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Abstract: The complexity of the Delaunay triangulation of n points distributed on a surface ranges from linear to quadratic. When the points are a deterministic good sample of a smooth compact generic surface, the size of the Delaunay triangulation is $O(n \log n)$ [2]. Using this result, we prove that when points are Poisson distributed on a surface under the same hypothesis, with intensity λ, the expected size is $O(\lambda \log^2 \lambda)$.

Key-words: 3D reconstruction, probabilistic analysis, point sampling

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Un échantillon de Poisson d'une surface est un bon échantillon.

Résumé : La complexité de la triangulation de Delaunay de n distribués sur une surface peut à priori varier de linéaire à quadratique. Lorsque les points sont un bon échantillon déterministe d'une surface générique férmée la taille de la triangulation est $O(n \log(n))$ [2]. Nous utilisons ce resultat pour prouver que, pour un processus de Poisson d'intensité λ sur une telle surface, la taille en espérance est $O(\lambda \log^2 \lambda)$.

Mots-clés : Reconstruction de surface, analyse probabiliste, échantillon de points

1 Introduction

While the complexity of the Delaunay triangulation of n points is strictly controlled in two dimensions to be between n and $2n$ triangles (depending on the size of the convex hull) the gap between the lower and upper bound ranges from linear to quadratic in dimension 3. The worst case is obtained using points on the moment curve[1] and the best case by using the center of spheres defining a packing.[2] Two relations connect the number of faces of different dimensions: the Euler relation says that the sum of the number of vertices and triangles is equal to the sum of the number of edges and tetrahedra, and counting the number of triangle-tetrahedra incidences yields that the number of tetrahedra is twice the number of triangles. Altogether, for a fixed number of vertices counting the number of edges, triangles, or tetrahedra are equivalent.

To get a more precise result on the size of the 3D Delaunay triangulation, it is possible to make different kinds of hypotheses on the point set. A first possibility is to assume a random distribution in 3D and if the points are evenly distributed, Dwyer proved that the expected size is $\Theta(n)$ [5]. But this hypothesis of random distribution is not relevant for all applications, for example when dealing with 3D reconstruction the Delaunay triangulation is an essential tool and it is much more natural to assume that the points are not distributed in space but on a surface [3]. If the points are evenly distributed on the boundary of a polyhedron, the expected size was proved to be $\Theta(n)$ in the convex case [8] and between $\Omega(n)$ and $O(n \log n)$ in the non convex case by Golin and Na [7].

Instead of using probabilistic hypotheses one can assume that the points are a good sampling of the surface, namely an (ϵ, η)-sample where any ball of radius ϵ centered on the surface contains at least one and at most η points of the point-set. Under such hypothesis Attali and Boissonnat proved that the complexity of the Delaunay triangulation of a polyhedron is linear [1]. Attali, Boissonnat, and Lieutier extend this result to smooth surfaces verifying some genericity hypotheses with an upper bound of $O(n \log n)$ [2]. The genericity hypothesis is crucial since Erickson proved that there exists good sample of a cylinder with a triangulation of size $\Omega(n\sqrt{n})$ [6]. This example by Erickson arrange the point set in a very special position on an helix, nevertheless, even with an unstructured point set it is possible to reach a supra-linear triangulation since Erickson, Devillers, and Goaoc proved that the triangulation of points evenly distributed on a cylinder has expected size $\Theta(n \log n)$ [4].

Contribution In this paper we prove that a Poisson sample of intensity λ on a smooth surface of finite area is an (ε, η)-sample for $\varepsilon = 3\sqrt{\frac{\log \lambda}{\lambda}}$ and $\eta = 1000 \log \lambda$ with high probability. Using the result of Attali, Boissonnat, and Lieutier, it yields that the complexity of the Delaunay triangulation of a Poisson sample of a generic surface is $O(\lambda \log^2 \lambda)$ loosing an extra logarithmic factor with respect to the case of good sampling (see Section 3).

[1] The moment curve is parameterized by (t, t^2, t^3). When computing the Delaunay triangulation of points on this curve, any pair of points define a Delaunay edge.
[2] The kissing number in 3D is 12, thus in such a point set, the number of edges is almost $6n$.

