Electronic Properties of Luttinger Liquid with Electron-Phonon Interaction

by

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Abstract

This thesis addresses a theoretical study of the problem of a single impurity embedded in a one-dimensional system of interacting electrons in presence of electron-phonon coupling. First we consider a system with a featureless point-like potential impurity, followed by the case of a resonant level hybridised with a Luttinger Liquid. The stress is made on a more fundamental problem of a featureless scatterer, for which two opposite limits in the impurity strength are considered: a weak scatterer and a weak link. We have found that, regardless of the transmission properties of phonons through the impurity, the scaling dimensions of the conductance in these limits obey the duality condition, $\Delta_{\mathrm{WS}}\Delta_{\mathrm{WL}} = 1$, known for the Luttinger Liquid in the absence of phonons. However, in the case when the strength of phonon scattering is correlated with electron scattering by the impurity, we find a nontrivial phase diagram with up to three fixed points and a possibility of a metal-insulator transition. We also consider the case of a weakly interacting electron-phonon system in the presence of a single impurity of an arbitrary scattering potential. In the problem of a resonant level attached to the Luttinger Liquid we show that the electron-phonon coupling significantly modifies the effective energy-dependent width of the resonant level in two different geometries, corresponding to the resonant and anti-resonant transmission in the Fermi gas.

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This thesis is dedicated to

my parents.

PUBLICATIONS

- Impurity Scattering in Luttinger Liquid with Electron-Phonon Coupling

  Alexey Galda, Igor V. Yurkevich, Igor V. Lerner

  Physical Review B 83, 041106(R) (2011), Editor's Suggestion

- Effect of electron-phonon coupling on transmission through Luttinger Liquid hybridized with resonant level

  Alexey Galda, Igor V. Yurkevich, Igor V. Lerner

  Europhysics Letters 93, 17009 (2011)

- Duality of Weak and Strong Scatterer in Luttinger liquid Coupled to Massless Bosons

  Igor V. Yurkevich, Alexey Galda, Oleg M. Yevtushenko, and Igor V. Lerner

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CHAPTER 1

INTRODUCTION

1.1 Motivation

Continuous progress in semiconductor technology over the past decades has stimulated great interest in studying transport properties of low-dimensional electron systems. Recent advances in the fabrication technology of two-dimensional electron gases have led to the creation of effectively one-dimensional systems in which electrons are confined within a narrow channel, called quantum wires. Such systems can be called one-dimensional when their width is smaller than electron's Fermi wavelength. Another important example of one-dimensional physics can be realised in quantum Hall systems with chiral edge states on either side of the Hall bar. Carbon nanotubes have also proved to be amongst systems which exhibit one-dimensional physics. Other experimental (quasi-)one-dimensional systems include organic Bechgaard salts and inorganic compounds (e.g. $NbSe_3$, TaS_3), as well as a highly anisotropic three-dimensional crystal $Li_{0.9}Mo_6O_{17}$ (purple bronze).

In the absence of any scattering mechanisms (interactions, imperfections, phonons, etc.) electron transport is ballistic and the system is expected to have a two-point conductance of $\frac{e^2}{h}$ per channel. In the context of one-dimensional systems it is known [1, 2] that upon

introducing a single impurity, the repulsive electron-electron interaction has a dramatic effect on electron transmission. Conductance tends to zero as a non-universal interaction-dependent power of temperature T, in the limit $T \to 0$. On contrary, provided the interaction is attractive, the backscattering of electrons by the impurity is proved to be irrelevant. These conclusions result in a phase diagram with two fixed points corresponding to an insulator (for repulsive electron-electron interaction) or an ideal metal (for attractive interaction).

Originally, the problem of a single impurity in an interacting one-dimensional system has only been solved [1, 2] in the limit of weak or strong impurity potentials. The exact solution for arbitrary impurity and interaction strength has proved to be extremely challenging. However, several successful attempts have been made to solve the problem for particular values of electron-electron interaction and arbitrary barrier strength. Those include the use of thermodynamic Bethe ansatz [3], Monte Carlo simulations [4] and refermionisation [5]. The solution is also available for a weakly interacting system [6, 7]. All these studies confirm the conjecture that only two fixed points exist for the single impurity problem regardless of the impurity potential and the interaction strength.

The main motivation of this thesis is to study the effects arising from the presence of electron-phonon coupling in the interacting one-dimensional spinless fermionic system described by the Luttinger Liquid theory. In particular, we consider a system of one-dimensional acoustic phonons linearly coupled to electron density. The topic of electron-phonon interactions in one-dimensional systems has been extensively studied in literature, both analytically and numerically [8, 9, 10, 11, 12, 13].

It is well-known [14, 15] that phonon-mediated electron-electron interaction leads to an attractive potential, which should modify the results obtained in the absence of phonons. In fact, the role of phonons should not be underestimated since even in the absence of bare electron-electron interaction the coupling to one-dimensional acoustic phonons by itself induces the characteristic Luttinger Liquid behaviour (power-law decay of various correlation

functions, etc.) [12]. The electron-phonon coupling in addition to the (Coulomb) electron-electron repulsion in the Luttinger Liquid is known to result in the formation of two polaron branches with different propagation velocities (see, for example, Refs. [12, 16, 17]). Therefore, it is of particular interest to study the problem of a single impurity embedded into such a two-component system. The case of a weak impurity has been partially addressed in Ref. [18]. Our work provides a rigorous analysis of both weak and strong backscattering limits. One of the main results of the thesis is the existence of a duality relation between the two limits, which has previously been shown only for systems without nonlocal phonon-mediated interaction.

1.2 Outline of the thesis

The presented thesis begins with a brief introduction into the field of strongly correlated electrons in one dimension. We outline the main differences and similarities between Luttinger Liquid and Fermi Liquid theories, which are commonly used to describe interacting electrons in one and higher dimensions, correspondingly. Following the introduction, in Chapter 2 we describe the main techniques used in this thesis: Keldysh formalism and functional bosonisation. Bosonisation is one of the most powerful tools in one-dimensional physics. More popular in literature is 'classical' bosonisation, also known as the operator approach, which originated in the 60's [19] and then evolved until it was eventually formulated as we know it now by Haldane [20]. However, within the framework of the single impurity problem, it is convenient for us to use the functional bosonisation approach instead. It is an alternative but equally powerful technique which originates from the work by Fogedby [21]. The way in which functional bosonisation is presented here is based on the paper by Lee and Chen [22, 23]. In particular, it allows for a so-called 'partial' bosonisation of the interacting system and turns out to be an excellent starting point for our calculations.

In Chapter 3 we build-up a foundation required for the following derivation of the main

results of this thesis. We first introduce a spinless Tomonaga-Luttinger model of interacting electrons and show its main features by deriving them with the help of functional bosonisation. We then provide an introduction into the single impurity problem, which has served as the main motivation for this work. We outline the main results by Kane and Fisher [1] for electron transport through a single scatterer and point out the duality existing between the weak and strong scattering limits. We then proceed on to the description of a system of interacting electrons coupled to phonons. We apply the functional bosonisation technique and develop an elegant way of accounting for the interactions in such a system exactly. At the end of the Chapter, when all the key ingredients for the main calculations are ready, we list all possible formulations of the single impurity problem which will be considered in the subsequent Chapters.

Chapter 4 is devoted to the case of a single impurity in two limits: weak scatterer and weak link. In both cases the scatterer is assumed to be transparent for acoustic phonons in the most relevant low-energy limit. Experimentally this can be realised in a system where the impurity oscillates together with the one-dimensional wire. In such case the phonon backscattering amplitude goes to zero for $\omega \to 0$. This type of impurity can also be modelled by a mass or spring constant different from the rest of the lattice. In addition, electron scattering impurity can be created without affecting the lattice at all, e.g., by creating a depletion region with a scanning tunnelling microscope tip. This way the system of phonons remains translation invariant. Alternatively, one can consider a phonon-nontransparent impurity. Since the main contribution to the electron transmission properties comes from the low-energy modes, this situation arises when the impurity is modelled, e.g., by a lattice atom pinned to the substrate or by any other external perturbation. The result is a perfect reflection of low-energy phonons [18], which are responsible for the logarithmic electron transmission corrections. This is the subject of Chapter 5. In these Chapters we derive perturbative renormalisation group flow equations for the corrections to electron transmission or reflection

amplitudes in all described situations. Whenever possible, we perform the calculations using the functional bosonisation method developed in previous Chapters. However, certain cases do not allow for its simple adaptation, which is when a 'brute force' method was used. An important result of these two Chapters is that the duality relation, which was previously only known for the standard Luttinger Liquid with local interaction, holds in presence of the phonon-mediated retarded electron-electron interaction.

If in Chapters 4 and 5 we considered various limits of possible electron and phonon scattering by an impurity, assuming they are independent, Chapter 6 addresses the case when the two are correlated and, in particular, are changed in parallel. Possible examples include bending a carbon nanotube or inducing a local structural change to a one-dimensional conductor with a tip of an atomic force microscope. We focus on the most realistic case when in the weak (strong) scattering limit both electron and phonon reflection (transmission) amplitudes are small. We propose the following experimental realisation of this model. A carbon nanotube can be bent by external forces at different angles. When almost undeformed, one can expect phonons to propagate freely through the system, while there is a possibility of a small electron backscattering. However, upon increasing the bending angle, after a certain point the strain in the nanotube leads to the formation of a kink in its lattice structure. This induces mixing of σ-and π-orbitals, which results in strong electron backscattering from the point of deformation [24]. The kink developed at high bending angles also suggests a strong phonon scattering. Therefore, in such a system the scattering of electrons and phonons by the impurity is no longer independent and can be changed in parallel. This scenario connects two previously considered limits of a weak scatterer from Chapter 4 and weak link from Chapter 5. The corresponding phase diagram differs qualitatively from the Kane-Fisher one and predicts a range of parameters of the system at which a metal-insulator transition becomes possible.

Up to this point the consideration was limited to only weak or strong electron scattering

by impurities which was essential for the analytic solubility. In Chapter 7 we focus on the problem of a single impurity of arbitrary strength in a weakly interacting system. In the phononless interacting Fermi gas it was first considered in Refs. [6, 7] where the authors presented the derivation of the temperature dependence of conductance through a barrier of arbitrary strength. Our calculations serve as a logical generalisation of their model by introducing an unavoidable in any realistic system longitudinal acoustic phonons into the system.

Finally, in Chapter 8 we consider the problem of electron transmission through a resonant level hybridised with a Luttinger Liquid. We first discuss a more common resonant-barrier geometry in which the resonant level is embedded into the system and leads to resonant transmission described by the Breit-Wigner formula with an energy-dependent effective level width. We derive and solve renormalisation group equations which give the effective width as a function of energy of electrons, both close to and away from the resonance. We also consider a dual geometry with a side-attached resonant level. In this case resonant reflection, rather than transmission, is observed. Renormalisation group analysis is used again to determine the scaling behaviour of the effective level width and, therefore, conducting properties of the system.

1.3   Fermi Liquid and Luttinger Liquid

It is often said that physics of interacting particles in one dimension is dramatically different from it in higher dimensions. This is only true, however, to a certain extent. Most of the 'unusual' properties of one-dimensional systems, in fact, have their counterparts in higher dimensions. The only difference being that they are more pronounced in, but not necessarily unique to, one dimension. This is the result of a smaller phase space in one dimension. Therefore, the difference between one and higher (throughout the text this term will refer to

systems with any dimension greater than one) dimensions is more quantitative than quali-
tative [25]. Yet, models which describe strongly correlated electrons in these two cases are
fundamentally different and this Section is devoted exactly to this topic.

Accounting for interactions in Fermi systems is not an easy task. It cannot be done
perturbatively since the ratio of magnitudes of the Coulomb interaction and the kinetic
energy of an electron in solids is of the order of $2 - 5$. In the late 50's Landau has developed
his phenomenological theory [26, 27, 28] which was the first successful model for studying
systems of interacting electrons. Landau's idea was to use Fermi gas as the foundation of his
theory. Upon adiabatically switching on interactions, Fermi gas turns into a Fermi Liquid in
which individual electrons and holes no longer play the role of elementary particles. Instead,
they become 'dressed' by density fluctuations in the Fermi sea around them and evolve into
fermionic 'quasiparticles', which is one of the key assumptions of the Fermi Liquid theory.
The rest is based around the idea of a one-to-one correspondence between the eigenstates
of a non-interacting Fermi gas and an interacting system of quasiparticles. A number of
textbooks provide an in-depth discussion of the Fermi Liquid theory [29, 30, 31].

Since, in fact, quasiparticles are not exact eigenstates of the interacting Hamiltonian,
they must have a finite lifetime. A simple calculation shows that, for instance, in three
dimensions the lifetime of a quasiparticle depends on its energy above the Fermi level ε
as $\tau \sim \varepsilon^{-2}$. In practice this means that quasiparticles are a good approximation for the
description of low-energy phenomena in higher dimensions. This also means that Landau's
theory is only applicable at sufficiently low temperatures, much lower than Fermi energy. In
metals the Fermi energy is normally about $3 - 5eV \sim 10^4 K$ which does not pose a serious
restriction. It should be stressed that the Fermi Liquid theory in not at all restricted to weak
interactions. Nevertheless, the residual interaction between quasiparticles is always weak
close to the Fermi surface (a direct consequence of the increase in their lifetime) and can be
treated perturbatively.

The transition from non-interacting electrons to weakly interacting quasiparticles results in renormalisation of kinematic parameters (effective mass, magnetic moment, etc.) and thermodynamic quantities (susceptibilities and specific heat) of the system. The leading terms in the low-temperature expansions of these quantities have the same T-dependence in the Fermi Liquid as in the Fermi gas. As it has been pointed out by Maslov [25], despite that, the next-to-leading terms are often singular. This uncovers the crucial role that interactions play in one-dimensional physics. Due to a small phase space granted to one-dimensional scattering phenomena this fact only reveals itself in the subleading terms. But one of the biggest differences between one- and higher-dimensional interacting systems is the following: even though single-particle excitations are well defined in the Fermi Liquid in higher dimensions, this is no longer true in the latter. Landau's Fermi Liquid theory breaks down in one dimension and a completely different model is required in this case.

Seemingly easier to deal with, interacting one-dimensional systems still pose a huge challenge when trying to calculate their collective properties. The problem comes from the fact that even weak interactions do not allow for a perturbative treatment. This follows from the absence of single particle excitations in one dimension mentioned earlier. In such a fermionic system any excitation becomes collective (and, more importantly, bosonic) simply because particles cannot avoid interacting strongly when passing through each other. Notice that, on contrary, quasiparticle excitations in two and three dimensions can propagate through the system without having to come very close to each other.

It was Tomonaga who first realised [32] that the problem of dealing with interactions in one dimension can be simplified greatly by replacing the quadratic electron dispersion relation with its linear approximation around the Fermi points. Having done that, one can then rewrite the full Hamiltonian in terms of cleverly defined bosonic density fluctuation operators. Tomonaga noticed that the part responsible for density-density interactions becomes bilinear in these bosonic fields, just like its free-fermion counterpart. As always, dealing
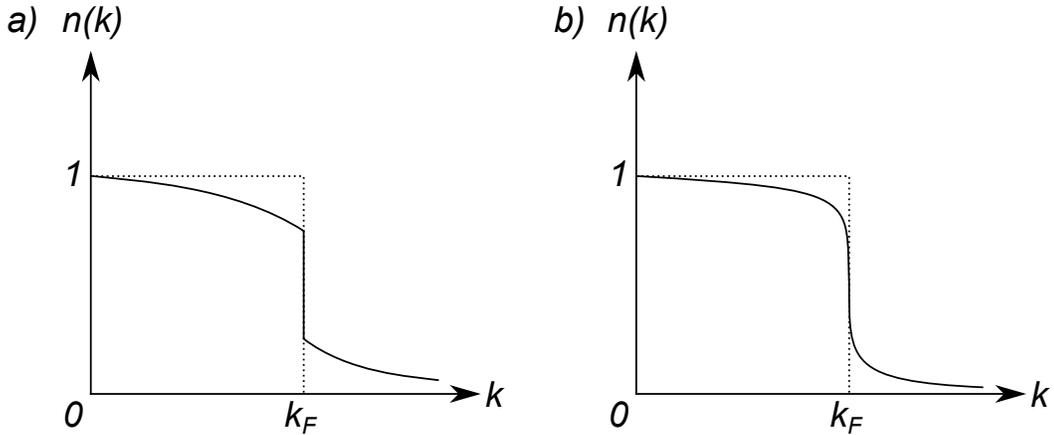
Figure 1.1: Zero-temperature particle momentum distribution function in a) Fermi Liquid, b) Luttinger Liquid. Dashed line corresponds to the non-interacting Fermi gas.

with a quadratic Hamiltonian is rather straightforward and calculation of observables simply becomes a technical exercise. The applicability of this approach is, however, limited to low energies (and, hence, low temperatures) where the curvature of the real dispersion law can be neglected. By analogy with the Fermi Liquid, this model is referred to as Luttinger Liquid (or Tomonaga-Luttinger model) after Luttinger who has written and solved in terms of Bloch sound waves [33] (incorrectly, as it turned out later) a similar model of interacting fermions to Tomomaga's.

Because the difference between one- and high-dimensional physics is so essential, it affects such fundamental characteristics as the particle's momentum distribution function and the spectral function. For comparison, the former is presented in Fig. 1.1 for both Fermi and Luttinger Liquids. Like in a Fermi gas, the occupation number in the Fermi Liquid has a discontinuity at the Fermi surface. The amplitude of the jump is, however, smaller than 1 and, roughly, represents which fraction of a bare electron is in a quasiparticle state. The
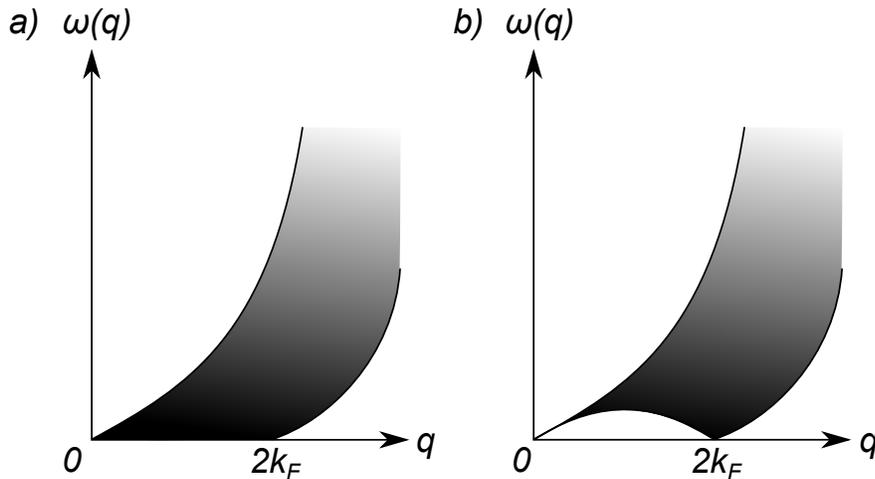
Figure 1.2: Particle-hole excitation spectrum in a) Fermi Liquid, b) Luttinger Liquid.

distribution function in a Luttinger Liquid in continuous but has a power-law dependence (similar to most other correlation functions in a Luttinger Liquid) around k_{F}:

$$n(k) \propto |k - k_{\mathrm{F}}|^{\gamma}, \tag{1.1}$$

where γ is some positive interaction-dependent constant. The absence of a jump at the Fermi surface again signals the absence of individual excitations - each electron participates only in collective excitations.

In one dimension Fermi surface is reduced to only two points, $q = \pm k_{\mathrm{F}}$, which has dramatic consequences on the spectrum of particle-hole excitations. In higher dimensions it is possible to excite a particle-hole pair of an arbitrary momentum because the Fermi surface is a connected set of points the distance between which in phase space can take any value between 0 and $2k_{\mathrm{F}}$. In one dimension, however, low energy excitations are only possible strictly near the points $\pm k_{\mathrm{F}}$ and, therefore, have momenta close to 0 or $2k_{\mathrm{F}}$, see Fig. 1.2. Small momentum excitations correspond to an electron being excited from just above to slightly

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below either of the Fermi points. Such bosonic particle-hole excitations in one dimension are objects with well defined energy and momentum and have a linear dispersion $\omega = v_{\mathrm{F}} q$. This is yet another confirmation of the bosonic nature of excitations in such systems.

