Submerging CSIDH

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Outline

One-way group action [Cou06, CLM+]

Group action

A **group** G acts on a **set** X.
$h * (g * x) = (h \cdot g) * x$

Easy

- Operations in G ;
- Action $g * x$, $g \in G$, $x \in X$.

Hard

- Find g from x and $x' = g * x$.

CSIDH

In the case of CSIDH [CLM+]

Set

Montgomery curves on $\mathbb{F}_p$:

$$E_A \ : y^2 = x^3 + Ax^2 + x \ .$$

Endomorphism Ring

- $End_{p^2}(E_A)$: Order of a quaternion algebra
- $End_p(E_A) = \mathbb{Z}[\pi] = \mathbb{Z}[\sqrt{-p}]$

Group

Isogenies between those curves, which correspond exactly to $\mathcal{C}\ell\mathcal{O}$ where $\mathcal{O} = \mathbb{Z}[\sqrt{-p}]$.

Parameters rationale

The base field is $\mathbb{F}_p$ for $p = 4\ell_1 \cdots \ell_u - 1$, with $\ell_1, \ldots \ell_u$ small primes.

It turns out that each ℓ_i gives an isogeny $[\mathfrak{l}_i]$ of **small** degree ℓ_i, very easy to compute (as $[\mathfrak{l}_i]^{-1}$).

$\mathcal{C}\ell\mathcal{O}$ is spanned by products of the form:

$$\prod_{i=1}^{u} [\mathfrak{l}_i]^{e_i}$$

for $e_i \in \{-m \ldots m\}$ and $2m + 1 \simeq p^{1/(2u)}$ ($\mathcal{C}\ell\mathcal{O}$ has $O(\sqrt{p})$ elements).

The one-way commutative group action!

Computing the action of $[\mathfrak{b}] = \Pi_{i=1}^{u} [\mathfrak{l}_i]^{e_i}$:

Apply successively *um* isogenies of degree $\leq \ell_u$.

Find $[\mathfrak{b}]$ such that $[\mathfrak{b}] \cdot E = E'$:

Compute an isogeny between two curves E and E'.

Commutative group action

$$[\mathfrak{b}] \cdot E = E' \Rightarrow \forall [\mathfrak{a}] \in \mathcal{C}\ell\mathcal{O}, [\mathfrak{a}\mathfrak{b}] \cdot E = [\mathfrak{a}] \cdot E'$$

CSIDH parameters for NIST security levels

Level	$\log_2 p$	# primes	Isogeny range	Estimated quantum query cost
NIST 1	512	74	5	2^{62}
NIST 3	1024	132	7	2^{94}
NIST 5	1792	209	10	2^{129}

Hidden Shift Algorithms

Hidden Shift

$$f(x) = g(x+s), x \in \mathbb{G}. \quad \text{Find } s$$

Quantum Algorithms

- $\mathcal{O}\left(8^{\sqrt{n}}\right)$ in $\mathbb{Z}/(2^n\mathbb{Z})$ [Kup05]
- $\mathcal{O}\left(8^{\sqrt{\log_2(N)}}\right)$ in $\mathbb{Z}/(N\mathbb{Z})$ [Kup05]
- $\widetilde{\mathcal{O}}\left(3^{\sqrt{2\log_3(N)}}\right)$ in $\mathbb{Z}/(N\mathbb{Z})$, N smooth [Kup05]
- $\widetilde{\mathcal{O}}\left(2^{\sqrt{2n\log_2(n)}}\right)$ in $\mathbb{Z}/(2^n\mathbb{Z})$, polynomial memory [Reg04]
- $\widetilde{\mathcal{O}}\left(2^{\sqrt{2\log_2(N)}}\right)$ in $\mathbb{Z}/(N\mathbb{Z})$, with QRAM [Kup13]
- $\widetilde{\mathcal{O}}\left(2^{\sqrt{2\log_2(N)\log_2(\log_2(N))}}\right)$ in $\mathbb{Z}/(N\mathbb{Z})$, polynomial memory [CJS14]
- $2^{\sqrt{2\log_2(3)n}}$ in $\mathbb{Z}/(2^n\mathbb{Z})$ [BNP18]

Hidden Shift

What we have

Hidden shift algorithm for $\mathbb{Z}/(2^n\mathbb{Z})$ that costs $2^{\sqrt{2\log_2(3)n}}$

What we need

Precise cost for a hidden shift algorithm for $\mathbb{Z}/(N\mathbb{Z})$

Hidden Shift in $\mathbb{Z}/(2^n\mathbb{Z})$

Oracle

$$\mathrm{O}: \begin{array}{lcl} |0\rangle\,|x\rangle\,|0\rangle & \mapsto & |0\rangle\,|x\rangle\,|f(x)\rangle \\ |1\rangle\,|x\rangle\,|0\rangle & \mapsto & |1\rangle\,|x\rangle\,|g(x)\rangle \end{array}$$