2 Notations, definitions, previous results

We consider a surface Σ embedded in $\mathbb{R}^3$, compact, smooth, oriented and without boundary. At a point $p \in \Sigma$, for a given orientation, we denote by $\kappa_1(p)$ and $\kappa_2(p)$ the principal curvatures at p with $\kappa_1(p) > \kappa_2(p)$. We assume that the curvature is bounded and define $\kappa_{\sup} = \sup_{p\in\Sigma} \max(|\kappa_1(p)|, |\kappa_2(p)|)$. We denote by $\sigma(c, R)$ the sphere of center p and radius R. We denote by $B(\sigma)$ the closed ball whose boundary is the sphere σ, by $\mathring{E}$ the interior of a set E and, for $p \in \Sigma$, by $D(p, R)$ the intersection between Σ and the $\mathring{B}(p, R)$. Abusively we call $D(p, R)$ a disk. For a discrete set X, we denote $\sharp\,(X)$ the cardinality of X. If X is a set of points, $\mathrm{Del}\,(X)$ denotes the Delaunay triangulation of X. In the $3D$ case, $\sharp\,(\mathrm{Del}\,(X))$ is the sum of the number of tetrahedra, triangles, edges and vertices belonging to the Delaunay triangulation.

Definition 1. We say that a set of points X is a spatial Poisson process, or Poisson sample, over E if, for all bounded regions $A \subset E$ measurable, the random variable $\sharp\,(X \cap A)$ has a Poisson distribution.

Without loss of generality, we assume that $\mathrm{Area}\,(\Sigma) = 1$ and consider X as a Poisson sample over Σ of intensity $\lambda > 0$.

We recall classical properties of a Poisson sample:

Proposition 2. *For two regions R and R' of Σ,*

- $\mathbb{P}\left[\sharp\,(X \cap R) = k\right] = \frac{(\lambda\,\mathrm{Area}(R))^k}{k!} e^{-\lambda\,\mathrm{Area}(R)}$*, in particular:* $\mathbb{P}\left[\sharp\,(X \cap R) = 0\right] = e^{-\lambda\,\mathrm{Area}(R)}$,
- $\mathbb{E}\left[\sharp\,(X \cap R)\right] = \lambda\,\mathrm{Area}\,(R)$
- $R \cap R' = \emptyset \Rightarrow \sharp\,(X \cap R)$ *and* $\sharp\,(X \cap R')$ *are independent random variables.*

We consider the same definition of genericity as Attali, Boissonnat and Lieutier, roughly: the set of points where one of the principal curvatures is locally maximal is a finite set of curves whose total length is bounded, and, the number of contacts of a medial ball with the surface is uniformly bounded by a constant.

Then we define what is a good-sampling of a surface and precise the result by Attali, Boissonnat and Lieutier.

Definition 3 (Good sample)**.** A point-set included in a surface is a (ϵ, η)-sample if any ball of radius ϵ centered on the surface contains at least one and at most η points of the sample.

Theorem 4 ([2])**.** *The 3D Delaunay triangulation of an (ϵ, η) sample of a generic smooth surface has complexity $O\left(\frac{\eta^2}{\epsilon^2}\log\frac{1}{\epsilon}\right)$.*

Proof. Looking more carefully at the result of Attali et al. [2, Eq.(14)], we notice that the actual complexity is smaller than $C(\frac{\eta}{\varepsilon})^2 \log(\varepsilon^{-1})$ for C being a constant of the surface. □

3 Is a random sample a good sample?

In a Poisson sampling of intensity λ on the surface, voids and accumulations of points may appear, thus such a sample is likely not to be a good sample with $\varepsilon^2 = \frac{1}{\lambda}$ and η constant. Nevertheless, it is possible to not consider η as a constant, namely, we take $\eta = \Theta(\log(\lambda))$

In a first lemma, we bound the area of $D(p, R)$, for any $p \in \Sigma$ and $R > 0$ sufficiently small.

Lemma 5. *Let Σ be a smooth surface of curvature bounded by $\kappa_{\sup}$, and consider $p \in \Sigma$ and $R > 0$ smaller than $\frac{1}{\kappa_{\sup}}$. The area of $D(p, R)$ is greater than $\frac{3}{4}\pi R^2$.*

Proof. The bound is obtained by considering the fact that the surface must stay in between the two tangent spheres of curvature κ_{sup} tangent to the surface at p. The tangent disk at p of radius $\frac{\sqrt{3}}{2}R > \frac{\sqrt{3}}{2}\frac{1}{\kappa_{\text{sup}}}$ is uncluded in the projection of $D(p,r)$ on the tangent plane and thus has a smaller area than $D(p,R)$. □

Lemma 6. *Let Σ be a $\mathcal{C}^3$ surface of curvature bounded by κ_{sup}.*
For R small enough, $Area(D(p,R)) < \frac{5}{4}\pi R^2$.