CHAPTER 2

TECHNIQUES USED

2.1 Keldysh technique

In this Section we give a brief introduction into the Keldysh technique. For a comprehensive review of the topic see Refs. [34, 35, 36]. Our main goal here is to provide a general idea about the method and introduce the notations which will be used throughout the thesis. We use the Keldysh formalism here as an alternative to the Matsubara technique.

The technique was developed by Keldysh in 1964 [37] in order to treat systems away from thermal equilibrium. For the description of the evolution of a system in time he introduced the concept of a closed contour. In addition to the evolution between the initial state of the system at $t_i = -\infty$ and some generally unknown final state at $t_f = t$ one considers the evolution back to the initial state. This allows one to avoid the complication of needing to know the final state of the system out-of-equilibrium. The price to pay is the double of degrees of freedom, because the time evolution now goes in the positive and then negative time direction, as in Fig. 2.1. When the time argument of a bosonic or fermionic field sits on the contour C_1 (C_2), we use a subscript '+' ('−').

Instead of a usual time-ordered Green function, in the Keldysh formalism one defines
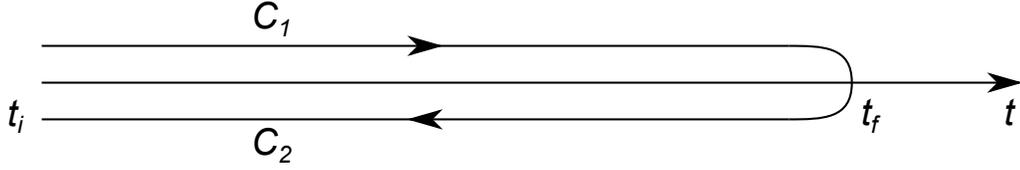
Figure 2.1: Closed Keldysh contour. Any time on the first part of the contour, C_1, going in the positive direction, is assumed to precede the time residing on the second part of the contour, C_2, going in the negative direction.

contour-ordered ones as follows:

$$\mathrm{i}G^T(t,t') = \langle \phi_+(t)\bar{\phi}_+(t')\rangle \,, \tag{2.1}$$

$$\mathrm{i}G^>(t,t') = \langle \phi_-(t)\bar{\phi}_+(t')\rangle \,, \tag{2.2}$$

$$\mathrm{i}G^<(t,t') = \langle \phi_+(t)\bar{\phi}_-(t')\rangle \,, \tag{2.3}$$

$$\mathrm{i}G^{\widetilde{T}}(t,t') = \langle \phi_-(t)\bar{\phi}_-(t')\rangle \,, \tag{2.4}$$

where G^T is a traditional time-ordered, or causal, Green function. The other three are called 'greater', 'lesser' and 'anti-time-ordered', correspondingly. Not all of the above four Green functions are independent. In fact, one can prove the following relation between them:

$$G^T + G^{\widetilde{T}} = G^> + G^< \,. \tag{2.5}$$

Various combinations of the three independent contour-oriented Green functions are possible. For instance, it is convenient to define the 'retarded' and 'advanced' Green functions due to their simple analytic structure:

$$G^R(t,t') = \Theta(t-t')\left[G^>(t,t') - G^<(t,t')\right] \,, \tag{2.6}$$

$$G^A(t,t') = \Theta(t'-t)\left[G^<(t,t') - G^>(t,t')\right] \,. \tag{2.7}$$

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Because it easier to manipulate with three independent Green functions rather than four dependent ones, one performs a so-called Keldysh rotation of the fields entering the Green functions. This transformation can be done is several different ways; in what follows we shall adapt the conventions and notations of [36]. For bosons the Keldysh rotation reads:

$$\phi^{\mathrm{cl}}(t) = \frac{1}{\sqrt{2}} \left[\phi_+(t) + \phi_-(t) \right] , \qquad \phi^{\mathrm{q}}(t) = \frac{1}{\sqrt{2}} \left[\phi_+(t) - \phi_-(t) \right] , \qquad (2.8)$$

where we have introduced two new fields, 'classical' and 'quantum'. In terms of these fields the Green function can be written as

$$- \mathrm{i} \langle \phi^\alpha(t) \bar\phi^\beta(t') \rangle = G^{\alpha\beta}(t,t') = \begin{pmatrix} G^K(t,t') & G^R(t,t') \\ G^A(t,t') & 0 \end{pmatrix} , \qquad \alpha, \beta = \{\mathrm{cl}, \mathrm{q}\} , \qquad (2.9)$$

where we have also introduced a non-hermitian 'Keldysh' component, G^K, which for bosons in case of thermal equilibrium is given by:

$$G^K(\omega) = (1 + 2n_{\mathrm{B}}(\omega)) \left[G^R(\omega) - G^A(\omega) \right] = \coth\left(\frac{\omega}{2T}\right) \left[G^R(\omega) - G^A(\omega) \right] . \qquad (2.10)$$

For fermions it is convenient to define the Keldysh rotation in a slightly different manner:

$$\psi_1(t) = \frac{1}{\sqrt{2}} \left[\psi_+(t) + \psi_-(t) \right] , \qquad \psi_2(t) = \frac{1}{\sqrt{2}} \left[\psi_+(t) - \psi_-(t) \right] , \qquad (2.11)$$

$$\bar\psi_1(t) = \frac{1}{\sqrt{2}} \left[\bar\psi_+(t) - \bar\psi_-(t) \right] , \qquad \bar\psi_2(t) = \frac{1}{\sqrt{2}} \left[\bar\psi_+(t) + \bar\psi_-(t) \right] . \qquad (2.12)$$

Notice that for Grassmann variables $\bar\psi$ and ψ are completely independent and unrelated.

The Green function for fermions in Keldysh formalism takes form

$$- \mathrm{i} \langle \psi_a(t) \bar\psi_b(t') \rangle = G_{ab}(t,t') = \begin{pmatrix} G^R(t,t') & G^K(t,t') \\ 0 & G^A(t,t') \end{pmatrix} , \qquad a, b = \{1, 2\} , \qquad (2.13)$$

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For the Keldysh component for fermions in thermal equilibrium one gets:

$$G^K(\varepsilon) = (1 - 2n_{\mathrm{F}}(\varepsilon)) \left[G^R(\varepsilon) - G^A(\varepsilon) \right] = \tanh\left(\frac{\varepsilon}{2T}\right) \left[G^R(\varepsilon) - G^A(\varepsilon) \right] . \qquad (2.14)$$

In ω-representation the retarded and advanced components are analytic in the upper and lower half-planes, respectively. We should make the following remark here. Whenever possible, in this thesis we try to specify the correct analyticity of retarded/advanced (R/A) components, which corresponds to the addition of $\pm i0$ to ω, correspondingly. The same applies to lesser/greater $(</>)$ components in t-representation, where we add $\pm i0$ to t. However, in certain cases these additional notations would have made the expressions difficult to read and are omitted, although implied.

2.2 Functional bosonisation

We present here the method of functional bosonisation and demonstrate how it can be used for calculating various correlation functions within Tomonaga-Luttinger model. The aim of this Section is to show how this alternative to standard operator bosonisation method can be applied in the simplest case of spinless interacting electrons in one-dimension without backscattering. Such complications of the model like impurities or phonons are omitted at this stage.

The success of the standard operator bosonisation in solving the Luttinger Liquid theory is due the fact that the Hamiltonian describing interactions in the system is quadratic in the same bosonic fields which are used to describe a free system. Hence, the interactions in this language could easily be taken into account be simply re-defining constants in the free Hamiltonian. This results in the interaction-dependent speed of plasmonic excitations in the system. Note that in the original fermionic representation the electron-electron interaction term is quartic in electronic fields (as opposed to the quadratic Hamiltonian of a free system)

which constitutes a huge problem when trying to calculate various properties of the system.

The key ingredient of the functional bosonisation method is the Hubbard-Stratonovich transformation, followed by a gauge transformation. The first one allows to eliminate the electron-electron interaction term by introducing an auxiliary bosonic field. Such procedure, however, results in a mixed fermion-boson term in the action which is then cancelled by the gauge transformation. This is also a common way of bosonisation in higher dimensions because the classical bosonisation is specific to one-dimensional problems by construction and requires major modifications otherwise.
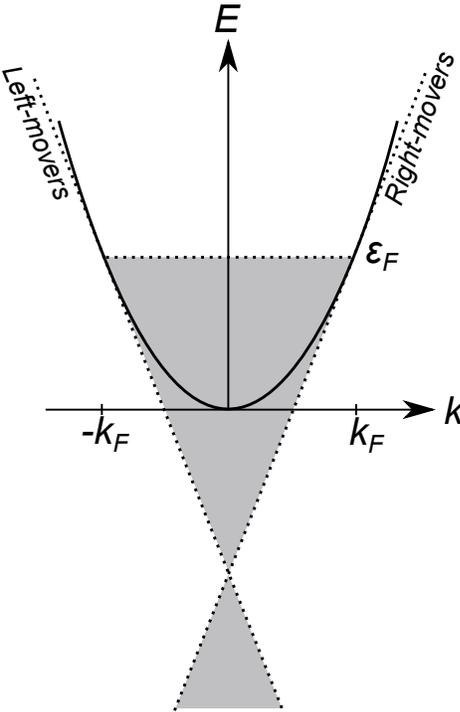
Figure 2.2: Linearisation of the electron spectrum in one dimension.

In order to allow for the solubility of the problem of interacting electrons in one dimension we need to linearise their spectrum around the Fermi points, as in Fig. 2.2. By doing so we

restrict ourselves to low-energy (and low-temperature) physics where this approximation is valid. The idea was first suggested by Tomonaga in his seminal paper in 1950 [32].

We should also note that by linearising the spectrum we impose a particle-hole symmetry on the system. This generally leads to inability of the theory to account for such effects as Coulomb drag, light absorption and thermoelectricity, for which the curvature in the dispersion relation is crucial. In our case this should not be the problem since we will not attempt to study any of those effects.

Another consequence of the linearisation procedure is the separation of electrons into left-and right-moving subsystems. Indeed, we can write the electron wave function as:

$$\psi(x,t) \simeq \psi_{\mathrm{R}}(x,t)\,\mathrm{e}^{ik_{\mathrm{F}}x} + \psi_{\mathrm{L}}(x,t)\,\mathrm{e}^{-ik_{\mathrm{F}}x}\,, \tag{2.15}$$

where the chiral wave functions $\psi_{\mathrm{R/L}}$ vary slowly on the scale of k_{F}^{-1}. In order for an electron to change its direction some mechanism of backscattering is required. In the absence of impurities backscattering is only possible via electron-electron interaction. We will assume, however, that the small momentum transfer part of the interaction is dominant, i.e. $V(q \simeq 0) \gg V(q \simeq 2k_{\mathrm{F}})$, which suppresses the probability of backscattering processes. This is generally true in real one-dimensional electronic systems due to interaction screening by charges in the system and in the leads to which it is connected [38]. In carbon nanotubes of large enough radius backscattering is also suppressed due to the Berry phase argument [39]. Also, as it was argued in Refs. [40, 41, 42], in metallic armchair nanotubes (N, N) the probability of two electrons to be close to each other scales as $1/N$, therefore, for $N \geq 10$ one can neglect the short-range part of the interaction, which in momentum representation corresponds to the processes with momentum transfer $2k_{\mathrm{F}}$, and, thereby, leave out of consideration backscattering and Umklapp scattering. We can also justify the suppression of Umklapp scattering by assuming incommensurate band filling. In fact, the elimination of

electron-electron backscattering was the key idea which allowed Larkin and Dzyaloshinskii to find the solution [43] of the Tomonaga-Luttinger model with long-range interaction of arbitrary strength.

Because we now have electrons of two chiralities, we introduce the following notations:

$$\boldsymbol{\psi} = \begin{pmatrix} \psi_\mathrm{R} \\ \psi_\mathrm{L} \end{pmatrix}, \qquad \boldsymbol{n} = \begin{pmatrix} n_\mathrm{R} \\ n_\mathrm{L} \end{pmatrix} \equiv \begin{pmatrix} \bar{\psi}_\mathrm{R}\psi_\mathrm{R} \\ \bar{\psi}_\mathrm{L}\psi_\mathrm{L} \end{pmatrix}, \qquad \partial_{\mathrm{R/L}} \equiv \partial_t \pm v_\mathrm{F}\partial_x. \tag{2.16}$$

In these notations we can write the total action for the Luttinger Liquid model as:

$$S_\mathrm{LL} = S_0 + S_\mathrm{int}, \tag{2.17}$$

where the free-electron part is described by the first term:

$$S_0 = \int \mathrm{d}x\mathrm{d}t\, \bar{\boldsymbol{\psi}}(x,t) \begin{pmatrix} i\partial_\mathrm{R} & 0 \\ 0 & i\partial_\mathrm{L} \end{pmatrix} \boldsymbol{\psi}(x,t), \tag{2.18}$$

while the interactions are given by the second term:

$$S_\mathrm{int} = -\frac{1}{2}\int \mathrm{d}x\mathrm{d}t\, \boldsymbol{n}^T(x,t)\hat{V}_0\,\boldsymbol{n}(x,t), \qquad \hat{V}_0 \equiv 2\pi v_\mathrm{F} \begin{pmatrix} g_4 & g_2 \\ g_2 & g_4 \end{pmatrix}. \tag{2.19}$$

We use Keldysh formalism [34, 36] described in Section 2.1 implying time integration along the Keldysh contour in Eq. (2.18) and onwards throughout the thesis. In Eq. (2.19) two assumptions have been made: first, the forward scattering electron-electron interaction only weakly depends on momentum and, therefore, can be approximated by a constant; and, second, depending on whether the interacting electrons have the same or different chiralities, the interaction constants can generally be different - we define them as $2\pi g_2$ and $2\pi g_4$,

correspondingly. These scattering amplitudes are just long-wave Fourier transforms of the direct (Coulomb) electron-electron interaction $V(x)$ for electrons of corresponding chiralities. Of course, $V(x)$ must be regularised at short distances, e.g. by a lattice parameter a. One possible realisation of the situation where $g_2 \neq g_4$ is the Quantum Hall Effect regime with electrons of different chiralities situated on the opposite edges of the Hall bar.

As an example we shall calculate the thermal Green function (2×2 diagonal matrix in R/L-space):

$$\mathcal{G}(x, x'; t, t') = \frac{\int \mathcal{D}[\bar{\boldsymbol{\psi}}, \boldsymbol{\psi}] \, \boldsymbol{\psi}(x, t) \bar{\boldsymbol{\psi}}(x', t') \, \exp\left\{ \mathrm{i} S_{\mathrm{LL}}[\bar{\boldsymbol{\psi}}, \boldsymbol{\psi}] \right\}}{\int \mathcal{D}[\bar{\boldsymbol{\psi}}, \boldsymbol{\psi}] \, \exp\left\{ \mathrm{i} S_{\mathrm{LL}}[\bar{\boldsymbol{\psi}}, \boldsymbol{\psi}] \right\}} . \tag{2.20}$$

We begin by applying the Hubbard-Stratonovich transformation to the exponential containing the interactions:

$$
\begin{aligned}
\exp\left\{ \mathrm{i} S_{\mathrm{int}} \right\} &= \exp\left\{ -\frac{\mathrm{i}}{2} \int \mathrm{d}x \mathrm{d}t \, \boldsymbol{n}^T(x, t) \hat{V}_0 \, \boldsymbol{n}(x, t) \right\} \\
&= \frac{\int \mathcal{D}[\boldsymbol{\phi}] \, \exp\left\{ \mathrm{i} \int \mathrm{d}x \mathrm{d}t \left[-\frac{1}{2} \boldsymbol{\phi}^T(x, t) \hat{V}_0^{-1} \, \boldsymbol{\phi}(x, t) - \mathrm{i} \boldsymbol{\phi}^T(x, t) \boldsymbol{n}(x, t) \right] \right\}}{\int \mathcal{D}[\boldsymbol{\phi}] \, \exp\left\{ -\frac{\mathrm{i}}{2} \int \mathrm{d}x \mathrm{d}t \, \boldsymbol{\phi}^T(x, t) \hat{V}_0 \, \boldsymbol{\phi}(x, t) \right\}} , \tag{2.21}
\end{aligned}
$$

where the introduced fields $\boldsymbol{\phi} = \begin{pmatrix} \phi_{\mathrm{R}} \\ \phi_{\mathrm{L}} \end{pmatrix}$ are bosonic in nature.

The expression for the Green function (2.20) now becomes:

$$\mathcal{G}(x, x'; t, t') = \frac{\int \mathcal{D}[\bar{\boldsymbol{\psi}}, \boldsymbol{\psi}, \boldsymbol{\phi}] \, \boldsymbol{\psi}(x, t) \bar{\boldsymbol{\psi}}(x', t') \, \exp\left\{ \mathrm{i} S[\bar{\boldsymbol{\psi}}, \boldsymbol{\psi}, \boldsymbol{\phi}] \right\}}{\int \mathcal{D}[\bar{\boldsymbol{\psi}}, \boldsymbol{\psi}, \boldsymbol{\phi}] \, \exp\left\{ \mathrm{i} S[\bar{\boldsymbol{\psi}}, \boldsymbol{\psi}, \boldsymbol{\phi}] \right\}} , \tag{2.22}$$

$$S[\bar{\boldsymbol{\psi}}, \boldsymbol{\psi}, \boldsymbol{\phi}] = \int \mathrm{d}x \mathrm{d}t \left\{ -\frac{1}{2} \boldsymbol{\phi}^T(x, t) \hat{V}_0^{-1} \, \boldsymbol{\phi}(x, t) + \sum_{\eta = \pm 1} \bar{\psi}_\eta(x, t) \left[\mathrm{i} \partial_\eta - \mathrm{i} \phi_\eta(x, t) \right] \psi_\eta(x, t) \right\} . \tag{2.23}$$

The action (2.23) is a functional of not only fermionic $\bar{\boldsymbol{\psi}}$ and $\boldsymbol{\psi}$ but also the auxiliary bosonic

fields ϕ. We now perform an integration over the Grassmann fields for which we notice that:

$$
\int \mathcal{D}[\bar{\psi}, \psi] \exp \left\{ i \sum_{\eta=\pm 1} \int \mathrm{d}x \mathrm{d}t \, \bar{\psi}_\eta(x,t) \left[i\partial_\eta - i\phi_\eta(x,t) \right] \psi_\eta(x,t) \right\}
$$

$$
= \det \left[\begin{pmatrix} \partial_{\mathrm{R}} - \phi_{\mathrm{R}}(x,t) & 0 \\ 0 & \partial_{\mathrm{L}} - \phi_{\mathrm{L}}(x,t) \end{pmatrix} \right] = \exp \left\{ \sum_{\eta=\pm 1} \mathrm{Tr} \ln \left[\partial_\eta - \phi_\eta(x,t) \right] \right\}
$$

$$
= \exp \left\{ \sum_{\eta=\pm 1} \left[\mathrm{Tr} \ln(\partial_\eta) - \sum_{n=1}^{\infty} \frac{1}{n} \mathrm{Tr} \left[ig_\eta(x,t)\phi_\eta(x,t) \right]^n \right] \right\} . \tag{2.24}
$$

By careful evaluation of the second term in the last expression one can show (see Appendix B) that the only non-zero contribution comes from $n = 2$, which yields a standard polarisation operator (a rigorous derivation of the polarisation operator is provided in Appendix A):

$$
\sum_{n=1}^{\infty} \frac{1}{n} \mathrm{Tr} \left[ig_\eta(x,t)\phi_\eta(x,t) \right]^n = -\frac{i}{2} \int \mathrm{d}x \mathrm{d}x' \mathrm{d}t \mathrm{d}t' \, \phi_\eta(x,t)\pi_\eta(x-x';t,t') \, \phi_\eta(x',t') , \tag{2.25}
$$

where

$$
\pi_\eta(x-x';t,t') = -ig_\eta(x-x';t,t')g_\eta(x'-x;t',t) , \tag{2.26}
$$

and the free electron Green function defined via the Fourier transform of its retarded component as

$$
g_\eta^{R/A}(\omega,q) = \frac{1}{\omega_\pm - \eta \, v_{\mathrm{F}} q} . \tag{2.27}
$$

The terms with $n > 2$ cancel, which has been first noted by Dzyaloshinskii and Larkin [43] and is often called the Loop Cancellation Theorem. As will be apparent below, effectively this fact means that the electron-electron interaction becomes screened in accordance with the random phase approximation, which turns out to be exact [44] within the framework of Tomonaga-Luttinger model.

