+\boldsymbol{\Phi}_0^\gamma[\boldsymbol{J}_{\text{inc}}] \end{Bmatrix}.$$

The integral operator matrix $\widehat{\boldsymbol{\mathcal{Z}}}_\gamma$ is then expanded as

$$\widehat{\boldsymbol{\mathcal{Z}}}_\gamma = \boldsymbol{\mathcal{Z}}^{(0)} + \gamma\boldsymbol{\mathcal{Z}}^{(1)} + O(\gamma^2) \tag{21}$$

where, recalling expansions (19), $\boldsymbol{\mathcal{Z}}^{(0)}$ and $\boldsymbol{\mathcal{Z}}^{(1)}$ are given by

$$\mathcal{Z}^{(0)} = \begin{bmatrix} \mathcal{A}_J^{(0)} & 0 & \mathcal{B}^{(0)} & \mathcal{B}^{(0)}\\ \mathcal{A}_J^{(0)} & \mathcal{Z}_J^{(0)} & \mathcal{B}^{(0)} & \mathcal{B}^{(0)}\\ \mathcal{B}^{(0)} & 0 & \mathcal{A}_M^{(0)} & \mathcal{A}_M^{(0)}\\ \mathcal{B}^{(0)} & 0 & \mathcal{A}_M^{(0)} & \mathcal{Z}_M^{(0)} \end{bmatrix},\qquad \boldsymbol{\mathcal{Z}}^{(1)} = \begin{bmatrix} \mathcal{A}_J^{(1)} & 0 & 0 & 0\\ \mathcal{A}_J^{(1)} & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

In particular, all integral operators featured in $\widehat{\boldsymbol{\mathcal{Z}}}_\gamma$ have a finite limit as $\gamma\to 0$ (this limit being nonzero in particular for all operators on the main diagonal), whereas in (14) $\mathcal{Z}_J$ diverges when $\gamma\to 0$. Likewise, the right-hand side $\boldsymbol{Y}$ in (14) has, by virtue of (18), the expansion

$$\boldsymbol{Y}_\gamma = \boldsymbol{Y}^{(0)} + O(\gamma^2), \qquad \boldsymbol{Y}^{(0)} = \begin{Bmatrix} \boldsymbol{\gamma}_\times^+ \boldsymbol{\Phi}_0^{(0)}[\boldsymbol{J}_{\text{inc}}] \\ \boldsymbol{\gamma}_\times^+ \boldsymbol{\Phi}_0^{(0)}[\boldsymbol{J}_{\text{inc}}] \\ \gamma_N^+ \boldsymbol{\Phi}_0^{(0)}[\boldsymbol{J}_{\text{inc}}] \\ \gamma_N^+ \boldsymbol{\Phi}_0^{(0)}[\boldsymbol{J}_{\text{inc}}] \end{Bmatrix}. \tag{22}$$

The following natural ansatz is then made for the unknown $\widehat{\boldsymbol{X}}_\gamma$ of problem (20):

$$\widehat{\boldsymbol{X}}_\gamma = \boldsymbol{X}^{(0)} + \gamma \boldsymbol{X}^{(1)} + \ldots, \qquad \boldsymbol{X}^{(0)} = \begin{Bmatrix} \boldsymbol{J}_{\text{L}}^{(0)} \\ \boldsymbol{J}_{\text{T}}^{(0)} \\ \boldsymbol{M}_{\text{L}}^{(0)} \\ \boldsymbol{M}_{\text{T}}^{(0)} \end{Bmatrix}, \quad \boldsymbol{X}^{(1)} = \begin{Bmatrix} \boldsymbol{J}_{\text{L}}^{(1)} \\ \boldsymbol{J}_{\text{T}}^{(1)} \\ \boldsymbol{M}_{\text{L}}^{(1)} \\ \boldsymbol{M}_{\text{T}}^{(1)} \end{Bmatrix}.$$

Inserting this ansatz, together with expansions (21) and (22), into problem (20) and setting to zero the resulting $O(1)$ and $O(\gamma)$ contributions, we find that $\boldsymbol{X}^{(0)}$ and $\boldsymbol{X}^{(1)}$ are sequentially governed by the following zeroth-order and first-order integral problems:

$$\begin{aligned} &\big\langle \widetilde{\boldsymbol{X}} \,,\, \mathcal{Z}^{(0)} \boldsymbol{X}^{(0)} \big\rangle_\times = \big\langle \widetilde{\boldsymbol{X}} \,,\, \boldsymbol{Y}^{(0)} \big\rangle_\times && \text{for all } \widetilde{\boldsymbol{X}} \in \mathbb{V}, \\ &\big\langle \widetilde{\boldsymbol{X}} \,,\, \mathcal{Z}^{(0)} \boldsymbol{X}^{(1)} \big\rangle_\times = \big\langle \widetilde{\boldsymbol{X}} \,,\, \boldsymbol{Y}^{(1)} - \mathcal{Z}^{(1)} \boldsymbol{X}^{(0)} \big\rangle_\times && \text{for all } \widetilde{\boldsymbol{X}} \in \mathbb{V}. \end{aligned} \tag{23}$$

Problem (23a) for $\boldsymbol{X}^{(0)}$ is found on inspection to coincide (after adjusting to the present notations) with the PMCHWT-type integral problem for the EC model established in [**14**]. Besides, since $G_0^{(1)}$ is a constant (see (17)), we have $\mathcal{A}_J^{(1)}[\boldsymbol{J}_{\text{L}}] = \boldsymbol{0}$ (for any $\text{div}_\Gamma$-free $\boldsymbol{J}_{\text{L}}$) and $\boldsymbol{\Phi}_0^{(1)}[\boldsymbol{J}_{\text{inc}}] = \boldsymbol{0}$ (because $\text{div}\, \boldsymbol{J}_{\text{inc}} = 0$ in $D$ and $\boldsymbol{J}_{\text{inc}} \cdot \boldsymbol{n} = 0$ on $\partial D$). Therefore $\boldsymbol{Y}^{(1)} - \boldsymbol{\mathcal{Z}}^{(1)} \boldsymbol{X}^{(0)} = \boldsymbol{0}$, implying that $\boldsymbol{X}^{(1)} = \boldsymbol{0}$.

**3.2. Resulting solution expansion and its justification.** The foregoing formal derivation yields the following expansion of the surface current densities:

$$\begin{Bmatrix} \boldsymbol{J}_{\text{L}} \\ \boldsymbol{J}_{\text{T}} \\ \boldsymbol{M}_{\text{L}} \\ \boldsymbol{M}_{\text{T}} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{J}_{\text{L}}^{(0)} \\ \gamma^2 \boldsymbol{J}_{\text{T}}^{(0)} \\ \boldsymbol{M}_{\text{L}}^{(0)} \\ \boldsymbol{M}_{\text{T}}^{(0)} \end{Bmatrix} + \begin{Bmatrix} O(\gamma^2) \\ O(\gamma^4) \\ O(\gamma^2) \\ O(\gamma^2) \end{Bmatrix} \tag{24}$$

where the leading terms solve the eddy current integral problem (23). To make more precise and justify the approximation order claimed in (24), we define the expansion error $\boldsymbol{\mathcal{E}}_\gamma$ on the rescaled surface current solution by

$$\boldsymbol{\mathcal{E}}_\gamma := \widehat{\boldsymbol{X}}_\gamma - (\boldsymbol{X}^{(0)} + \gamma \boldsymbol{X}^{(1)}).$$

Then, setting $\widehat{\boldsymbol{X}}_\gamma = \boldsymbol{\mathcal{E}}_\gamma + \boldsymbol{X}^{(0)} + \gamma \boldsymbol{X}^{(1)}$ in (20), the expansion error is found to solve the integral problem

$$\text{find } \boldsymbol{\mathcal{E}}_\gamma \in \mathbb{V} \text{ such that:} \quad \big\langle \widetilde{\boldsymbol{X}} \,,\, \widehat{\boldsymbol{\mathcal{Z}}}_\gamma \boldsymbol{\mathcal{E}}_\gamma \big\rangle_\times = \big\langle \widetilde{\boldsymbol{X}} \,,\, \boldsymbol{D}_\gamma \big\rangle_\times \quad \text{for all } \widetilde{\boldsymbol{X}} \in \mathbb{V}$$
$$\text{with} \quad \boldsymbol{D}_\gamma := \boldsymbol{Y}_\gamma - \widehat{\boldsymbol{\mathcal{Z}}}_\gamma \big( \boldsymbol{X}^{(0)} + \gamma \boldsymbol{X}^{(1)} \big). \tag{25}$$

Using these definitions, we state our main result in the following theorem, whose proof is given in Sec. 4:

**Theorem 1.** *The rescaled surface currents* $\widehat{\boldsymbol{X}}_\gamma$ *introduced in* (20) *admit the expansion*

$$\widehat{\boldsymbol{X}}_\gamma = \boldsymbol{X}^{(0)} + \boldsymbol{\mathcal{E}}_\gamma,$$

*where* $\boldsymbol{X}^{(0)}$ *solves the eddy current integral problem* (23) *and the expansion error* $\boldsymbol{\mathcal{E}}_\gamma$ *verifies*

$$\|\boldsymbol{\mathcal{E}}_\gamma\|_{\mathbb{V}} \le C\gamma^2$$

*for some constant* $C > 0$*. As a result, the expansions* (24) *of the surface currents* $\boldsymbol{J}_L, \boldsymbol{J}_T, \boldsymbol{M}_L, \boldsymbol{M}_T$ *hold in the sense of the* $\|\cdot\|_{\boldsymbol{\mathcal{V}}}$ *norm.*

