On subspace trails cryptanalysis

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Outline

The AES

NIST standard since 2001, SPN on 10 rounds, 128-bit blocks [DR02].

$$x = \begin{pmatrix} x_0 & x_4 & x_8 & x_{12} \\ x_1 & x_5 & x_9 & x_{13} \\ x_2 & x_6 & x_{10} & x_{14} \\ x_3 & x_7 & x_{11} & x_{15} \end{pmatrix} \in \mathbb{F}_{2^8}^{16}$$

S-box $\begin{cases} \mathbb{F}_{2^8} & \rightarrow & \mathbb{F}_{2^8} \\ x_i & \mapsto & y_i \end{cases}$

$$SR(y) = \begin{pmatrix} y_0 & y_4 & y_8 & y_{12} \\ y_5 & y_9 & y_{13} & y_1 \\ y_{10} & y_{14} & y_2 & y_6 \\ y_{15} & y_3 & y_7 & y_{11} \end{pmatrix}$$

ShiftRows SR

$$MC(z) = \begin{pmatrix} 2 & 3 & 1 & 1 \\ 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 3 \\ 3 & 1 & 1 & 2 \end{pmatrix} \times z$$

MixColumns MC

The AES and the distinguisher of [GRR17]

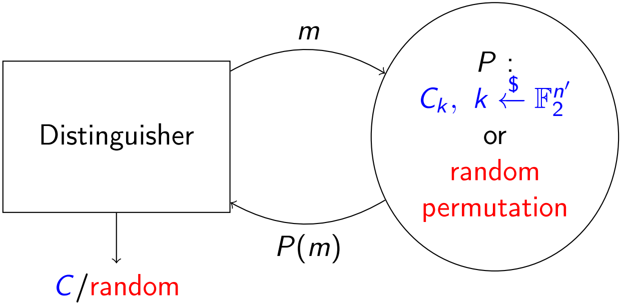
- The AES
- The distinguisher of Grassi, Rechberger and Rønjom

Proof for the distinguisher

Conclusion

What is a distinguisher ?

Let C_k be a cipher with key k,



Distinguisher $\rightarrow$ attack (on more rounds).

Grassi, Rechberger and Rønjom at Eurocrypt 2017 [GRR17]
$\rightarrow$ $C = 5$ AES rounds.

Some definitions...

$$\mathbb{K} = \mathbb{F}_{2^8} \qquad \begin{pmatrix} x_0 & x_4 & x_8 & x_{12} \\ x_1 & x_5 & x_9 & x_{13} \\ x_2 & x_6 & x_{10} & x_{14} \\ x_3 & x_7 & x_{11} & x_{15} \end{pmatrix} \in \mathcal{M}_4(\mathbb{K}) \qquad x_i \in \mathbb{K}$$

$$\begin{pmatrix} x_0 & 0 & 0 & 0 \\ x_1 & 0 & 0 & 0 \\ x_2 & 0 & 0 & 0 \\ x_3 & 0 & 0 & 0 \end{pmatrix} \in \mathcal{C}_0$$

Columns
$\mathcal{C}_i = \mathrm{vect}_{\mathbb{K}}(e_{0,i}, e_{1,i}, e_{2,i}, e_{3,i})$

$$\begin{pmatrix} 0 & x_0 & 0 & y_0 \\ 0 & x_1 & 0 & y_1 \\ 0 & x_2 & 0 & y_2 \\ 0 & x_3 & 0 & y_3 \end{pmatrix} \in \mathcal{C}_{\{1,3\}}$$

$I \subseteq [\![0, 3]\!]$:
$\mathcal{C}_I = \bigoplus_{i \in I} \mathcal{C}_i$.

$$\begin{pmatrix} x_0 & 0 & 0 & 0 \\ 0 & x_1 & 0 & 0 \\ 0 & 0 & x_2 & 0 \\ 0 & 0 & 0 & x_3 \end{pmatrix} \in \mathcal{D}_0,$$

Diagonals:
$\mathcal{D}_i = SR^{-1}(\mathcal{C}_i)$

$$\begin{pmatrix} x_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_1 \\ 0 & 0 & x_2 & 0 \\ 0 & x_3 & 0 & 0 \end{pmatrix} \in \mathcal{ID}_0,$$

