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Abstract

In this paper, the problem of fixed-time stabilization of constant-parameter reaction-diffusion partial differential equations by means of continuous boundary time-varying feedbacks is considered. Moreover, the time of convergence can be prescribed in the design. The design of time-varying feedbacks is carried out based on the backstepping approach. Using a suitable target system with a time varying-coefficient, one can state that the resulting kernel of the backstepping transformation is time-varying and rendering the control feedback to be time-varying as well. Explicit representations of the kernel solution in terms of *generalized Laguerre polynomials* and *modified Bessel functions* are derived. The fixed-time stability property is then proved. A simulation example is presented to illustrate the main results.

Key words: Linear reaction-diffusion systems; backstepping method; time-varying feedbacks; fixed-time stabilization; generalized Laguerre polynomials.

1 Introduction

The problem of finite-time stabilization and estimation has been widely considered for many years in the framework of linear and nonlinear ordinary differential equations (ODEs) (see e.g. [11,3,27,12,31]). The need to meet some performance, time constraints and precision has highly motivated the stabilization and estimation in finite time.

For infinite dimensional systems, namely partial differential equations (PDEs), finite-time concepts have also become an attractive research area but have not achieved yet a sufficient level of maturity as in the finite-dimensional case. It is known, however, that finite-time convergence may be a natural phenomena in PDEs. As a matter of example, some nonlinear parabolic equations may face finite-time extinction, provided some necessary and sufficient conditions on the in-domain absorption term, as reported in [10]. Finite-time extinction property can be also realized for hyperbolic systems of conservation laws as reported in [21]. Both examples are very motivating and strongly support the fact that finite-time property can appear naturally or can be established/imposed by means of control actions. In this regard, for hyperbolic PDE systems, some contributions on stabilization in finite-time can be highlighted: see for instance [1,6] where the backstepping method is employed to design boundary controllers. The main idea is to select suitable target systems that meet naturally the finite-time property. Then, the problem of stabilization is turned to a solvability problem and well-posedness issues for kernels of the underlying backstepping transformation (Volterra and/or Fredholm type). For more general classes of infinite dimensional systems, we point out homogeneity arguments as developed in [26]. The designed control law steers any solution of the closed-loop system to zero in a finite time and whose design procedure is based on the concept of generalized homogeneity of operators in Banach/Hilbert spaces.

Besides this, some recent results deal with finite-time control and the null controllability for linear parabolic PDEs. In [23] for instance, by following some sliding-mode techniques, a distributed power-fractional control strategy is used to stabilize an unperturbed heat equation in finite-time. In [25], a piecewise linear boundary control with a state dependent switching law is designed to boost the convergence of the state of the heat equation to zero in finite-time. The latter is highly inspired by [7] where the null controllability and finite-time sta-

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bilization of the heat equation are deeply studied. The main idea builds on the design of time-varying feedbacks via the backstepping approach. It is worth recalling that backstepping method has been used as standard tool for design feedback laws for stabilization of PDEs (exponentially, for the most part). The first continuous backstepping control was proposed for the heat equation in [4]. Then, closed-form controllers were introduced in [28]. In [2] and [19] some generalizations are presented for coupled reaction-diffusion systems with constant parameters. More recently, [33] and [8] consider a more general case of coupled reaction-diffusion systems with spatially-varying coefficients. A rigorous study on well-posedness issues of kernel transformation is also carried out.

The motivations for fixed/finite-time control and observation for PDEs are in the same way of those for the finite-dimensional systems. In particular, since many complex systems are described in parabolic PDE setting, convergence while meeting time constraints or just realizing the well-known separation principle are central issues, which can be coped when addressing finite-time concepts. Synthesis of controllers to achieve these goals would bring more challenges than exponential stabilization. These considerations are perfectly relevant to very important applications. To mention a few: 1) large networks of multi-agent systems whenever a continuum model may be posed in terms of diffusion parabolic equations. In fact, finite-time deployment and formation control have attracted a lot of attention. Some recent results on finite-time deployment of multi-agent systems are reported e.g. in [13]. 2) Tubular chemical reactors [18], in which precision in time may be required to control the concentration along the reactor. 3) Thermal control of solid propellant rockets (see e.g. [28, Section D] and the reference therein) whose operation duration may be confined to a finite-time interval (e.g. for space rockets or tactical missile guidance applications). These examples under parabolic PDE setting along with the above considerations strongly motivate the development of finite-time concepts with *late-lumped* based control and observation strategies.

This paper is then devoted to the fixed-time stabilization of a scalar reaction-diffusion system. Highly inspired by [7,25] and [29], we propose continuous boundary time-varying feedbacks for finite-time stabilization in a prescribed time; often refereed in this paper as *fixed-time stabilization*. The concept of fixed-time differs from finite-time whenever the settling time is independent of the initial condition as it is in our study. It is worth recalling that for linear hyperbolic PDEs, fixed-time convergence may be achieved at a prescribed time independent of the initial conditions but dependent on the transport velocities. There is however a minimal control time to be respected due to the propagation nature of characteristic solutions. One of the main features in linear hyperbolic PDEs is that there is no need to use time-varying feedbacks to control in fixed-time whereas for linear parabolic PDEs, the use of time-varying feedbacks really constitutes a key tool.

This work borrows the idea of prescribed time control recently studied in [30,31] for normal-form nonlinear systems by using time-varying feedbacks based on a scaling transformation of the states with blowing up functions. Prior to this, a similar approach to stabilize the underlying nonlinear system was proposed in [17] by means of instantaneous impulse actuation according to the Schwartz' distribution theory where the use of delta-wise description is utilized. Moreover, fully in line with this description, for parabolic PDEs, point-wise sensing and actuation are developed in [22].

The main contribution of our work relies on the use of a continuous boundary time-varying feedback obtained via the backstepping transformation (whose kernel is time-varying) for stabilization in a fixed-time. A suitable target system, with a time-dependent coefficient is employed to come up with the time-varying kernel. It makes our approach considerably different w.r.t for example [25]. We provide an explicit representation of the kernel solution which involves a suitable dynamic variable whose solution blows up in prescribed finite-time. When solving the kernel system, classical orthogonal polynomial such as *generalized Laguerre polynomials* and their properties (see e.g. [32,15]) turn out to be involved. A relationship with special functions such as modified and nonmodified Bessel functions is also discussed.

This paper is organized as follows. In Section 2, we present the problem and the backstepping approach with time-varying kernels. Section 3 provides the main result on fixed-time stabilization. In Section 4 we discuss the choice of the blow up function. Section 5 provides a numerical example to illustrate the main results. Finally, conclusions and perspectives are given in Section 6.
Notations $\mathbb{R}^+$ will denote the set of nonnegative real numbers. The set of all functions $g : [0,1] \to \mathbb{R}$ such that $\int_0^1 g(x)^2 dx < \infty$ is denoted by $L^2((0,1),\mathbb{R})$ and is equipped with the norm $\|\cdot\|_{L^2((0,1),\mathbb{R})}$. $\Gamma(\cdot)$ denotes the Gamma function. $I_m(\cdot)$, $J_m(\cdot)$ with $m \in \mathbb{Z}$, denote the modified Bessel and (nonmodified) Bessel functions of the first kind, respectively. $L_m^{(\alpha)}(\cdot)$ denotes the generalized Laguerre polynomial. Finally, $\binom{n}{k} := \frac{n!}{k!(n-k)!}, k = 1, 2, .., n$ denotes the binomial coefficients.