**3.3. Expansion of related quantities.** Upon using the Helmholtz–Hodge decomposition of $\boldsymbol{J}$ in the Stratton–Chu integral representation formulas (11), recalling the definitions (9) of the Maxwell potentials and invoking Theorem 1, the following expansions are found for $\boldsymbol{E}_\gamma$ in $\Omega_0$ and $\Omega_1$:

$$\boldsymbol{E}_{0,\gamma} = \mathrm{i}\frac{\gamma\xi}{L}\sqrt{\frac{\mu_0}{\varepsilon_0}}\Big( -\boldsymbol{\Psi}_0^{(0)}[\boldsymbol{J}_{\mathrm{L}}^{(0)}] - \frac{L^2}{\xi^2}\boldsymbol{\nabla}\Psi_0^{(0)}[\mathrm{div}_S\boldsymbol{J}_{\mathrm{T}}^{(0)}] - \mathbf{rot}\,\boldsymbol{\Psi}_0^{(0)}[\boldsymbol{M}^{(0)}] + \boldsymbol{\Phi}_0^{(0)}[\boldsymbol{J}_{\mathrm{inc}}] + O(\gamma^2)\Big), \tag{26a}$$

$$\boldsymbol{E}_{1,\gamma} = \mathrm{i}\frac{\gamma\xi}{L}\sqrt{\frac{\mu_0}{\varepsilon_0}}\Big( \mu_{\mathrm{r}}\boldsymbol{\Psi}_1^{(0)}[\boldsymbol{J}_{\mathrm{L}}^{(0)}] + \mathbf{rot}\,\boldsymbol{\Psi}_1^{(0)}[\boldsymbol{M}^{(0)}] + O(\gamma^2)\Big). \tag{26b}$$

Since $\mathrm{i}\gamma\xi\sqrt{\mu_0/\varepsilon_0}/L = \mathrm{i}\omega\mu_0$ and $\gamma^2 = \omega\varepsilon_0/\sigma$, the above expansions, which hold pointwise, are consistent with the estimates $\|\boldsymbol{E}_{\mathrm{EC}}\|_2 = O(\omega)$, $\|\boldsymbol{E}_\omega - \boldsymbol{E}_{\mathrm{EC}}\|_2 = O(\omega^2)$ given in **[23]** (which focus on the frequency dependence). Likewise, the magnetic field $\boldsymbol{H}_0$ in $\Omega_0$ is readily found to verify the expansion

$$\boldsymbol{H}_{0,\gamma} = \mathbf{rot}\,\Big( -\boldsymbol{\Psi}_0^{(0)}[\boldsymbol{J}_{\mathrm{L}}^{(0)}] - \mathbf{rot}\,\boldsymbol{\Psi}_0^{(0)}[\boldsymbol{M}^{(0)}] + \boldsymbol{\Phi}_0^{(0)}[\boldsymbol{J}_{\mathrm{inc}}]\Big) + O(\gamma^2). \tag{26c}$$

Eddy current nondestructive testing exploits measurements of the impedance variation

$$\Delta Z_\gamma := \frac{1}{I^2}\int_\Gamma \big(\boldsymbol{M}_\gamma\cdot\mathbf{rot}\,\boldsymbol{E}_{\mathrm{inc}} + \boldsymbol{J}_\gamma\cdot\boldsymbol{E}_{\mathrm{inc}}\big)\,\mathrm{d}S, \tag{27}$$

(where $I$ is the current intensity in the excitating coil) which reflects the electromagnetic field perturbation (relative to $\boldsymbol{E}_{\mathrm{inc}}, \boldsymbol{H}_{\mathrm{inc}}$) induced by the conducting body $\Omega_1$. The asymptotic expansion of $\Delta Z_\gamma$ is readily found, using (7) and expansions (24), to be

$$\Delta Z_\gamma = \Delta Z_{\mathrm{EC}} + O(\gamma^3)$$
$$\text{with}\quad \Delta Z_{\mathrm{EC}} = \frac{\mathrm{i}\xi\gamma}{I^2L}\sqrt{\frac{\mu_0}{\varepsilon_0}}\int_\Gamma\big(\boldsymbol{M}_{\mathrm{T}}^{(0)}\cdot\mathbf{rot}\,\boldsymbol{\Phi}_0^{(0)}[\boldsymbol{J}_{\mathrm{inc}}] + \boldsymbol{J}_{\mathrm{L}}^{(0)}\cdot\boldsymbol{\Phi}_0^{(0)}[\boldsymbol{J}_{\mathrm{inc}}]\big)\,\mathrm{d}S, \tag{28}$$

having used that $\boldsymbol{\Phi}_0^{(1)}[\boldsymbol{J}_{\mathrm{inc}}] = \mathbf{0}$.

*Remark* 2. Although it contributes only at order $O(\gamma^2)$ to $\boldsymbol{J}$, the component $\boldsymbol{J}_{\mathrm{T}}$ participates to the leading-order approximation of $\boldsymbol{E}_0$ (but does not to those of $\boldsymbol{E}_1$, $\boldsymbol{H}$ and $\Delta Z$).

**4. Proof of Theorem 1.** The proof needs some estimates for expansions of the operators associated to the potentials $\boldsymbol{\Phi}_\ell^\gamma$ and $\boldsymbol{\Psi}_\ell^\gamma$ defined by (6) and (8), respectively, with $\kappa_\ell = \kappa_\ell(\gamma)$. In particular, it exploits the fact that the single-layer potentials (8) can be expressed using the volume potentials (6) and the dual Dirichlet trace mapping, so that all $\gamma$-dependent estimates occur in operator norm bounds for volume potential expansions. The latter are particular instances of pseudo-differential operators (PDOs), see e.g. **[15,** Chaps. 6,7] or **[12]**. As such their continuity properties between Sobolev spaces are known (e.g. **[12,** Thm. 11 of Chap. 2]), but those results do not provide information on how continuity constants depend on parameters (such as $\gamma$ here). Lemma 3 gives $\gamma$-dependent estimates for relevant differences of volume potentials; its proof rests on ideas from PDO theory but presented in more elementary, self-contained and explicit terms that allow to investigate the required dependencies in $\gamma$. We first state Lemma 3, then give the resulting needed estimates in Lemmas 4 and 5 before returning to the proof of Theorem 1. The proofs of the lemmas are finally provided in separate subsections.

**Lemma 3.** *The volume potential differences* $D^{(0)}\boldsymbol{\Phi}_0^\gamma := \boldsymbol{\Phi}_0^\gamma - \boldsymbol{\Phi}_0^{(0)}$, $D^{(1)}\boldsymbol{\Phi}_0^\gamma := \boldsymbol{\Phi}_0^\gamma - \boldsymbol{\Phi}_0^{(0)} - \gamma\boldsymbol{\Phi}_0^{(1)}$ *and* $D^{(0)}\boldsymbol{\Phi}_1^\gamma := \boldsymbol{\Phi}_1^\gamma - \boldsymbol{\Phi}_1^{(0)}$, *where* $\boldsymbol{\Phi}_\ell^\gamma$ *is the volume potential defined by* (6) *with* $\kappa_\ell = \kappa_\ell(\gamma)$, *are continuous* $\boldsymbol{H}^{-1}_{comp}(\mathbb{R}^3) \to \boldsymbol{H}^3_{loc}(\mathbb{R}^3)$ *linear operators. Moreover, the corresponding operator norms satisfy for some* $C > 0$ *the estimates*

$$\textit{(a)}\ \ \|D^{(0)}\boldsymbol{\Phi}_0^\gamma\| \le C\gamma, \quad \textit{(b)}\ \ \|D^{(1)}\boldsymbol{\Phi}_0^\gamma\| \le C\gamma^2, \quad \textit{(c)}\ \ \|D^{(0)}\boldsymbol{\Phi}_1^\gamma\| \le C\gamma^2.$$

*In addition,* $\boldsymbol{\nabla}D^{(0)}\boldsymbol{\Phi}_\ell^\gamma$ *and* $\mathbf{rot}\,D^{(0)}\boldsymbol{\Phi}_\ell^\gamma$ *are continuous* $\boldsymbol{H}^{-1}_{comp}(\mathbb{R}^3) \to \boldsymbol{H}^2_{loc}(\mathbb{R}^3)$ *linear operators whose norms satisfy for some* $C > 0$ *the estimates*

$$\textit{(d)}\ \ \|\boldsymbol{\nabla}D^{(0)}\boldsymbol{\Phi}_\ell^\gamma\| \le C\gamma^2, \quad \textit{(e)}\ \ \|\mathbf{rot}\,D^{(0)}\boldsymbol{\Phi}_\ell^\gamma\| \le C\gamma^2.$$

**Lemma 4.** *For $\ell = 0, 1$, let $D\boldsymbol{\Psi}_\ell^\gamma := \boldsymbol{\Psi}_\ell^\gamma - \boldsymbol{\Psi}_\ell^0$, where $\boldsymbol{\Psi}_\ell^\gamma$ is the Helmholtz surface potential defined by* (8) *with $\kappa_\ell = \kappa_\ell(\gamma)$. $D\boldsymbol{\Psi}_\ell^\gamma$ defines continuous $\boldsymbol{\mathcal{V}} \to \boldsymbol{H}_{loc}(\mathbf{rot}\,, \mathbb{R}^3 \backslash \Gamma)$ and $\boldsymbol{\mathcal{V}} \to \boldsymbol{H}_{loc}(\mathbf{rot}\,\mathbf{rot}\,, \mathbb{R}^3 \backslash \Gamma)$ operators. Moreover, in both cases, there exists a constant $C$ such that for all $\gamma$ small enough*

$$\text{(a)}\ \ \big\| \boldsymbol{\gamma}_\times D^{(0)} \boldsymbol{\Psi}_0^\gamma \big\| \le C\gamma, \qquad \text{(b)}\ \ \big\| \boldsymbol{\gamma}_\times D^{(1)} \boldsymbol{\Psi}_0^\gamma \big\| \le C\gamma^2, \qquad \text{(c)}\ \ \big\| \boldsymbol{\gamma}_\times D^{(0)} \boldsymbol{\Psi}_1^\gamma \big\| \le C\gamma^2$$