After these manipulations the identity (2.24) becomes:

$$\int \mathcal{D}[\bar{\psi}, \psi] \, \exp\left\{ i \sum_{\eta=\pm 1} \left[\int \mathrm{d}x\mathrm{d}t \, \bar{\psi}_\eta(x,t) i\partial_\eta \psi_\eta(x,t) \right.\right.$$
$$\left.\left. + \frac{1}{2} \int \mathrm{d}x\mathrm{d}x'\mathrm{d}t\mathrm{d}t' \, \phi_\eta(x,t)\pi_\eta(x-x';t,t')\phi_\eta(x',t') \right] \right\}. \tag{2.28}$$

Let us note that the modification of the action from Eq. (2.23) to Eq. (2.28) is equivalent to performing the following gauge transformation of the fields $\psi_\eta(x,t)$:

$$\psi_\eta(x,t) \to \psi_\eta(x,t) \, e^{i\theta_\eta(x,t)}, \qquad i\partial_\eta \theta_\eta(x,t) = \varphi_\eta(x,t), \tag{2.29}$$

which was aimed at cancelling the coupling term between electron density and bosonic field ϕ in Eq. (2.23). Therefore, expression (2.25) is the Jacobian of this transformation, which is explicitly calculated in Appendix B.

Now, using the results (2.24)-(2.28), we can rewrite the original expression for electron Green function (2.22) with the help of some auxiliary Green function $\tilde{\mathcal{G}}(x, x'; t, t'; [\phi])$:

$$\mathcal{G}(x, x'; t, t') = \frac{\int \mathcal{D}[\phi] \, \tilde{\mathcal{G}}(x, x'; t, t'; [\phi]) \, \exp\{iS[\phi]\}}{\int \mathcal{D}[\phi] \, \exp\{iS[\phi]\}}, \tag{2.30}$$

$$S[\phi] = \frac{1}{2} \int \mathrm{d}x\mathrm{d}x'\mathrm{d}t\mathrm{d}t' \, \phi^T(x,t) \left[\hat{V}_0^{-1} - \hat{\pi} \right] \phi(x',t'), \tag{2.31}$$

where

$$\tilde{\mathcal{G}}(x, x'; t, t'; [\phi]) = \frac{\int \mathcal{D}[\bar{\psi}, \psi] \, \psi(x,t)\bar{\psi}(x',t') \, \exp\left\{ i \sum_{\eta=\pm 1} \int \mathrm{d}x\mathrm{d}t \, \bar{\psi}_\eta(x,t) \left[i\partial_\eta - i\phi_\eta(x,t) \right] \psi_\eta(x,t) \right\}}{\int \mathcal{D}[\bar{\psi}, \psi] \, \exp\left\{ i \sum_{\eta=\pm 1} \int \mathrm{d}x\mathrm{d}t \, \bar{\psi}_\eta(x,t) \left[i\partial_\eta - i\phi_\eta(x,t) \right] \psi_\eta(x,t) \right\}}. \tag{2.32}$$

The physical meaning of the auxiliary $\tilde{\mathcal{G}}$ is the Green function of a free electron in external

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potential $\phi(x,t)$ and its components must satisfy the equations:

$$[\mathrm{i}\partial_t + \eta v_{\mathrm{F}}\partial_x - \mathrm{i}\phi_\eta(x,t)]\,\tilde{\mathcal{G}}_\eta(x,x';t,t;[\phi(x,t)]) = \delta(x-x')\delta(t-t')\,. \qquad (2.33)$$

These equations have the following solutions:

$$\tilde{\mathcal{G}}_\eta(x,x';t,t;[\phi_\eta]) = g_\eta(x-x';t,t')\mathrm{e}^{\mathrm{i}\theta_\eta(x,t)-\mathrm{i}\theta_\eta(x',t')}\,, \qquad (2.34)$$

where g_η are chiral free electron Green functions (see Appendix F) and the gauge fields $\theta_\eta(x,t)$ are determined by:

$$\mathrm{i}\partial_\eta\theta_\eta(x,t) = \phi_\eta(x,t)\,. \qquad (2.35)$$

Therefore, in order to calculate the original interacting electron Green function $\mathcal{G}$ one needs to take the auxiliary electron Green function $\tilde{\mathcal{G}}$ from Eq. (2.34) and average it over the fields $\boldsymbol{\theta}$ with the Gaussian action (2.31) using (2.35):

$$\mathcal{G}_\eta(x,x';t,t) = g_\eta(x-x';t,t')\mathrm{e}^{\mathrm{i}\theta_\eta(x,t)-\mathrm{i}\theta_\eta(x',t')}\,, \qquad (2.36)$$

In fact, the outlined method is by no means restricted to a single-particle Green function. One can calculate an arbitrary correlation function with any number of fermionic operators in Eq.(2.32) with the help of Wick's theorem. Since after the Hubbard-Stratonovich transformation the fermionic action becomes quadratic, any observable which consists of multiple fermi operators ψ can be represented as a product of the auxiliary Green functions $\tilde{\mathcal{G}}[\phi]$ which then are averaged over the fields ϕ with the quadratic action (2.31). This technique reduces the original problem of interacting electrons to the trivial evaluation of Gaussian integrals.

CHAPTER 3

MODEL UNDER CONSIDERATION

The aim of this Chapter is to show in action the general approach which we will use later to tackle the problem of a single impurity embedded into interacting one-dimensional electron-phonon system. First, we re-derive some known results for the pure Luttinger Liquid and outline the solution of a single impurity problem by Kane and Fisher [1]. We then proceed onto the next ingredient which is the description of a translation invariant interacting electron-phonon system. This Chapter is concluded with the discussion of all possible formulations of the single impurity problem which we then consider in Chapters 4 and 5.

3.1 Pure Luttinger Liquid

This Section is devoted to the derivation of some of the fundamental properties of Luttinger Liquids. We consider an infinite system of chiral spinless fermions interacting via the potential (2.19). We begin with calculating the single-particle Green function following the method described in Section 2.2. All we need to do in order to calculate chiral Green functions of interacting electrons is to average Eq. (2.34) with quadratic action:

$$S[\phi] = \frac{1}{2} \sum_{\eta,\eta'=\pm 1} \int \mathrm{d}x \mathrm{d}x' \mathrm{d}t \mathrm{d}t' \, \theta_\eta(x,t) \, g_\eta \Big[\hat{V}_0^{-1} - \hat{\pi} \Big] g_{\eta'} \, \theta_{\eta'}(x',t') \,, \qquad (3.1)$$

where we took into account the relation between fields ϕ_η and θ_η, Eq. (2.35. $\hat{V}_0$ is the matrix of electron-electron scattering amplitudes defined in Eq.(2.19) and $\hat{\pi}$ refers to the polarisation operators derived in Appendix A:

$$
\hat{\pi} = \begin{pmatrix} \pi_{\mathrm{R}} & 0 \\ 0 & \pi_{\mathrm{L}} \end{pmatrix} , \tag{3.2}
$$

with $\pi_\eta(x,t)$ defined by Eq. (2.26):

$$
\pi_{\mathrm{R}}^{R/A}(\omega, q) = \frac{q}{2\pi(\omega_\pm - v_{\mathrm{F}}q)} , \qquad \pi_{\mathrm{L}}^{R/A}(\omega, q) = -\frac{q}{2\pi(\omega_\pm + v_{\mathrm{F}}q)} , \qquad \omega_\pm \equiv \omega \pm \mathrm{i}0 . \tag{3.3}
$$

By putting everything together, we eventually obtain the following expressions:

$$
\langle \theta_\eta \theta_{\eta'} \rangle = \frac{\mathrm{i}V_{\eta\eta'}}{g_\eta \, g_{\eta'}} \equiv \mathrm{i}U_{\eta\eta'} , \qquad \hat{U} = \begin{pmatrix} U_{\mathrm{RR}} & U_{\mathrm{RL}} \\ U_{\mathrm{LR}} & U_{\mathrm{LL}} \end{pmatrix} . \tag{3.4}
$$

$$
U_{\mathrm{RR}}^{R/A}(\omega, q) = \frac{2\pi}{q}\left[-\frac{1}{\omega_\pm - v_{\mathrm{F}}q} + \frac{1 + \gamma_f/2}{\omega_\pm - vq} - \frac{\gamma_f/2}{\omega_\pm + vq} \right] , \tag{3.5}
$$

$$
U_{\mathrm{LL}}^{R/A}(\omega, q) = \frac{2\pi}{q}\left[\frac{1}{\omega_\pm + v_{\mathrm{F}}q} + \frac{\gamma_f/2}{\omega_\pm - vq} - \frac{1 + \gamma_f/2}{\omega_\pm + vq} \right] , \tag{3.6}
$$

$$
U_{\mathrm{RL}}^{R/A}(\omega, q) = \frac{2\pi}{q}\frac{\gamma_b}{2}\left[\frac{1}{\omega_\pm - vq} - \frac{1}{\omega_\pm + vq} \right] , \tag{3.7}
$$

where v is the renormalised by interactions speed of plasmonic excitations in the Luttinger Liquid,

$$
v = v_{\mathrm{F}}\sqrt{(1 + g_4)^2 - g_2^2} , \tag{3.8}
$$

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and $\gamma_{f/b}$ are interaction-dependent constants corresponding to forward/backward scattering:

$$\gamma_f = \frac{(1-K)^2}{2K} \,, \qquad \gamma_b = \frac{1-K^2}{2K} \,, \tag{3.9}$$

$$K = \sqrt{\frac{1 + g_4 - g_2}{1 + g_4 + g_2}} \,. \tag{3.10}$$

Here K is the Luttinger parameter which characterises strength of interactions in a Luttinger Liquid: $K = 1$ in a noninteracting system, $K < 1$ if the interaction between electrons is repulsive, and $K > 1$ for attractive interaction.

By utilising the following correspondence between retarded/advanced components in (ω, q)-representation and lesser/greater components in (x, t)-representation,

$$-i\frac{2\pi}{q}\frac{1}{\omega_\pm - vq} \;\leftrightarrow\; \ln \sinh \pi T \left(t_\pm - \frac{x}{v}\right) = l\left(t_\pm - \frac{x}{v}\right) , \tag{3.11}$$

$$i\frac{2\pi}{q}\frac{1}{\omega_\pm + vq} \;\leftrightarrow\; \ln \sinh \pi T \left(t_\pm + \frac{x}{v}\right) = l\left(t_\pm + \frac{x}{v}\right) , \tag{3.12}$$

we finally obtain an expression for the electron Green function:

$$g_\eta^<(x,t) = -\frac{T}{2v_{\mathrm{F}}}\frac{1}{\sinh \pi T \left(t - \eta\frac{x}{v_{\mathrm{F}}} + i0\right)} \,, \tag{3.13}$$

$$\mathcal{G}_\eta^<(x,t) = g_\eta^<(x,t)e^{iU_{\eta\eta}^<(x,t) - iU_{\eta\eta}^<(0,0)} \,, \tag{3.14}$$

$$\mathcal{G}_\eta^<(x,t) = -\frac{T}{2v_{\mathrm{F}}}\frac{1}{\sinh \pi T \left(t - \eta\frac{x}{v} + i0\right)}$$

$$\times \left[\frac{(\pi T t_0)^2}{\sinh \pi T \left(t - \eta\frac{x}{v} + it_0\right)\sinh \pi T \left(t + \eta\frac{x}{v} + it_0\right)^2}\right]^{\frac{\gamma_f}{2}} \,, \tag{3.15}$$

where $t_0 \sim \varepsilon_{\mathrm{F}}^{-1}$ is a small cut-off and the sign of $i0$ is reversed for greater ($>$) components. The contribution in brackets is the direct consequence of the interactions (in their absence $\gamma_f = 0$). This is one of the examples of the non-universal power-law behaviour of correlation functions in the Luttinger Liquid.

3.2 Impurity in a Luttinger Liquid

The purpose of this Section is to briefly outline the solution of the single impurity problem in a spinless Luttinger Liquid, as presented originally in the seminal paper by Kane and Fisher [1]. The authors have considered two opposite limits in the impurity strength, weak scatterer and weak link. We will focus on the first limit and then prove its duality with the weak link case.

First of all, the authors in Ref. [1] used the classical operator bosonisation approach, which allows to rewrite the free fermionic fields $\psi_{0\,\eta}(x,t)$ as follows:

$$\psi_{0\,\eta}(x,t) = \frac{F_\eta}{\sqrt{2\pi a}} e^{i\phi_\eta(x,t)} \,. \tag{3.16}$$

Here F_η is a Klein factor and a is a short distance cut-off (lattice parameter). Klein factors will not be important at any point in this thesis and will be omitted in what follows. Together with the gauge fields responsible for interactions, Eq. (2.29), the above expression gives the following fully bosonised representation of interacting electron wave functions:

$$\psi_\eta(x,t) \propto e^{i\Theta_\eta(x,t)} \,, \qquad \Theta_\eta(x,t) \equiv \phi_\eta(x,t) + \theta_\eta(x,t) \,. \tag{3.17}$$

It is then natural to use instead of Θ_η their following combinations:

$$\tilde{\theta}(x,t) = \frac{1}{2} \left[\Theta_R(x,t) - \Theta_L(x,t) \right] \,, \tag{3.18}$$

$$\tilde{\phi}(x,t) = \frac{1}{2} \left[\Theta_R(x,t) + \Theta_L(x,t) \right] \,, \tag{3.19}$$

which are canonically conjugate bosonic fields. Their spatial derivatives are proportional to

the fluctuations of the total electron density and the current:

$$\rho(x) = \rho_{\rm R}(x) + \rho_{\rm L}(x) = \frac{1}{\pi}\partial_x \tilde{\theta}(x)\,, \tag{3.20}$$

$$j(x) = \rho_{\rm R}(x) - \rho_{\rm L}(x) = \frac{1}{\pi}\partial_x \tilde{\phi}(x)\,. \tag{3.21}$$

It can be shown [25, 45] that the Luttinger Liquid action can be written in terms of the field $\tilde{\theta}$ as follows:

$$S_{\rm LL} = \frac{1}{2\pi v K} \int {\rm d}x {\rm d}t \left\{ \left[\partial_t \tilde{\theta}(x,t)\right]^2 - \left[v\partial_x \tilde{\theta}(x,t)\right]^2 \right\}\,, \tag{3.22}$$

with the speed of plasmonic excitations, v, and Luttinger parameter K, defined by Eqs. (3.8) and (3.10), correspondingly. Alternatively, one can use the dual representation in terms of fields $\tilde{\phi}$ by substituting $\tilde{\theta} \to \tilde{\phi}$ and $K \to 1/K$ in Eq. (3.22).

Now, let us introduce a weak scatterer at the origin. The corresponding term can be written as:

$$S_{\rm ws} = \int {\rm d}x {\rm d}t \, \mathcal{V}(x) n(x,t)\,, \tag{3.23}$$

with the local impurity potential $\mathcal{V}(x) \simeq v\, r_0\, \delta(x)$. Here r_0 is a small electron reflection amplitude from the impurity. Upon substituting Eq. (3.16), the action takes form:

$$S_{\rm ws} = \lambda_0 \int {\rm d}t \, \cos 2\tilde{\theta}(t)\,, \qquad \lambda_0 \sim \frac{v\, r_0}{\pi a}\,, \tag{3.24}$$

where $\tilde{\theta}(t) \equiv \tilde{\theta}(x=0,t)$. λ_0 plays a role of a bare impurity strength, whose scaling at low energies (temperatures) is analysed by means of the renormalisation group analysis. The forward scattering terms originating from Eq. (3.23) can be eliminated by redefining the field $\tilde{\theta}(t)$ [45]. We can integrate out all the fields $\tilde{\theta}(x \neq 0, t)$ in action (3.22) and obtain the

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effective action:

$$S_{\text{ws}}^{\text{eff}} = \frac{1}{2} \int \mathrm{d}t\mathrm{d}t' \, \tilde{\theta}(t) \mathcal{G}^{-1}(t-t')\tilde{\theta}(t') + \lambda_0 \int \mathrm{d}t \, \cos 2\tilde{\theta}(t) \,, \tag{3.25}$$

where $\mathcal{G}(t-t')$ the autocorrelation function of the field $\tilde{\theta}(t)$, by taking the Fourier transform of which one arrives at the definition of the scaling dimension of the problem, Δ:

$$\mathcal{G}^R(\omega) = \frac{\pi}{2\mathrm{i}} \frac{\Delta_{\text{ws}}}{\omega + \mathrm{i}0} \,. \tag{3.26}$$

For the weak scatterer problem $\Delta_{\text{ws}} = K$. The scaling dimension plays an important role in the renormalisation group analysis of the backscattering term in the action. One can derive the flow equation for $\lambda(E)$ by analogy with the derivation in Appendix D:

$$\partial_l \lambda(E) = (1 - \Delta_{\text{ws}})\lambda(E) \,, \qquad l \equiv \ln(E/E_0) \,, \tag{3.27}$$

which gives the scaling of the effective impurity strength at low energies $\varepsilon = E$:

$$\lambda(\varepsilon) = \lambda_0 \left(\frac{\varepsilon}{E_0} \right)^{K-1} \,. \tag{3.28}$$

Since $K < 1$ for repulsive electron-electron interaction, even arbitrarily weak scatterer is relevant and suppresses low-temperature conductance of the system:

$$G(T) = \frac{e^2}{h} \left(1 - \xi \, T^{2K-2} \right) \,, \tag{3.29}$$

where ξ is some temperature-independent constant.

Let us now turn to the opposite limit of a strong scatterer, or weak link. In this case one considers two semi-infinite Luttinger Liquids connected by a tunneling link. This adds the term $t_{\text{wl}} \cos 2\tilde{\phi}(t)$ to the Luttinger Liquid action. Here $\tilde{\phi}(t) \equiv \tilde{\phi}(x=0,t)$ and t_{wl} is the

overlap matrix element proportional to the small electron tunneling amplitude between the subsystems, t_0. One can also show [1] that the action describing two semi-infinite Luttinger Liquids after integrating out all the fields $\tilde{\phi}(x \neq 0, t)$ takes the following form:

$$S_{\text{LL}}(x=0) = \frac{1}{2} \int \mathrm{d}t\mathrm{d}t' \, \tilde{\phi}(t)\tilde{\mathcal{G}}^{-1}(t - t')\tilde{\phi}(t') , \qquad (3.30)$$

where

$$\tilde{\mathcal{G}}^R(\omega) = \frac{\pi}{2\mathrm{i}} \frac{\Delta_{\text{WL}}}{\omega + \mathrm{i}0} , \qquad \Delta_{\text{WL}} = \frac{1}{K} . \qquad (3.31)$$

This gives the following effective action of the weak link problem:

$$S_{\text{WL}}^{\text{eff}} = \frac{1}{2} \int \mathrm{d}t\mathrm{d}t' \, \tilde{\phi}(t)\tilde{\mathcal{G}}^{-1}(t - t')\tilde{\phi}(t') + t_{\text{wl}} \int \mathrm{d}t \, \cos 2\tilde{\phi}(t) . \qquad (3.32)$$

Therefore, in the weak link limit one obtains exactly the same form of the action as in the weak scattering case, but with $\tilde{\theta}(t) \to \tilde{\phi}(t)$ and the scaling dimension $\Delta_{\text{ws}} = K \to \Delta_{\text{WL}} = 1/K$.

The renormalisation group analysis of the weak link term yields

$$t_{\text{wl}}(\varepsilon) = t_{\text{wl}}(E_0) \left(\frac{\varepsilon}{E_0} \right)^{\frac{1}{K}-1} . \qquad (3.33)$$

When $K < 1$ the tunneling term is irrelevant in the sense of renormalisation group. The conductance across the weak link is suppressed and scales at low temperatures as:

$$G(T) \propto \frac{e^2}{h} T^{\frac{2}{K}-2} . \qquad (3.34)$$

Notice that as we lower the temperature, the initially weak impurity will at some point increase beyond the applicability of the perturbation theory. At temperatures $T \lesssim E_0 \, r_0^{\frac{1}{1-K}}$ one enters the weak link regime and a crossover between the expressions (3.29) and (3.34) is expected.