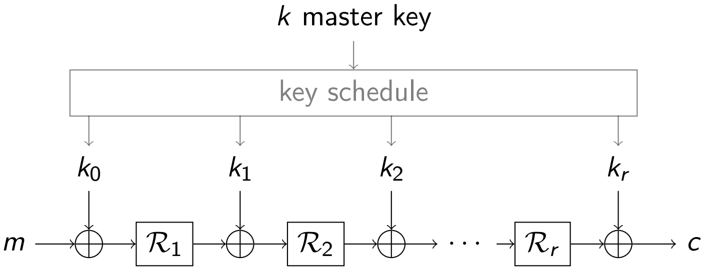
Anti-diagonals:
$\mathcal{ID}_i = SR(\mathcal{C}_i)$

$$\begin{pmatrix} 2 \cdot x_0 & x_1 & x_2 & 3 \cdot x_3 \\ x_0 & x_1 & 3 \cdot x_2 & 2 \cdot x_3 \\ x_0 & 3 \cdot x_1 & 2 \cdot x_2 & x_3 \\ 3 \cdot x_0 & 2 \cdot x_1 & x_2 & x_3 \end{pmatrix} \in \mathcal{M}_0.$$

Mixed:
$\mathcal{M}_i = MC(\mathcal{ID}_i)$

$$\mathcal{D}_I \overbrace{\xrightarrow{\mathcal{S}} \mathcal{D}_I \xrightarrow{SR} \mathcal{C}_I \xrightarrow{MC}}^{\mathcal{R}} \mathcal{C}_I \overbrace{\xrightarrow{\mathcal{S}} \mathcal{C}_I \xrightarrow{SR} \mathcal{ID}_I \xrightarrow{MC}}^{\mathcal{R}} \mathcal{M}_I$$

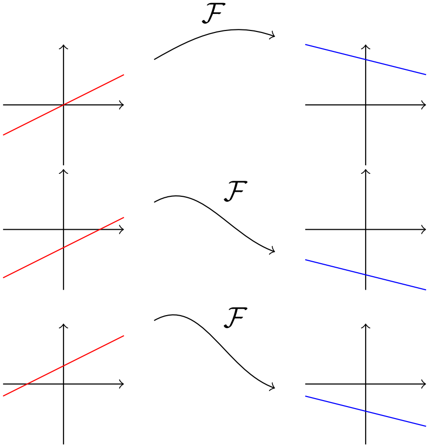
The AES is a key-alternating blockcipher



Subspace trails

Definition ([LTW18])

We have $U \overset{\mathcal{F}}{\rightrightarrows} V$ if $\forall a \in \mathbb{K}^N, \exists b \in \mathbb{K}^N : \mathcal{F}(U + a) \subseteq V + b$.



Examples:

- $\{0\} \overset{\mathcal{F}}{\rightrightarrows} \{0\}$
- $U \overset{\mathcal{F}}{\rightrightarrows} \mathbb{K}^N$
- $\mathcal{D}_I \overset{\mathcal{R}}{\rightrightarrows} \mathcal{C}_I$
- $\mathcal{C}_I \overset{\mathcal{R}}{\rightrightarrows} \mathcal{M}_I$

$$\mathcal{D}_0 \overset{\mathcal{R}}{\rightrightarrows} \mathcal{C}_0$$

$\forall a, \forall x,$

$$\begin{pmatrix} x_0 & 0 & 0 & 0 \\ 0 & x_1 & 0 & 0 \\ 0 & 0 & x_2 & 0 \\ 0 & 0 & 0 & x_3 \end{pmatrix} \xrightarrow{+a} \begin{pmatrix} x_0 + a_0 & * & * & * \\ * & x_1 + a_1 & * & * \\ * & * & x_2 + a_2 & * \\ * & * & * & x_3 + a_3 \end{pmatrix}$$

$$\xrightarrow{\mathcal{S}} \begin{pmatrix} y_0 & * & * & * \\ * & y_1 & * & * \\ * & * & y_2 & * \\ * & * & * & y_3 \end{pmatrix} \xrightarrow{SR} \begin{pmatrix} y_0 & * & * & * \\ y_1 & * & * & * \\ y_2 & * & * & * \\ y_3 & * & * & * \end{pmatrix}$$

$$\xrightarrow{MC} \begin{pmatrix} \vdots & * & * & * \\ \vdots & * & * & * \\ MC(y) & * & * & * \\ \vdots & * & * & * \end{pmatrix}$$