*and*

$$\text{(d)}\ \ \big\| \boldsymbol{\gamma}_\times \boldsymbol{\nabla} D^{(0)} \boldsymbol{\Psi}_\ell^\gamma \big\| \le C\gamma^2, \qquad \text{(e)}\ \ \big\| \boldsymbol{\gamma}_N D^{(0)} \boldsymbol{\Psi}_\ell^\gamma \big\| \le C\gamma^2.$$

**Lemma 5.** *(i) The right-hand side $\boldsymbol{Y}_\gamma$ of problem* (14) *verifies (with the second equality stemming from Lemma 2)*

$$\|\boldsymbol{Y}_\gamma - \boldsymbol{Y}^{(0)} - \gamma \boldsymbol{Y}^{(1)}\|_{\mathbb{V}} = \|\boldsymbol{Y}_\gamma - \boldsymbol{Y}^{(0)}\|_{\mathbb{V}} = O(\gamma^2).$$

*(ii) The first-order expansion* (21) *of the operator matrix $\widehat{\boldsymbol{X}}_\gamma$ and the corresponding zeroth-order expansion hold, with expansion error estimates given in terms of the $\mathcal{L}(\mathbb{V})$ operator norm by*

$$\text{(a)}\ \ \|\widehat{\boldsymbol{\mathcal{Z}}}_\gamma - \boldsymbol{\mathcal{Z}}^{(0)}\| = O(\gamma), \qquad \text{(b)}\ \ \|\widehat{\boldsymbol{\mathcal{Z}}}_\gamma - \boldsymbol{\mathcal{Z}}^{(0)} - \gamma \boldsymbol{\mathcal{Z}}^{(1)}\| = O(\gamma^2).$$

*Proof of Theorem 1.* Recalling that the expansion error solves problem (25), the claimed estimate of $\|\boldsymbol{\mathcal{E}}_\gamma\|$ can be proved by showing that there exists $\gamma_0 > 0$ such that (i) $\widehat{\boldsymbol{\mathcal{Z}}}_\gamma$ is boundedly invertible, the bound being uniform for $\gamma \in [0, \gamma_0[$ and (ii) $\|\boldsymbol{D}_\gamma\|_{\mathbb{V}} \le C\gamma^2$ for all $\gamma \in [0, \gamma_0[$.

Regarding item (i), we first observe that $\boldsymbol{\mathcal{Z}}^{(0)} : \boldsymbol{\mathcal{V}} \to \boldsymbol{\mathcal{V}}$ is elliptic (up to notational adjustments, this is Theorem 12 of [**14**]) and hence boundedly invertible. We can therefore write

$$\widehat{\boldsymbol{\mathcal{Z}}}_\gamma = \boldsymbol{\mathcal{Z}}^{(0)} \big[ \boldsymbol{\mathcal{I}} + \boldsymbol{\mathcal{K}}_\gamma \big] \qquad \text{with}\ \ \boldsymbol{\mathcal{K}}_\gamma := (\boldsymbol{\mathcal{Z}}^{(0)})^{-1} \big( \widehat{\boldsymbol{\mathcal{Z}}}_\gamma - \boldsymbol{\mathcal{Z}}^{(0)} \big).$$

By the second part of Lemma 5, we have $\|\boldsymbol{\mathcal{K}}_\gamma\|_{\mathbb{V}\to\mathbb{V}} \le C\gamma$ for some $C > 0$ and any small enough $\gamma$. Therefore there exists $\gamma_0 > 0$ and $C_{\mathcal{K}} < 1$ such that $\|\boldsymbol{\mathcal{K}}_\gamma\|_{\mathbb{V}\to\mathbb{V}} \le C_{\mathcal{K}}$ for any $\gamma < \gamma_0$. Using a standard Neumann series argument, $\boldsymbol{\mathcal{I}} + \boldsymbol{\mathcal{K}}_\gamma$ is therefore invertible with $\|(\boldsymbol{\mathcal{I}} + \boldsymbol{\mathcal{K}}_\gamma)^{-1}\|_{\mathbb{V}\to\mathbb{V}} \le (1 - C_{\mathcal{K}})^{-1}$ for any $\gamma < \gamma_0$. Concluding, $\widehat{\boldsymbol{\mathcal{Z}}}_\gamma$ is boundedly invertible, uniformly for $\gamma \in [0, \gamma_0[$.

To address item (ii), recasting $\boldsymbol{D}_\gamma$ as

$$\begin{aligned} \boldsymbol{D}_\gamma &= \boldsymbol{Y}_\gamma - \big( \widehat{\boldsymbol{\mathcal{Z}}}_\gamma - \boldsymbol{\mathcal{Z}}^{(0)} - \gamma \boldsymbol{\mathcal{Z}}^{(1)} \big) \boldsymbol{X}^{(0)} - \big( \boldsymbol{\mathcal{Z}}^{(0)} + \gamma \boldsymbol{\mathcal{Z}}^{(1)} \big) \boldsymbol{X}^{(0)} \\ &= \big( \boldsymbol{Y}_\gamma - \boldsymbol{Y}^{(0)} - \gamma \boldsymbol{Y}^{(1)} \big) - \big( \widehat{\boldsymbol{\mathcal{Z}}}_\gamma - \boldsymbol{\mathcal{Z}}^{(0)} - \gamma \boldsymbol{\mathcal{Z}}^{(1)} \big) \boldsymbol{X}^{(0)} \end{aligned}$$

(having used that $\boldsymbol{X}^{(1)} = \mathbf{0}$) and invoking Lemma 5, we directly obtain that $\|\boldsymbol{D}_\gamma\|_{\mathbb{V}} \le C\gamma^2$ for all $\gamma \in [0, \gamma_0[$ (with $C = C(\gamma_0)$) since $\boldsymbol{X}^{(0)} \in \boldsymbol{\mathcal{V}}$. The proof of Theorem 1 is complete. ☐

**4.1. Proof of Lemma 3.** We first consider the easier-to-address case of estimate (c). The fundamental solution $G(\boldsymbol{z}; \kappa_1(\gamma))$ is in $L^1(\mathbb{R}^3)$ for any $\gamma$ (since it is only weakly singular at the origin, and its exponential decay for large $|\boldsymbol{z}|$ is ensured by $\Im\big(\kappa_1(\gamma)\big) > 0$, see (16)). Its Fourier transform is easily found (e.g. by transforming the governing equation $-\big[\Delta + \kappa_1(\gamma)^2\big] G(\cdot; \kappa_1(\gamma)) = \delta$) to be given by

$$\mathcal{F}\big[G(\cdot; \kappa_1(\gamma))\big](\boldsymbol{\zeta}) = \frac{1}{\kappa_1(\gamma)^2 - |\boldsymbol{\zeta}|^2}, \tag{29}$$

where $\boldsymbol{\zeta} \in \mathbb{R}^3$ is the Fourier conjugate variable of $\boldsymbol{z}$. By the classical form of the Fourier convolution theorem, we then have $\mathcal{F}\big(\boldsymbol{\Phi}_1^\gamma[\boldsymbol{u}]\big) = \mathcal{F}\big(G(\boldsymbol{z}; \kappa_1(\gamma)) \mathcal{F}(\boldsymbol{u})$ for any (scalar- or vector-valued) $C_0^\infty(\mathbb{R}^3)$ density $\boldsymbol{u}$. For some negative integer $m$ and real $s$, we can therefore write

$$(1 + |\boldsymbol{\zeta}|^2)^{s-m} \big| \mathcal{F}\big( D^{(0)} \boldsymbol{\Phi}_1^\gamma[\boldsymbol{u}] \big) \big|^2 = K(\boldsymbol{\zeta}) \times (1 + |\boldsymbol{\zeta}|^2)^s \big| \mathcal{F}(\boldsymbol{u}) \big|^2 \tag{30}$$

with (recalling (29))

$$K(\boldsymbol{\zeta}) := (1 + |\boldsymbol{\zeta}|^2)^{-m} \big| \mathcal{F}\big( G(\cdot; \kappa_1(\gamma)) \big) - \mathcal{F}\big( G(\cdot; \kappa_1(0)) \big) \big|^2(\boldsymbol{\zeta}) = \frac{(1 + |\boldsymbol{\zeta}|^2)^{-m} \big| \kappa_1(0) - \kappa_1(\gamma) \big|^2}{\big| \kappa_1(\gamma)^2 - |\boldsymbol{\zeta}|^2 \big|^2 \, \big| \kappa_1(0)^2 - |\boldsymbol{\zeta}|^2 \big|^2}$$

Let $\gamma_0>0$. Recalling (16) and noting that the function $\gamma\mapsto\kappa_1(\gamma)$ is differentiable for $\gamma\in[0,\gamma_0]$, we have

$$K(\boldsymbol{\zeta})\le C\gamma^4$$

for some constant $C=C(m)>0$ and any $m\ge -4$. Using this for $m=-4$ in (30) and integrating the resulting inequality over $\boldsymbol{\zeta}\in\mathbb{R}^3$, we obtain

$$\big\| D^{(0)}\boldsymbol{\Phi}_1^{\gamma}[\boldsymbol{u}]\big\|_{s+4}\le C\gamma^2\|\boldsymbol{u}\|_s$$

($\|\cdot\|_s$ being the $H^s$ norm), from which estimate (c) follows from the density of $C_0^\infty(\mathbb{R}^3)$ in $H^s_{\text{comp}}(\mathbb{R}^3)$ and by choosing $s=-1$.