The above results show the duality between the limits of weak and strong impurity scattering in Luttinger Liquid:

$$\Delta_{\mathrm{WS}}\Delta_{\mathrm{WL}} = 1. \tag{3.35}$$

In fact, one can the understand the origin of the duality, at least in the case of a system with local electron-electron interaction, by performing the 'unfolding' transformation of the weak link geometry described in Appendix E. It modifies the two half-infinite systems on either side of the impurity into a translation invariant Luttinger Liquid, as in the weak scattering limit. Under this transformation the impurity tunneling action $S_{\mathrm{WL}} = t_{\mathrm{wl}}\bar{\psi}_{1\mathrm{L}}(t)\psi_{2\mathrm{R}}(t) + \mathrm{c.c.}$, where indices 1 and 2 label the left and right subsystems, becomes the same as the weak scattering action $S_{\mathrm{WS}} = t_{\mathrm{wl}}\bar{\psi}_{\mathrm{R}}(t)\psi_{\mathrm{L}}(t) + \mathrm{c.c.}$, with t_{wl} playing the role of the weak impurity strength. Although the interaction in the system after unfolding is no longer short-ranged or even translation invariant, this problem is easy to cure [45] in the Luttinger Liquid case: the interaction can be formally removed from the model by rescaling $\tilde{\theta} \to \tilde{\theta}/\sqrt{K}$ (and $\tilde{\phi} \to \tilde{\phi}/\sqrt{K}$, to keep them canonically conjugate) before unfolding. This is another way to see the duality of the two limits.

Let us remark here that the above argument relies on the locality of the electron-electron interaction in the Luttinger Liquid. It is not possible to remove nonlocal interactions by a simple rescaling of the fields. Therefore, an interaction of a general form would violate the above statement that upon unfolding two half-infinite subsystems become equivalent to a translation invariant system.

3.3 Electron-phonon liquid

In this Section we provide a general description of the mixed electron-phonon system employing the functional bosonisation formalism in the form developed in Refs. [46, 47, 48] and described in Section 2.2. This approach allows us to include the electron-phonon interaction

into the system which leads to electrons dressing with phonons, i.e. the formation of polarons, after which we can bosonise the action. Prior to considering an embedded scatterer we describe the polaron formation in a translation invariant Luttinger Liquid, reproducing the known results [12, 48] in form convenient for further considerations.

We consider a model of one-dimensional acoustic phonons linearly coupled to the electron density. In particular, the electron-phonon interaction originates from local polarisation produced by lattice vibrations (for the derivation see Ref. [29], p. 77). The phonon spectrum is assumed to be linear with a cutoff at the Debye frequency $\omega_D = c\, q_D$. The phonon part of the action can be written as:

$$S_{ph} = -\frac{1}{2} \int dx\, dt\, dx'dt'\, \phi(x,t)\, \mathcal{D}_0^{-1}(x-x';t,t')\phi(x',t')\,, \tag{3.36}$$

Free phonon propagator $\mathcal{D}_0$ of the phonon field ϕ can be defined by its retarded component

$$\mathcal{D}_0^R(\omega,q) = \pi v_F \alpha_{ph} \frac{\omega_q^2}{\omega_+^2 - \omega_q^2}\,, \qquad \omega_q = c\, q\,, \tag{3.37}$$

where c is the sound velocity, α_{ph} is the dimensionless electron-phonon coupling and $\omega_+ \equiv \omega + i0$.

The electron-phonon interaction term

$$S_{el\text{-}ph} = \int dx dt\, \phi(x,t)n(x,t) \tag{3.38}$$

describes coupling of the phonon field to the total electron density, which is a sum of densities of both movers, $n \equiv n_R + n_L$. This is a legitimate approximation in absence of electron backscattering due to both electron-electron (see page 16) and electron-phonon interactions. The latter is justified if there can be no phonons of momentum $\sim 2k_F$ and energy $\sim T$ such that an electron can be scattered by them between vicinities of the Fermi points, i.e. at temperatures $T \ll c\, k_F$. Notice that in Eq. (3.38) we have incorporated the coupling strength

α_{ph} into the field ϕ for simplicity of notations.

The electron part of the system is described within the Luttinger Liquid formalism discussed in Section 3.1 with the linearised wave function (2.15) and the action term (2.17):

$$S_{\mathrm{LL}} = \sum_{\eta=\pm 1} \int \mathrm{d}x\mathrm{d}t\, \bar{\psi}_\eta(x,t)\, g_\eta^{-1}(x,t)\, \psi_\eta(x,t) - \frac{1}{2} \int \mathrm{d}x\mathrm{d}t\, \boldsymbol{n}(x,t)\, \hat{V}_0\, \boldsymbol{n}(x,t)\,. \tag{3.39}$$

The first step is to integrate out the phonon field in the standard way which results in substituting the dynamical coupling

$$\hat{V}(x,t) = \hat{V}_0 + \mathcal{D}_0(x,t)(\hat{1} + \hat{\tau}_1)\,, \qquad \hat{\tau}_1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.40}$$

for the screened Coulomb interaction $\hat{V}_0$ in the Luttinger Liquid action:

$$S_{\mathrm{LL}} = \sum_{\eta=\pm 1} \int \mathrm{d}x\mathrm{d}t\, \bar{\psi}_\eta(x,t)\, \mathrm{i}\partial_\eta \psi_\eta(x,t) - \frac{1}{2} \int \mathrm{d}x\mathrm{d}t\mathrm{d}x'\mathrm{d}t'\, \boldsymbol{n}(x,t)\hat{V}(x,x';t,t')\boldsymbol{n}(x',t')\,. \tag{3.41}$$

The following few steps exactly repeat the treatment presented in Section 2.2, i.e. the Hubbard-Stratonovich transformation followed by the gauge transformation. The former one decouples the interaction term in the action (3.41) and results in the mixed fermionic-bosonic action in terms of the auxiliary bosonic field $\boldsymbol{\varphi}$ minimally coupled to $\boldsymbol{\psi}$:

$$\begin{aligned} S_{\mathrm{eff}} = &-\frac{1}{2} \int \mathrm{d}x\mathrm{d}t\mathrm{d}x'\mathrm{d}t'\, \boldsymbol{\varphi}(x,t)\, \hat{V}^{-1}(x,x';t,t')\, \boldsymbol{\varphi}(x',t') \\ &+ \sum_{\eta=\pm 1} \int \mathrm{d}x\mathrm{d}t\, \bar{\psi}_\eta(x,t)\, [\mathrm{i}\partial_\eta - \mathrm{i}\varphi_\eta(x,t)]\, \psi_\eta(x,t)\,. \end{aligned} \tag{3.42}$$

Now the coupling term can be cancelled out by the gauge transformation

$$\psi_\eta(x,t) \to \psi_\eta(x,t)\, \mathrm{e}^{\mathrm{i}\theta_\eta(x,t)}\,, \qquad \mathrm{i}\partial_\eta \theta_\eta(x,t) = \varphi_\eta(x,t)\,. \tag{3.43}$$

The Jacobian of this transformation results (see Appendix B) in substituting $\hat{V}^{-1} - \hat{\pi}$ for $\hat{V}^{-1}$ in Eq. (3.42), where $\hat{\pi}$ is the one-loop electronic polarisation operator.

After the gauge transformation (3.43) the effective action takes form

$$
\begin{aligned}
S_{\text{eff}} = \sum_{\eta=\pm 1} &\int \mathrm{d}x \mathrm{d}t \, \bar{\psi}_\eta(x,t) \, g_\eta^{-1}(x,t) \, \psi_\eta(x,t) \\
&- \frac{1}{2} \int \mathrm{d}x \mathrm{d}t \mathrm{d}x' \mathrm{d}t' \, \boldsymbol{\varphi}(x,t) \left[\hat{V}^{-1}(x,x';t,t') - \hat{\pi}(x-x';t,t') \right] \boldsymbol{\varphi}(x',t') .
\end{aligned}
\tag{3.44}
$$

The phase θ_η is related to the auxiliary field $\boldsymbol{\varphi}$ by

$$
\theta_\eta(x,t) = \int \mathrm{d}x \mathrm{d}t \, g_\eta^B(x-x';t,t') \, \varphi_\eta(x',t') ,
\tag{3.45}
$$

where g_η^B is the bosonic Green functions that resolves Eq. (3.43). Its retarded component coincides with the free fermionic g_η^R (while the Keldysh components are naturally different).

The first part of the action (3.44) describes free non-interacting electrons whereas the second part is quadratic in the field ϕ and, consequently, in the fields θ. Hence, all information about the interaction is contained in the correlation function of the θ fields. The Green function of electrons interacting with the medium (polarons) is not gauge invariant with respect to the transformation (3.43) and can be cast as:

$$
G_\eta(x,x';t,t') = g_\eta(x-x';t,t') \, \mathrm{e}^{\mathrm{i}U_{\eta\eta}(x-x';t,t')} ,
\tag{3.46}
$$

$$
\mathrm{i}U_{\eta\eta'}(x-x';t,t') = \langle \theta_\eta(x,t) \, \theta_{\eta'}(x',t') \rangle ,
\tag{3.47}
$$

The retarded Fourier component of the correlation function $U_{\eta\eta'}$ (Green function of a com-

posite boson) describing the effect of interactions is found as

$$U_{\eta\eta'}^R(\omega, q) = \frac{\omega_+ + \eta' v_{\mathrm{F}} q}{\omega_+ - \eta\, v_{\mathrm{F}} q} \frac{(\omega_+^2 - \omega_q^2)(\mathcal{D}_0 + 2\pi v_{\mathrm{F}} g_{3+\eta\eta'})}{(\omega_+^2 - v_+^2 q^2)\,(\omega_+^2 - v_-^2 q^2)}$$
$$+ \frac{\delta_{\eta\eta'}}{\omega_+ - \eta v_{\mathrm{F}} q} \frac{2 v_{\mathrm{F}} q (g_4 - g_2)\left(\omega_+^2 - \omega_q^2\right)[\mathcal{D}_0 + 2\pi v_{\mathrm{F}}(g_4 + g_2)]}{(\omega_+^2 - v_+^2 q^2)\,(\omega_+^2 - v_-^2 q^2)}, \tag{3.48}$$

where $\omega_q = c|q|$ is the dispersion relation of acoustic phonons in the Debye model. Notice that this cumbersome expression can be simplified greatly for $g_4 = g_2 = V_0/2\pi v_{\mathrm{F}}$:

$$U_{\eta\eta'}^R(\omega, q) = \frac{\omega_+ + \eta' v_{\mathrm{F}} q}{\omega_+ - \eta\, v_{\mathrm{F}} q} \frac{V_0(\omega_+^2 - \omega_q^2) + \pi v_{\mathrm{F}} \alpha_{\mathrm{ph}} \omega_q^2}{(\omega_+^2 - v_+^2 q^2)\,(\omega_+^2 - v_-^2 q^2)}. \tag{3.49}$$

As usual, the denominator (poles of the correlation function) provides us with the dispersion law of excitations in the system. Here $v_\pm$ are velocities of the composite bosonic modes:

$$v_\pm^2 = \frac{v^2 + c^2}{2} \pm \sqrt{\left(\frac{v^2 - c^2}{2}\right)^2 + \alpha\, v^2 c^2}, \qquad \alpha \equiv \frac{\alpha_{\mathrm{ph}} K v_{\mathrm{F}}}{v} = \frac{\alpha_{\mathrm{ph}}}{1 + g_2 + g_4}. \tag{3.50}$$

Here v is the speed of plasmonic excitations in the phononless Luttinger Liquid, given by Eq. (3.8). A simple analysis of the expression (3.50) shows that without phonons ($c = 0$) one has $v_- = 0$ and $v_+ = v$, so that in this case $U_{\eta\eta'}$ reduces to the usual LL plasmonic propagator (3.4). The same is, obviously, true if we uncouple phonons from the Luttinger Liquid ($\alpha_{\mathrm{ph}} = 0$, as well as for $\omega > \omega_{\mathrm{D}}$).

We assume the parameter α in Eq. (3.50) obeying the inequality $\alpha < 1$ to avoid the Wentzel-Bardeen instability [49, 50] corresponding to $v_-^2 < 0$ (with the threshold shifted from $\alpha_{\mathrm{ph}} = 1$ for a pure electron-phonon model to $\alpha_{\mathrm{ph}} = 1 + g_2 + g_4 > 1$ due to the presence of the Coulomb repulsion).

The velocities $v_\mp$ of the slow and fast composite bosonic modes in Eq. (3.50) obey the

following inequalities:

$$v_- < c, \, v < v_+ \,. \tag{3.51}$$

These modes mean that Luttinger Liquid in the presence of the electron-phonon coupling becomes two-component. Notice also that this behaviour does not require any Coulomb interaction and is solely triggered by phonons. Phonon-mediated electron-electron interaction is sufficient for the appearance of the Luttinger Liquid state with its power-law decay of various correlation functions.

The Wentzel-Bardeen instability mentioned above is a remarkable feature of coupled electron-phonon systems in one dimension. It was discovered early in the 50's [49, 50] when such systems were considered to be candidates for superconductivity. It was found that strong enough electron-phonon coupling leads to a collapse of the system. More specifically, it is the forward scattering processes (the only ones we consider) of electrons on phonons which cause this effect. One can see it directly from the Eq. (3.50). When α approaches its critical value of 1 the velocity of the slow bosonic mode v_- becomes zero, which means that it no longer cost any energy to create excitations in this mode which is somewhat similar to the infrared catastrophe in Quantum Electrodynamics. Indeed, when calculating the average number of particles in this mode one finds that it diverges as $\alpha \to 1$ [51]. Extremely large number of bosons causes the system to collapse. For instance, the compressibility of the system becomes negative at $\alpha > 1$ [52], i.e. at sufficiently attractive effective electron-electron interaction. This indicates that the Wentzel-Bardeen singularity is a signature of a possible phase transition into the superconducting regime in such systems, which is also supported by the fact that long-ranged superconducting correlations become dominant as attractive interaction due to phonons overcomes repulsive Coulomb interaction [53].

3.4 Problem formulations

We have already accepted the fact that in order to be able to employ the (perturbative) renormalisation group analysis for solving a single impurity problem in a system with arbitrarily strong interactions we need a small parameter. This can be either electron backscattering or tunnelling amplitude, which corresponds to the weak scatterer or weak link limits in impurity strength, correspondingly. In Chapter 7 we will consider a different approach when the impurity potential can be arbitrarily strong and the role of a small parameter will be played by interactions. Until then we will be restricted by these two limits only.

When adding phonons into the system it is important that we establish how the impurity affects propagation of phonons. It has been already mentioned that the logarithmic corrections, which are to be analysed by means of the renormalisation group theory, come from low-energy contributions. With respect to phonons this means that it is sufficient to consider only two limiting cases: when propagation of long-wavelength phonons is not affected by the scatterer at all (phonon-transparent impurity), and the opposite situation when they are completely reflected by the scatterer, which effectively cuts the phonon system into two independent parts (phonon-nontransparent impurity). This is because in the limit $\omega \to 0$ translational invariance for phonons with energy ω can either be broken or not. If it breaks, we deal with the phonon-nontransparent impurity, and visa versa. Therefore, we have four different situations to consider, which is the purpose of the following two Chapters.

A single impurity can be modelled in many different ways. Examples of possible defects in a lattice include atoms of a different mass, structural deformation of the lattice affecting its elastic properties, and pinning of an individual atom of the lattice with some elasticity constant [18, 54], as well as their combinations. Other examples include models of various flaws in quantum wires [55, 56]. What all these and many other possible defects have in common is that phonon conductivity is generally energy-dependent, i.e. the phonon reflection amplitude is a function of its energy and, therefore, our assumptions of either complete

transmission or reflection may, at first, seem rather primitive and unrealistic. The point is that the actual energy dependence of the phonon reflection amplitude is not of huge importance. What is important is its small-energy which can only be either 1 or 0 for the specified above reason.

We should note that, in fact, it is possible to solve the problem with an arbitrary phonon reflection amplitude [57]. Moreover, the duality relation between the weak and strong electron scattering limits still holds in this case. However, for the purpose of this thesis, we will only restrict ourselves to a phonon-transparent and phonon-nontransparent impurities.

Regardless of the phonon transmission properties through the impurity, the electron scattering terms can be written as follows. In the weak scattering limit we get:

$$S_{\mathrm{ws}} = \lambda_0 \int \mathrm{d}t\, \bar{\psi}_{\mathrm{R}}(t)\psi_{\mathrm{L}}(t) + \text{c.c.}, \qquad \psi_\eta(t) \equiv \psi_\eta(x=0,t)\,, \qquad \eta = \mathrm{R/L}\,, \qquad (3.52)$$

with λ_0 being the bare impurity strength. It introduces a small probability of electron scattering by the featureless local impurity embedded at $x = 0$.

In the opposite limit we model the system as effectively two independent electron subsystems separated by a strong scatterer or a weak tunnelling link. The weak link action contains the tunnelling term connecting the two halves of the Luttinger Liquid, labeled by $a = 1, 2$:

$$S_{\mathrm{WL}} = t_{\mathrm{wl}} \int \mathrm{d}t\, \bar{\psi}_1 \psi_2 + \text{c.c.}\,, \qquad \psi_a = \psi_{a\mathrm{R}} + \psi_{a\mathrm{L}}\,, \qquad (3.53)$$

with the bare hopping amplitude t_{wl}.

In both limits all operators in the scattering action are local. The terms are analysed with the help of renormalisation group theory. In particular, we will be interested in how the coupling constants λ and t_{wl} scale as one lowers the high-energy cut-off in the model while keeping the action the same, i.e. at the fixed point. Since temperature can play the role of the cut-off, one immediately obtains effective low-temperature behaviour of the system.

CHAPTER 4

PHONON-TRANSPARENT IMPURITY

In this Chapter of the thesis we provide the solution of the problem of a single featureless quantum impurity embedded in a Luttinger Liquid with electron-phonon interaction. The system under consideration consists of one-dimensional spinless (or spin-polarised) electrons linearly coupled to one-dimensional acoustic phonons, as described in Section 3.3. Here and everywhere below, the impurity will be assumed to be placed at the origin with electrons and phonons existing on a full axis. The quantum impurity considered in this Chapter is assumed to be perfectly transparent for phonons, allowing for an unperturbed phonon-mediated interaction between electrons on the opposite sides from the impurity. This can be achieved, for instance, in the case of a defect vibrating together with the one-dimensional wire. Another possible realisation involves a local depletion of electron density, which would only affect electron transport and not phonons. In contrast to this, the subject of the following Chapter is a phonon-nontransparent impurity.

Following Kane and Fisher [1, 58], we consider two opposite limits in the impurity strength with regards to electron scattering: a weak electron backscatterer and a weak tunnelling link. Both limits allow for an analytically exact solution within the random phase approximation, meaning that when deriving later the renormalisation group flow equations we only take into account random phase approximation diagrams of all orders in electron-electron and

electron-phonon interaction. All other diagrams may be nonzero but are marginally smaller since they do not contain logarithmically divergent terms. Random phase approximation is asymptotically exact in the limit $q_0/k_{\mathrm{F}} \to 0$, where q_0 is the maximum transferred momentum in electron-electron interaction [25]. We solve both the weak scatterer and weak link cases analytically by calculating the full polarisation operator of the system, which in the language of Feynman diagrams is a sum of any number of bare polarisation operators (derived in Appendix A) connected in a sequential order by the effective electron-electron interaction line. The full polarisation operator is then used to compute corresponding correlation functions which eventually leads to the calculation of the scaling dimension of the impurity as a perturbation. The scaling dimension governs the direction of the renormalisation group flows. We call it the brute force approach.

In addition to the brute force method which is applicable to all formulations of the single impurity problem discussed in Section 3.4, the weak scatterer limit considered in this Chapter can also be solved in a more elegant way by using the functional bosonisation technique from Section 3.3 (see also Refs. [46, 47, 48]). In each Section of this Chapter we have a goal of deriving the renormalisation group equation and, in particular, focus on the coefficients in those equations which govern the direction of renormalisation group flows. Once we have calculated the flow equations for both the weak scatterer and weak link limits, we proceed to their discussion in the final Section of this Chapter.