Proof. Let $z = f(x,y) := \frac{1}{2}\kappa_1 x^2 + \frac{1}{2}\kappa_2 y^2 + O(x^3+y^3)$ be the Monge patch [9] at a point p on Σ. We denote by $d\sigma$ an element of surface and by $A(p,R)$ the projection of $D(p,R)$ on the xy-plane, since on $D(p,R)$ the slope of the normal to Σ is bounded, we have:

$$Area(D(p,R)) = \int_{D(p,R)} d\sigma = \int\int_{A(p,R)} \sqrt{1 + (\frac{\partial f}{\partial x}(x,y))^2 + (\frac{\partial f}{\partial y}(x,y))^2 dxdy}$$

That is smaller than $\int\int_{x^2+y^2\leq R^2} \sqrt{1 + (\frac{\partial f}{\partial x}(x,y))^2 + (\frac{\partial f}{\partial y}(x,y))^2 dxdy}$, since $D(p,R) \subset B(p,R)$.

Since f is $\mathcal{C}^3$ and $\frac{\partial f}{\partial x}(x,y) \sim \kappa_1 x$, we can say that there exists a neighborhood of p on which $|\frac{\partial f}{\partial x}| \leq \sqrt{2}\kappa_1|x| \leq \sqrt{2}\kappa_{sup}|x|$, i.e., $\left(\frac{\partial f}{\partial x}\right)^2 \leq 2(\kappa_{sup}x)^2$. Applying the same for y, and turning to polar coordinates, we get:

$$Area(D(p,R)) \leq \int_{\theta=0}^{2\pi}\int_{r=0}^{R} r\sqrt{1+2(r\kappa_{\text{sup}})^2}drd\theta = \frac{\pi}{3}\frac{(2(R\kappa_{\text{sup}})^2+1)^{\frac{3}{2}}-1}{\kappa_{\text{sup}}^2}$$

Noticing that $(a+1)^{\frac{3}{2}} - 1 = a\frac{a+\sqrt{a+1}+2}{\sqrt{a+1}+1} \leq \frac{15}{8}a$ for $a < 1$, we can conclude that for any R small enough,

$$Area(D(p,R)) \leq \frac{\pi}{3}\frac{\frac{15}{4}(R\kappa_{\text{sup}})^2}{\kappa_{\text{sup}}^2} = \frac{5}{4}\pi R^2.$$

□

Lemma 7. *Let Σ be a $\mathcal{C}^3$ surface with $Area(\Sigma) = 1$. Let M_R be a maximal set of k_R disjoint disks $D(p_i,R)$ on Σ. If R is small enough then $k_R \leq \frac{4}{3\pi R^2}$.*

Proof. Thanks to lemma 5, for R small enough, we have $D(p,R) \geq \frac{3}{4}\pi R^2$. Thus:

$$k_R \cdot \frac{3}{4}\pi R^2 \leq \sum_{i=1}^{i=k_R} \text{Area}\left(D(p_i,R)\right) \leq \text{Area}\left(\Sigma\right) = 1,$$

and we can deduce the following bound: $k_R \leq \frac{4}{3\pi R^2}$. □

Lemma 8. *Let X be a Poisson sample of intensity λ distributed on a $\mathcal{C}^3$ smooth closed surface Σ of area 1. If λ is large enough, the probability that there exists $p \in \Sigma$ such that $D\left(p, 3\sqrt{\frac{\log\lambda}{\lambda}}\right)$ does not contain any point of X is $O(\lambda^{-2})$.*

Proof. We prove that a Poisson sample has no void of radius $3\sqrt{\frac{\log\lambda}{\lambda}}$ with probability $O(\lambda^{-2})$. In a first part we use a packing argument. On the one hand, for any $\varepsilon > 0$ small enough and given a maximal set $M_{\varepsilon/3}$ and any point $p \in \Sigma$, the disk $D(p,\varepsilon)$ contains entirely one of the disks

$D(p_i, \frac{\varepsilon}{3})$ belonging to $M_{\varepsilon/3}$. Indeed, by maximality of $M_{\varepsilon/3}$, the disk $D(p, \varepsilon/3)$ intersects a disk of $M_{\varepsilon/3}$ whose diameter is $2\varepsilon/3$ so $D(p, \varepsilon)$ contains it entirely. On the other hand, remember from lemma 6 that if ε is small enough then $Area(D(p, R)) \leq \frac{4}{3}\pi R^2$. Then we can bound the probability of existence of an empty disk for ε small enough:

$$\begin{aligned}
\mathbb{P}\left[\exists p \in \Sigma,\ \sharp\left(X \cap D(p, \varepsilon)\right) = 0\right] &\leq \mathbb{P}\left[\exists i < k_{\varepsilon/3},\ \sharp\left(X \cap D(p_i, \varepsilon/3)\right) = 0\right] \\
&\leq k_{\varepsilon/3}\, \mathbb{P}\left[\sharp\left(X \cap D(c, \varepsilon/3)\right) = 0\right] \text{ for a point } c \text{ on } \Sigma \\
&\leq \frac{4}{3\pi(\varepsilon/3)^2} e^{-\lambda \frac{4}{3}\pi(\frac{\varepsilon}{3})^2} = \frac{12}{\pi\varepsilon^2} e^{-\lambda\frac{4\pi\varepsilon^2}{27}}.
\end{aligned}$$

By taking $\varepsilon = 3\sqrt{\frac{\log\lambda}{\lambda}}$ we get:

$$\mathbb{P}\left[\exists p \in \Sigma,\ \sharp\left(X \cap D(p, 3\sqrt{\frac{\log\lambda}{\lambda}})\right) = 0\right] \leq \frac{12\lambda}{9\pi\log\lambda} e^{-\frac{4*9\pi\log\lambda}{27}} = O(\lambda^{-2}).$$

□

We have proved that when a Poisson sample is distributed on a surface, the points cover sufficiently the surface, i.e., there is no large empty disk on the surface with high probability. Now we have to verify the other property of a good sample, namely, a Poisson sample does not create large concentration of points in a small area.

Lemma 9. *Let X be a Poisson sample of intensity λ distributed on a $\mathcal{C}^3$ closed surface of area 1. If λ is large enough, the probability that there exists $p \in \Sigma$ such that $D(p, 3\sqrt{\frac{\log\lambda}{\lambda}})$ contains more that $1000\log(\lambda)$ points of X is $O(\lambda^{-2})$.*

Proof. Consider a M_ε maximal set, we can notice that for any $p \in \Sigma$, the disk $D(p, \varepsilon)$ with $p \in \Sigma$ is entirely contained in one disk $D(p_i, 3\varepsilon)$ that is an augmented disk of M_ε. Indeed, by maximality of M_ε, the disk $D(p, \varepsilon)$ intersects a disk from M_ε say $D(p_j, \varepsilon)$ so $D(p_j, 3\varepsilon)$ contains entirely $D(p, \varepsilon)$.

Then we can bound the probability of existence of a disk containing more than η points:

$$\begin{aligned}
\mathbb{P}\left[\exists p \in \Sigma,\ \sharp\left(X \cap D(p, \varepsilon)\right) > \eta\right] &\leq \mathbb{P}\left[\exists i < k_\varepsilon,\ \sharp\left(X \cap D(p_i, 3\varepsilon)\right) > \eta\right] \\
&\leq k_\varepsilon\, \mathbb{P}\left[\sharp\left(X \cap D(c, 3\varepsilon)\right) > \eta\right] \text{ for a point } c \text{ on } \Sigma \\
&\leq \frac{4}{3\pi\varepsilon^2}\, \mathbb{P}\left[\sharp\left(X \cap D(c, 3\varepsilon)\right) > \eta\right]
\end{aligned}$$

We use a Chernoff inequality [10] to bound $\mathbb{P}\left[\sharp\left(X \cap D(c, 3\varepsilon)\right) > \eta\right]$: If V follows a Poisson law of mean v_0, then $\forall v > v_0$,

$$P(V > v) \leq e^{v - v_0}\left(\frac{v_0}{v}\right)^v.$$

From lemmas 5 and 6, we have that: $\frac{27}{4}\pi\varepsilon^2 \leq Area(D(c, 3\varepsilon)) \leq \frac{45}{4}\pi\varepsilon^2$ for ε small enough. Consequently we can say that the expected number of points v_0 in $D(c, 3\varepsilon)$ verifies $\frac{27}{4}\lambda\pi\varepsilon^2 \leq v_0 \leq \frac{45}{4}\lambda\pi\varepsilon^2$.