*Estimates (a), (b).* We now address estimates (a), (b), where $\kappa_0(\gamma)=\gamma\xi L^{-1}\in\mathbb{R}$ and $\kappa_0(0)=0$. Setting for notational convenience $\kappa:=\kappa_0(\gamma)$, we define the integral operators $H^k_\kappa$ $(k=0,1)$ as $H^{(k)}_\kappa=\kappa^{-k-1}D^{(k)}\boldsymbol{\Phi}_0^{\gamma}$. Since $\kappa_0(\gamma)=O(\gamma)$, proving the $\boldsymbol{H}^{-1}_{\text{comp}}(\mathbb{R}^3)\to\boldsymbol{H}^3_{\text{loc}}(\mathbb{R}^3)$ estimates (a), (b) then amounts to showing that, for some $\bar\kappa>0$, $H^{(k)}_\kappa$ and for each compact subset $K$ of $\mathbb{R}^3$ and $\phi\in C_0^\infty(\mathbb{R}^3)$, we have

$$\|H^{(k)}_\kappa[\phi\boldsymbol{u}]\|_3\le C\|\boldsymbol{u}\|_{-1}\qquad\text{for all }\boldsymbol{u},\ \operatorname{supp}(\boldsymbol{u})\subset K\qquad(k=0,1),\tag{31}$$

uniformly in $\kappa$ for $\kappa\in[0,\bar\kappa[$. Again it is sufficient to assume that $\boldsymbol{u}\in C_0^\infty(\mathbb{R}^3;\mathbb{R}^3)$.

The (convolution) operators $H^{(k)}_\kappa$ $(k=0,1)$ have kernel functions $h^{(k)}_\kappa(\boldsymbol{z})$ given by

$$h^{(k)}_\kappa(\boldsymbol{z})=|\boldsymbol{z}|^k h^k(\kappa|\boldsymbol{z}|)\qquad\text{with}\quad h^0(t)=\frac{1}{4\pi t}\big(e^{\mathrm{i}t}-1\big),\quad h^1(t)=\frac{1}{4\pi t^2}\big(e^{\mathrm{i}t}-1-\mathrm{i}t\big).\tag{32}$$

Both functions $h^k$ are $C^\infty(]0,+\infty[)$; moreover $|h^k(t)|\to 0$ as $t\to+\infty$ while $h^0(t)\to\mathrm{i}$ and $h^1(t)\to -\frac12$ as $t\to 0^+$. Consequently, both functions $[0,\infty[\supset t\mapsto|h^k(t)|$ are bounded, and this bound of course does not depend on $\kappa$, an observation that will play an important role.

The wavenumber $\kappa$ being real, the kernel functions are not in $L^1(\mathbb{R}^3)$ due to insufficient decay at infinity. Letting $\psi$ be a fixed $C_0^\infty$ cut-off function such that $\psi(z)=1$ in a neighborhood of $z=0$ and $\psi(z)=0$ for $|z|>1$, we decompose the operators $H^{(k)}_\kappa$ in the form

$$\begin{aligned}H^{(k)}_\kappa[\boldsymbol{u}](\boldsymbol{x})&=N^{(k)}_\kappa[\boldsymbol{u}]+R^{(k)}_\kappa[\boldsymbol{u}]\\&=\int_{\mathbb{R}^3}h^{(k)}_\kappa(\boldsymbol{x}-\boldsymbol{x}')\psi(|\boldsymbol{x}-\boldsymbol{x}'|)\boldsymbol{u}(\boldsymbol{x}')\,\mathrm{d}\boldsymbol{x}'+\int_{\mathbb{R}^3}h^{(k)}_\kappa(\boldsymbol{x}-\boldsymbol{x}')\big[1-\psi(|\boldsymbol{x}-\boldsymbol{x}'|)\big]\boldsymbol{u}(\boldsymbol{x}')\,\mathrm{d}\boldsymbol{x}'\end{aligned}$$

and examine separately the relevant boundedness of each resulting operator.

We begin with operators $N^{(k)}_\kappa$. Their convolution kernels are in $L^1(\mathbb{R}^3)$, so that the classical form of the Fourier convolution theorem again applies. Setting $\boldsymbol{v}^{(k)}:=H^{(k)}_\kappa[\boldsymbol{u}]$, we thus have

$$\mathcal{F}\big(\boldsymbol{v}^{(k)}\big)=a^{(k)}_\kappa\mathcal{F}(\boldsymbol{u}),\qquad a^{(k)}_\kappa(\boldsymbol{\zeta}):=\mathcal{F}\big[h^{(k)}_\kappa\psi(|\cdot|)\big](\boldsymbol{\zeta})=\int_{\mathbb{R}^3}e^{-\mathrm{i}\boldsymbol{\zeta}\cdot\boldsymbol{z}}h^{(k)}_\kappa(\boldsymbol{z})\psi(|\boldsymbol{z}|)\,\mathrm{d}\boldsymbol{z}.\tag{33}$$

Due to the previously-mentioned boundedness of functions $h^k(t)$, both symbols $\boldsymbol{\zeta}\mapsto a^{(k)}_\kappa(\boldsymbol{\zeta})$ are readily found to be bounded uniformly in $\kappa$; moreover, we have (see details at end of proof)

$$|a^{(0)}_\kappa(\boldsymbol{\zeta})|=\frac{\kappa}{(\kappa^2-|\boldsymbol{\zeta}|^2)|\boldsymbol{\zeta}|^2}+o(|\boldsymbol{\zeta}|^{-4}),\qquad|a^{(1)}_\kappa(\boldsymbol{\zeta})|=\frac{1}{(\kappa^2-|\boldsymbol{\zeta}|^2)|\boldsymbol{\zeta}|^2}+o(|\boldsymbol{\zeta}|^{-4})\qquad|\boldsymbol{\zeta}|\to\infty.\tag{34}$$

Consequently, there exists $C>0$ such that $|a^{(k)}_\kappa(\boldsymbol{\zeta})|\le C(1+|\boldsymbol{\zeta}|^2)^{-2}$ for all $\kappa$. To show that operators $N^{(k)}_\kappa$ verify an estimate of the form (31), we note (following the proof of **[12,** Thm. 11 of Chap. 2] and since $\mathcal{F}(\phi\boldsymbol{v})=(2\pi)^{-3}\mathcal{F}(\phi)\star\mathcal{F}(\boldsymbol{v})$) that

$$(1+|\boldsymbol{\eta}|^2)^{3/2}\mathcal{F}\big(\phi\boldsymbol{v}^{(k)}\big)(\boldsymbol{\eta})=\int_{\mathbb{R}^3}K(\boldsymbol{\eta},\boldsymbol{\zeta})(1+|\boldsymbol{\zeta}|^2)^{-1/2}\mathcal{F}(\boldsymbol{u})(\boldsymbol{\zeta})\,\mathrm{d}\boldsymbol{\zeta}$$

with

$$K(\boldsymbol{\eta},\boldsymbol{\zeta})=(2\pi)^{-3}(1+|\boldsymbol{\eta}|^2)^{3/2}(1+|\boldsymbol{\zeta}|^2)^{1/2}a^{(k)}_\kappa(\boldsymbol{\zeta})\mathcal{F}[\phi](\boldsymbol{\eta}-\boldsymbol{\zeta})$$

and the sought estimate amounts to showing that the kernel $K(\boldsymbol{\eta},\boldsymbol{\zeta})$ defines a (uniformly in $\kappa$) bounded $L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ operator. Due to the estimates $|a_\kappa^{(k)}(\boldsymbol{\zeta})| \leq C(1+|\boldsymbol{\zeta}|^2)^{-2}$ (see above) and $\big|\mathcal{F}[\phi](\boldsymbol{\xi})\big| \leq C(1+|\boldsymbol{\xi}|^2)^{-N}$ for any $N$ (by virtue of $\phi \in C_0^\infty(\mathbb{R}^3)$), we have

$$\big|K(\boldsymbol{\eta},\boldsymbol{\zeta})\big| \leq C(1+|\boldsymbol{\eta}-\boldsymbol{\zeta}|^2)^{-N}(1+|\boldsymbol{\eta}|^2)^{3/2}(1+|\boldsymbol{\zeta}|^2)^{-3/2}.$$

Applying Peetre's inequality $1+|\boldsymbol{\eta}|^2 \leq 2(1+|\boldsymbol{\eta}-\boldsymbol{\zeta}|^2)(1+|\boldsymbol{\zeta}|^2)$, we deduce

$$\big|K(\boldsymbol{\eta},\boldsymbol{\zeta})\big| \leq C(1+|\boldsymbol{\eta}-\boldsymbol{\zeta}|^2)^{-N+3/2}$$

so that, picking any $N>3$, we have

$$\int_{\mathbb{R}^3} \big|K(\boldsymbol{\eta},\boldsymbol{\zeta})\big| \,\mathrm{d}\boldsymbol{\eta} \leq C_N, \quad \int_{\mathbb{R}^3} \big|K(\boldsymbol{\eta},\boldsymbol{\zeta})\big| \,\mathrm{d}\boldsymbol{\zeta} \leq C_N \qquad \text{with} \quad C_N := C\int_{\mathbb{R}^3} (1+|\boldsymbol{\xi}|^2)^{-N+3/2}\,\mathrm{d}\boldsymbol{\xi}.$$

The above inequalities constitute the verification of the Schur test (see e.g. [**12,** Lemma 10 of Chap. 2]), implying that the kernel $K$ defines a bounded $L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ operator and that moreover its operator norm is not larger than $C_N$, i.e. is bounded uniformly in $\kappa$.