4.1   Weak Scatterer

First, we provide the solution of the weak scatterer problem based on the functional bosonisation method. For the phonon-transparent impurity this technique is only applicable in the weak scatterer case for the following reason. At the very beginning of Section 3.3 when integrating out the phonon fields ϕ we assumed that the electron-electron and electron-

phonon interactions are translation invariant. We then introduced an electron scattering term which allows for a small probability of an incident electron to be backscattered by the impurity at $x = 0$. Since the impurity is phonon-transparent, its presence does not affect any of the interaction mechanism considered. On contrary, the weak link regime implies that although electrons on the opposite sides of the impurity are forbidden to interact directly since the interaction within the Luttinger Liquid model is short-ranged (on the scale of electron Fermi wavelength), the phonon-mediated interaction between them is still possible. This difference between the two types of electron-electron interactions, direct Coulomb and phonon-mediated, makes it nontrivial to treat them simultaneously by only introducing one pair of chiral bosonic fields $\theta_{R/L}$ like in Eq. (3.43).

4.1.1   Functional bosonisation solution

We begin with the effective action (3.42) and add the corresponding scattering term, Eq. (3.52). The gauge transformation, Eq. (3.43), replaces λ_0 in Eq. (3.52) with

$$\lambda(t) = \lambda_0 \, e^{i[\theta_L(t) - \theta_R(t)]} \,, \qquad \theta_\eta(t) \equiv \theta_\eta(x = 0, t) \,. \tag{4.1}$$

Integrating out all $x \neq 0$ fields leads to an action quadratic in $\theta_\eta(t)$ with the scattering term (4.1) as in the phononless problem [1, 58]. The only difference is that the local (in x) correlation function is governed by the composite bosonic modes, Eq. (3.48), with the retarded Fourier component

$$\langle \theta_\eta(-\omega) \, \theta_{\eta'}(\omega) \rangle^R = i \int \frac{dq}{2\pi} \, U^R_{\eta\eta'}(\omega, q) \equiv \frac{\pi \gamma_{\eta\eta'}}{\omega_+} \,, \tag{4.2}$$

where the correlation matrix $\gamma_{\eta\eta'}$ is found from the straightforward integration above and can be expressed as follows:

$$\gamma_{\mathrm{RR}} = \gamma_{\mathrm{LL}} = \frac{(\beta r + 1)(1 + K^2) - \alpha}{\sqrt{\beta^2 + 2\beta r + 1}} - 1\,, \tag{4.3}$$

$$\gamma_{\mathrm{RL}} = \gamma_{\mathrm{LR}} = \frac{(\beta r + 1)(1 - K^2) - \alpha}{\sqrt{\beta^2 + 2\beta r + 1}}\,, \tag{4.4}$$

where $\beta \equiv v/c$ and $r \equiv \sqrt{1 - \alpha}$.

The renormalisation group analysis for the weak scattering case is exactly the same as that in Refs. [1, 58] (for more details on the procedure see Appendix D). The only change is the appearance of the dimensionless matrix $\gamma_{\eta\eta'}$ in Eq. (4.2).

The central point of many calculations in this thesis is to determine the scaling dimension of the problem, Δ. It is defined as follows:

$$\langle \theta_{\mathrm{R}}(-\omega)\,\theta_{\mathrm{R}}(\omega)\rangle^R - \langle \theta_{\mathrm{R}}(-\omega)\,\theta_{\mathrm{L}}(\omega)\rangle^R = \frac{\pi(\Delta - 1)}{\omega_+}\,. \tag{4.5}$$

We should note here that if the system was bosonised fully, as described on p. 26, then the definition would correspond to:

$$\langle \Theta_{\mathrm{R}}(-\omega)\,\Theta_{\mathrm{R}}(\omega)\rangle^R - \langle \Theta_{\mathrm{R}}(-\omega)\,\Theta_{\mathrm{L}}(\omega)\rangle^R = \frac{\pi\Delta}{\omega_+}\,. \tag{4.6}$$

In our case this yields:

$$\Delta_{\mathrm{WS1}} = K\frac{\beta + r^{-1}}{\sqrt{\beta^2 + 2\beta r + 1}}\,, \tag{4.7}$$

where the subscript '1' denotes the phonon-transparent impurity case, whereas the scaling dimensions in the next Chapter will have an index '2'.

One can see that in the absence of phonons ($\alpha = 0$) one recovers the standard expression $\Delta_{\mathrm{WS}} = K$ for the weak scatterer limit in a Luttinger Liquid. As a result of electron-phonon

coupling, the renormalisation group equation for the impurity strength λ acquires a different prefactor at energies below the Debye energy:

$$\partial_l \lambda(E) = \begin{cases} (1-K)\lambda(E), & E > \omega_{\mathrm{D}}, \\ (1-\Delta_{\mathrm{WS1}})\lambda(E), & E < \omega_{\mathrm{D}}. \end{cases} \tag{4.8}$$

Here E is a running cutoff and $l \equiv \ln E_0/E$ with $E_0 \sim \varepsilon_{\mathrm{F}}$ being the bandwidth.

4.1.2 Brute force method

Here we provide a straightforward calculation of the full polarisation operator in the random phase approximation, followed by the calculation of the scaling dimension of the problemcalculation of , which is our final goal in this Section. This calculation is analogous to the one provided in the previous Section. Its main advantage is that the same scheme can also be applied to the weak link case, however, not without some minor modifications.

The unperturbed polarisation operator (see Appendix A) is given by

$$\hat{\pi}^{-1}(\omega; q, q') = 2\pi v_{\mathrm{F}} \left(\frac{\omega \hat{\tau}_3}{v_{\mathrm{F}} q} - 1 \right) 2\pi \delta(q - q') \tag{4.9}$$

In the matrix notations (in R/L-space) the electron-electron interaction takes form (see Eq. (2.19)):

$$\hat{V}_0(q, q') = 2\pi v_{\mathrm{F}} \left[g_4 \hat{1} + g_2 \hat{\tau}_1 \right] 2\pi \delta(q - q'). \tag{4.10}$$

The matrices $\hat{1}, \hat{\tau}_{1/3}$ we use here are defined as follows:

$$\hat{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \hat{\tau}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \hat{\tau}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.11}$$

The electron-phonon interaction in the case of a phonon-transparent impurity is transla-

tion invariant and the same between electrons of both chiralities:

$$\hat{\mathcal{D}}(\omega; q, q') = \mathcal{D}_0(\omega, q) \left[\hat{1} + \hat{\tau}_1\right] 2\pi\delta(q - q'), \qquad (4.12)$$

where $\mathcal{D}_0$ is the free phonon propagator defined by its retarded component

$$\mathcal{D}_0^R(\omega, q) = \pi v_{\text{F}} \alpha_{\text{ph}} \frac{c^2 q^2}{\omega_+^2 - c^2 q^2}. \qquad (4.13)$$

In the random phase approximation the full polarisation operator is given by

$$\hat{\Pi}_0^{-1}(\omega; q, q') = \hat{\pi}^{-1}(\omega; q, q') - \hat{V}_0(q, q') - \hat{\mathcal{D}}(\omega; q, q')$$

$$= (2\pi)^2 v_{\text{F}} \left[\left(\frac{\omega\hat{\tau}_3}{v_{\text{F}} q} - (1 + \tilde{g}_4)\,\hat{1}\right) - \tilde{g}_2\hat{\tau}_1\right], \qquad (4.14)$$

where

$$\tilde{g}_2 \equiv g_2 + \frac{\mathcal{D}_0(\omega, q)}{2\pi v_{\text{F}}}, \tilde{g}_4 \equiv g_4 + \frac{\mathcal{D}_0(\omega, q)}{2\pi v_{\text{F}}}. \qquad (4.15)$$

From the Eq. (4.14) we find the polarisation operator by simple inversion (see Appendix C for the details of the procedure):

$$\hat{\Pi}_0(\omega; q, q') = \frac{q}{\omega^2 - V_q^2 q^2} \begin{pmatrix} \omega + v_{\text{F}} q(1 + \tilde{g}_4) & -v_{\text{F}} q\, \tilde{g}_2 \\ -v_{\text{F}} q\, \tilde{g}_2 & -\omega + v_{\text{F}} q(1 + \tilde{g}_4) \end{pmatrix} \delta(q - q'), \qquad (4.16)$$

where $V_q^2 \equiv v^2 \left[1 + K\mathcal{D}_0(\omega, q)/\pi v\right]$, and v and K are defined as in Eqs. (3.8), (3.10).

Having found the full polarisation operator, we can now proceed to calculating the scaling dimension of the problem, which will enter the renormalisation group equation for the impurity strength.

The correlation function matrix $\hat{U}(\omega; q, q')$ defined in Eq. (3.47) is related to the full

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polarisation operator as follows:

$$\hat{U}(\omega; q, q') = \frac{(2\pi)^2}{q\,q'} \hat{\tau}_3 \hat{\Pi}_0(\omega; q, q') \hat{\tau}_3\,. \tag{4.17}$$

Since we are only interested in local correlations, we need to integrate over all momenta:

$$\hat{U} = \int \frac{\mathrm{d}q\mathrm{d}q'}{(2\pi)^2} \hat{U}(\omega; q, q')\,, \tag{4.18}$$

which gives us the following retarded component of the local correlation matrix:

$$\hat{U}^R = 2\pi \int \frac{\mathrm{d}q}{2\pi} \frac{q}{\omega_+^2 - V_q^2 q^2} \begin{pmatrix} \omega + v_{\mathrm{F}} q (1 + \tilde{g}_4) & v_{\mathrm{F}} q\,\tilde{g}_2 \\ v_{\mathrm{F}} q\,\tilde{g}_2 & -\omega + v_{\mathrm{F}} q (1 + \tilde{g}_4) \end{pmatrix} \tag{4.19}$$

The definition of the scaling dimension, Eq. (4.5), can be rewritten as:

$$\Delta = \frac{\mathrm{i}\omega}{2\pi} \mathrm{tr} \left[(1 - \hat{\tau}_1)\,\hat{U} \right]\,, \tag{4.20}$$

which yields

$$\Delta_{\mathrm{WS1}} = K \frac{\beta + r^{-1}}{\sqrt{\beta^2 + 2\beta r + 1}}\,. \tag{4.21}$$

It is in perfect agreement with Eq. (4.7) and, consequently, leads to the same renormalisation group equation for the impurity strength λ as found in the previous Section, Eq. (4.8).

4.2   Weak Link

This Section is devoted to the opposite limit when the introduced local electron scattering potential is strong and effectively cuts the wire into two halves, meaning that electrons cannot penetrate through the impurity. This results in the electron-phonon liquid being split into two subsystems on opposite sides of the impurity. These semi-infinite subsystems can,

however, interact with each other via phonons.

4.2.1   Brute force method

In order to employ the method from Section 4.1.2 we find it convenient to perform the unfolding procedure described in detail in Appendix E and also in Refs. [59, 60]. The idea is to modify the system of right- and left-moving electrons on a half-axis into a translation invariant system of chiral electrons on a full axis, provided the boundary conditions are taken care of. This becomes possible if we assume that the electron wave function acquires no phase shift upon reflection. We then unfold the system to the right from the origin into right-movers, and electrons to the left from origin - into left-movers. This transformation corrupts the electron-electron and electron-phonon interactions in the following way (see Appendix E for details):

$$\hat{V}_0(q, q') = (2\pi)^2 v_{\mathrm{F}} \left[g_4 \delta(q - q') + g_2 \delta(q + q') \right] \hat{1} \, . \tag{4.22}$$

Notice that the off-diagonal elements are zero. This is due to the fact that the $x < 0$ and $x > 0$ subsystems of electrons (which do not interact between each other via local interaction) became electrons of opposite chiralities after the unfolding procedure. The phonon-mediated interaction between η and η' chiral modes in x-representation is given by

$$\mathcal{D}_{\eta\eta'}(x, x') = \mathcal{D}_0(\eta|x| - \eta'|x'|) \, . \tag{4.23}$$

We can rewrite it in the following form which is more suitable for the Fourier transformation:

$$\hat{\mathcal{D}}(x, x') = \left[\mathcal{D}_0(x - x') + \mathcal{D}_0(x + x') \right] \hat{1} + \mathcal{D}_0(|x| + |x'|)(\hat{\tau}_1 - \hat{1}) \, . \tag{4.24}$$

$$\mathcal{D}_0(|x| + |x'|) = \pi v_{\mathrm{F}} \alpha_{\mathrm{ph}} \frac{\omega}{2\mathrm{i}c} \mathrm{e}^{\mathrm{i}\frac{\omega}{c}(|x| + |x'|)} \, . \tag{4.25}$$

The fact that the argument of $\mathcal{D}_0(|x|+|x'|)$ is strongly positive allows us to rewrite the whole interaction term in q-representation as follows:

$$\hat{\mathcal{D}}(\omega; q, q') = 2\pi \left[\mathcal{D}_0(\omega, q)\delta(q - q') + \mathcal{D}_0(\omega, q)\delta(q + q') \right] \hat{1} + \widetilde{\mathcal{D}}(\omega; q, q')(\hat{\tau}_1 - \hat{1}). \tag{4.26}$$

$$\mathcal{D}_0(\omega, q)^R = \pi v_{\mathrm{F}} \alpha_{\mathrm{ph}} \frac{c^2 q^2}{\omega_+^2 - c^2 q^2}, \tag{4.27}$$

$$\widetilde{\mathcal{D}}^R(\omega; q, q') = \underbrace{2\pi v_{\mathrm{F}} \mathrm{i} \alpha_{\mathrm{ph}} \omega^3 c}_{B_1} \frac{1}{\Omega^2(q)\Omega^2(q')}, \tag{4.28}$$

where $\Omega^2(q) \equiv \omega_+^2 - c^2 q^2$.

Having performed the 'unfolding' trick and determined the corresponding modifications of the interactions, Eqs. (4.22), (4.24), we now have a translation invariant system of interacting chiral electrons, analogous to the one solved in Section 4.1.2. Therefore, we can repeat the same calculations just with different form of interactions.

We shall find the full polarisation operator (again, in the random phase approximation) in two steps. First, we find an 'intermediate' polarisation operator, $\hat{\Pi}_0$, which does not include the $\widetilde{\mathcal{D}}$ term:

$$\hat{\Pi}_0^{-1}(\omega; q, q') = \hat{\pi}^{-1}(\omega; q, q') - \hat{V}_0(q, q') - 2\pi \left[\mathcal{D}_0(\omega, q)\delta(q - q') + \mathcal{D}_0(\omega, q)\delta(q + q') \right] \hat{1}$$

$$= (2\pi)^2 v_{\mathrm{F}} \left[\left(\frac{\omega \hat{\tau}_3}{v_{\mathrm{F}} q} - (1 + \tilde{g}_4)\,\hat{1} \right) \delta(q - q') - \tilde{g}_2 \hat{\tau}_1 \delta(q + q') \right], \tag{4.29}$$

where we follow the notations from Section 4.1.2. After the inversion:

$$\hat{\Pi}_0(\omega; q, q') = \frac{q}{\omega^2 - V_q^2 q^2} \begin{pmatrix} [\omega + v_{\mathrm{F}} q(1 + \tilde{g}_4)]\,\delta(q - q') & -v_{\mathrm{F}} q\,\tilde{g}_2\,\delta(q + q') \\ -v_{\mathrm{F}} q\,\tilde{g}_2\,\delta(q + q') & [-\omega + v_{\mathrm{F}} q(1 + \tilde{g}_4)]\,\delta(q - q') \end{pmatrix}.$$

$$\tag{4.30}$$

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For the full polarisation operator we can write:

$$\hat{\Pi}^{-1}(\omega; q, q') = \hat{\Pi}_0^{-1}(\omega; q, q') - (\hat{\tau}_1 - \hat{1})\widetilde{\mathcal{D}}(\omega; q, q') \,. \tag{4.31}$$

which leads to the following integral equation:

$$\hat{\Pi}(q, q') = \hat{\Pi}_0(q, q') + \int \frac{\mathrm{d}q_1 \mathrm{d}q_2}{(2\pi)^2} \hat{\Pi}_0(q, q_1)(\hat{\tau}_1 - \hat{1})\widetilde{\mathcal{D}}(q_1, q_2)\hat{\Pi}(q_2, q') \,. \tag{4.32}$$

In what follows we shall omit the ω-dependence of operators when it is not essential, for simplicity of notations. It is quite remarkable that in the case of one-dimensional acoustic phonons the kernel $\widetilde{\mathcal{D}}(q_1, q_2)$ is separable in terms of variables q_1 and q_2, according to Eq. (4.28). It is only due to this fact that we are able to solve the integral equation (4.32). Below we briefly outline the solution.

After multiplying by $1/\Omega^2(q)$ and integrating over momenta q, we obtain:

$$\hat{\mathbb{C}}(q') = \hat{\mathbb{A}}(q') + B_1 \int \frac{\mathrm{d}q}{2\pi} \frac{1}{\Omega^2(q)} \int \frac{\mathrm{d}q_1}{2\pi} \hat{\mathbb{A}}(q)(\hat{\tau}_1 - \hat{1}) \int \frac{\mathrm{d}q_2}{2\pi} \hat{\mathbb{C}}(q) \,, \tag{4.33}$$

where we have introduced the following notations:

$$\hat{\mathbb{A}}(q') \equiv \int \frac{\mathrm{d}q}{2\pi} \frac{\hat{\Pi}_0(q, q')}{\Omega^2(q)} \,, \tag{4.34}$$

$$\hat{\mathbb{C}}(q') \equiv \int \frac{\mathrm{d}q}{2\pi} \frac{\hat{\Pi}(q, q')}{\Omega^2(q)} \,. \tag{4.35}$$

Eq. (4.33) allows us to express $\hat{\mathbb{C}}(q')$ in terms of $\hat{\mathbb{A}}(q')$ and then substitute it back into Eq. (4.32), thus obtaining the full polarisation operator in random phase approximation:

$$\hat{\Pi}(q, q') = \hat{\Pi}_0(q, q') + B_1 \hat{\mathbb{A}}(q)(\hat{\tau}_1 - \hat{1}) \left[\hat{1} - B_1 \int \frac{\mathrm{d}q}{2\pi} \frac{\hat{\mathbb{A}}(q)}{\Omega^2(q)}(\hat{\tau}_1 - \hat{1}) \right]^{-1} \hat{\mathbb{A}}(q') \,. \tag{4.36}$$

Following then the same steps and notation as in Section 4.1.2 (Eqs. 4.17 – 4.20), we arrive at the final expression for the scaling dimension:

$$\Delta_{\text{WL1}} = \frac{1}{K} \frac{\beta + r}{\sqrt{\beta^2 + 2\beta r + 1}} - \frac{1}{K} \frac{\frac{\alpha\beta}{1+\beta r}}{\sqrt{\beta^2 + 2\beta r + 1}} \,, \tag{4.37}$$

where the first term comes from the 'intermediate' polarisation operator $\hat{\Pi}_0$. This result can be rewritten as:

$$\Delta_{\text{WL1}} = \frac{1}{K} \frac{\sqrt{\beta^2 + 2\beta r + 1}}{\beta + r^{-1}} \,. \tag{4.38}$$

The renormalisation group analysis of the tunnelling term, Eq. (3.53), leads to the equation which is again almost the same as in the phononless case but with a different scaling dimension of the problem at $E < \omega_{\text{D}}$:

$$\partial_l t(E) = \begin{cases} (1 - K^{-1}) \, t(E)\,, & E > \omega_{\text{D}}\,, \\[2mm] (1 - \Delta_{\text{WL1}}) \, t(E)\,, & E < \omega_{\text{D}}\,. \end{cases} \tag{4.39}$$

4.3 Discussion

In the previous Sections of this Chapter we have derived perturbative renormalisation group equations, Eqs. (4.8) and (4.39), for the weak impurity strength $\lambda(\varepsilon)$ and small tunnelling amplitude $t(\varepsilon)$ under the assumption that in both cases the electron scattering impurity does not break translational invariance for phonons. The equations should be solved with the initial conditions:

$$\lambda(E = E_0) = \lambda_0\,, \qquad t(E = E_0) = t_0\,. \tag{4.40}$$

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The solutions for $\varepsilon < \omega_{\mathrm{D}}$ are given by:

$$\frac{\lambda(\varepsilon)}{\lambda_0} = \left(\frac{\varepsilon}{\omega_{\mathrm{D}}}\right)^{\Delta_{\mathrm{WS1}}-1} \left(\frac{\omega_{\mathrm{D}}}{E_0}\right)^{K-1}, \tag{4.41}$$

$$\frac{t(\varepsilon)}{t_0} = \left(\frac{\varepsilon}{\omega_{\mathrm{D}}}\right)^{\Delta_{\mathrm{WL1}}-1} \left(\frac{\omega_{\mathrm{D}}}{E_0}\right)^{\frac{1}{K}-1}. \tag{4.42}$$

The scaling behaviour is solely determined by the values of the scaling dimensions, Δ_{WS} and Δ_{WL}. We drop the index '1' in Δ's in the remainder of this Section, since what follows will also apply to the case of a phonon-nontransparent impurity considered in Chapter 5 (even though the scaling dimensions will be given by different expressions, the duality discussed below will hold). If $\Delta > 1$ then the analysed perturbation is irrelevant at low energies (temperatures). In the opposite case, when $\Delta > 1$, it becomes relevant, which is indicated by the divergence of the corresponding coupling constant at $\varepsilon \to 0$. The renormalisation group analysis (also called a 'poor man's scaling') presented here provides only a qualitative prediction, which is, however, enough to build a general understanding of the system in question and draw its phase diagram.