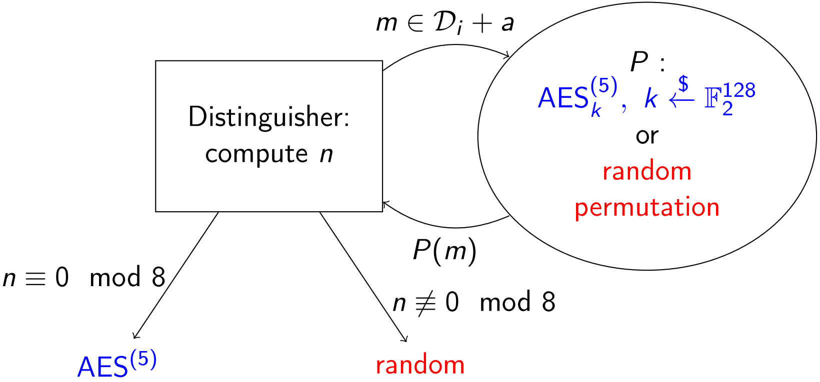
The distinguisher

Theorem ([GRR17])

Let $a \in \mathcal{M}_4(\mathbb{K}),\ i \in [\![0,3]\!],\ J \subseteq [\![0,3]\!]$. *We define*

$$n = \#\{\ \{p^0, p^1\} \in \mathcal{P}^2(\mathcal{D}_i + a) \mid \mathcal{R}^5(p^0) + \mathcal{R}^5(p^1) \in \mathcal{M}_J\}.$$

Then $n \equiv 0 \mod 8$.



A key lemma

Lemma ([GRR17])

Let $a \in \mathcal{M}_4(\mathbb{K}),\ I \subset [\![0,3]\!],\ J \subseteq [\![0,3]\!]$. *We define*

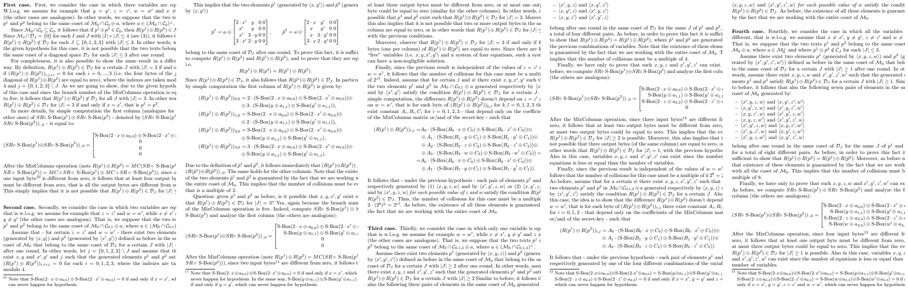
$$n = \#\{\ \{p^0, p^1\} \in \mathcal{P}^2(\mathcal{M}_I + a) \mid \mathcal{R}(p^0) + \mathcal{R}(p^1) \in \mathcal{D}_J \}.$$

Then $n \equiv 0 \mod 8$.

$$\overbrace{\mathcal{D}_I \overset{\mathcal{R}}{\rightrightarrows} \mathcal{C}_I \overset{\mathcal{R}}{\rightrightarrows} \mathcal{M}_I}^{2} \overbrace{\overset{\text{Lemma}}{\overset{\mathcal{R}}{\dashrightarrow}}}^{1} \overbrace{\mathcal{D}_J \overset{\mathcal{R}}{\rightrightarrows} \mathcal{C}_J \overset{\mathcal{R}}{\rightrightarrows} \mathcal{M}_J}^{2}$$

Proof

In the original paper [GRR17]:



Our contribution starts here

- Search for the underlying property ;
- write a better proof for it to come out;
- generalize ?