We now turn to the operators $R_\kappa^{(k)}$, whose convolution kernels $k_\kappa^{(k)} := h_\kappa^{(k)}\big[1-\psi(\cdot)\big]$ are $C^\infty(\mathbb{R}^3)$ functions. We will this time estimate Sobolev norms $\|\cdot\|_s$ without using the Fourier transform. First, let $\varphi_K$ be a $C_c^\infty(\mathbb{R}^3)$ cut-off function such that $\varphi_K=1$ in a neighborhood of the compact set $K$; then for any $\boldsymbol{u}$ such that $\text{supp}(\boldsymbol{u})\subset K$ we have $\boldsymbol{u}=\varphi_K\boldsymbol{u}$ and

$$\phi(\boldsymbol{x})R_\kappa^{(k)}[\boldsymbol{u}](\boldsymbol{x}) = \int_{\mathbb{R}^3} K^k(\boldsymbol{x},\boldsymbol{x}')\,\boldsymbol{u}(\boldsymbol{x}')\,\mathrm{d}\boldsymbol{x}', \qquad K^k(\boldsymbol{x},\boldsymbol{x}') := \phi(\boldsymbol{x})k_\kappa^{(k)}(\boldsymbol{x}-\boldsymbol{x}')\varphi_K(\boldsymbol{x}').$$

Using again the boundedness of functions $h^k(t)$ over $\mathbb{R}^+$, we have $\big\|K^k(\boldsymbol{x},\boldsymbol{x}')\big\| \leq \big\||\boldsymbol{x}-\boldsymbol{x}'|^k\phi(\boldsymbol{x})\varphi_K(\boldsymbol{x}')\big\|$, and therefore

$$\int_{\mathbb{R}^3} \big\|K^k(\boldsymbol{x},\boldsymbol{x}')\big\|\,\mathrm{d}\boldsymbol{x} \leq C_1^k, \qquad \int_{\mathbb{R}^3} \big\|K^k(\boldsymbol{x},\boldsymbol{x}')\big\|\,\mathrm{d}\boldsymbol{x}' \leq C_2^k,$$

with the constants $C_1^k, C_2^k$ given by (recalling that $\phi \in C_c^\infty(\mathbb{R}^3)$)

$$C_1^k = B^k\int_{\mathbb{R}^3} \phi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}', \qquad C_2^k = B^k\int_{\mathbb{R}^3} \varphi_K(\boldsymbol{x}')\,\mathrm{d}\boldsymbol{x}', \qquad B := \sup_{\boldsymbol{x}\in\text{supp}(\phi),\,\boldsymbol{x}'\in\text{supp}(\varphi_K)} |\boldsymbol{x}-\boldsymbol{x}'|.$$

The Schur test therefore shows that kernels $K^k$ define bounded $L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ operators, and that in fact $\big\|\phi R_\kappa^{(k)}[\boldsymbol{u}]\big\|_0 \leq B^k\sqrt{C_1^kC_2^k}\|\boldsymbol{u}\|_0$ (so the operator norm bound is again uniform in $\kappa$). Since $k_\kappa^{(k)}(\boldsymbol{x}-\boldsymbol{x}')$ is a $C^\infty(\mathbb{R}^3)$ kernel, we can proceed similarly for estimating the seminorms $\big\|\boldsymbol{\nabla}^m\big(\phi R_\kappa^{(k)}[\boldsymbol{u}]\big)\big\|_0$ for any differentiation order $m$. For example, with $m=1$, we have

$$\boldsymbol{\nabla}\big(\phi R_\kappa^{(k)}[\boldsymbol{u}]\big)(\boldsymbol{x}) = \int_{\mathbb{R}^3} k_\kappa^{(k)}(\boldsymbol{x}-\boldsymbol{x}')\big[(\varphi_K\boldsymbol{u})(\boldsymbol{x}')\boldsymbol{\nabla}\phi(\boldsymbol{x}) + \phi(\boldsymbol{x})\boldsymbol{\nabla}(\varphi_K\boldsymbol{u})(\boldsymbol{x}')\big]\,\mathrm{d}\boldsymbol{x}',$$

(after integration by parts) and $\boldsymbol{\nabla}^m\big(\phi R_\kappa^{(k)}[\boldsymbol{u}]\big)$ is easily seen to be given by a similar, lengthier, formula involving $k_\kappa^{(k)}(\boldsymbol{x}-\boldsymbol{x}')$ and derivatives up to order $m$ in its cofactor. Applying the Schur test to each additive component of $\boldsymbol{\nabla}^m\big(\phi R_\kappa^{(k)}[\boldsymbol{u}]\big)$ then shows the existence of a $\kappa$-independent constant $A_m$ such that

$$\big\|\boldsymbol{\nabla}^m\big(\phi R_\kappa^{(k)}[\boldsymbol{u}]\big)\big\|_0 \leq A_m\|\boldsymbol{u}\|_m,$$

which implies in turn the existence of $\kappa$-independent constants $C'_m, C_m$ such that

$$\big\|\phi R_\kappa^{(k)}[\boldsymbol{u}]\big\|_m \leq C'_m\|\boldsymbol{u}\|_m \leq C_m\|\boldsymbol{u}\|_{-1},$$

the last equality resulting from the well-known continuous injection of $\boldsymbol{H}^m(\mathbb{R}^3)$ into $\boldsymbol{H}^{-1}(\mathbb{R}^3)$. The desired estimate (31) for $R_\kappa^{(k)}$ follows, and this completes the proof of estimates (a), (b).

Finally, estimates (d), (e) are direct consequences of estimates (a), (b) since $\|\boldsymbol{\nabla}\boldsymbol{u}\|_2 \leq \|\boldsymbol{u}\|_3$ for any $\boldsymbol{u}\in\boldsymbol{H}^3_{\text{loc}}(\mathbb{R}^3)$. This completes the proof of the Lemma.

*Proof of symbol decay* (34). The symbols $a_\kappa^{(k)}$ introduced in (33), being Fourier transform of radial functions (see e.g. [**24,** Thm. 3.3]), can be expressed as

$$a_\kappa^{(k)}(\boldsymbol{\zeta}) = \frac{4\pi}{\zeta}\int_0^\infty \sin(\zeta z) z^{k+1} h^k(\kappa z)\psi(z)\,\mathrm{d}z \tag{35}$$

(with $z := |\boldsymbol{z}|$ and $\zeta := |\boldsymbol{\zeta}|$), which decay as $\zeta \to +\infty$. Moreover, using definitions (32) of functions $h^k$ and elementary calculus, we find the primitives

$$\begin{aligned}\int \sin(\zeta z) z h^0(\kappa z)\,\mathrm{d}z = f_\kappa^{(0)}(z) &= \frac{1}{2\kappa}\Big[\frac{e^{\mathrm{i}z(\kappa-\zeta)}}{\kappa-\zeta} - \frac{e^{\mathrm{i}z(\kappa+\zeta)}}{\kappa+\zeta} + \frac{2\cos(\zeta z)}{\zeta^2}\Big],\\ \int \sin(\zeta z) z^2 h^1(\kappa z)\,\mathrm{d}z = f_\kappa^{(1)}(z) &= \frac{1}{\kappa} f_\kappa^{(0)}(z) + \frac{\mathrm{i}}{\kappa}\Big[\frac{z\cos(\zeta z)}{\zeta^2} - \frac{\sin(\zeta z)}{\zeta^3}\Big].\end{aligned}$$

Integrating by parts the representations (35) and since $\psi(0)=1$, $\psi(\infty)=0$, the symbols $a_\kappa^{(k)}$ become

$$a_\kappa^{(k)}(\boldsymbol{\zeta}) = -f_\kappa^{(k)}(0) - \int_0^\infty f_\kappa^{(k)}(z)\psi'(z)\,\mathrm{d}z. \tag{36}$$

The above integral term can then be further subjected to arbitrarily many iterated integration by parts (IBPs), each IBP increasing the decay rate in $\zeta$ of the resulting integrand while all subsequently arising non-integral terms vanish since $\psi^{(m)}(0)=\psi^{(m)}(\infty)=0$ $(m\geq 1)$. The integral term of (36) is therefore found to decay faster than any negative power of $\zeta$. On the other hand, we easily find

$$f_\kappa^{(0)}(0) = \kappa f_\kappa^{(1)}(0) = \frac{\kappa}{\zeta^2(\kappa^2-\zeta^2)}$$

which, used in (36), completes the proof of the leading large-$|\boldsymbol{\zeta}|$ behavior (34) of the symbols $a_\kappa^{(k)}$. We note in passing that, as Fourier transforms of compactly-supported functions, the symbols $a_\kappa^{(k)}$ are $C^\infty$ functions; in particular they are well defined if $|\boldsymbol{\zeta}|=\kappa$, formulas (34) notwithstanding.

**4.2. Proof of Lemma 4.** The single-layer potential $\boldsymbol{\Psi}_\ell^\gamma$ has (see [**22,** Def. 3.1.5]) the representation $\boldsymbol{\Psi}_\ell^\gamma = \boldsymbol{\Phi}_\ell^\gamma \boldsymbol{\gamma}_0'$, where $\boldsymbol{\Phi}_\ell^\gamma$ is the volume potential (6) with $\kappa_\ell=\kappa_\ell(\gamma)$ and $\boldsymbol{\gamma}_0'$ is the dual of the Dirichlet trace mapping $\boldsymbol{\gamma}_0$. The claimed mapping properties and estimates (a)–(e) then result directly from combining (i) the known continuity of $\boldsymbol{\gamma}_0' : \boldsymbol{H}^{-1/2}(\Gamma) \to \boldsymbol{H}^{-1}_{\mathrm{comp}}(\mathbb{R}^3)$ and the mapping properties of $\boldsymbol{\gamma}_\times^\pm$ recalled in Lemma 1, and (ii) the mapping properties and estimates for relevant volume potential differences established in Lemma 3 (recalling that $\boldsymbol{\gamma}_N^\pm = \boldsymbol{\gamma}_\times^\pm \circ \mathbf{rot}$ ).

**4.3. Proof of Lemma 5.** *Part (i):* since $\boldsymbol{Y}_\gamma - \boldsymbol{Y}^{(0)} - \gamma\boldsymbol{Y}^{(1)}$ uses $D^{(1)}\boldsymbol{\Phi}_0^\gamma[\boldsymbol{J}_{\mathrm{inc}}]$ (see (22)), part (i) follows from estimates (b), (e) of Lemma 3 and the continuity properties of traces recalled in Lemma 1.