Let us analyse the derived scaling dimensions in the weak and strong scattering limits, Eqs. (4.21) and (4.38). It is important to notice that

$$\Delta_{\mathrm{WS}}\Delta_{\mathrm{WL}} = 1, \tag{4.43}$$

like in the phononless Luttinger Liquid. The duality between the weak and strong scattering limits in the Luttinger Liquid with instantaneous short-ranged interaction (see Section 3.2) has been known since the work [61]. However, it comes as a big surprise to us that it extends even to retarded non-local interactions, i.e. way beyond the standard Tomonaga-Luttinger model. We have seen that the nonlocal interaction terms are largely affected by the unfolding, therefore making any connection between the limits not trivial, let alone the duality. We
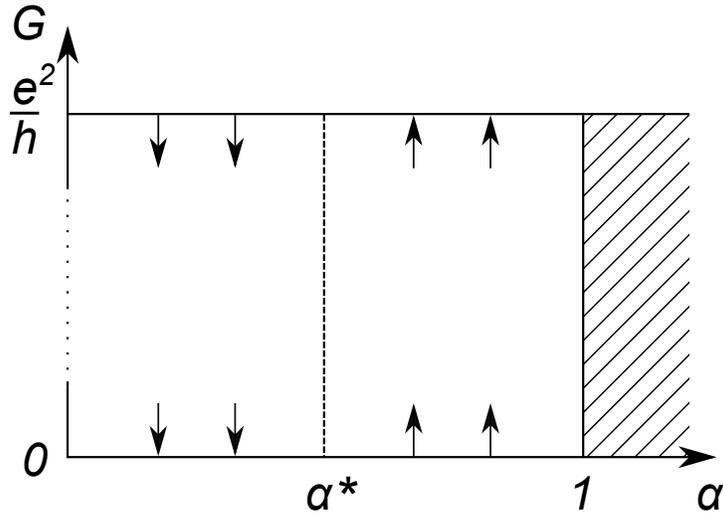
Figure 4.1: Renormalisation group flows of uncorrelated electron and phonon scattering from impurity. A transition between an insulator $(G \to 0)$ and ideal metal $(G \to e^2/h)$ occurs at α^*, which depends on phonon parameters.

expect there to be some underlying intrinsic symmetry which ensures this relation.

The duality, Eq. (4.43), guarantees that at any fixed parameter of the system (instantaneous electron-electron interaction, strength of electron-phonon coupling, Fermi and sound velocities) either weak scattering or weak tunneling is always irrelevant, with the other one being relevant at the same time. Exactly the same holds in the phononless case, except that now the transition from between metallic and insulating behaviour occurs not at $K = 1$ but rather depends on the parameters listed above, see Fig. 4.1. The hatched area is the region of parameters which must be avoided due to the Wentzel-Bardeen instability. As one can see, when effective electron-electron interaction is repulsive, $\alpha < \alpha^*$, the system is in the insulating state. Regardless of how weak (strong) the introduced scatterer is, the renormalisation group analysis proves that the effective scattering (tunneling) amplitude is renormalised to infinity (zero) at low-energies, which suppresses electron conductance. Exactly the opposite is true for stronger electron-phonon coupling, $\alpha^* < \alpha < 1$, when the effective electron-electron interaction changes sign and becomes attractive. The values of α^* depend on the phonon

transparency of the impurity and, generally, can only be calculated numerically. Analytic expressions are provided in Chapter 6 for the limit of weak interactions.

CHAPTER 5

PHONON-NONTRANSPARENT IMPURITY

This Chapter is devoted to impurities which are impenetrable for soft low-energy one-dimensional acoustic phonons. We again consider two opposite limits in electron scattering amplitude, weak scatterer and weak link. Naturally, it will be assumed that electrons cannot interact with each other when they are on different sides of the impurity because neither direct short-ranged interaction nor phonon-mediated one allows for this.

Both limits in this Chapter are treated by the same brute force method which we used in the previous Chapter. It turns out that we can also successfully apply the functional boson-isation description of electron-phonon liquid in the weak link case after a slight modification of the system geometry (unfolding).

It is important to notice that because phonons are now reflected back at the origin (with no change in phase - for simplicity), the phonon-mediated part of the interaction in such a system has two contributions. Electrons on one side of the impurity can not only interact via an exchange of direct but also through reflected phonons. Due to this fact the phonon propagator can be written as:

$$\mathcal{D}(x, x'; t, t') = [\mathcal{D}_0(x - x'; t, t') + \mathcal{D}_0(x + x'; t, t')]\, \Theta(x\, x')\,, \tag{5.1}$$

where $\Theta(x)$ is the Heaviside step function, which is introduced in order to allow phonon-

mediated interaction only between electrons on one side of the impurity.

5.1 Weak Scatterer

The weak scatter limit considered in this Section differs from the one solved in previous Chapter only the the form of phonon-mediated electron-electron interaction. Because we assume complete reflection of phonons off the impurity, for exactly the same reason as in Section 4.2 we only provide calculations using the brute force method. The use of functional bosonisation (in the form developed earlier) is complicated by the fact that electrons can propagate freely through the system, with only a small backscattering probability at $x = 0$, whereas phonons are restricted to a half-line. Therefore, the description of translation invariant interacting electron-phonon system does not allow for a simple adaptation.

5.1.1 Brute force method

The polarisation bubble and direct electron-electron interaction term have the same form as in Eqs. (4.9), (4.10). The phonon-mediated electron-electron interaction, Eq. (5.1) can be rewritten as follows:

$$\hat{\mathcal{D}}(x, x') = \left[\mathcal{D}_0(x - x') + \widetilde{\mathcal{D}}(x, x')\right] (\hat{1} + \hat{\tau}_1),\qquad(5.2)$$

which allows for a simpler Fourier transform. Here we used the following notations:

$$\widetilde{\mathcal{D}}(x, x') = \Theta(x\,x')\mathcal{D}_0(x + x') - \Theta(-x\,x')\mathcal{D}_0(x - x'),\qquad(5.3)$$

$$\widetilde{\mathcal{D}}^R(\omega; q, q') = -\underbrace{2\pi\mathrm{i}\omega\alpha_{\mathrm{ph}}v_{\mathrm{F}}c^3}_{B_2}\frac{qq'}{\Omega^2(q)\Omega^2(q')}.\qquad(5.4)$$

As in Section 4.2.1, we shall calculate the full polarisation operator (in the random phase approximation) in two steps. First, we find an intermediate polarisation operator $\hat{\Pi}_0$ which does not include the $\widetilde{\mathcal{D}}(q, q')$ part of the interaction. This $\hat{\Pi}_0$ is exactly the same as the full polarisation operator for the weak scattering problem with a phonon-transparent impurity, Eq. (4.16):

$$\hat{\Pi}_0(\omega; q, q') = \frac{q}{\omega^2 - V_q^2 q^2} \begin{pmatrix} \omega + v_{\mathrm{F}} q(1 + \tilde{g}_4) & -v_{\mathrm{F}} q\, \tilde{g}_2 \\ -v_{\mathrm{F}} q\, \tilde{g}_2 & -\omega + v_{\mathrm{F}} q(1 + \tilde{g}_4) \end{pmatrix} \delta(q - q'). \tag{5.5}$$

The second step now is to include the $\widetilde{D}$ term which leads to a similar integral equation for $\hat{\Pi}(q, q')$ as in Section 4.2.1:

$$\hat{\Pi}(q, q') = \hat{\Pi}_0(q, q') + \int \frac{\mathrm{d}q_1 \mathrm{d}q_2}{(2\pi)^2} \hat{\Pi}_0(q, q_1)(\hat{1} + \hat{\tau}_1)\widetilde{\mathcal{D}}(q_1, q_2)\hat{\Pi}(q_2, q'). \tag{5.6}$$

Since the kernel $\widetilde{\mathcal{D}}(q_1, q_2)$ is separable in terms of variables q_1 and q_2, we can solve the Eq. (5.6) in the similar fashion as in Section 4.2.1. For the full polarisation operator we obtain the following expression:

$$\hat{\Pi}(q, q') = \hat{\Pi}_0(q, q') - B_2\, \hat{\mathbb{A}}(q)(\hat{1} + \hat{\tau}_1) \left[\hat{1} + B_2 \int \frac{\mathrm{d}q}{2\pi} \frac{q}{\Omega^2(q)} \hat{\mathbb{A}}(q)(\hat{1} + \hat{\tau}_1) \right]^{-1} \hat{\mathbb{A}}(q'), \tag{5.7}$$

where

$$\hat{\mathbb{A}}(q) \equiv \int \frac{\mathrm{d}q'}{2\pi} \frac{q'}{\Omega^2(q')} \hat{\Pi}_0(q, q'). \tag{5.8}$$

Again, following the same steps and notation as in Section 4.1.2 (Eqs. 4.17 − 4.20), for the scaling dimension we get the following expression:

$$\Delta_{\mathrm{WS2}} = K \frac{\beta + r^{-1}}{\sqrt{\beta^2 + 2\beta r + 1}} - K \frac{\frac{\alpha\beta}{r(\beta + r)}}{\sqrt{\beta^2 + 2\beta r + 1}} = K \frac{\sqrt{\beta^2 + 2\beta r + 1}}{\beta + r}, \tag{5.9}$$

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where the first term comes from the intermediate polarisation operator $\hat{\Pi}_0$.

5.2   Weak Link

The presence of a strong phonon-nontransparent impurity at the origin effectively cuts the whole system into two uncoupled electron-phonon liquids. As it has been mentioned before, it is possible to solve this case by applying the functional bosonisation technique after a certain change in the system's geometry. The transformation required [59, 60] modifies a system of right- and left-moving particles on a half-axis into a system of chiral particles living on a full axis. This procedure is possible because wave functions of the initial movers in the system (we first consider the $x > 0$ system) are not independent but, in fact, are related via the boundary condition $\psi_R(x, t) = \psi_L(-x, t)$, where in order not to overcomplicate the problem we have assumed that the wave functions acquire a zero phase shift upon reflection. Hence, we can just unfold our $x > 0$ system and end up with interacting right-moving electrons on a full axis. The same procedure can be applied to the $x < 0$ system transforming it into a system of left-movers living on entire axis, see Appendix E. In the end, our initial system turns into a system of right- and left-moving species with interaction only between electrons of the same chirality.

5.2.1   Functional bosonisation solution

The weak link action term to be analysed is given by Eq. (3.53). We perform the unfolding transformation and introduce the weak link fields $\theta_\eta^{\mathrm{wl}}(x, t)$ as follows:

$$\theta_R^{\mathrm{wl}}(x, t) = \begin{cases} \theta_R(x, t), & x < 0; \\ \theta_L(-x, t), & x > 0, \end{cases} \qquad \theta_L^{\mathrm{wl}}(x, t) = \begin{cases} \theta_R(-x, t), & x > 0; \\ \theta_L(x, t), & x < 0. \end{cases} \qquad (5.10)$$

In this Section we prefer to work in (ω, q)-representation. For the sake of simplicity in this Section we assume that $g_4 = g_2 = V_0/2\pi v_{\mathrm{F}}$. In the following Section where we treat the same problem with a brute force approach and this assumption will be lifted. After the described unfolding the screened effective electron-electron interaction, $V \equiv \left[(V_0 + \mathcal{D}_0)^{-1} - \pi\right]^{-1}$, becomes:

$$V_{\mathrm{wl}}(\omega; q, q') = V(\omega, q) \, 2\pi \left[\delta(q + q') + \delta(q - q')\right]. \tag{5.11}$$

Notice that we no longer require matrix notations here, since the interaction connects electrons of the same chirality only. The correlation function of the new weak link fields, $iU_{\mathrm{wl}} = \langle\theta_\eta^{\mathrm{wl}}\theta_\eta^{\mathrm{wl}}\rangle$, can be now written as:

$$\begin{aligned}
U_{\mathrm{wl}}^R(\omega; q, q') &= g_{\mathrm{R}}^R(\omega, q) g_{\mathrm{R}}^R(\omega, q') V_{\mathrm{wl}}^R(\omega; q, q') \\
&= 2\pi \left[U_{\mathrm{RR}}^R(\omega, q)\delta(q + q') + U_{\mathrm{RL}}^R(\omega, q)\delta(q - q')\right].
\end{aligned} \tag{5.12}$$

We note that V_{wl} is the interaction between only electrons of the same chirality. Electrons of opposite chiralities represent electrons form the opposite sides of the impurity in the original problem (before unfolding), which could not interact between each other by construction. The U_{RR} and U_{RL} correlators were defined in Eq. (3.48).

The tunnelling action, Eq. (3.53), upon unfolding turns into the weak scattering action, Eq. (3.52), with t_0 playing the role of a bare impurity strength λ_0:

$$S_{\mathrm{wl}} = t_0 \int dt \, \bar{\psi}_{\mathrm{R}}^{\mathrm{wl}} \psi_{\mathrm{L}}^{\mathrm{wl}} + \mathrm{c.c.}, \qquad \psi_\eta^{\mathrm{wl}} = \psi_\eta e^{i\theta_\eta^{\mathrm{wl}}}. \tag{5.13}$$

After this point, the renormalisation group analysis of the tunnelling term is equivalent

to the one described in Section 4.1.1, except that the scaling exponent changes from Δ_{WS1},

$$i \int \frac{\mathrm{d}q}{2\pi} \left[U^R_{\mathrm{RR}}(\omega, q) - U^R_{\mathrm{RL}}(\omega, q) \right] \equiv \frac{\pi \Delta_{\mathrm{WS1}}}{\omega_+} \tag{5.14}$$

to Δ_{WL2}, defined by a similar equation:

$$i \int \frac{\mathrm{d}q}{2\pi} \left[U^R_{\mathrm{RR}}(\omega, q) + U^R_{\mathrm{RL}}(\omega, q) \right] \equiv \frac{\pi \Delta_{\mathrm{WL2}}}{\omega_+} , \tag{5.15}$$

which is the consequence of a different form of interaction, Eq. (5.11).

After simple algebraic manipulations, one obtains the following result for Δ_{WL2}:

$$\Delta_{\mathrm{WL2}} = \frac{1}{K} \frac{\beta + r}{\sqrt{\beta^2 + 2\beta r + 1}} . \tag{5.16}$$

The corresponding renormalisation group equation reads:

$$\partial_l t(E) = \begin{cases} (1 - K^{-1}) \, t(E), & E > \omega_{\mathrm{D}} , \\ \\ (1 - \Delta_{\mathrm{WL2}}) \, t(E), & E < \omega_{\mathrm{D}} . \end{cases} \tag{5.17}$$

5.2.2   Brute force method

The direct calculation of the total polarisation in the random phase approximation after unfolding in this case is very straightforward. The direct electron-electron interaction can be written as in Eq. (4.22):

$$\hat{V}_0(q, q') = (2\pi)^2 v_{\mathrm{F}} \left[g_4 \delta(q - q') + g_2 \delta(q + q') \right] \hat{1} , \tag{5.18}$$

whereas the phonon-mediated interaction, also only between electrons of the same chirality, takes the following form:

$$\hat{\mathcal{D}}(x, x') = [\mathcal{D}_0(x - x') + \mathcal{D}_0(x + x')] \,\hat{1}\,, \tag{5.19}$$

$$\hat{\mathcal{D}}(\omega; q, q') = 2\pi \left[\mathcal{D}_0(\omega, q)\delta(q - q') + \mathcal{D}_0(\omega, q)\delta(q + q')\right] \hat{1}\,. \tag{5.20}$$

Notice that it coincides with the phonon-mediated interaction in Section 4.2.1, except without the $\widetilde{\mathcal{D}}(q, q')$ part. This allows us to write the expression for the scaling dimension straight away by just taking the first part of Eq. (4.37):

$$\Delta_{\text{WL2}} = \frac{1}{K} \frac{\beta + r}{\sqrt{\beta^2 + 2\beta r + 1}}\,. \tag{5.21}$$

We again notice that, as for the phonon-transparent impurity in Chapter 4, that

$$\Delta_{\text{WS}}\Delta_{\text{WL}} = 1\,. \tag{5.22}$$

The discussion of this remarkable property, as well as the phase diagram, provided in Section 4.3 are, therefore, fully applicable to this situation.

CHAPTER 6

CORRELATED ELECTRON AND PHONON SCATTERING

So far when considering electron and phonon scattering off a single impurity we assumed them to be uncorrelated. However, in certain experimental situations this may not be the case. It is possible that phonon scattering properties also change in the crossover between the weak scatterer and weak link limits. For instance, there has been a numerical study [24] of a carbon nanotube which was a subject to structural deformations. It has been found that local bending of a nanotube past a certain limit has a dramatic effect on its electric properties. In particular, as a result of the excessive stress in its lattice structure upon bending, the nanotube develops a kink (see Fig. 6.1) which results in strong mixing of σ- and π-orbitals, which leads to a sudden drop in conductance. Therefore, we suggest that this local deformation (controlled by the bending angle) can be regarded as a single impurity with a tunable strength.

The authors of [24], however, did not include in their model any mechanical vibrations which are normally present in real systems. We have already established in previous Chapters that electron coupling to low-energy acoustic vibrations affects electron transport properties. In carbon nanotubes longitudinal stretching modes can be described in terms of acoustic phonons. The effects of transverse twisting modes, as well as the retardation effects due to
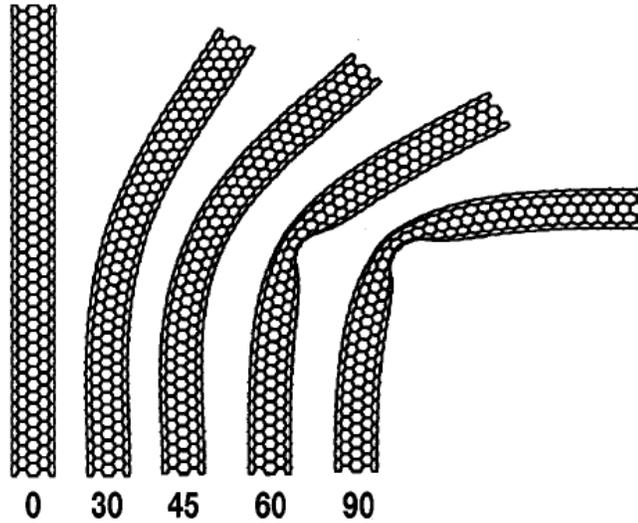
Figure 6.1: Bending of a carbon nanotube at angles between 0 and 90 [24]. A kink in lattice structure is formed at high bending angles, which can reflect electrons and phonons.

breathing modes, can be neglected at low energies [62]. We argue that defects in nanotube's atomic structure would also affect propagation of phonons. In particular, at small bending angles we can assume that the defect is transparent for phonons. Whereas, the formation of a kink can lead to partial or even complete reflection of low-energy acoustic phonons. These assumptions help us create a model which connects the weak electron scattering limit in presence of unperturbed one-dimensional phonons (Section 4.1) with the limit of strong electron scattering off a phonon-nontransparent impurity from Section 5.2.