Sampling

$$\mathrm{O}\left(\frac{1}{2^{(n+1)/2}}\sum_{i=0}^{2^n}(|0\rangle + |1\rangle)\,|i\rangle\,|0\rangle\right) = \frac{1}{2^{(n+1)/2}}\sum_{f(x)}(|0\rangle\,|x\rangle + |1\rangle\,|x+s\rangle)\,|f(x)\rangle$$

Quantum Fourier Transform

$$|\psi_\ell\rangle = |0\rangle + \exp\left(2i\pi s\frac{\ell}{2^n}\right)|1\rangle\,,\ell$$

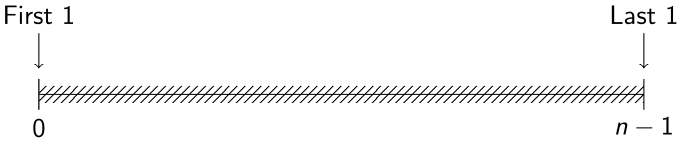
Combining the qubits

Targets

$$\begin{aligned} |\psi_{2^{n-1}}\rangle &= |0\rangle + (-1)^s |1\rangle \\ |\psi_{2^{n-2}}\rangle &= |0\rangle + (-1)^{\lfloor s/2 \rfloor} \exp\left(2i\pi \frac{s \mod 2}{4}\right) |1\rangle \\ &\dots \end{aligned}$$

Combination

$$(\ell_1, \ell_2) \mapsto \ell_1 \pm \ell_2 \mod 2^n$$



$\ell \in [0; 2^n)$

Combining the qubits

Targets

$$
\begin{aligned}
|\psi_{2^{n-1}}\rangle &= |0\rangle + (-1)^s |1\rangle \\
|\psi_{2^{n-2}}\rangle &= |0\rangle + (-1)^{\lfloor s/2 \rfloor} \exp\left(2i\pi \frac{s \mod 2}{4}\right) |1\rangle \\
&\dots
\end{aligned}
$$

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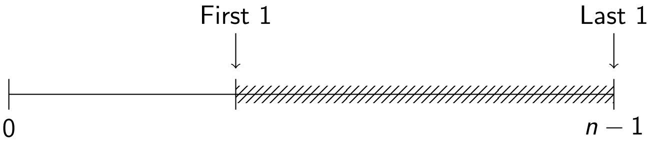

$\ell \in 2^a[0; 2^{n-a})$

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Combining the qubits

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&\dots
\end{aligned}
$$

Combination

$$(\ell_1, \ell_2) \mapsto \ell_1 \pm \ell_2 \mod 2^n$$

$\ell \in \{0, 2^{n-1}\}$

Hidden Shift in $\mathbb{Z}/(N\mathbb{Z})$

Situation

Elements	$\|\psi_\ell\rangle = \|0\rangle + \exp\left(2i\pi s\frac{\ell}{N}\right)\|1\rangle$
Targets	$\bigotimes_{i=0}^{n}\|\psi_{2^i}\rangle \simeq QFT\|t\rangle, \frac{t}{2^n} \simeq \frac{s}{N}$
Combination	$(\ell_1, \ell_2) \mapsto \ell_1 \pm \ell_2 \mod N$

Hidden Shift in $\mathbb{Z}/(N\mathbb{Z})$

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Combination	$(\ell_1, \ell_2) \mapsto \ell_1 \pm \ell_2$ in $\mathbb{Z}$

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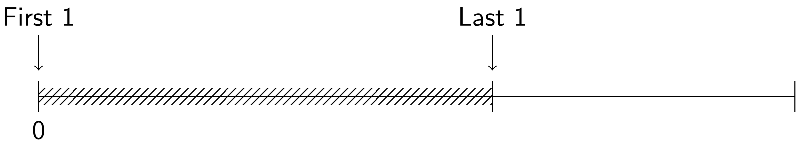

$\ell \in [0; N)$

Hidden Shift in $\mathbb{Z}/(N\mathbb{Z})$

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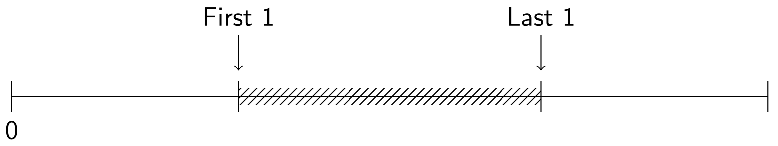