*Part (ii).* We first note that the operator matrix $\widehat{\boldsymbol{\mathcal{Z}}}_\gamma - \boldsymbol{\mathcal{Z}}^{(0)} - \gamma\boldsymbol{\mathcal{Z}}^{(1)}$ involves the operator differences

$$\begin{aligned}
&\text{(a)} & \mathcal{A}_M - \mathcal{A}_M^{(0)} &= \{\boldsymbol{\gamma}_\times\}\big(\kappa_0^2\boldsymbol{\Psi}_0^\gamma + \mu_{\mathrm{r}}^{-1}\kappa_1^2 D^{(0)}\boldsymbol{\Psi}_1^\gamma\big)\\
&\text{(b)} & \mathcal{Z}_M - \mathcal{Z}_M^{(0)} &= \mathcal{A}_M - \mathcal{A}_M^{(0)} + \{\boldsymbol{\gamma}_\times\}\big[\big(\boldsymbol{\nabla} D^{(0)}\boldsymbol{\Psi}_0^\gamma + \mu_{\mathrm{r}}^{-1}\boldsymbol{\nabla} D^{(0)}\boldsymbol{\Psi}_1^\gamma\big)\circ \mathrm{div}_S\big]\\
&\text{(c)} & \mathcal{B} - \mathcal{B}^{(0)} &= \{\boldsymbol{\gamma}_N\}\big(D^{(0)}\boldsymbol{\Psi}_0^\gamma + D^{(0)}\boldsymbol{\Psi}_1^\gamma\big)\\
&\text{(d)} & \mathcal{A}_J - \mathcal{A}_J^{(0)} - \gamma\mathcal{A}_J^{(1)} &= \{\boldsymbol{\gamma}_\times\}\big(D^{(1)}\boldsymbol{\Psi}_0^\gamma + \mu_{\mathrm{r}} D^{(0)}\boldsymbol{\Psi}_1^\gamma\big)\\
&\text{(e)} & \gamma^2\mathcal{Z}_J - \mathcal{Z}_J^{(0)} &= \gamma^2\mathcal{A}_J^{(0)} + \xi^{-2}L^2\{\boldsymbol{\gamma}_\times\}\boldsymbol{\nabla} D^{(0)}\boldsymbol{\Psi}_0^\gamma\circ\mathrm{div}_S + \gamma^2\kappa_1^{-2}\mu_{\mathrm{r}}\{\boldsymbol{\gamma}_\times\}\boldsymbol{\nabla}\boldsymbol{\Psi}_1^{(0)}\circ\mathrm{div}_S,
\end{aligned} \tag{37}$$

and the operators $\gamma^2\mathcal{B}$, $\gamma^2\mathcal{A}_J$; see (20) and (21). Lemma 4 implies that the operator differences (37) all define continuous $\boldsymbol{\mathcal{V}} \to \boldsymbol{\mathcal{V}}$ operators whose norm is $O(\gamma^2)$ for $\gamma$ small enough (recalling for (37a) that in addition $\kappa_0^2=\xi^2\gamma^2$). Moreover, we have $\mathcal{B} = (\mathcal{B}-\mathcal{B}^{(0)}) + \mathcal{B}^{(0)}$ with $\mathcal{B}^{(0)} = \{\boldsymbol{\gamma}_N\}(\boldsymbol{\Psi}_0^{(0)} + \boldsymbol{\Psi}_1^{(0)})$. The potentials $\boldsymbol{\Psi}_\ell^{(0)}$ define continuous $\boldsymbol{\mathcal{V}} \to \boldsymbol{H}_{\mathrm{loc}}(\mathbf{rot}\,\mathbf{rot}\,,\Omega_0)$ and $\boldsymbol{\mathcal{V}} \to \boldsymbol{H}(\mathbf{rot}\,\mathbf{rot}\,,\Omega_1)$ operators. Together with Lemma 1, this implies that $\mathcal{B}^{(0)} : \boldsymbol{\mathcal{V}} \to \boldsymbol{\mathcal{V}}$ is bounded, and therefore that $\big\|\gamma^2\mathcal{B}\big\| = \gamma^2\big\|\big(\mathcal{B}-\mathcal{B}^{(0)}\big) + \mathcal{B}^{(0)}\big\| \leq C\gamma^2$ for any small enough $\gamma$. A similar argument allows to show that $\big\|\gamma^2\mathcal{A}_J\big\| \leq C\gamma^2$. This completes the proof of estimate (b).

Then, since $\mathcal{A}_J^{(1)}$ is the only nonzero operator appearing in $\boldsymbol{\mathcal{Z}}^{(1)}$ (see (21)), the proof of estimate (a) is nearly identical. The operator matrix $\widehat{\boldsymbol{\mathcal{Z}}}_\gamma - \boldsymbol{\mathcal{Z}}^{(0)}$ involves the same quantities as before, except for (37d) which reduces to $\mathcal{A}_J - \mathcal{A}_J^{(0)} = -\{\boldsymbol{\gamma}_\times\}\big(D^{(0)}\boldsymbol{\Psi}_0^\gamma + \mu_{\rm r} D^{(0)}\boldsymbol{\Psi}_1^\gamma\big)$. Estimates (a) and (c) of Lemma 4 imply that $\|\mathcal{A}_J - \mathcal{A}_J^{(0)}\| = O(\gamma)$, with all other constituents of $\widehat{\boldsymbol{\mathcal{Z}}}_\gamma - \boldsymbol{\mathcal{Z}}^{(0)}$ having $O(\gamma^2)$ norms, from which estimate (a) follows.

## 5. Numerical experiments.

**5.1. BE discretization.** We summarize the salient points of our BE implementation for solving EC or Maxwell versions of the PMCHWT integral problem. More comprehensive descriptions of BE methods for electromagnetic transmission problems can be found in e.g. **[13, 26, 28]**. Let $\Phi := \big\{\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_N\big\}$ denote the set of usual $H$-div conforming basis functions (see e.g. [**28**]) for a boundary element (BE) mesh made of $E$ flat triangular elements; $N$ is the number of edges, with $2N = 3E$ for the mesh of a simply-connected surface. The Hodge decomposition is achieved at the discrete level by defining the sub-family $\Phi_{\rm L}$ of (divergence-free) *loop* functions and the (supplementary) sub-family $\Phi_{\rm T}$ of *tree* functions as linear combinations of the original basis functions: with suitable coefficient matrices $L$ and $T$, we then have $\Phi_{\rm L} = L\Phi$ and $\Phi_{\rm T} = T\Phi$. The approximation $\boldsymbol{J}_h$ of the current density $\boldsymbol{J}$ is then sought in the form

$$\boldsymbol{J}_h = X_J^{\rm t}\Phi, \qquad \text{with} \quad X_J = L^{\rm t}X_{J,{\rm L}} + T^{\rm t}X_{J,{\rm T}}$$

where $X_J \in \mathbb{C}^N$ is the column vector of degrees of freedom (DOFs) for $\boldsymbol{J}_h$ and $X_{J,{\rm L}}, X_{J,{\rm T}}$ are the column vectors of corresponding loop and tree DOFs for $\boldsymbol{J}_{h,{\rm L}}$ and $\boldsymbol{J}_{h,{\rm T}}$ (with similar definitions for the corresponding quantities $X_{M,{\rm L}}, X_{M,{\rm T}}$ associated with $\boldsymbol{M}$). Moreover, the Hodge decomposition is also applied to the test functions. Then, the DOF subvectors $X_{J,{\rm L}}, X_{J,{\rm T}}, X_{M,{\rm L}}, X_{M,{\rm T}}$ for the EC integral problem solve the uncoupled systems

$$\begin{bmatrix} LA_J^{(0)}L^{\rm t} & LB_\star^{(0)}L^{\rm t} & LB^{(0)}T^{\rm t} \\ LB_\star^{(0)}L^{\rm t} & LA_M^{(0)}L^{\rm t} & LA_M^{(0)}T^{\rm t} \\ TB^{(0)}L^{\rm t} & TA_M^{(0)}L^{\rm t} & TZ_M^{(0)}T^{\rm t} \end{bmatrix} \begin{Bmatrix} X_{J,{\rm L}} \\ X_{M,{\rm L}} \\ X_{M,{\rm T}} \end{Bmatrix} = \begin{Bmatrix} LY_J^{(0)} \\ LY_M^{(0)} \\ TY_M^{(0)} \end{Bmatrix} \tag{38a}$$

and

$$\big[TZ_J^{(0)}T^{\rm t}\big]X_{J,{\rm T}} = \gamma^2 T\big(Y_J^{(0)} - \big[A_J^{(0)}L^{\rm t}\big]X_{J,{\rm L}} - \big[B^{(0)}L^{\rm t}\big]X_{M,{\rm L}} - \big[B^{(0)}T^{\rm t}\big]X_{M,{\rm T}}\big).$$

In the above block systems, we define the matrices $K$ associated with integral operators $\mathcal{K}$ by their entries $K_{ij} := \big\langle \boldsymbol{\phi}_i\,,\,\mathcal{K}\boldsymbol{\phi}_j\big\rangle_\times$ (with $\mathcal{K} = \mathcal{A}_J^{(0)}, \mathcal{A}_M^{(0)}, \mathcal{Z}_J^{(0)}, \mathcal{Z}_M^{(0)}, \mathcal{B}^{(0)}, \mathcal{B}_\star^{(0)}$). With the exception of $\mathcal{B}_\star^{(0)}$, to be defined shortly thereafter, the foregoing operators are as introduced in (19), so that the various entries $K_{ij}$ can be expressed (with details left to the reader) as linear combinations of

$$\big\langle \boldsymbol{\phi}_i\,,\,\{\boldsymbol{\gamma}_\times\}\boldsymbol{\Psi}_\ell[\boldsymbol{\phi}_j]\big\rangle_\times = \int_\Gamma \boldsymbol{\phi}_i(\boldsymbol{x}) \cdot \int_\Gamma G(\boldsymbol{x}-\boldsymbol{x}';\kappa_\ell)\boldsymbol{\phi}_j(\boldsymbol{x}')\,{\rm d}S(\boldsymbol{x}')\,{\rm d}S(\boldsymbol{x}),$$

$$\big\langle \boldsymbol{\phi}_i\,,\,\{\boldsymbol{\gamma}_\times\}\boldsymbol{\nabla}\boldsymbol{\Psi}_\ell[{\rm div}_S\boldsymbol{\phi}_j]\big\rangle_\times = -\int_\Gamma {\rm div}_S\boldsymbol{\phi}_i(\boldsymbol{x}) \int_\Gamma G(\boldsymbol{x}-\boldsymbol{x}';\kappa_\ell){\rm div}_S\boldsymbol{\phi}_j(\boldsymbol{x}')\,{\rm d}S(\boldsymbol{x}')\,{\rm d}S(\boldsymbol{x}),$$

$$\big\langle \boldsymbol{\phi}_i\,,\,\{\boldsymbol{\gamma}_\times\}\textbf{rot}\,\boldsymbol{\Psi}_\ell[\boldsymbol{\phi}_j]\big\rangle_\times = \int_\Gamma \boldsymbol{\phi}_i(\boldsymbol{x}) \cdot \int_\Gamma \boldsymbol{\nabla}G(\boldsymbol{x}-\boldsymbol{x}';\kappa_\ell)\times\boldsymbol{\phi}_j(\boldsymbol{x}')\,{\rm d}S(\boldsymbol{x}')\,{\rm d}S(\boldsymbol{x}).$$