Since electron and phonon scattering amplitudes are now changed simultaneously, the weak and strong scattering limits are no longer dual. Indeed, the duality relation, Eq. (4.43), observed in Chapters 4 and 5 relied on the fact that all parameters of the system (strengths of short-ranged electron-electron and electron-phonon interactions, ratio of Fermi and sound velocities, phonon reflection amplitude) stay fixed in the crossover between the weak scatterer and weak link limits. Now, however, this is not the case since the phonon reflection amplitude is correlated with the electron backscattering amplitude. This fact changes system's behaviour qualitatively.

The scaling dimensions in the two opposite limits are now given by:

$$\Delta_{\mathrm{WS}} = K \frac{\beta + r^{-1}}{\sqrt{\beta^2 + 2\beta r + 1}} \,, \tag{6.1}$$

$$\Delta_{\mathrm{WL}} = \frac{1}{K} \frac{\beta + r}{\sqrt{\beta^2 + 2\beta r + 1}} \,. \tag{6.2}$$

It can be shown analytically that three principally different situations are possible:

1. $\Delta_{\mathrm{WS}} < 1$ and $\Delta_{\mathrm{WL}} > 1$;

2. $\Delta_{\mathrm{WS}} > 1$ and $\Delta_{\mathrm{WL}} < 1$;

3. $\Delta_{\mathrm{WS}} > 1$ and $\Delta_{\mathrm{WL}} > 1$.

The appearance of the third possibility is the main result of this Chapter. We have proved in Chapters 4 and 5 that due the duality between the weak and strong scattering regimes, one can expect either the first or the second situation from the ones listed above. The former makes the system under consideration an insulator, whereas the later is responsible for metallic behaviour. The situation when both Δ's are greater than 1 means that both weak and strong scatterers are irrelevant. Therefore, on the phase diagram at some intermediate impurity strength there should be an unstable fixed point as depicted in Fig. 6.2(a). The appearance of the Wentzel-Bardeen instability in the system at $\alpha > 1$, in fact, can make the region of parameters responsible for the metallic behaviour inaccessible, see Fig. 6.2(b).

The parameters α_1 and α_2 on Fig. 6.2 determine the points where the corresponding renormalisation groups flows change their direction to the opposite. They are the solutions of the following equations:

$$\Delta_{\mathrm{WS}}(\alpha_1) = 1\,, \qquad \Delta_{\mathrm{WL}}(\alpha_2) = 1\,. \tag{6.3}$$

These equations do not allow for a simple analytic solution, which, nevertheless, can be found

numerically. In the limit of weak electron-phonon coupling, $\alpha \ll 1$, one obtains the following expansions:

$$\Delta_{\text{WS}} = K \left[1 + \frac{1 + 2\beta}{2(1 + \beta)^2} \alpha \right] + O(\alpha^2) , \tag{6.4}$$

$$\Delta_{\text{WL}} = \frac{1}{K} \left[1 - \frac{K^2}{2(1 + \beta)^2} \alpha \right] + O(\alpha^2) , \tag{6.5}$$

which yield for the solutions of the Eqs. (6.3):

$$\alpha_1 \simeq \frac{2(1 - K)(1 + \beta)^2}{K(1 + 2\beta)^2} , \tag{6.6}$$

$$\alpha_2 \simeq \frac{2(1 - K)(1 + \beta)^2}{K^2} . \tag{6.7}$$

Notice that $\alpha_1 < \alpha_2$, which is, in fact, a general statement for arbitrary coupling constants, according to our analysis. In this particular case, the initial assumption that $\alpha \ll 1$ also means that $(1 - K) \ll K$, i.e., the solutions (6.6)–(6.7) require both electron-electron and electron-phonon interaction to be weak. This, of course, is expected, since the Eqs. Eqs provide a condition of the exact balance between the forces of direct electron-electron repulsion and phonon-mediated electron-electron attraction.

The fact that $\alpha_1 < \alpha_2$ is a simple consequence of the following two observations: monotonic dependence of both Δ_{WS} and Δ_{WL} on α and that $\Delta_{\text{WS}} > 1/\Delta_{\text{WL}}$ (see Eq. (5.9), where $\Delta_{\text{WS}} \equiv \Delta_{\text{WS1}}$ and $\Delta_{\text{WL}} \equiv \Delta_{\text{WL2}}$). As a result, the intermediate fixed point on Fig. 6.2 is always unstable.
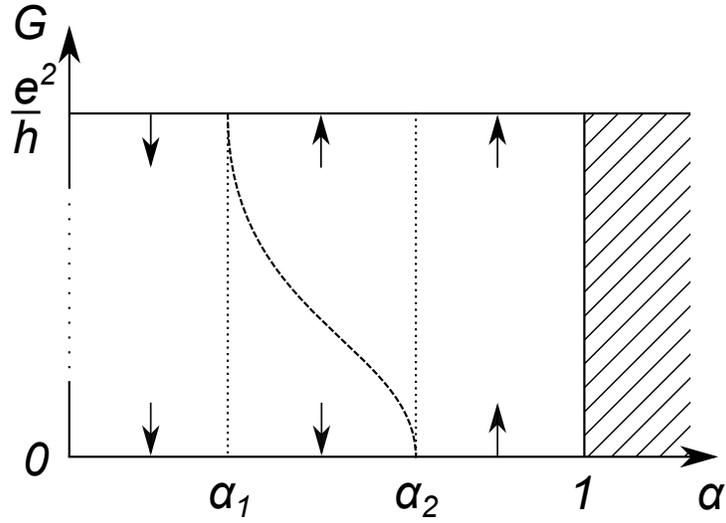
(a)
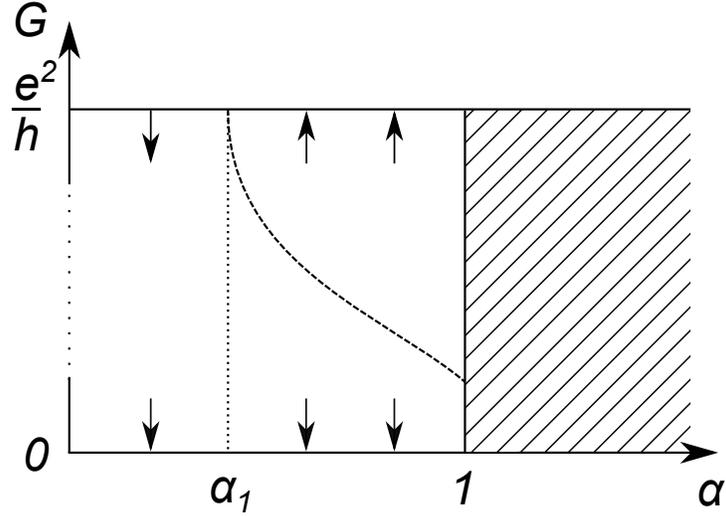
(b)

Figure 6.2: Renormalisation group flows for different α when electron and phonon scattering from the impurity are correlated. We assume that both are almost fully transmitted through the impurity in the weak scattering limit and almost fully backscattered in the weak link one; in this case Δ_{ws} and Δ_{wL} equal to 1 at different values of α, say $\alpha_{1,2}$. For $\alpha_1 < \alpha < \min\{\alpha_2, 1\}$ the insulator $(G = 0)$ and metal $(G = e^2/h)$ fixed points are both stable so that an unstable fixed point should exist at finite $G < e^2/h$ corresponding to a metal-insulator transition at some intermediate value of the bare backscattering. Depending on a value of $\beta \equiv v/c$, such a line might (a) end at $G = 0$ dividing the phase diagram in the regions of insulator, $\alpha \leq \alpha_1$, metal, $\alpha_2 \leq \alpha < 1$, or metal-insulator transition, $\alpha_1 < \alpha < \alpha_2$; or (b) end at the Wentzel-Bardeen instability line, $\alpha = 1$, in which case the purely metallic region is absent.

CHAPTER 7

POTENTIAL SCATTERING IN WEAKLY INTERACTING ELECTRON-PHONON LIQUID

This Chapter is devoted to the problem of weakly interacting electrons scattering from a quantum impurity of arbitrary strength. Here, as before, we consider a system spinless interacting electrons coupled to a system of one-dimensional acoustic phonons. The electron-electron and electron-phonon interactions are considered to be small ($1 - K \ll 1$, $\alpha_{\mathrm{ph}} \ll 1$) which allows us to employ the perturbation theory in both interaction potentials. Our main goal is to calculate the first order correction to the transmission amplitude through the impurity. We show how electron scattering parameters are renormalised by the interactions, thereby solving the problem considered previously in Ref. [6] for a more general type of interaction. We should stress here that the impurity strength is arbitrary not only for electrons but also for phonons. The renormalisation procedure used here is the same poor man's scaling which is used throughout the thesis (see Appendix D for its description and derivation).

The focus of this Chapter is on the modification of the result for spinless electrons presented in Ref. [6] by the presence of electron-phonon coupling. Following the aforementioned authors, in this Chapter we do not require that $V(q \simeq 2k_{\mathrm{F}}) \ll V(q \simeq 0)$ for electron-electron interaction, i.e. we can include local electron-electron backscattering term explicitly. In accordance with the notations of previous Chapters, we define $V(q = 2k_{\mathrm{F}}) \equiv 2\pi g_1$. The direct

short-ranged (on the scale of k_F^{-1}) electron-electron interaction is effectively local, unlike the retarded phonon-mediated one. We still assume that the temperature is low enough such that no electron-phonon backscattering is possible, since in this case there can be no phonons in the system with large enough momenta to change the direction of electrons.

7.1 Corrections in the phononsless problem

We begin by deriving the first order corrections to reflection and transmission coefficients due to local electron-electron interaction. In presence of a single impurity placed at $x = 0$ the full electron action can be written as:

$$S = S_0 + S_{\text{int}} \,, \tag{7.1}$$

$$S_0 = \sum_{\eta,\eta'=\pm 1} \int \mathrm{d}x\mathrm{d}t \, \bar{\psi}_\eta(x,t) G^{-1}_{0\,\eta\eta'}(x,t) \psi_{\eta'}(x,t) \,, \tag{7.2}$$

$$S_{\text{int}} = -\pi v_F \sum_{\eta=\pm 1} \int \mathrm{d}x\mathrm{d}t \, \Big[g_4 n_\eta^2(x,t) + g_2 n_\eta(x,t) n_{-\eta}(x,t)$$

$$+ g_1 \bar{\psi}_\eta(x,t) \psi_{-\eta}(x,t) \bar{\psi}_{-\eta}(x,t) \psi_\eta(x,t) \Big] \,, \tag{7.3}$$

where

$$\hat{G}_0(\varepsilon; x, x') = \hat{g}(\varepsilon; x - x') + \mathrm{i} v_F \hat{g}(\varepsilon; x, 0) \hat{T} \, \hat{g}(\varepsilon; 0, x') \,. \tag{7.4}$$

The three sectors of low-energy electron-electron interactions in one dimension are depicted in Fig. 7.1.

The T-matrix in Eq. (7.4) is related to the scattering matrix S by $\hat{T} \equiv \hat{S} - \hat{1}$ with the S-matrix defined as:

$$\hat{S} = \begin{pmatrix} t_0 & r_0 \\ r_0 & t_0 \end{pmatrix} \,. \tag{7.5}$$
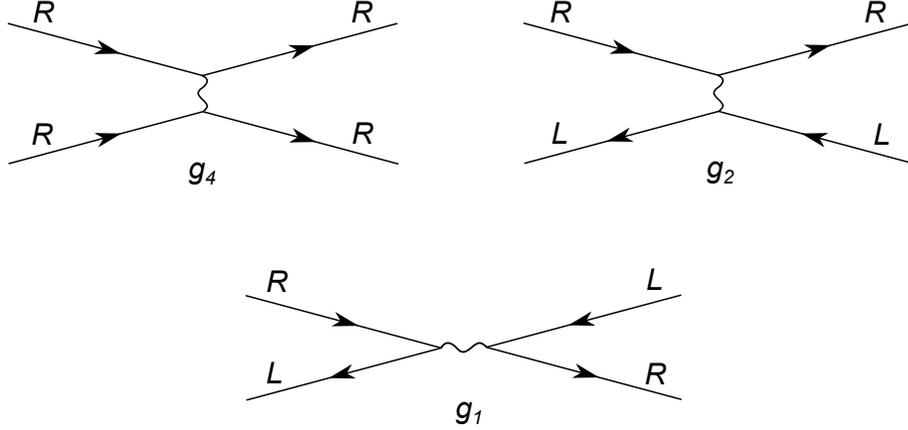
Figure 7.1: Possible electron-electron scattering sectors (spin-irrelevant): g_4 corresponds to the forward scattering ($q \sim 0$) of electrons of the same chirality, g_2 - forward scattering ($q \sim 0$) of electrons of opposite chirality, g_1 - electron-electron backscattering ($q \sim 2k_\text{F}$). Right(left) - moving electrons are denoted by corresponding indices.

We assume that the transmission and reflection amplitudes for electrons of each chirality are the equal. In the first order perturbation theory we calculate all possible pair-correlations in the interaction term (7.3). This gives the following self-energy matrix:

$$
\hat{\Sigma} \equiv \begin{pmatrix} \Sigma_{\text{RR}} & \Sigma_{\text{RL}} \\ \Sigma_{\text{LR}} & \Sigma_{\text{LL}} \end{pmatrix} = 2\pi v_\text{F}(g_2 - g_1) \begin{pmatrix} \langle \bar{\psi}_\text{L}\psi_\text{L} \rangle & \langle \psi_\text{R}\bar{\psi}_\text{L} \rangle \\ \langle \psi_\text{L}\bar{\psi}_\text{R} \rangle & \langle \bar{\psi}_\text{R}\psi_\text{R} \rangle \end{pmatrix}
$$
$$
= 2\pi v_\text{F}(g_2 - g_1) \begin{pmatrix} -iG^<_{0\,\text{LL}} & iG^<_{0\,\text{RL}} \\ iG^<_{0\,\text{LR}} & -iG^<_{0\,\text{RR}} \end{pmatrix}. \tag{7.6}
$$

Here we employ the Keldysh formalism notations from Section 2.1.

Using the fact that $G^<(\varepsilon; x, x') = -n_\text{F}(\varepsilon)\Delta G(\varepsilon; x, x')$, where $\Delta G \equiv G^R - G^A$, we get the following expression for the self-energy matrix:

$$
\hat{\Sigma}(x) = iv_\text{F}(g_2 - g_1) \int d\varepsilon'\, n_\text{F}(\varepsilon') \begin{pmatrix} \Delta G_{0\,\text{LL}}(\varepsilon'; x, x) & -\Delta G_{0\,\text{RL}}(\varepsilon'; x, x) \\ -\Delta G_{0\,\text{LR}}(\varepsilon'; x, x) & \Delta G_{0\,\text{RR}}(\varepsilon'; x, x) \end{pmatrix}. \tag{7.7}
$$

Substituting the expression for $\Delta \hat{G}_0(\varepsilon; x, x)$ (see Appendix F) and absorbing the constant in the diagonal terms into the chemical potential, we obtain the final expression for the self-energy:

$$
\hat{\Sigma}^R(x) = (g_2 - g_1) \int d\varepsilon' n_{\rm F}(\varepsilon') \begin{pmatrix} 0 & -[\theta(x) r_0 + \theta(-x) \bar{r}_0] e^{i\frac{2\varepsilon'_+}{v_{\rm F}} x} \\ -[\theta(-x) r_0 + \theta(x) \bar{r}_0] e^{-i\frac{2\varepsilon'_+}{v_{\rm F}} x} & 0 \end{pmatrix},
$$

(7.8)

where $\varepsilon'_+ \equiv \varepsilon' + i0$. The first-order correction to Green function can be written as

$$
\delta \hat{G}_0^R(\varepsilon; x, x') = \int d\tilde{x}\, \hat{G}_0^R(\varepsilon; x, \tilde{x}) \hat{\Sigma}^R(\tilde{x}) \hat{G}_0^R(\varepsilon; \tilde{x}, x') .
$$

(7.9)

We shall first calculate the correction to the transmission amplitude t_0 by evaluating $\delta G_{0\,{\rm RR}}^R(\varepsilon; x \to +\infty, x' \to -\infty)$:

$$
\delta G_{0\,{\rm RR}}^R(\varepsilon; x, x') = \int d\tilde{x}\, \Big[\underbrace{G_{0\,{\rm RR}}^R \Sigma_{\rm RR}^R G_{0\,{\rm RR}}^R}_{(1)} + \underbrace{G_{0\,{\rm RL}}^R \Sigma_{\rm LR}^R G_{0\,{\rm RR}}^R}_{(2)} + \underbrace{G_{0\,{\rm RR}}^R \Sigma_{\rm RL}^R G_{0\,{\rm LR}}^R}_{(3)} + \underbrace{G_{0\,{\rm RL}}^R \Sigma_{\rm LL}^R G_{0\,{\rm LR}}^R}_{(4)} \Big]
$$

(7.10)

Straightforward calculations give:

$$
(1) = (4) = 0 ,
$$

(7.11)

$$
(2) = \frac{e^{i\frac{\varepsilon_+}{v_{\rm F}}(x - x')}}{i v_{\rm F}} \frac{g_2 - g_1}{2} t_0 |r_0|^2 \ln \frac{|\varepsilon|}{\varepsilon_{\rm F}} ,
$$

(7.12)

$$
(3) = \frac{e^{i\frac{\varepsilon_+}{v_{\rm F}}(x - x')}}{i v_{\rm F}} \frac{g_2 - g_1}{2} t_0 |r_0|^2 \ln \frac{|\varepsilon|}{\varepsilon_{\rm F}} .
$$

(7.13)

The total correction to t_0 reads:

$$\delta t_0 = (g_2 - g_1)t_0|r_0|^2 \ln \frac{|\varepsilon|}{\varepsilon_F} \,. \tag{7.14}$$

The correction to r_0 can be calculated by evaluating $\delta G^R_{0\,RL}(\varepsilon; x \to +\infty, x' \to +\infty)$:

$$\delta G^R_{0\,RL}(\varepsilon; x, x') = \int d\tilde{x} \left[\underbrace{G^R_{0\,RR}\Sigma^R_{RR}G^R_{0\,RL}}_{(1)} + \underbrace{G^R_{0\,RL}\Sigma^R_{LR}G^R_{0\,RL}}_{(2)} + \underbrace{G^R_{0\,RR}\Sigma^R_{RL}G^R_{0\,LR}}_{(3)} + \underbrace{G^R_{0\,RL}\Sigma^R_{LL}G^R_{0\,LL}}_{(4)} \right]$$

$$\tag{7.15}$$

$$(1) = (4) = 0 \,, \tag{7.16}$$

$$(2) = \frac{e^{i\frac{\varepsilon_+}{v_F}(x-x')}}{iv_F} \frac{g_2 - g_1}{2} r_0|r_0|^2 \ln \frac{|\varepsilon|}{\varepsilon_F} \,, \tag{7.17}$$

$$(3) = \frac{e^{i\frac{\varepsilon_+}{v_F}(x-x')}}{iv_F} \frac{g_2 - g_1}{2} (\bar{r}_0 t_0^2 - r_0) \ln \frac{|\varepsilon|}{\varepsilon_F} \,, \tag{7.18}$$

which yields:

$$\delta r_0 = -(g_2 - g_1)r_0|t_0|^2 \ln \frac{|\varepsilon|}{\varepsilon_F} \,. \tag{7.19}$$

The expressions (7.14) and (7.19) coincide with the results in Ref. [6].

7.2 Corrections due to phonons

The main advantage of the approach we use in this Chapter is that it is no longer necessary to consider only weak or strong limit in impurity strength, both for electrons and phonons. The price to pay is the limit of weak interactions in the system due to the perturbative nature of calculations. As for electrons, we can introduce arbitrary phonon scattering coefficients:

transmission amplitude τ and reflection amplitude ρ. It can be shown that the condition of continuity (at $x = 0$) of the Green function of atomic displacement fields imposes the following constraint:

$$\tau - \rho = 1 \, , \tag{7.20}$$

where, generally, ρ is a complex number. The phonon scattering amplitudes enter our model as bare parameters of the system and are not affected by the renormalisation procedure.

In presence of the impurity the phonon propagator can be written as:

$$\mathcal{D}(x, x') = \mathcal{D}_0(x - x') - \rho \, \mathrm{sgn}(xx') \mathcal{D}_0(|x| + |x'|) \, . \tag{7.21}$$

Notice that the limit of a phonon-transparent (phonon-nontransparent) impurity corresponds to $\rho = 0$ ($\rho = -1$).