2 Problem description

Our approach builds on continuous boundary time-varying control functionals that are bounded. We are interested in injecting more and more energy to the system until reaching the equilibrium in finite-time. Moreover, the time of convergence can be prescribed in the control design. The main ideas are highly inspired by those in [7] and [30]. Let us consider the following scalar reaction-diffusion system with constant coefficients:

$$u_t(t,x) = \theta u_{xx}(t,x) + \lambda u(t,x) \quad (1)$$
$$u(t,0) = 0 \quad (2)$$
$$u(t,1) = U(t) \quad (3)$$

and initial condition:

$$u(0,x) = u_0(x) \quad (4)$$

where $\theta > 0$ and $\lambda \in \mathbb{R}$. $u : [0,T) \times [0,1] \to \mathbb{R}$ is the system state where $T > 0$ is given, and will be called from now *prescribed time*. In addition, $U(t) \in \mathbb{R}$ is the control input which will be from now a *time-varying feedback* having the functional form

$$U(t) = \mathcal{K}(t)[u(t,\cdot)](1) \tag{5}$$

with

$$\mathcal{K}(t)[u(t,\cdot)](1) = \int_0^1 k(1,y,t)u(t,y)dy \tag{6}$$

which will be characterized later on.

2.1 Backstepping transformation and time-varying kernel equations

In this work, we aim at providing a boundary control, designed via the backstepping approach, in order to steer the state of the system (1)-(4) to zero in a prescribed time T. As aforementioned, the key ingredient in this framework is the use of time-varying feedbacks. Consequently, the invertible Volterra integral transformation is chosen to depend on time. It is given as follows,

$$\begin{aligned} w(t,x) &= u(t,x) - \int_0^x k(x,y,t)u(t,y)dy \\ &= \mathcal{K}(t)[u(t,\cdot)](x) \end{aligned} \tag{7}$$

rendering the kernel time-varying. The aim is to transform the system (1)-(4) into the following target system:

$$w_t(t,x) = \theta w_{xx}(t,x) - c(t)w(t,x) \tag{8}$$
$$w(t,0) = 0 \tag{9}$$
$$w(t,1) = 0 \tag{10}$$

with initial condition:

$$w_0(x) = u_0(x) - \int_0^x k(x,y,0)u_0(y)dy \tag{11}$$

where $w : [0,T) \times [0,1] \to \mathbb{R}$ is the target system state. Note that different to exponentially stable target reaction-difussion systems found in the literature, in the right-hand side of (8), there is a time-dependent parameter $c(t)$ that will be designed to achieve fixed-time stability.

Following the standard methodology to find the kernel equations and by taking into account the time-varying dependence, it can be shown that by introducing (7) into (8)-(10), using the Leibnitz differentiation rule, integrating by parts and using the the boundary conditions, the original system is transformed into the target system with the kernel of the transformation (7) satisfying the following PDE system:

$$k_{xx}(x,y,t) - k_{yy}(x,y,t) = \gamma(t)k(x,y,t) + \tfrac{1}{\theta}k_t(x,y,t) \tag{12}$$
$$k(x,0,t) = 0 \tag{13}$$
$$\frac{d}{dx}k(x,x,t) = -\tfrac{1}{2}\gamma(t) \tag{14}$$

provided that

$$c(t) = -\lambda + \gamma(t)\theta \tag{15}$$

where k is defined on the domain $\mathcal{T} = \{(x,y,t) \in \mathbb{R}^2 \times [0,T) : 0 \le y \le x \le 1\}$ and γ can be chosen to be a smooth positive time-varying scalar function defined on $[0,T)$.

Under a suitable characterization of γ, the fixed-time stabilization problem is now related to the problem of solvability of kernel equations (12)-(14). Solving them and choosing $U(t)$ given in (5) we realize the backstepping transformation.

Remark 1. *Note that the right-hand side of* (12) *contains the partial derivative of the kernel w.r.t time. It brings an additional source of complexity when solving the PDE kernel system. However, it is worth remarking that more general cases, where the reaction term is both space- and time-varying dependent, have been rigorously addressed in e.g. [14] and [34]. In these works, the solutions of the PDE kernel (PIDE kernel, respectively) are determined by means of the method of integral operators. They constitute even a more general approach than the seminal work [5], where solvability is guaranteed under the assumption of time analyticity of the reaction term. Having said that, the PDE kernel* (12)-(14) *is much simpler and can be seen as a particular case while having the reaction term only time-depend; thus well-posendess would immediately follows. Furthermore, the problem with time-dependent reaction term has been already addressed in [29] where series solutions are obtained. We intend to exploit the latter work.*

2.2 Solution of the PDE kernel (12)-(14)

Let us choose γ in (12)-(14) to be the solution that satisfies the following scalar nonlinear ordinary differential equation:

$$\dot{\gamma}(t) = \frac{2}{\gamma_0 T}\gamma^{3/2}(t), \quad \gamma(0) = \gamma_0^2 > 0 \tag{16}$$

where $T > 0$ is given and is going to be the prescribed time. It is straightforward to verify that the solution to (16) is as follows:

$$\gamma(t) = \frac{\gamma_0^2 T^2}{(T-t)^2} \tag{17}$$

This solution is monotonically increasing and blows up at time T [1].

The well-posedness of (12)-(14),(16) will follow by explicitly finding a closed-form analytical solution. To that end, let us first state a relevant result (well-known in the framework of orthogonal polynomials; see e.g. [32]).

[1] A discussion about the choice of the power degree of (17) will be provided in Section 4.

Proposition 1. *Let $L_n^{(\alpha)}(\cdot)$ a generalized Laguerre polynomial be defined by*

$$L_n^{(\alpha)}(p) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-p)^k}{k!} \tag{18}$$

and let $J_\alpha(\cdot)$ a Bessel function of the first kind of order α be defined by

$$J_\alpha(p) = \sum_{k=0}^{\infty} \frac{(-1)^k (p/2)^{\alpha+2k}}{k!\Gamma(k+\alpha+1)} \tag{19}$$

where $\Gamma(\cdot)$ denotes the Gamma function. Then, the following relation holds:

$$\sum_{n=0}^{\infty} \frac{v^n L_n^{(\alpha)}(p)}{\Gamma(n+\alpha+1)} = (vp)^{-\alpha/2} e^v J_\alpha(2\sqrt{vp}) \tag{20}$$