$\ell \in 2^a[0; 2^{a'})$

Hidden Shift in $\mathbb{Z}/(N\mathbb{Z})$

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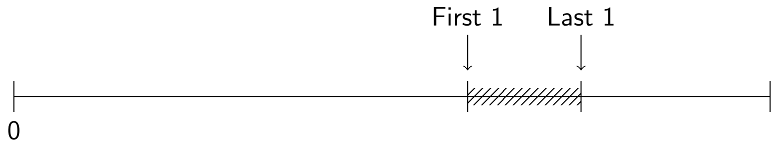

$\ell \in 2^b[0; 2^{b'})$

Hidden Shift in $\mathbb{Z}/(N\mathbb{Z})$

Situation

Elements	$\lvert\psi_\ell\rangle = \lvert 0\rangle + \exp\left(2i\pi s \frac{\ell}{N}\right)\lvert 1\rangle$
Targets	$\bigotimes_{i=0}^{n} \lvert\psi_{2^i}\rangle \simeq QFT\lvert t\rangle$, $\frac{t}{2^n} \simeq \frac{s}{N}$
Combination	$(\ell_1, \ell_2) \mapsto \ell_1 \pm \ell_2$ in $\mathbb{Z}$

$\ell \in \{0, 2^c\}$

Cost for CSIDH

Final complexity

- Around $5 \times 2^{1.8\sqrt{\log_2(N)}}$ (simulated!) queries to f and g and quantum memory
- Log. overhead for classical time and memory

Costs for CSIDH ($\log_2$)

$\log_2(p)$	n	Our Hidden Shift query cost	Query cost estimation from [CLM+]
512	256	32.5	62
1024	512	44.5	94
1792	896	57.5	129

Computing a group action

Objectives

Target

Find an **efficient** procedure to compute:

$$[\mathfrak{g}] \cdot E$$

where E is a CSIDH curve and $[\mathfrak{g}] \in \mathcal{C}\ell\mathcal{O}$, in superposition over the whole group $\mathcal{C}\ell\mathcal{O}$.

General situation

Direct computation of $[\mathfrak{g}] \cdot E$ is expensive

In CSIDH

Computing the $[\mathfrak{l}_i] \cdot E$ is cheap

Cost reduction

Strategy

- Decompose $[\mathfrak{g}] = \prod [\mathfrak{l}_i]^{e_i}$
- Ensure $(e_1, \ldots, e_k)$ is small
- Compute $\prod [\mathfrak{l}_i]^{e_i}$

In Practice

- Precompute a short basis B of $\{(e_1, \ldots, e_k) | \prod [\mathfrak{l}_i]^{e_i} = 1\}$ (BKZ-20)
- Quantumly decompose $[\mathfrak{g}]$ over $[\mathfrak{l}_i]$ (Shor)
- Reduce the size of the exponents using B (Babai)
- Compute the isogeny

Overhead between 2^5 and 2^8 theoretically, heuristically between 2 and 5.

Ordinary curves

The Couveignes–Rostovtsev–Stolbunov scheme

- In general, in the ordinary case, one can find ideal classes to span $\mathcal{C}\ell\mathcal{O}$, but they cost much more.
- Taking $u = \frac{1}{2}\frac{\log p}{\log 3}$:

$$\mathcal{C}\ell\mathcal{O} \simeq \{[\mathfrak{l}_1]^{e_1} \cdots [\mathfrak{l}_u]^{e_u}, e_i \in \{-1, 0, 1\}\} \ .$$

Two choices

- Keep this basis: the dimension increases. The approximation factor remains good in practice: 2^3 for $\log_2 p = 512$ to 2^4 for $\log_2 p = 1024$. could increase up to 2^{15} (in practice better).
- Take a smaller dimension and bigger exponents (asymptotically better) [BFJ16, BJI18].

De Feo–Kieffer–Smith's scheme [FKS18]

Intermediate situation. Products are of the form:

$$[\mathfrak{l}_1]^{e_1} \cdots [\mathfrak{l}_u]^{e_u} \cdots [\mathfrak{l}_{u+v}]^{e_{u+v}} \quad .$$

The e_i have different ranges $-m_i \dots m_i$ and some **must be** positive.

We can adapt!

- Take the weights m_i into account in $\mathcal{L}$.
- Adapt the CVP instance to force some coordinates to be positive.

Overhead 2^5 w.r.t a classical group action (better than taking a naïve decomposition).
$\Rightarrow 2^{38}$ equivalent classical group actions for 56-bit parameters proposed in [FKS18].

Conclusion

Conclusion

We have estimated the cost of Kuperberg's algorithm.

- To reach the NIST security levels in **queries**, parameters should be multiplied by 4.

We have estimated the time to attack CSIDH.

- To reach the NIST security levels in **time**, parameters should be doubled to tripled.

Level	Original $\log_2 p$	Corrected $\log_2 p$
NIST 1	512	900
NIST 3	1024	2500
NIST 5	1792	5000

Thank you!

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