Step 1: equivalence relation between pairs

In $\mathcal{M}_0$,

$$\left\{ \begin{pmatrix} 2\cdot x_0 & x_1 & z_2 & 3\cdot z_3 \\ x_0 & x_1 & 3\cdot z_2 & 2\cdot z_3 \\ x_0 & 3\cdot x_1 & 2\cdot z_2 & z_3 \\ 3\cdot x_0 & 2\cdot x_1 & z_2 & z_3 \end{pmatrix}, \begin{pmatrix} 2\cdot y_0 & y_1 & z_2 & 3\cdot z_3 \\ y_0 & y_1 & 3\cdot z_2 & 2\cdot z_3 \\ y_0 & 3\cdot y_1 & 2\cdot z_2 & z_3 \\ 3\cdot y_0 & 2\cdot y_1 & z_2 & z_3 \end{pmatrix} \right\}$$

$$\sim$$

$$\left\{ \begin{pmatrix} 2\cdot x_0 & y_1 & w_2 & 3\cdot w_3 \\ x_0 & y_1 & 3\cdot w_2 & 2\cdot w_3 \\ x_0 & 3\cdot y_1 & 2\cdot w_2 & w_3 \\ 3\cdot x_0 & 2\cdot y_1 & w_2 & w_3 \end{pmatrix}, \begin{pmatrix} 2\cdot y_0 & x_1 & w_2 & 3\cdot w_3 \\ y_0 & x_1 & 3\cdot w_2 & 2\cdot w_3 \\ y_0 & 3\cdot x_1 & 2\cdot w_2 & w_3 \\ 3\cdot y_0 & 2\cdot x_1 & w_2 & w_3 \end{pmatrix} \right\}$$

$$\left\{ \begin{pmatrix} 2\cdot x_0 & x_1 & z_2 & 3\cdot z_3 \\ x_0 & x_1 & 3\cdot z_2 & 2\cdot z_3 \\ x_0 & 3\cdot x_1 & 2\cdot z_2 & z_3 \\ 3\cdot x_0 & 2\cdot x_1 & z_2 & z_3 \end{pmatrix}, \begin{pmatrix} 2\cdot y_0 & y_1 & z_2 & 3\cdot z_3 \\ y_0 & y_1 & 3\cdot z_2 & 2\cdot z_3 \\ y_0 & 3\cdot y_1 & 2\cdot z_2 & z_3 \\ 3\cdot y_0 & 2\cdot y_1 & z_2 & z_3 \end{pmatrix} \right\}$$

$$\sim$$

$$\left\{ \begin{pmatrix} 2\cdot x_0 & y_1 & w_2 & 3\cdot w_3 \\ x_0 & y_1 & 3\cdot w_2 & 2\cdot w_3 \\ x_0 & 3\cdot y_1 & 2\cdot w_2 & w_3 \\ 3\cdot x_0 & 2\cdot y_1 & w_2 & w_3 \end{pmatrix}, \begin{pmatrix} 2\cdot y_0 & x_1 & w_2 & 3\cdot w_3 \\ y_0 & x_1 & 3\cdot w_2 & 2\cdot w_3 \\ y_0 & 3\cdot x_1 & 2\cdot w_2 & w_3 \\ 3\cdot y_0 & 2\cdot x_1 & w_2 & w_3 \end{pmatrix} \right\}$$

Definition

Let $\{p^0, p^1\}$ a pair of states from $\mathcal{M}_I + a$. The information set K of the pair $\{p^0, p^1\}$ is $\{k \in [\![0,3]\!] \mid \exists i \in I : x_{i,k} \neq y_{i,k}\}$.

It is $K = \{0, 1\}$ in the example.

$$\left\{ \begin{pmatrix} 2 \cdot x_0 & x_1 & z_2 & 3 \cdot z_3 \\ x_0 & x_1 & 3 \cdot z_2 & 2 \cdot z_3 \\ x_0 & 3 \cdot x_1 & 2 \cdot z_2 & z_3 \\ 3 \cdot x_0 & 2 \cdot x_1 & z_2 & z_3 \end{pmatrix}, \begin{pmatrix} 2 \cdot y_0 & y_1 & z_2 & 3 \cdot z_3 \\ y_0 & y_1 & 3 \cdot z_2 & 2 \cdot z_3 \\ y_0 & 3 \cdot y_1 & 2 \cdot z_2 & z_3 \\ 3 \cdot y_0 & 2 \cdot y_1 & z_2 & z_3 \end{pmatrix} \right\}$$

$$\sim$$

$$\left\{ \begin{pmatrix} 2 \cdot x_0 & y_1 & w_2 & 3 \cdot w_3 \\ x_0 & y_1 & 3 \cdot w_2 & 2 \cdot w_3 \\ x_0 & 3 \cdot y_1 & 2 \cdot w_2 & w_3 \\ 3 \cdot x_0 & 2 \cdot y_1 & w_2 & w_3 \end{pmatrix}, \begin{pmatrix} 2 \cdot y_0 & x_1 & w_2 & 3 \cdot w_3 \\ y_0 & x_1 & 3 \cdot w_2 & 2 \cdot w_3 \\ y_0 & 3 \cdot x_1 & 2 \cdot w_2 & w_3 \\ 3 \cdot y_0 & 2 \cdot x_1 & w_2 & w_3 \end{pmatrix} \right\}$$