Proof. See [32, Chapter 5 p. 102] ■

Lemma 1. *Let $T > 0$ be given and γ satisfy (16). The problem (12)-(14) has a well-posed C^∞ solution on $\mathcal{T}$, given by*

$$\begin{aligned} k(x,y,t) = &-\frac{y}{2}\gamma(t)\sum_{n=0}^{\infty} \frac{(\sqrt{\gamma(t)}(x^2-y^2))^n}{4^n (T\gamma_0\theta)^n (n+1)!} \\ &\times L_n^{(1)}\left(-(T\gamma_0\theta)\sqrt{\gamma(t)}\right) \end{aligned} \tag{21}$$

Proof. Following the main lines of [29, Section 5], we look for a solution of the form:

$$k(x,y,t) = -\tfrac{y}{2} e^{-\theta\int_0^t \gamma(\tau)d\tau} f(z,t), \quad z = \sqrt{\tfrac{x^2-y^2}{\theta}} \tag{22}$$

Introducing (22) into (12)-(14), we straightforwardly obtain that $f(z,t)$ satisfies the following nonlinear parabolic PDE:

$$f_t(z,t) = f_{zz}(z,t) + \tfrac{3}{z} f_z(z,t) \tag{23}$$

with boundary conditions,

$$f_z(0,t) = 0, \quad f(0,t) = \gamma(t) e^{\theta\int_0^t \gamma(\tau)d\tau} =: F(t) \tag{24}$$

where $F(t)$ is a C^∞ function (whose n-th derivative w.r.t time is denoted by $F^{(n)}(t) := \frac{d^n}{dt^n}F(t)$). Since we deal with an infinitely differentiable function $\gamma(t)$, the solution to (23)-(24) can be found in [24] and it is as follows:

$$f(z,t) = \sum_{n=0}^{\infty} \frac{z^{2n}}{4^n n!(n+1)!} F^{(n)}(t) \tag{25}$$

The time derivative of F, with γ satisfying (16), is given by

$$F^{(1)}(t) = \tfrac{1}{T\gamma_0}\gamma^{3/2}(t)\left(T\gamma_0\theta\gamma^{1/2}(t) + 2\right) e^{\theta\int_0^t \gamma(\tau)d\tau} \tag{26}$$

Computing few more time derivatives of F we get,

$$\begin{aligned} F^{(2)}(t) =& \tfrac{1}{(T\gamma_0)^2}\gamma^2(t)\Big((T\gamma_0\theta)^2\gamma(t) \\ &+ 6(T\gamma_0\theta)\gamma^{1/2}(t) + 6\Big) e^{\theta\int_0^t \gamma(\tau)d\tau} \end{aligned}$$

$$\begin{aligned} F^{(3)}(t) =& \tfrac{1}{(T\gamma_0)^3}\gamma^{5/2}(t)\Big((T\gamma_0\theta)^3\gamma^{3/2}(t) + 12(T\gamma_0\theta)^2\gamma(t) \\ &+ 36(T\gamma_0\theta)\gamma^{1/2}(t) + 24\Big) e^{\theta\int_0^t \gamma(\tau)d\tau} \\ F^{(4)}(t) =& \tfrac{1}{(T\gamma_0)^4}\gamma^3(t)\Big((T\gamma_0\theta)^4\gamma^2(t) + 20(T\gamma_0\theta)^3\gamma^{3/2}(t) \\ &+ 120(T\gamma_0\theta)^2\gamma(t) + 240(T\gamma_0\theta)\gamma^{1/2}(t) \\ &+ 120\Big) e^{\theta\int_0^t \gamma(\tau)d\tau} \end{aligned}$$

Iterating the computations we can observe a recursive pattern which involves a *generalized Laguerre polynomial* and thereby, it leads to the following formula:

$$\begin{aligned} F^{(n)}(t) =& \tfrac{1}{(T\gamma_0)^n}\gamma(t)\gamma^{n/2}(t) n! \\ &\times L_n^{(1)}\left(-(T\gamma_0\theta)\gamma^{1/2}(t)\right) e^{\theta\int_0^t \gamma(\tau)d\tau} \end{aligned} \tag{27}$$

Let us prove by induction that (27) is correct for all $n > 0$. For $n = 1$, the result is immediate and yields (26). Now, we assume the result is valid for n. Then, we prove it for $n+1$.

To that end, we shall use some properties of generalized Laguerre polynomials. In particular, the recurrence and the derivative formulas. For any $\alpha > -1$, these are as follows, respectively [32]:

$$L_{n+1}^{(\alpha)}(p) = \tfrac{1}{n+1}\Big((2n+\alpha+1-p)L_n^{(\alpha)}(p) - (n+\alpha)L_{n-1}^{(\alpha)}(p)\Big) \tag{28}$$

$$\frac{d}{dp}L_n^{(\alpha)}(p) = p^{-1}\left(nL_n^{(\alpha)}(p) - (n+\alpha)L_{n-1}^{(\alpha)}(p)\right) \tag{29}$$

Let us take for simplicity $p = T\gamma_0\theta\gamma^{1/2}(t)$.

Therefore, computing the time derivative of F^n, we get

$$\begin{aligned} &F^{(n+1)}(t) = \\ &\Big(\tfrac{(n+2)n!}{(T\gamma_0)^{n+1}}\gamma^{n/2+3/2}(t)L_n^{(1)}(-p) + \\ &\tfrac{n!(-p)}{(T\gamma_0)^{n+1}}\gamma^{n/2+3/2}(t)\frac{d}{dp}L_n^{(1)}(-p) \\ &\quad + \tfrac{n!p}{(T\gamma_0)^{n+1}}\gamma^{n/2+3/2}(t)L_n^{(1)}(-p)\Big) e^{\theta\int_0^t \gamma(\tau)d\tau} \end{aligned}$$

with $\frac{d}{dp}L_n^{(1)}(-p) = (-p)^{-1}\big(nL_n^{(1)}(-p) - (n+1)L_{n-1}^{(1)}(-p)\big)$ by virtue of (29). Thus,

$$\begin{aligned} &F^{(n+1)}(t) = \\ &\Big(\tfrac{(n+2)n!}{(T\gamma_0)^{n+1}}\gamma^{n/2+3/2}(t)L_n^{(1)}(-p) + \\ &\tfrac{n!}{(T\gamma_0)^{n+1}}\gamma^{n/2+3/2}(t)\big(nL_n^{(1)}(-p) - (n+1)L_{n-1}^{(1)}(-p)\big) \\ &\quad + \tfrac{n!p}{(T\gamma_0)^{n+1}}\gamma^{n/2+3/2}(t)L_n^{(1)}(-p)\Big) e^{\theta\int_0^t \gamma(\tau)d\tau} \end{aligned}$$

which is simplified as follows

$$F^{(n+1)}(t) = \tfrac{1}{(T\gamma_0)^{n+1}}\gamma^{n/2+3/2}(t)n!\Big(2(n+1)L_n^{(1)}(-p) + pL_n^{(1)}(-p) - (n+1)pL_{n-1}^{(1)}(-p)\Big)e^{\theta\int_0^t \gamma(\tau)d\tau} \tag{30}$$

According to the recurrence formula (28) applied to (30), we have

$$F^{(n+1)}(t) = \tfrac{1}{(T\gamma_0)^{n+1}}\gamma^{n/2+3/2}(t)(n+1)! \times L_{n+1}^{(1)}(-p)e^{\theta\int_0^t \gamma(\tau)d\tau} \tag{31}$$

Then, it is proved by induction that (27) holds for all $n > 0$.