Then we apply the above Chernoff bound with $v = \frac{45}{4}e\pi\lambda\varepsilon^2$ (chosen for the convenience of the calculus)

$$\begin{aligned}\mathbb{P}\left[\sharp\left(X\cap D(c,3\varepsilon)\right)>\frac{45}{4}e\pi\lambda\varepsilon^2\right] &\leq e^{\frac{45}{4}e\pi\lambda\varepsilon^2-v_0}\left(\frac{v_0}{\frac{45}{4}e\pi\lambda\varepsilon^2}\right)^{\frac{45}{4}e\pi\lambda\varepsilon^2}\\ &\leq e^{\frac{45}{4}e\pi\lambda\varepsilon^2-\frac{27}{4}\pi\lambda\varepsilon^2}\left(\frac{\frac{45}{4}\pi\lambda\varepsilon^2}{\frac{45}{4}e\pi\lambda\varepsilon^2}\right)^{\frac{45}{4}e\pi\lambda\varepsilon^2}\\ &= e^{-\frac{27}{4}\pi\lambda\varepsilon^2}\end{aligned}$$

So for $\varepsilon = 3\sqrt{\frac{\log\lambda}{\lambda}}$ and $\eta = \frac{45}{4}e\pi\lambda\varepsilon^2 = \frac{405}{4}e\pi\log\lambda$, we have:

$$\mathbb{P}\left[\exists p\in\Sigma,\ \sharp\left(X\cap D(p,3\sqrt{\tfrac{\log\lambda}{\lambda}})\right)>\frac{405}{4}e\pi\log\lambda\right]\leq\frac{4\lambda}{27\pi\log\lambda}e^{-\frac{243}{4}\pi\log\lambda}=O(\lambda^{-189})$$

Since $\frac{405}{4}e\pi < 1000$, it is sufficient for our purpose to say:

$$\mathbb{P}\left[\exists p\in\Sigma,\ \sharp\left(X\cap D(p,3\sqrt{\tfrac{\log\lambda}{\lambda}})\right)>1000\log(\lambda)\right]=O(\lambda^{-2})$$

□

Theorem 10. *On a $\mathcal{C}^3$ closed surface, a Poisson sample of intensity λ large enough is a $(3\sqrt{\frac{\log\lambda}{\lambda}}, 1000\log\lambda)$-sample with probability $1-O(\lambda^{-2})$.*

Proof. From lemmas 8 and 9, we have that a Poisson sample is not a $(3\sqrt{\frac{\log\lambda}{\lambda}}, 1000\log\lambda)$-sample with probability $O(\lambda^{-2})$. □

Theorem 11. *For λ great enough, the Delaunay triangulation of a point set Poisson distributed with intensity λ on a closed smooth generic surface of area 1 has $O(\lambda\log^2\lambda)$ expected size.*

Proof. Given a Poisson sample X we distinguish two cases:

- If X is a good sample, i.e., an (ε,η)-sample with $\varepsilon = 3\sqrt{\frac{\log\lambda}{\lambda}}$ and $\eta = 1000\log\lambda$, we apply Attali et al. result and get a complexity bounded by $(\frac{\eta}{\varepsilon})^2\log(\varepsilon^{-1})$ up to a constant that is $O(\lambda\log^2\lambda)$.
- If X is not a good sample, which arises with small probability by Lemma 10, we bound the triangulation size by the quadratic bound:

$$\begin{aligned}\sum_{k\in\mathbb{N}}k^2\,\mathbb{P}\left[\sharp(X)=k\right] &= \sum_{k\in\mathbb{N}}k^2\frac{\lambda^k}{k!}e^{-\lambda}\\ &= \lambda\sum_{k=1}^{\infty}k\frac{\lambda^{k-1}}{(k-1)!}e^{-\lambda}\\ &= \lambda\left(\sum_{k=1}^{\infty}(k-1)\frac{\lambda^{k-1}}{(k-1)!}e^{-\lambda}+\sum_{k=1}^{\infty}\frac{\lambda^{k-1}}{(k-1)!}e^{-\lambda}\right)\\ &= \lambda\left(\lambda\sum_{k=2}^{\infty}\frac{\lambda^{k-2}}{(k-2)!}e^{-\lambda}+1\right)\\ &= \lambda(\lambda+1)\\ &= O(\lambda^2)\end{aligned}$$

Combining the two results, we get

$$\begin{aligned}\mathbb{E}\left[\sharp\left(\mathrm{Del}\left(X\right)\right)\right] &= \mathbb{E}\left[\sharp\left(\mathrm{Del}\left(X\right)\right)|X \text{ good sample}\right]\mathbb{P}\left[X \text{ good sample}\right] \\ &\quad + \mathbb{E}\left[\sharp\left(\mathrm{Del}\left(X\right)\right)|X \text{ not good sample}\right]\mathbb{P}\left[X \text{ not good sample}\right] \\ &\le O\left(\lambda\log^2\lambda\right)\times 1 + O(\lambda^2)\times O\left(\frac{1}{\lambda^2}\right) = O(\lambda\log^2\lambda)\end{aligned}$$

□

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