Definition

Let $P = \{p^0, p^1\},\ Q = \{q^0, q^1\} \in \mathcal{P}^2(\mathcal{M}_I + a)$. We have $P \sim Q$ if:

- K is the information set of $P \Rightarrow K$ is the information set of Q.
- $\forall k \in K, \exists b \in \{0,1\} : \forall i \in I, q^0_{i,k} = p^b_{i,k}$ et $q^1_{i,k} = p^{1-b}_{i,k}$.

$\sim$ is an equivalence relation on $\mathcal{P}^2(\mathcal{M}_I + a)$.

Lemma

The function

$$f: \begin{array}{ccc} \mathcal{P}^2(\mathcal{M}_I + a) & \longrightarrow & \mathcal{M}_4(\mathbb{K}) \\ \{p^0, p^1\} & \longmapsto & \mathcal{R}(p^0) + \mathcal{R}(p^1) \end{array}$$

is constant on the equivalence classes of $\sim$.

Proposition

Let $\mathfrak{C}$ *be an equivalence class* K. *Then*

$$\#\mathfrak{C} = 2^{|K|-1+8|I|(4-|K|)} \equiv 0 \mod 8.$$

Lemma

If

$$n = \#\{ \{p^0, p^1\} \in \mathcal{P}^2(\mathcal{M}_I + a) \mid \mathcal{R}(p^0) + \mathcal{R}(p^1) \in \mathcal{D}_J\},$$

then $n \equiv 0 \mod 8$.

Proof.

$$\begin{aligned} n &= \# f^{-1}(\mathcal{D}_J) \\ &= \sum_{\mathfrak{C} \in \mathcal{P}^2(\mathcal{M}_I + a)/\sim} \#(f^{-1}(\mathcal{D}_J) \cap \mathfrak{C}) \\ &= \sum_{\mathfrak{C} \in \mathcal{P}^2(\mathcal{M}_I + a)/\sim} 1_{\tilde{f}(\mathfrak{C}) \in \mathcal{D}_J} \# \mathfrak{C} \\ &\equiv 0 \mod 8 \end{aligned}$$

□

What about the branch number ?

In [GRR17], the proof needs maximal branch number. But...

Proposition ([GRR16])

Let $I, J \subseteq [\![0,3]\!]$ and b be the differential branch number of MC. Then

$$|I| + |J| < b \quad \Rightarrow \quad \mathcal{D}_I \cap \mathcal{M}_J = \{0\}$$

If $\{p^0, p^1\} \in \mathcal{P}^2(\mathcal{M}_I + a)$ has information set K,

$$p^0 + p^1 \in \mathcal{C}_K \text{ and then } \mathcal{R}(p^0) + \mathcal{R}(p^1) \in \mathcal{M}_K.$$

If $|K| < b - |J|, \mathcal{M}_K \cap \mathcal{D}_J = \{0\}$ and $\mathcal{R}(p^0) + \mathcal{R}(p^1) \notin \mathcal{D}_J$.

Lemma

If

$$n = \#\{\ \{p^0, p^1\} \in \mathcal{P}^2(\mathcal{M}_I + a) \mid \mathcal{R}(p^0) + \mathcal{R}(p^1) \in \mathcal{D}_J\},$$

then $n \equiv 0 \mod 8$.