Hence, from (22) and (25) along with (27) we finally obtain that

$$k(x,y,t) = -\tfrac{y}{2}\gamma(t)\sum_{n=0}^{\infty}\tfrac{(\sqrt{\gamma(t)}(x^2-y^2))^n}{4^n(T\gamma_0\theta)^n(n+1)!} \times L_n^{(1)}\left(-(T\gamma_0\theta)\sqrt{\gamma(t)}\right)$$

that is (21). This concludes the proof. ∎

In the sequel, we provide a further closed-form kernel in terms of the modified Bessel function of the first kind. It will be helpful for numerical tractability purposes and even more importantly, to establish a qualitative analysis for the boundedness of the backstepping transformation; truly necessary in the fixed-time stability result.

Theorem 1. *The system* (12)-(14),(16) *has a well-posed* C^∞ *solution on* $\mathcal{T}$*, given by*

$$k(x,y,t) = -y\gamma(t)e^{\frac{\sqrt{\gamma(t)}(x^2-y^2)}{4T\gamma_0\theta}}\frac{I_1\left(\sqrt{\gamma(t)(x^2-y^2)}\right)}{\sqrt{\gamma(t)(x^2-y^2)}} \tag{32}$$

Proof. A straightforward application of Proposition 1 to (21), with $v = \frac{\sqrt{\gamma(t)}(x^2-y^2)}{4T\gamma_0\theta}$ and knowing that $\Gamma(n+2) = (n+1)!$ as well as $I_1(p) = \frac{1}{\sqrt{-1}}J_1(\sqrt{-1}p)$, leads to the relation (32). It concludes the proof. ∎

Note that in this framework, kernel gain increases more and more meaning that convergence to the equilibrium is increasingly faster as t goes to T; but the control function remains bounded as it is going to be stated in the main result later on.

Remark 2. *According to* (16)*,* $\dot{\gamma}(t) \to 0$ *and* $\gamma(t) \to \gamma_0^2$ *as* $T \to \infty$ *uniformly on any compact interval of time. In this case, from* (32)*,* $e^{\frac{\sqrt{\gamma(t)}(x^2-y^2)}{4T\gamma_0\theta}} \to 1$ *as* $T \to \infty$ *and therefore we derive,*

$$k(x,y) = -y\gamma^*\frac{I_1\left(\sqrt{\gamma^*(x^2-y^2)}\right)}{\sqrt{\gamma^*(x^2-y^2)}} \tag{33}$$

with $\gamma^* = \gamma_0^2$.

An alternative way to obtain (33) *is by remarking that if* $\gamma(t) \equiv \gamma^*$ *is constant, we would have that* $F(t) = \gamma^* e^{\theta\gamma^* t}$ *(in* (24)*). Therefore, by computing iteratively its time derivative we get*

$$F^{(n)}(t) = \theta^n(\gamma^*)^{n+1}e^{\theta\gamma^* t}$$

Replacing it into (25) *together with* (22) *yields*

$$k(x,y,t) = -\tfrac{y}{2}\gamma^*\sum_{n=0}^{\infty}\tfrac{((x^2-y^2)/\theta)^n}{4^n n!(n+1)!}\theta^n(\gamma^*)^n$$

which corresponds to (33)*. This is the closed-form kernel, originally found in [28] for the exponential stabilization of constant-parameter reaction-diffusion systems under the backstepping approach.*

2.3 Inverse transformation and time-varying kernel equations

The analysis of fixed-time stability of the closed-loop system requires the study of the inverse backstepping transformation which is determined explicitly. Indeed, it is given by

$$\begin{aligned} u(t,x) &= w(t,x) + \int_0^x l(x,y,t)w(t,y)dy \\ &= \mathcal{L}(t)[w(t,\cdot)](x) \end{aligned} \tag{34}$$

whose kernel $l(x,y,t)$, can be shown (by following the standard procedure) to satisfy the following PDE system:

$$l_{xx}(x,y,t) - l_{yy}(x,y,t) = -\gamma(t)l(x,y,t) + \tfrac{1}{\theta}l_t(x,y,t) \tag{35}$$

$$l(x,0,t) = 0 \tag{36}$$

$$\frac{d}{dx}l(x,x,t) = -\tfrac{1}{2}\gamma(t) \tag{37}$$

provided that $c(t) = -\lambda + \gamma(t)\theta$ and l is defined on the domain $\mathcal{T} = \{(x,y,t) \in \mathbb{R}^2 \times [0,T) : 0 \le y \le x \le 1\}$. As before, γ can be chosen to be a smooth positive time-varying scalar function defined on $[0,T)$. In this framework, it will be as in (16).

The time-varying feedback (5) can equivalently be written under the following functional form

$$U(t) = \mathcal{L}(t)[w(t,\cdot)](1) \tag{38}$$

with

$$\mathcal{L}(t)[w(t,\cdot)](1) = \int_0^1 l(1,y,t)w(t,y)dy \tag{39}$$

2.4 Solution of the PDE kernel (35)-(37)

Lemma 2. *Let $T>0$ be given and γ satisfying* (16). *The problem* (35)-(37) *has a well-posed C^∞ solution on $\mathcal{T}$, given by*

$$\begin{aligned} l(x,y,t) = &-\frac{y}{2}\gamma(t)\sum_{n=0}^{\infty}\frac{(\sqrt{\gamma(t)}(x^2-y^2))^n}{4^n(T\gamma_0\theta)^n(n+1)!} \\ &\times L_n^{(1)}\left(T\gamma_0\theta\sqrt{\gamma(t)}\right) \end{aligned} \tag{40}$$

Proof. We proceed similarly to the proof of Lemma 1. Let us just point out that we seek for a solution of the form

$$l(x,y,t) = -\tfrac{y}{2}e^{\theta\int_0^t\gamma(\tau)d\tau}g(z,t), \quad z=\sqrt{\tfrac{x^2-y^2}{\theta}} \tag{41}$$

where g satisfies

$$g_t(z,t) = g_{zz}(z,t)+\tfrac{3}{z}g_z(z,t)$$

with boundary conditions

$$g_z(0,t)=0, \quad g(0,t)=\gamma(t)e^{-\theta\int_0^t\gamma(\tau)d\tau} =: G(t) \tag{42}$$

where $G(t)$ is a C^∞ function (whose n-th derivative w.r.t time is denoted by $G^{(n)}(t) := \frac{d^n}{dt^n}G(t)$). As before, the solution to the nonlinear parabolic PDE admits a power series representation,

$$g(z,t) = \sum_{n=0}^{\infty}\frac{z^{2n}}{4^n n!(n+1)!}G^{(n)}(t) \tag{43}$$

Computing iteratively the time derivative of G and realizing the pattern which leads to the generalized Laguerre polynomials, one gets, for all $n>0$

$$\begin{aligned} G^{(n)}(t) =& \tfrac{1}{(T\gamma_0)^n}\gamma(t)\gamma^{n/2}(t)n! \\ &\times L_n^{(1)}\left((T\gamma_0\theta)\gamma^{1/2}(t)\right)e^{-\theta\int_0^t\gamma(\tau)d\tau} \end{aligned} \tag{44}$$

The proof by induction follows the same lines as before.