Likewise, the vectors $Y_J^{(0)}$, $Y_M^{(0)}$ appearing in the right-hand sides of (38a,b) are defined by their entries

$$\big(Y_J^{(0)}\big)_j = \big\langle \boldsymbol{\phi}_i\,,\,\boldsymbol{\gamma}_\times^+\boldsymbol{\Phi}_0^{(0)}[\boldsymbol{J}_{\rm inc}]\big\rangle_\times = \int_\Gamma \boldsymbol{\phi}_i(\boldsymbol{x})\cdot\boldsymbol{\Phi}_0^{(0)}[\boldsymbol{J}_{\rm inc}](\boldsymbol{x})\,{\rm d}S(\boldsymbol{x}),$$

$$\big(Y_M^{(0)}\big)_j = \big\langle \boldsymbol{\phi}_i\,,\,\boldsymbol{\gamma}_N^+\boldsymbol{\Phi}_0^{(0)}[\boldsymbol{J}_{\rm inc}]\big\rangle_\times = \int_\Gamma \boldsymbol{\phi}_i(\boldsymbol{x})\cdot\textbf{rot}\,\boldsymbol{\Phi}_0^{(0)}[\boldsymbol{J}_{\rm inc}](\boldsymbol{x})\,{\rm d}S(\boldsymbol{x}).$$

The impedance variation $\Delta Z_{\rm EC}$ predicted by the EC model is then evaluated by using the solution of (38a,b) in the discretized version of (28), to obtain

$$\Delta Z_{\rm EC} = \frac{{\rm i}\xi\gamma}{I^2 L}\sqrt{\frac{\mu_0}{\varepsilon_0}}\big(\big[L^{\rm t}X_{M,{\rm L}} + T^{\rm t}X_{M,{\rm T}}\big]^{\rm t}Y_M^{(0)} + \big[L^{\rm t}X_{J,{\rm L}}\big]^{\rm t}Y_J^{(0)}\big).$$

The approximation method outlined above is also applied, in similar fashion, to the BE discretization of the Maxwell integral problem (14); we omit the details for brevity. The impedance variation $\Delta Z_\gamma$ predicted by the Maxwell model is then evaluated using (27) in discretized form, using the DOFs $X_J, X_M$ associated with the BE solution of (14).

If $\Gamma$ (or a connected component of $\Gamma$) is simply connected, each loop function $\boldsymbol{\phi}^{\rm L}$ is in fact equal on each element to the surface curl of a continuous and piecewise-linear function $\varphi$:

$$\boldsymbol{\phi}^{\rm L} = \big(\boldsymbol{\nabla}_{S}\varphi\big)\times\boldsymbol{n} = \textbf{rot}\,(\widetilde{\varphi}\boldsymbol{n})\big|_{\Gamma},$$

with $\widetilde{\varphi}$ denoting the extension of $\varphi$ by a constant along the normal direction in a tubular neighborhood of $\Gamma$. Moreover, since $\textbf{rot}\,\textbf{rot}\,\boldsymbol{\Psi}_\ell[\boldsymbol{u}] = \kappa_\ell^2\boldsymbol{\Psi}_\ell[\boldsymbol{u}] + \boldsymbol{\nabla}\boldsymbol{\Psi}_\ell[\text{div}_S\boldsymbol{u}]$, an integration by parts yields $\big\langle \boldsymbol{\phi}^{\rm L}_i\,,\,\{\boldsymbol{\gamma}_\times\}\textbf{rot}\,\boldsymbol{\Psi}^{(0)}_0[\boldsymbol{\phi}^{\rm L}_j]\big\rangle_\times = 0$ for any $(i,j)$ (since $\kappa^{(0)}_0=0$ and, by construction, $\text{div}_S\boldsymbol{\phi}^{\rm L}=0$). Since this theoretically vanishing contribution may in practice pollute the discrete solution process, it is subtracted from each medium's contribution to the double-layer operator $\mathcal{B}$ (restricted to the loop trial and test spaces), i.e. we introduce the modified double-layer operators

$$\mathcal{B}_\star = \{\boldsymbol{\gamma}_N\}\big(\boldsymbol{\Psi}^{(0)}_1 - \boldsymbol{\Psi}^{(0)}_0\big) \qquad \text{(EC case)}$$
$$\mathcal{B}_\star = \{\boldsymbol{\gamma}_N\}\big((\boldsymbol{\Psi}_1 - \boldsymbol{\Psi}^{(0)}_0) + (\boldsymbol{\Psi}_0 - \boldsymbol{\Psi}^{(0)}_0)\big) \qquad \text{(Maxwell case)}$$

for the EC and Maxwell cases, respectively. Observe that (i) $\mathcal{B}_\star = \mathcal{B}$ for the loop–loop interaction terms and (ii) $\mathcal{B}_\star$ only involves nonsingular integrals (due to $\boldsymbol{\nabla}G(\boldsymbol{r};\kappa_\ell) - \boldsymbol{\nabla}G(\boldsymbol{r};0)$ being bounded at $\boldsymbol{r}=\boldsymbol{0}$).

If (a connected component of) $\Gamma$ is not simply connected, "local" loop basis functions must be supplemented by "global" loop basis functions **[10]**, so that

$$L\mathcal{B}_\star L^{\rm t} = \begin{bmatrix} L_{\rm loc}\mathcal{B}_\star L^{\rm t}_{\rm loc} & L_{\rm loc}\mathcal{B}_\star L^{\rm t}_{\rm glob} \\ L_{\rm glob}\mathcal{B}_\star L^{\rm t}_{\rm loc} & L_{\rm glob}\mathcal{B}L^{\rm t}_{\rm glob}\end{bmatrix},$$

with the matrices $L_{\rm loc}$ and $L_{\rm glob}$ defining the linear combinations of basis functions that yield the local and global loop functions. Finally, an integration-by-parts argument similar to that made above shows that $\big\langle \boldsymbol{\phi}^{\rm L}_i\,,\,\boldsymbol{\gamma}^+_N\boldsymbol{\Phi}^{(0)}_0[\boldsymbol{J}_{\rm inc}]\big\rangle_\times = 0$. As a consequence, we have $L_{\rm loc}Y^{(0)}_M = 0$ (for the EC case) and $L_{\rm loc}Y_M = L_{\rm loc}\big(Y_M - Y^{(0)}_M\big)$ (for the Maxwell case).

**5.2. Numerical results.** We now demonstrate on a test configuration the main results of the foregoing asymptotic analysis. The conducting body $\Omega_1$ occupies an ellipsoidal domain with semiaxes $(1, 0.8, 0.6)$ m, centered at the coordinate origin $O=(0,0,0)$ and inclined (relative to the fixed frame $Oxyz$) by means of a 30° rotation about $Oy$. The boundary element mesh is created by deforming the mesh of a unit sphere made of 4608 triangles. The source field is emitted by an annular coil (centered at $O$, with an internal radius of 4m and a square section of size 0.05 m) made of a single spire carrying a 1 A current, so that the current density $\boldsymbol{J}_{\rm inc}$ is of magnitude 400 $\text{Am}^{-2}$. Resulting electric and magnetic fields will be evaluated on the spheres $S_0 \subset \Omega_0$ (center $O$, radius 1.2 m) and $S_1 \subset \Omega_1$ (center $O$, radius 0.4 m), see Fig. 2.

Values for the frequency $f = \gamma\xi/2\pi L\sqrt{\varepsilon_0\mu_0}$ and the conductivity $\sigma = \xi\sqrt{\varepsilon_0}/L\gamma\sqrt{\mu_0}$ are selected so that $10^{-5} \le \gamma \le 10$ and $\xi = 1$ (we set $L = 1$m). Maxwell solutions are compared to their asymptotic approximations. The absolute and relative differences (in discrete $L^\infty(\Gamma)$ norm) between the surface currents $\boldsymbol{J}, \boldsymbol{M}$ for the Maxwell solutions and their asymptotic approximation, shown in Fig. 3, corroborates the expansion (24). Corresponding comparisons for the evaluation of $\boldsymbol{H}$ (Fig. 4) and $\boldsymbol{E}$ (Fig. 5) on the outer and inner evaluation surfaces reproduce the asymptotic behavior predicted by (26a), (26b) and (26c); in addition, Fig. 5a also shows (unconnected symbols) the unacceptable relative approximation error committed on $\boldsymbol{E}_0$ by (mistakenly) omitting the contribution of $\boldsymbol{J}_{\rm T}$ to $\boldsymbol{E}_{0,\rm EC}$ due to this component being of higher order in $\gamma$ on $\Gamma$. Finally, the absolute and relative differences between $\Delta Z_\gamma$ and $\Delta Z_{\rm EC}$, plotted in Fig. 6, validate the expansion (28) of $\Delta Z_\gamma$. We conclude with an illustrative plot of the surface current magnitude (in terms of $|\Im(\boldsymbol{M})|$) for $\gamma = 10^{-3}$ (Fig. 7).