Proof.

$$n = \sum_{\mathfrak{C} \in \mathcal{P}^2(\mathcal{M}_I + a) / \sim} 1_{\tilde{f}(\mathfrak{C}) \in \mathcal{D}_J} \#\mathfrak{C}$$

$$= \sum_{h=0}^{4} \sum_{\mathfrak{C} : |K(\mathfrak{C})| = h} 1_{\tilde{f}(\mathfrak{C}) \in \mathcal{D}_J} \#\mathfrak{C}$$

$$= \sum_{h=b-|J|}^{4} \sum_{\mathfrak{C} : |K(\mathfrak{C})| = h} 1_{\tilde{f}(\mathfrak{C}) \in \mathcal{D}_J} \#\mathfrak{C}$$

Definition

Let $V \subseteq \mathbb{K}^N$ be a $\mathbb{K}$-subspace. We say V is compatible with $\mathcal{S}$ if it has a basis g such that its matrix in basis f is of the form:

$$\begin{pmatrix}
\begin{matrix} * & \cdots & * \\ \vdots & \lambda_{0,\ell,i} & \vdots \\ * & \cdots & * \end{matrix} & 0 & 0 \\
0 & \begin{matrix} * & \cdots & * \\ \vdots & \lambda_{k,\ell,i} & \vdots \\ * & \cdots & * \end{matrix} & 0 \\
0 & 0 & \begin{matrix} * & \cdots & * \\ \vdots & \lambda_{h-1,\ell,i} & \vdots \\ * & \cdots & * \end{matrix} \\
0 & 0 & 0
\end{pmatrix}$$

$$\begin{matrix} \uparrow & \qquad \uparrow & \qquad \uparrow \\ g_{0,i} & \qquad g_{k,i} & \qquad g_{h-1,i} \end{matrix}$$

$$\begin{pmatrix} 2 & & & \\ 1 & & & \\ 1 & & & \\ 3 & & & \\ & 1 & & \\ & 1 & & \\ & 3 & & \\ & 2 & & \\ & & 1 & \\ & & 3 & \\ & & 2 & \\ & & 1 & \\ & & & 3 \\ & & & 2 \\ & & & 1 \\ & & & 1 \end{pmatrix}$$

$\mathcal{M}_0$ is compatible with $\mathcal{S}_{AES}$.
Likewise, $\mathcal{M}_0 \cap \mathcal{C}_{0,1}$ is compatible with $\mathcal{S}_{AES}$.

$$\begin{pmatrix} 2 \cdot x_0 & x_1 & 0 & 0 \\ x_0 & x_1 & 0 & 0 \\ x_0 & 3 \cdot x_1 & 0 & 0 \\ 3 \cdot x_0 & 2 \cdot x_1 & 0 & 0 \end{pmatrix} \in \mathcal{M}_0 \cap \mathcal{C}_{0,1}.$$

Midori

Midori, presented at Asiacrypt 2015 [BBI+15].
Goal: low energy consumption.

- $\mathcal{R} : \mathbb{F}_2^{128} \rightarrow \mathbb{F}_2^{128}$
- S-box: $\mathbb{F}_{2^8} \rightarrow \mathbb{F}_{2^8}$
- $\mathcal{L}$:
  - ShuffleCell *SC* (more complex ShiftRows)
  - MixColumns *MC*

$$M_{MC} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

$$\overbrace{\mathcal{D}'_I \overset{\mathcal{R}}{\rightrightarrows} \mathcal{C}_I \overset{\mathcal{R}}{\rightrightarrows} \mathcal{M}'_I}^{2} \quad \overbrace{\overset{\text{Genaralized Lemma}}{\overset{\mathcal{R}}{\dashrightarrow}}}^{1} \quad \overbrace{\mathcal{D}'_J \overset{\mathcal{R}}{\rightrightarrows} \mathcal{C}_J \overset{\mathcal{R}}{\rightrightarrows} \mathcal{M}'_J}^{2}$$

What now ?

- The generalization can be useful (the distinguisher can be easily transposed)
  but cannot give better results!
- Working on subspace trails [LTW18].

$$\overbrace{\mathcal{D}'_I \overset{\mathcal{R}}{\rightrightarrows} \mathcal{C}_I \overset{\mathcal{R}}{\rightrightarrows} \mathcal{M}'_I}^{2} \quad \overbrace{\underset{\overset{\mathcal{R}}{\dashrightarrow}}{\text{Genaralized Lemma}}}^{1} \quad \overbrace{\mathcal{D}'_J \overset{\mathcal{R}}{\rightrightarrows} \mathcal{C}_J \overset{\mathcal{R}}{\rightrightarrows} \mathcal{M}'_J}^{2}$$

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