From (41) and (43) along with (44), we finally obtain [2]

$$l(x,y,t) = -\tfrac{y}{2}\gamma(t)\sum_{n=0}^{\infty}\tfrac{(\sqrt{\gamma(t)}(x^2-y^2))^n}{4^n(T\gamma_0\theta)^n(n+1)!}L_n^{(1)}\left((T\gamma_0\theta)\sqrt{\gamma(t)}\right)$$

This concludes the proof. ∎

Let us again provide a closed-form kernel in terms now of the Bessel function. This form will be instrumental for the main result since its proof will exploit suitable boundedness of kernels as well as the fixed-time stability property of the target system. A qualitative analysis of the kernels will allow then to establish the equivalence between the L^2-norm of the original system and the target.

Theorem 2. *The system* (35)-(37),(16) *has a well-posed C^∞ solution on $\mathcal{T}$, given by*

$$l(x,y,t) = -y\gamma(t)e^{\frac{\sqrt{\gamma(t)}(x^2-y^2)}{4T\gamma_0\theta}}\frac{J_1\left(\sqrt{\gamma(t)(x^2-y^2)}\right)}{\sqrt{\gamma(t)(x^2-y^2)}} \tag{45}$$

Proof. Similarly to the proof of Theorem 1, an immediate application of Propositions 1 to (40) yields the result. ∎

Remark 3. *As in Remark 2, if $\dot\gamma(t)\to 0$ and $\gamma(t)\to\gamma_0^2$ as $T\to\infty$ uniformly on any compact interval of time, then, from* (45), $e^{\frac{\sqrt{\gamma(t)}(x^2-y^2)}{4T\gamma_0\theta}} \to 1$ *as $T\to\infty$, therefore*

$$l(x,y,t) = -y\gamma^*\frac{J_1(\sqrt{\gamma^*(x^2-y^2)})}{\sqrt{\gamma^*(x^2-y^2)}} \tag{46}$$

with $\gamma^*=\gamma_0^2$. This is the closed-form kernel for the inverse transformation that was originally found in [28].

Alternatively, if we consider the case when $\gamma(t)\equiv\gamma^*$ is constant, therefore, $G(t)=\gamma^*e^{-\theta\gamma^*t}$ (in (42)*). Computing iteratively its time derivative we get*

$$G^{(n)}(t) = (-1)^n\theta^n(\gamma^*)^{n+1}e^{-\theta\gamma^*t}$$

From (43) *and* (41) *one can also derive* (46).

2.5 Well-posedness of the closed-loop system (1)-(4)

It follows by standard arguments from the well-posedness of the target system and the bounded invertibility of the related transformations. Indeed, the linear operator $c(t)$ in (8)-(10) is chosen to be differentiable by virtue of (15) and (17) defined on any time interval $[0,T^*]$ with $T^*<T$ and $T^*\to T$. In such a case, one can apply the results of e.g. [20, Theorem 4.8]; thus, the existence and uniqueness of classical solutions to (8)-(10), on the operation time interval $[0,T^*]$, for initial conditions $w^0\in H^2((0,1),\mathbb{R}))\cap H_0^1((0,1),\mathbb{R}))$, hold. Therefore, since transformation (7) is proved to be bounded invertible, then the existence and uniqueness of classical solutions of (1)-(4) are guaranteed, for initial conditions $u_0\in H^2((0,1),\mathbb{R})$ satisfying the zero order compatibility conditions $u(0,0)=0$ and $u(0,1)=U(0)$. The notion of well-posedness for $t\geq T$ is not addressed in this work. Possible ways of analysis to deal with this issue may be based on ideas of [18, Section 2.2.3].

3 Main result: fixed-time stabilization

Let us state the main result for fixed-time stabilization using the obtained time-varying kernels.

[2] Note that the only change with respect to k is the sign into the Laguerre function argument.

Theorem 3. *Let $\theta, T > 0$ be fixed. If γ_0 is chosen such that*

$$T\gamma_0\theta > \frac{1}{2} \tag{47}$$

then, the time-varying feedback controller

$$U(t) = \int_0^1 k(1, y, t)u(t, y)dy \tag{48}$$

with $k(1, y, t)$ as in (32) *(at $x = 1$), stabilizes the system* (1)-(4) *in a prescribed T, i.e. for any initial condition $u_0 \in L^2((0, 1), \mathbb{R})$, it holds*

$$\|u(t, \cdot)\|_{L^2((0,1),\mathbb{R})} \to 0 \quad as \quad t \to T \tag{49}$$

Moreover, $U(t)$ remains bounded and $|U(t)| \to 0$ as $t \to T$.

Proof. We use first the target system (8)-(11) to prove fixed-time stability in L^2-norm. We establish then the equivalence between norms for the original system and the target one under suitable boundedness of the related transformations. Consider the following Lyapunov function candidate, $V : L^2((0, 1), \mathbb{R}) \to \mathbb{R}^+$,

$$V(w) = \tfrac{1}{2}\int_0^1 w^2(x)dx \tag{50}$$

Computing the time derivative along the solutions of (8)-(11), performing integration by parts, and using the boundary conditions, yields

$$\dot{V} = -2\theta\|w_x(t, \cdot)\|^2_{L^2((0,1),\mathbb{R})} - c(t)\int_0^1 w^2dx \tag{51}$$

Thus,

$$\begin{aligned}\dot{V} &\leq -2\theta\|w_x(t, \cdot)\|^2_{L^2((0,1),\mathbb{R})} - 2c(t)V \\ &\leq -2c(t)V\end{aligned} \tag{52}$$

Since $c(t) = \gamma(t)\theta - \lambda$, and by the Grönwall's lemma we obtain that

$$V(w(t, \cdot)) \leq e^{-2\theta\int_0^t \gamma(\tau)d\tau + 2\lambda t}V(w_0) \tag{53}$$

In addition, denoting $\zeta(t) := e^{-2\theta\int_0^t \gamma(\tau)d\tau}$ and using (17), one can straightforwardly obtain that

$$\zeta(t) := e^{-2\theta\gamma_0 T\sqrt{\gamma(t)}}e^{2\theta\gamma_0 T\sqrt{\gamma(0)}} \tag{54}$$

which is a monotonically decreasing function having the properties $\zeta(0) = 1$ and $\zeta(T) = 0$ (this function may be refereed as a smooth " bump-like " function; see [30] for further information).