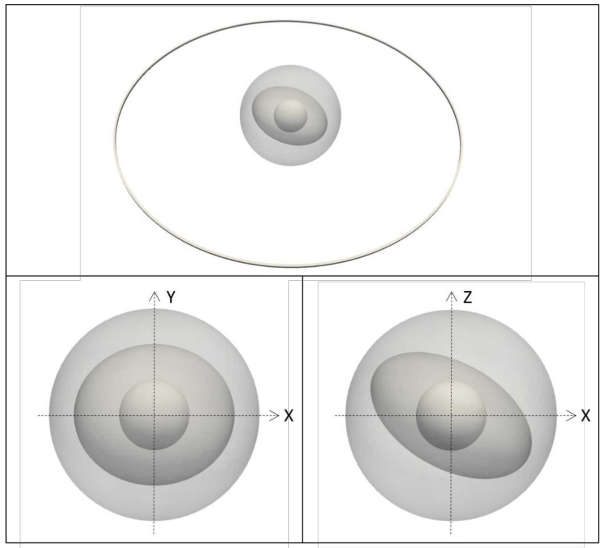
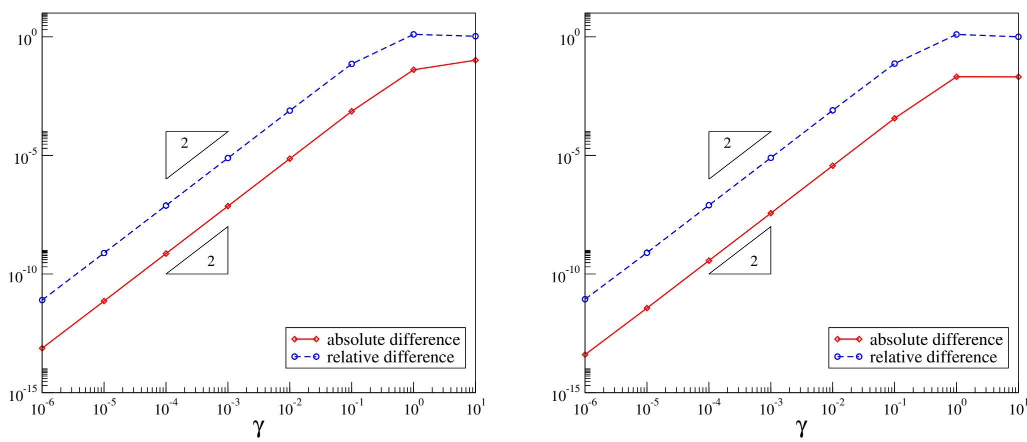



***Figure 2.*** *Sketch of the example configuration showing the coil (top panel only), the ellipsoidal conducting part, and the two spherical evaluation surfaces.*

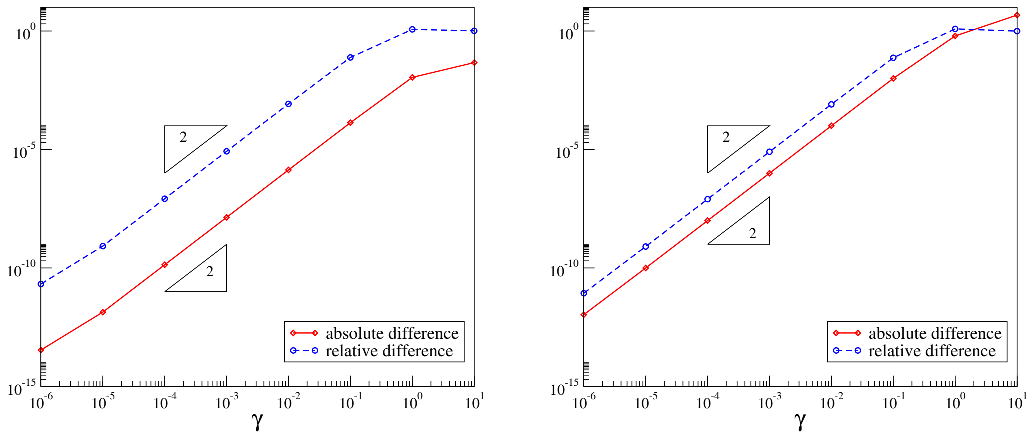



*(a) Current density $\boldsymbol{J}$ on $\Gamma$* *(b) Current density $\boldsymbol{M}$ on $\Gamma$*

***Figure 3.*** *Absolute and relative differences (in discrete $L^\infty(\Gamma)$ norm) between Maxwell solutions on $\Gamma$ and their asymptotic approximation*

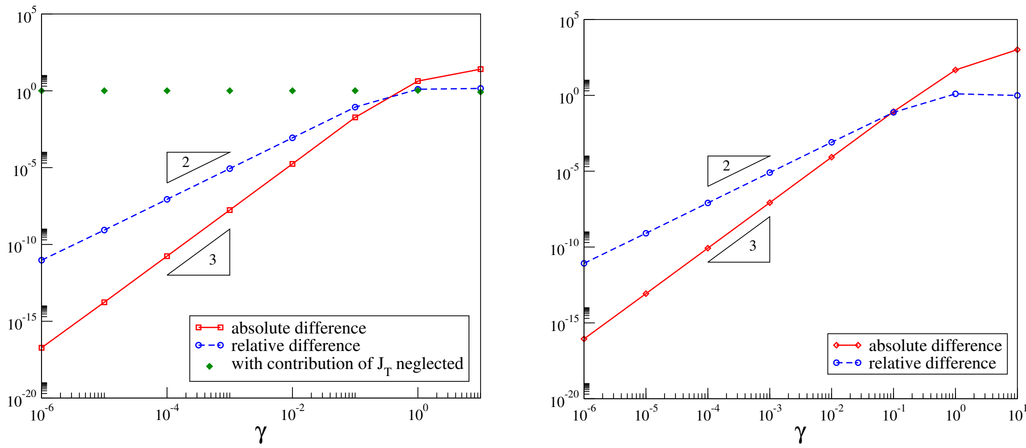
**6. Concluding remarks.** In this article, we have sought and established the limiting form of the (PMCHWT) integral formulation for the Maxwell transmission problem involving a spatially-bounded conductor in air or vacuum as $\gamma := \sqrt{\omega\varepsilon_0/\sigma} \to 0$ while $\xi := L\sqrt{\omega\sigma\mu_0}$ remains bounded. These conditions pertain to situations simultaneously involving low frequencies and high conductivity ($\gamma$ small) while the skin depth $d$ remains fixed ($\xi$ set to a nonzero value), and are distinct from a low-frequency approximation. The derivation and mathematical justification of the asymptotic results, where in particular the leading approximations of the surface currents are found to solve the EC PMCHWT integral problem of **[14]**, constitutes the main theoretical contribution of this work.

In addition to bringing insight into the mathematical relationship between the EC and Maxwell models for the transmission problem, this study allows to quantify the quality of the EC model as an approximation of the Maxwell model (the relative residuals on electromagnetic fields being $O(\gamma^2) = O(\omega/\sigma)$) and paves the way for the derivation of higher-order approximations (e.g. by setting up the governing problem for $\boldsymbol{X}^{(2)}$). Moreover, the revealed disparities in how the various sub-operators scale with $\gamma$ pinpoint a suitable rescaling of the Maxwell PMCHWT, such that all diagonal sub-operators




*(a) Outer evaluation surface $S_0$* *(b) Inner evaluation surface $S_1$*

***Figure 4.*** *Absolute and relative differences (in discrete $L^\infty(S)$ norm) between $\boldsymbol{H}$ evaluated using the exact Stratton–Chu representation and its asymptotic approximation*




*(a) Outer evaluation surface $S_0$* *(b) Inner evaluation surface $S_1$*

***Figure 5.*** *Absolute and relative differences (in discrete $L^\infty(S)$ norm) between $\boldsymbol{E}$ evaluated using the exact Stratton–Chu representation and its asymptotic approximation. In the left panel, the unconnected symbols correspond to the relative error committed by omitting the contribution of $\boldsymbol{J}_T$ (of higher order on $\Gamma$) to $\boldsymbol{E}_{0,EC}$*

are $O(1)$ in $\gamma$; this feature is exploited (without mathematical analysis) in the form of a simple block-SOR algorithm in [**27**].

Future work includes extending this asymptotic methodology, and finding the correct limiting problem, for configurations additionally involving a nearby weakly conducting but permeable object (e.g. a ferrite core in the EC probe).

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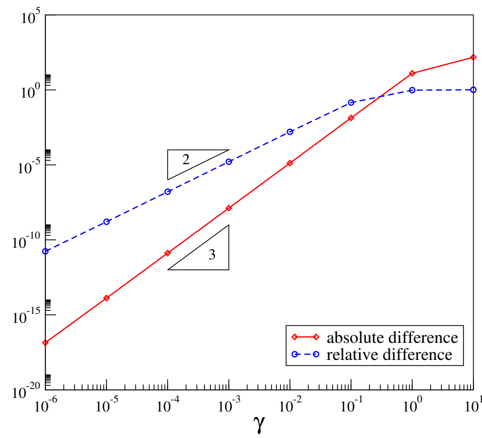


***Figure 6.*** *Absolute and relative differences on the Maxwell and asymptotic evaluations of the impedance variation* $\Delta Z$

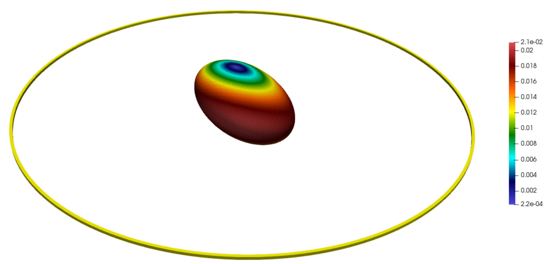

***Figure 7.*** *Plot of* $|\Im(\boldsymbol{M})|$ *on* $\Gamma$ *for* $\gamma = 10^{-3}$


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