Hence, for all $t \in [0, T)$,

$$V(w(t, \cdot)) \leq \zeta(t)e^{2\lambda t}V(w_0) \tag{55}$$

therefore,

$$\|w(t, \cdot)\|_{L^2((0,1),\mathbb{R})} \leq \sqrt{\zeta(t)}e^{\lambda t}\|w_0\|_{L^2((0,1),\mathbb{R})} \tag{56}$$

from which one can conclude that $\|w(t, \cdot)\|_{L^2((0,1),\mathbb{R})} \to 0$ as $t \to T$.

On the other hand, since kernel k is continuous on $\mathcal{T}$, it holds, for each $t \in [0, T)$, that $\|\mathcal{K}(t)\|_\infty \leq M_k(t)$ where $\|\mathcal{K}(t)\|_\infty = \sup_{0\leq y\leq x\leq 1}|k(x, y, t)|$. Similarly, it holds, for each $t \in [0, T)$, that $\|\mathcal{L}(t)\|_\infty \leq M_l(t)$ where $\|\mathcal{L}(t)\|_\infty = \sup_{0\leq y\leq x\leq 1}|l(x, y, t)|$. Using (7), the estimate of $\mathcal{K}$ and the Cauchy-Schwarz inequality, we have

$$\|w(t, \cdot)\|_{L^2((0,1),\mathbb{R})} \leq (1 + M_k(t))\|u(t, \cdot)\|_{L^2((0,1),\mathbb{R})} \tag{57}$$

Similarly, using (34) and the estimate of $\mathcal{L}$, we have

$$\|u(t, \cdot)\|_{L^2((0,1),\mathbb{R})} \leq (1 + M_l(t))\|w(t, \cdot)\|_{L^2((0,1),\mathbb{R})} \tag{58}$$

In order to characterize both M_k and M_l, let us first exploit some known estimates for the following functions [7,25]:

$$\left|y\gamma(t)\frac{I_1\left(\sqrt{\gamma(t)(x^2-y^2)}\right)}{\sqrt{\gamma(t)(x^2-y^2)}}\right| \leq e^{c_k\sqrt{\gamma(t)}} \tag{59}$$

for some positive c_k independent of γ.

$$\left|y\gamma(t)\frac{J_1\left(\sqrt{\gamma(t)(x^2-y^2)}\right)}{\sqrt{\gamma(t)(x^2-y^2)}}\right| \leq c_l\gamma(t) \tag{60}$$

for some positive c_l independent of γ. [3]

Therefore, from (32) and (45) together with (59) and (60), suitable candidates for M_k and M_l can be derived:

$$M_k(t) = e^{\sqrt{\gamma(t)}\left(\frac{1}{4T\gamma_0\theta} + c_k\right)} \tag{61}$$

whereas

$$M_l(t) = c_l\gamma(t)e^{\frac{\sqrt{\gamma(t)}}{4T\gamma_0\theta}} \tag{62}$$

From (57) at $t = 0$, it holds

$$\|w_0\|_{L^2((0,1),\mathbb{R})} \leq (1 + M_k(0))\|u_0\|_{L^2((0,1),\mathbb{R})} \tag{63}$$

Then, combining (56), (58) and (63), we get

$$\begin{aligned}\|u(t, \cdot)\|_{L^2((0,1),\mathbb{R})} \leq& (1 + M_l(t))\sqrt{\zeta(t)} \\ &\times e^{\lambda T}(1 + M_k(0))\|u_0\|_{L^2((0,1),\mathbb{R})}\end{aligned} \tag{64}$$

Note that thanks to (54) and (62), the term $M_l(t)\sqrt{\zeta(t)}$ is given by the following relation:

$$M_l(t)\sqrt{\zeta(t)} = c_l\gamma(t)e^{\left(\frac{1}{4T\gamma_0\theta} - T\gamma_0\theta\right)\sqrt{\gamma(t)}}e^{\theta\gamma_0 T\sqrt{\gamma(0)}} \tag{65}$$

[3] The functions on the left hand of side of (59) and (60) would correspond (for every fixed $t \in [0, T)$) to (33) and (46), respectively; whose bounds are already established in the literature. We are particularly interested in a less conservative upper bound as in (60) in order to further take advantage of the linearity w.r.t γ.

In light of (47), we obtain that (65) converges to zero as t goes to T. This is certainly true since the exponential decreasing term governed by $e^{-\alpha_0\sqrt{\gamma(t)}}$ (with $\alpha_0 := \frac{4T^2\gamma_0^2\theta^2-1}{4T\gamma_0\theta} > 0$) dominates the linear increasing term governed by $\gamma(t)$.

Then, from (54), (64) and (65), one can conclude that $\|u(t,\cdot)\|_{L^2((0,1),\mathbb{R})} \to 0$ as $t \to T$. The well-posedness of the closed-loop solution holds by the arguments in Subsection 2.5 as the related kernels of the transformations are bounded due to (57)-(58) with (61)-(62). Finally, it remains to show that the control input is bounded. Moreover, its convergence to zero in finite-time is guaranteed as well. It is sufficient to see that $U(t)$ given by (48), can equivalently be rewritten as (38) due to the inverse of the backstepping transformation (34) (at $x = 1$), by

$$U(t) = \int_0^1 l(1, y, t)w(t, y)dy$$

with l as in (45). Therefore, the following estimate holds, for all $t \in [0, T)$,

$$|U(t)| \leq M_l(t)\|w(t,\cdot)\|_{L^2((0,1),\mathbb{R})} \tag{66}$$

Using (56), (62) and (63) we obtain

$$\begin{aligned}|U(t)| \leq & c_l\gamma(t)e^{-\alpha_0\sqrt{\gamma(t)}}e^{\theta\gamma_0 T\sqrt{\gamma(0)}} \\ & \times e^{\lambda T}(1 + M_k(0))\|u_0\|_{L^2((0,1),\mathbb{R})}\end{aligned}$$

Following the same arguments as before, we conclude that $|U(t)| \to 0$ as $t \to T$. This completes the proof. ■

Remark 4. *Besides condition* (47) *in Theorem 3, there is no strong restriction on the choice of γ_0 for the previous fixed-time stability result. Nevertheless, it would be desirable that $c(t) > 0$ for all $t \in [0, T)$. This is because one could reduce high control effort during the transient. Therefore, it is sufficient to choose γ_0 such that $\gamma_0 > \frac{\lambda}{\theta}$. Hence, due to the monotonicity of γ, it holds that, for all $t \in [0, T)$, $c(t) > 0$.*

Remark 5. *Throughout the paper, we have established that the time of convergence T can be fixed or prescribed independent of initial conditions of the system. Then, it is worth remarking that in the limiting case as T is chosen arbitrary small, i.e. $T \to 0$, the resulting actuation turns out to be impulsive. Therefore a rigorous framework, under generalized control in the sense of distributions as done in [16,22], could be suitable to deal with well-posedness issues and with practical implementations.*

Comments on the robustness w.r.t uncertainties: Let us just point out two cases:

(1) Robustness w.r.t uncertain time-varying reaction coefficient, i.e. slightly reformulate (1) as $u_t(t, x) = \theta u_{xx}(t, x) + \tilde{\lambda}(t)u(t, x)$ with $\tilde{\lambda}(t)$ a continuous bounded function. It can be shown that the same steps of the proof of Theorem 3 apply, hence the time-varying feedback controller (48) stabilizes the the closed-loop system in a fixed-time. The method remains insensitive w.r.t the underlying uncertainty.

(2) Robustness w.r.t distributed uncertainty. Suppose that (1) is reformulated as $u_t(t, x) = \theta u_{xx}(t, x) + \lambda u(t, x) + \psi(t, x)$ with ψ a sufficiently smooth function. This case is more critical and robustness may not be guaranteed unless some strong conservative assumption on ψ is imposed. Indeed, under the backstepping transformation (7) with the obtained time-varying kernel, the dynamics of the target systems reads as $w_t(t, x) = \theta w_{xx}(t, x) - c(t)w(t, x) + \psi(t, x) - \int_0^x k(x, y, t)\psi(t, y)dy$. Performing a Lyapunov analysis on the perturbed target system similar to the proof of Theorem 3, we could conclude that if ψ is such that $\int_0^1 \left((\psi(t, x))^2 + \left(\int_0^x k(x, y, t)\psi(t, y)dy\right)^2\right)dx \leq \frac{1}{2}\gamma^\mu(t)$, $\mu \in (0, 1)$, then the target perturbed system is fixed-time stable. The conservatism may rely on the uncertainty vanishing also in fixed-time.

A detailed analysis of robustness is out of the scope of this paper and is left for future investigations.

4 Discussion about the choice of the blow up function

A general blow up function may be formulated as $\gamma(t) = \frac{(\gamma_0 T)^{1+\varepsilon}}{(T-t)^{1+\varepsilon}}$ satisfying the following ODE:

$$\dot{\gamma}(t) = \tfrac{1+\varepsilon}{\gamma_0 T}\gamma^{(2+\varepsilon)/(1+\varepsilon)}(t), \quad \gamma(0) = \gamma_0^{1+\varepsilon} > 0 \tag{67}$$

where $\varepsilon \geq 0$ is a design parameter. In this work, the degree ε has been carefully selected (i.e. $\varepsilon = 1$ which yields (16)) in such a way that the main result holds as it has been proved in the previous section.

Nevertheless, it is important to emphasize that not all blow up functions may work in this framework. Let us consider one counterexample to illustrate that the proof of Theorem 3 cannot be easily repeated: consider (67) with $\varepsilon = 0$. Following the same lines of the proof of Theorem 3, it holds that the target system (8)-(11) is fixed-time stable in L^2-norm since, from (53), the term $\zeta(t) := e^{-2\theta\int_0^t \gamma(\tau)d\tau}$ would be of the following form:

$$\zeta(t) = (T - t)^{2T\gamma_0\theta}(T)^{-2T\gamma_0\theta} \tag{68}$$

from which together with (56), one can immediately observe the fixed-time convergence to zero as t goes to T.

On the other hand, some issues arise in establishing the fixed-time convergence of the L^2-norm of the original system:

If we select the blow up function (67) with $\varepsilon = 0$, the kernels of the direct and inverse transformation (7), (34) are no longer given by (32) and (45), respectively. It can be established however that, by carefully handling the power series (25) and (43) and rewriting them in terms of the so-called *Kummer confluent hypergeometric function* along with its connection with *generalized Laguerre polynomials*, the kernels of the direct and inverse transformation, are rather given as follows, respectively [9]:

$$k(x,y,t) = -\frac{y}{2}\gamma(t)e^{\frac{\gamma(t)(x^2-y^2)}{4N}}\frac{1}{N}L_{N-1}^{(1)}\left(-\frac{\gamma(t)(x^2-y^2)}{4N}\right) \tag{69}$$

$$l(x,y,t) = -\frac{y}{2}\gamma(t)\frac{1}{N}L_{N-1}^{(1)}\left(\frac{\gamma(t)(x^2-y^2)}{4N}\right) \tag{70}$$

where, $N := T\gamma_0\theta$ which may be enforced to belong to $\mathbb{N}_{>0}$. Using a well-established upper estimate of generalized Laguerre functions as reported e.g. in [15] (more precisely $|L_n^{(\alpha)}(p)| \leq \binom{n+\alpha}{n}e^{p/2}, \quad \text{for} \quad \alpha \geq 0, \quad p \geq 0$); one can derive, from (70) that

$$|l(x,y,t)| \leq \gamma(t)e^{\gamma(t)/8N} \tag{71}$$

The problematic issue lies on the fact that if $M_l(t)$ (as in the proof of Theorem 3) is set as $M_l(t) = \gamma(t)e^{\gamma(t)/8N}$ and is replaced into (64) along with $\zeta(t)$ given by (68), we would have that

$$\begin{aligned}\|u(t,\cdot)\|_{L^2} \leq &\left(\frac{(T-t)^N}{(T)^N} + \frac{N(T-t)^{N-1}e^{1/8\theta(T-t)}}{\theta(T)^N}\right)\\ &\times e^{\lambda T}(1+M_k(0))\|u_0\|_{L^2}\end{aligned} \tag{72}$$

However, it can be clearly noticed that the term on the right-hand side blows up due to the dominance of the exponential term $e^{1/8\theta(T-t)}$. Thus, the fixed-time convergence to zero of the original system is *not* guaranteed.

5 Simulations

We illustrate the results of Section 3 by considering a scalar reaction-diffusion system with $\theta = 1$, $\lambda = 11$ and initial condition $u(0,x) = 10.25x(1-x)$ satisfying the zero order compatibility conditions.. Note that, in open loop (e.g. $U(t) = 0$), the system is unstable. For numerical simulations, the state of the system has been discretized by divided differences on a uniform grid with the step $h = 0.02$ for the space variable. The discretization with respect to time was done using the *implicit Euler scheme* with step size $\tau = h^2$. The continuous boundary time-varying feedback was implemented by taking advantage of the obtained closed-form solution for the kernel gain.
The parameter γ_0 is chosen to be $\gamma_0 = 3.3$. The selected prescribed time is $T = 1$. Hence condition (47) holds and Theorem 3 applies. Figure 1 shows the numerical solutions of the closed-loop system (1)-(4); more precisely, on the left, the system is subject to a traditional boundary control feedback $U(t)$ with control gain (33) with $\gamma^* = \gamma_0^2$ which stabilizes the system exponentially. On the right, the system is stabilized in fixed-time according to Theorem 3, i.e. by means of a boundary time-varying feedback with kernel gains (32). Figure 2 shows the time evolution of L^2- norm of the closed-loop system plotted in logarithmic scale to better illustrate that with time-varying feedbacks the closed-loop system converges in a prescribe time given by $T = 1$. It can be observed that the convergence to zero is faster than using linear control for exponential stabilization (red-dashed line). Similarly, Figure 3 shows the time evolution of Lyapunov function V (in (50)) plotted in logarithmic scale to illustrate the convergence to zero in fixed-time for the target system. Moreover, we performed simulations for three different prescribed times, $T = 0.6$, $T = 1$ and $T = 1.5$. Figure 4 illustrates the evolution of the L^2-norms of the closed-loop system with three different initial conditions for the aforementioned three prescribed times. The numerical result clearly shows that convergence to zero in prescribed time is independent of initial conditions of the reaction-diffusion system.

6 Conclusion

In this paper, the problem of fixed-time stabilization of boundary controlled reaction-diffusion PDEs has been considered. By means of time-varying feedbacks it has been proved that one can steer any solution of the closed-loop system to zero in a prescribed time. To come up with a time-varying feedback, we used a time varying-kernel for the backstepping transformation along with a suitable blow up function. While solving the kernel equations, some special functions such as the generalized Laguerre functions and the modified Bessel functions and their properties came into play. The proof of fixed-time convergence to zero is carried by means of Lyapunov techniques on the target system. Suitable estimates for relating the L^2-norm of the original system and the target one are derived by virtue of the obtained closed-form kernels.
It is worth mentioning that the results of this paper could be extended to coupled reaction-diffusion systems with constant parameters. However, in order to obtain closed-form solutions, some assumptions on the structure of the kernel matrix and the diffusion coefficients need to be imposed [9]. A natural question may rise for the case of coupled reaction-diffusion systems with space varying coefficients which motivates the study of the arising kernels characterization and the estimation of the growth-in time from the method of successive approximations.

This work leaves some other open questions. We expect to carry out a more detailed study on the robustnesses with respect to uncertainties and external boundary disturbances. In line with Remark 5, it would be interesting to study robustness with respect to actuation and sensing (in the ISS sense) by formulating the problem with delta-functions as a standardized system in distributions, based on [22].

We will also explore kernel solution characterizations for the blow up function (67) with different powers and study whether the fixed-time convergence is preserved. New patterns for the kernel solution characterization will need to be found.

Finally, fixed-time observers for reaction-diffusion equations are currently under investigation.

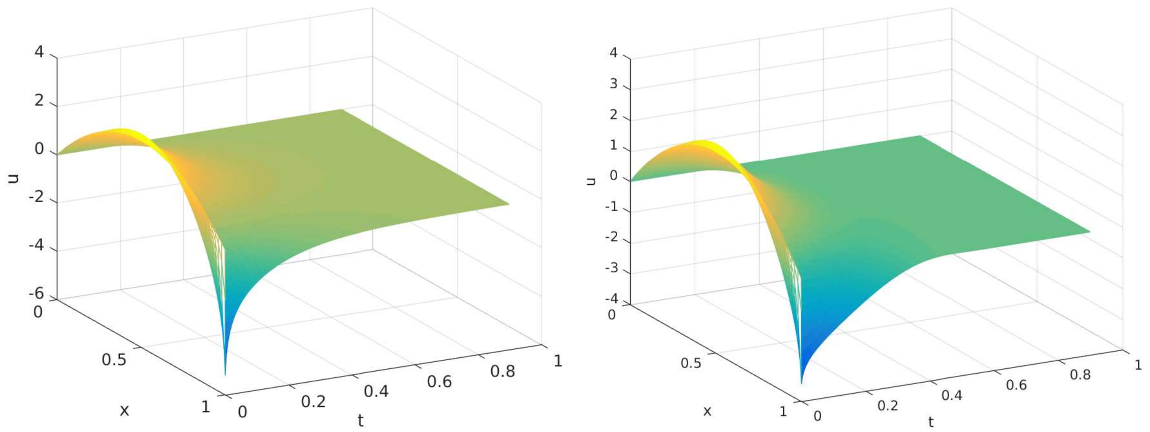

Figure 1. Numerical solution of the closed-loop system with boundary controller U (whose kernel gain is (33) for exponential stabilization (left) and boundary time-varying controller U (whose kernel gain is (32)) for fixed-time stabilization with prescribed time $T = 1$ (right).

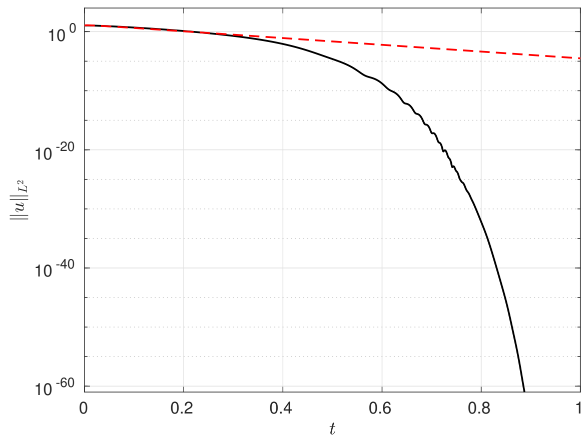

Figure 2. Evolution of the L^2-norm of the closed-loop system (logarithmic scale) with time-varying feedback (black line) for a prescribed time $T = 1$ and linear control feedback (red dashed line) for exponential stabilization.

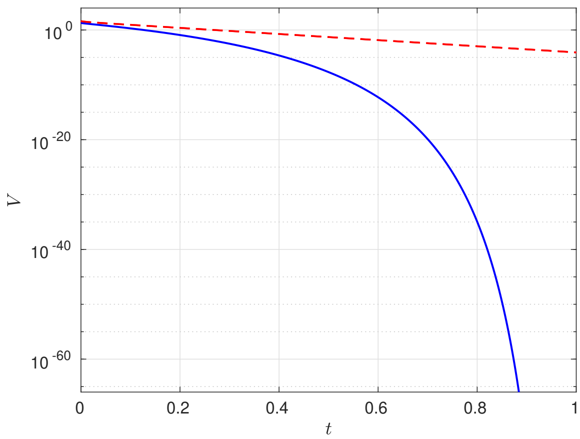

Figure 3. Evolution of the Lyapunov function V (logarithmic scale) for the fixed-time stable target system (blue line) and for exponential stable target system (red dashed line).

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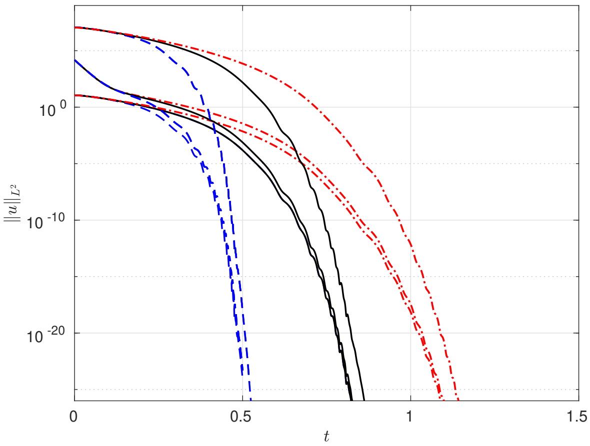

Figure 4. Evolution of the L^2-norm of the closed-loop system (logarithmic scale) with time-varying feedbacks for different initial conditions and prescribed times: $T = 0.6$ (blue dashed line), $T = 1$ (black line) and $T = 1.5$ (red dotted dashed line) .

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