We choose to work in (ε, x)-representation, therefore, it is convenient for us to rewrite the translation invariant phonon-mediated electron-electron interaction term as follows, which allows for a simple Fourier transform:

$$\mathcal{D}_0(\omega, q) = \pi v_{\mathrm{F}} \alpha_{\mathrm{ph}} \frac{c^2 q^2}{\omega^2 - c^2 q^2} = -\pi v_{\mathrm{F}} \alpha_{\mathrm{ph}} + \frac{\omega^2}{\omega^2 - c^2 q^2} \, , \tag{7.22}$$

$$\mathcal{D}_0(\omega; x - x') = -\pi v_{\mathrm{F}} \alpha_{\mathrm{ph}} \delta(x - x') + \underbrace{\pi v_{\mathrm{F}} \alpha_{\mathrm{ph}} \frac{\omega}{2\mathrm{i}c} \mathrm{e}^{\mathrm{i}\frac{\omega}{c}|x-x'|}}_{D(\omega; x-x')} \, . \tag{7.23}$$

The first term on the right-hand side in Eq. (7.23) appears to behave as a short-range contribution due to the δ-function dependence on coordinate. It can be analysed on the same ground as the local interaction term in the previous Section. We can take it into account by simply replacing g_2 by $(g_2 - \alpha_{\mathrm{ph}}/2)$ in Eqs. (7.14) and (7.19). The second term in Eq. (7.23), $D(\omega; x - x')$, is responsible for the retardation effects and is clearly nonlocal. First-order perturbation theory corrections due to this interaction term contain a Fock term

only. The Hartree contribution is zero since this term vanishes at $\omega = 0$.

As in the previous Section, our aim is to calculate the correction to electron self-energy and, in particular, we are interested in its retarded component, $\Sigma^R(x, x'; t, t')$, because eventually we want to compute the correction to electron Green function, $\delta G_0^R(\varepsilon; x, x')$, which is given by:

$$\delta \hat{G}_0^R(\varepsilon; x, x') = \int \mathrm{d}x_1 \mathrm{d}x_2 \, \hat{G}_0^R(\varepsilon; x, x_2) \hat{\Sigma}^R(\varepsilon; x_2, x_1) \hat{G}_0^R(\varepsilon; x_1, x') \,, \qquad (7.24)$$

where

$$
\begin{aligned}
\hat{\Sigma}^R(\varepsilon; x_2, x_1) =\ & \mathrm{i} \int \frac{\mathrm{d}\omega}{2\pi} \left[\Theta_\omega D^R(\omega; x_2 - x_1) + \Theta_{-\omega} D^A(\omega; x_2 - x_1) \right] \\
& \times \left[\Theta_{\varepsilon-\omega} \hat{G}_0^R(\varepsilon - \omega; x_2, x_1) + \Theta_{\omega-\varepsilon} \hat{G}_0^A(\varepsilon - \omega; x_2, x_1) \right] \\
=\ & \mathrm{i} \left[\int_{-\infty}^{0} \frac{\mathrm{d}\omega}{2\pi} D^A(\omega; x_2 - x_1) \hat{G}_0^R(\varepsilon - \omega; x_2, x_1) \right. \\
& + \underbrace{\int_{0}^{\varepsilon} \frac{\mathrm{d}\omega}{2\pi} D^R(\omega; x_2 - x_1) \hat{G}_0^R(\varepsilon - \omega; x_2, x_1)}_{\text{does not contribute into RG}} \\
& \left. + \int_{\varepsilon}^{+\infty} \frac{\mathrm{d}\omega}{2\pi} D^R(\omega; x_2 - x_1) \hat{G}_0^A(\varepsilon - \omega; x_2, x_1) \right]. \qquad (7.25)
\end{aligned}
$$

Without loss of generality, we assume that $\varepsilon > 0$ in Eq. (7.25). Since in the end we will performing renormalisation group analysis of the found corrections by varying an introduced by hand high-energy cut-off, the second integral in Eq. (7.25) will not contribute, since ε is considered to be much smaller than the Fermi energy. For the same reason we can omit integrals running between $-\varepsilon$ and 0. We shall ignore all those contributions in what follows.

It also important to notice that the two remaining contributions in Eq. (7.25) have a slightly different nature. The first term which contains $\hat{G}_0^R(\varepsilon - \omega; x_2, x_1)$ describes an electron with positive energy $\varepsilon + |\omega|$ which emits a phonon at point x_1 and then absorbs it back at

x_2 without any change in direction throughout the process. On contrary, the term with $\hat{G}_0^A(\varepsilon - \omega; x_2, x_1)$ describes and electron which upon emitting a photon at x_1 gives out an energy ω which is greater than its whole energy, thereby turning into a hole and propagating in the opposite direction from the initial one until eventually absorbing the phonon back and recovering its initial direction of motion at point $x_2 < x_1$. Propagating electrons and holes are depicted in Figs. 7.2 and 7.3 by solid and dashed lines, correspondingly.

7.2.1 Correction to the transmission amplitude

In this Section we calculate the correction δt to the transmission amplitude t. In order to obtain it we consider the correction to the Green function:

$$\delta G_{0\,\mathrm{RR}}^R(\varepsilon; x, x') \equiv \delta t\, G_{0\,\mathrm{RR}}^R(\varepsilon; x, x'), \quad \text{when} \quad x \to +\infty\,, \; x' \to -\infty\,. \qquad (7.26)$$

From the Eq. (7.24) we can derive all contributions into δt which can be represented as Feynman diagrams shown in Fig. 7.2. The integration over x_1 and x_2 is performed from x' to x, which are the initial and final coordinates assumed to be at $\mp\infty$, correspondingly. We also assume that all interactions are adiabatically 'switched off' at infinity as a regularisation procedure in order to avoid diverging integrals.

Whenever an electron (hole) is passes through or gets reflected by the impurity at $x = 0$, the corresponding expression acquires a factor of t_0 or r_0 ($\bar{t}_0$ or $\bar{r}_0$), correspondingly. Similarly, for transmitted and reflected phonons one gets a factor of τ and $-\rho$ (or $\bar{\tau}$ and $-\bar{\rho}$, depending on whether the phonon propagator is retarded or advanced).

It turns out that diagrams (1a) and (1b), as well as (2a) and (2b), together give a zero contribution. Also, the terms (4a) and (4b) are equal by symmetry. The calculation of all

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diagrams from the Fig. 7.2 is provided below:

$$(1a) + (1b) = (2a) + (2b) = 0 \,, \tag{7.27}$$

$$(3a) = -\alpha_{\mathrm{ph}} t_0 \bar{\tau} \frac{\beta_{\mathrm{F}}}{4(\beta_{\mathrm{F}} + 1)^2} \int\limits_{-\infty}^{-\varepsilon} \frac{\mathrm{d}\omega}{\omega} \,, \tag{7.28}$$

$$(3b) = -\alpha_{\mathrm{ph}} t_0 |t_0|^2 \tau \frac{\beta_{\mathrm{F}}}{4(\beta_{\mathrm{F}} + 1)^2} \int\limits_{\varepsilon}^{+\infty} \frac{\mathrm{d}\omega}{\omega} \,, \tag{7.29}$$

$$(4a) = (4b) = -\alpha_{\mathrm{ph}} t_0 |r_0|^2 \left[\frac{\beta_{\mathrm{F}}}{4(\beta_{\mathrm{F}} + 1)} - \rho \frac{\beta_{\mathrm{F}}}{4(\beta_{\mathrm{F}} + 1)^2} \right] \int\limits_{\varepsilon}^{+\infty} \frac{\mathrm{d}\omega}{\omega} \,, \tag{7.30}$$

$$(5) = -\alpha_{\mathrm{ph}} \bar{t}_0 r_0^2 \tau \frac{\beta_{\mathrm{F}}}{4(\beta_{\mathrm{F}} + 1)^2} \int\limits_{\varepsilon}^{+\infty} \frac{\mathrm{d}\omega}{\omega} \,. \tag{7.31}$$

The S-matrix elements are related via $t_0 \bar{r}_0 + \bar{t}_0 r_0 = 0$ due to its unitarity. Using this fact together with Eq. (7.20) and introducing a high-energy cut-off $E_0 \sim \varepsilon_{\mathrm{F}}$ in preparation for the renormalisation group analysis of the total correction, we obtain:

$$\delta t = \alpha_{\mathrm{ph}} t_0 |r_0|^2 \left[\left[\rho - \bar{\rho} - 2|r_0|^2 (1 + 2\rho) \right] \frac{\beta_{\mathrm{F}}}{4(\beta_{\mathrm{F}} + 1)^2} + 2|r_0|^2 \frac{\beta_{\mathrm{F}}}{4(\beta_{\mathrm{F}} + 1)} \right] \ln \frac{|\varepsilon|}{E_0} \,. \tag{7.32}$$

Or, when put together with the correction due to the short-range electron-electron interaction, Eq. (7.14), also taking into account the local term from Eq. (7.23):

$$\delta t = \left\{ \left(g_2 - g_1 - \frac{\alpha_{\mathrm{ph}}}{2} \right) |r_0|^2 + \frac{\alpha_{\mathrm{ph}}}{2} \left[\left[\rho - \bar{\rho} - 2|r_0|^2 (1 + 2\rho) \right] \frac{\beta_{\mathrm{F}}}{2(\beta_{\mathrm{F}} + 1)^2} + |r_0|^2 \frac{\beta_{\mathrm{F}}}{\beta_{\mathrm{F}} + 1} \right] \right\}$$
$$\times t_0 \ln \frac{|\varepsilon|}{E_0} \,. \tag{7.33}$$

We present Eq. (7.33) it as a generalisation of the result in Ref. [6] for spinless electrons.

7.2.2 Correction to the reflection amplitude

Here we calculate the correction δr due to the phonon-mediated interaction following essentially the same scheme as in the previous Section. We calculate

$$\delta G_{0\,\mathrm{RL}}^{R}(\varepsilon; x, x') \equiv \delta r \, G_{0\,\mathrm{RL}}^{R}(\varepsilon; x, x'), \quad \text{when} \quad x, x' \to +\infty. \tag{7.34}$$

Using the Eq. (7.24) we obtain all contributions into δr which are shown in Fig. 7.3. The integration over x_1 and x_2 is performed between x' and x, which are initial and final coordinates assumed to be at $+\infty$. We again demand that all interactions are adiabatically switched off at infinity.

The corrections from Fig. 7.3 yield:

$$(1a) + (1b) = (2a) + (2b) = 0, \tag{7.35}$$

$$(3a) = -\alpha_{\mathrm{ph}} r_0 \left[\frac{\beta_{\mathrm{F}}}{4(\beta_{\mathrm{F}} + 1)} - \bar{\rho} \frac{\beta_{\mathrm{F}}}{4(\beta_{\mathrm{F}} + 1)^2} \right] \int\limits_{-\infty}^{-\varepsilon} \frac{\mathrm{d}\omega}{\omega}, \tag{7.36}$$

$$(3b) = -\alpha_{\mathrm{ph}} r_0 |r_0|^2 \left[\frac{\beta_{\mathrm{F}}}{4(\beta_{\mathrm{F}} + 1)} - \rho \frac{\beta_{\mathrm{F}}}{4(\beta_{\mathrm{F}} + 1)^2} \right] \int\limits_{\varepsilon}^{+\infty} \frac{\mathrm{d}\omega}{\omega}, \tag{7.37}$$

$$(4a) = (4b) = -\alpha_{\mathrm{ph}} r_0 |t_0|^2 \tau \frac{\beta_{\mathrm{F}}}{4(\beta_{\mathrm{F}} + 1)^2} \int\limits_{\varepsilon}^{+\infty} \frac{\mathrm{d}\omega}{\omega}, \tag{7.38}$$

$$(5) = -\alpha_{\mathrm{ph}} \bar{r}_0 t_0^2 \left[\frac{\beta_{\mathrm{F}}}{4(\beta_{\mathrm{F}} + 1)} - \rho \frac{\beta_{\mathrm{F}}}{4(\beta_{\mathrm{F}} + 1)^2} \right] \int\limits_{\varepsilon}^{+\infty} \frac{\mathrm{d}\omega}{\omega}. \tag{7.39}$$

After introducing a high-energy cut-off E_0 and putting everything together for the total

correction we get

$$\delta r = -\left\{\left(g_2 - g_1 - \frac{\alpha_{\text{ph}}}{2}\right)|t_0|^2 + \frac{\alpha_{\text{ph}}}{2}\left[[\rho - \bar{\rho} - 2|t_0|^2(1+2\rho)]\frac{\beta_{\text{F}}}{2(\beta_{\text{F}}+1)^2} + |t_0|^2\frac{\beta_{\text{F}}}{\beta_{\text{F}}+1}\right]\right\}$$
$$\times r_0 \ln \frac{|\varepsilon|}{E_0} . \tag{7.40}$$

7.3   Comparison with the Weak Scatterer and Weak Link results

In Chapters 4 and 5 we have derived renormalisation group equations for the impurity strength, λ, and electron tunnelling amplitude, t, for both phonon-transparent and phonon-nontransparent impurities. In both cases at low energies they scale as a power-law with a non-universal power defined by the scaling dimension of the problem:

$$\lambda(\varepsilon) \sim \left(\frac{|\varepsilon|}{E_0}\right)^{\Delta_{\text{WS}}-1} , \tag{7.41}$$

$$t(\varepsilon) \sim \left(\frac{|\varepsilon|}{E_0}\right)^{\Delta_{\text{WL}}-1} . \tag{7.42}$$

Let us see how this scaling behaviour relates to the results derived earlier in this Chapter. If we assume in the above expressions that electron-electron and electron-phonon interactions are weak, i.e. $g_2, g_4, \alpha_{\text{ph}} \ll 1$, we can expand the scaling dimensions in powers of the interaction strengths up to the first order:

$$\Delta_{\text{WS1}}^{(1)} - 1 = -\left(\Delta_{\text{WL1}}^{(1)} - 1\right) = -g_2 + \alpha_{\text{ph}}\frac{2\beta_{\text{F}} + 1}{2(\beta_{\text{F}} + 1)^2} , \tag{7.43}$$

$$\Delta_{\text{WS2}}^{(1)} - 1 = -\left(\Delta_{\text{WL2}}^{(1)} - 1\right) = -g_2 + \alpha_{\text{ph}}\frac{1}{2(\beta_{\text{F}} + 1)^2} . \tag{7.44}$$

At the same time, if we take, for instance, Eq. (7.33) and assume that $t_0 \ll 1 (r_0 \simeq 1)$ (weak link limit), then the coefficient in the curly brackets is equal to:

$$\{..\} = \begin{cases} g_2 - g_1 - \alpha_{\mathrm{ph}} \frac{2\beta_{\mathrm{F}}+1}{2(\beta_{\mathrm{F}}+1)^2}, & \text{when } \rho = 0\,; \\[2ex] g_2 - g_1 - \alpha_{\mathrm{ph}} \frac{1}{2(\beta_{\mathrm{F}}+1)^2}, & \text{when } \rho = -1\,, \end{cases} \tag{7.45}$$

which are exactly the coefficients $(\Delta_{\mathrm{WL\,1,2}} - 1)$ from Eq. (7.43). Of course, we have omitted the local backscattering amplitude g_1 here, which was not included in Chapters 4 and 5.

By analogy, if we set $r_0 \ll 1 (t_0 \simeq 1)$ (weak scattering limit) in Eq. (7.40), then we get exactly the same coefficient as in Eq. (7.45). If we take into account that $r(\varepsilon) \sim \lambda(\varepsilon)$ in the weak scattering limit, then by solving the flow equations (7.33) and (7.40) one obtains exactly the same scaling behaviour as given by Eqs. (7.41) and (7.42). Therefore, our results obtained for the system with arbitrarily strong electron-electron and electron-phonon interactions but either weak or strong impurity strength exactly match the results of this Chapter for arbitrary impurity potential in a weakly interacting system. Naturally, the agreement can be seen in the region of parameters where both the interactions and electron backscattering amplitude (or tunnelling amplitude) are small simultaneously.

To summarise the results of this Chapter, we have derived first order (in interaction strength) corrections to transmission and reflection amplitudes through an impurity of arbitrary strength, Eqs. (7.33) and (7.40). Phonon reflection amplitude, ρ, is also assumed to be arbitrary. One can check that $\delta(|t|^2) + \delta(|r|^2) = 0$, as it should be, which serves as another indication that the corrections δt and δr have been found correctly.
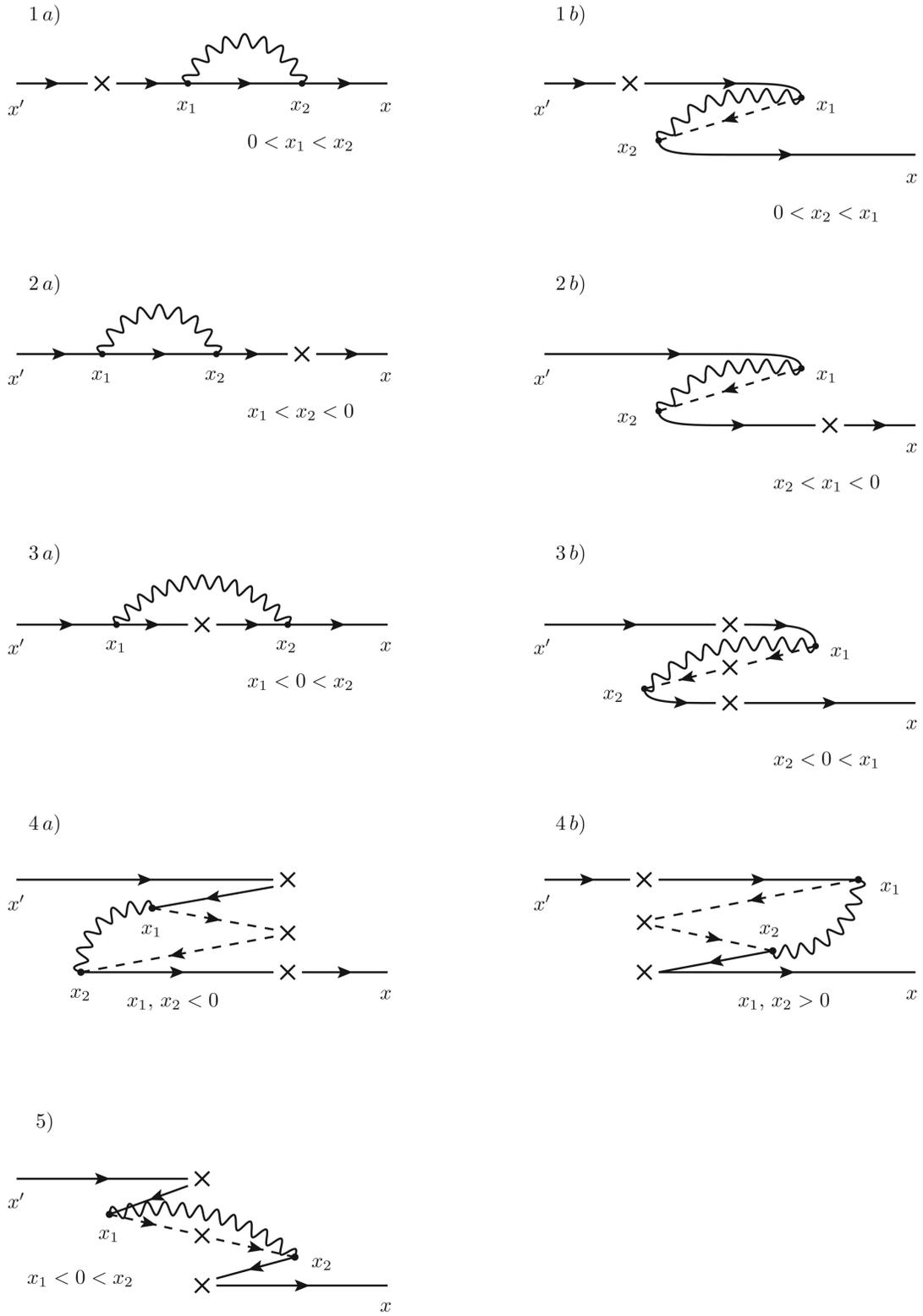
Figure 7.2: First order corrections to the transmission amplitude.

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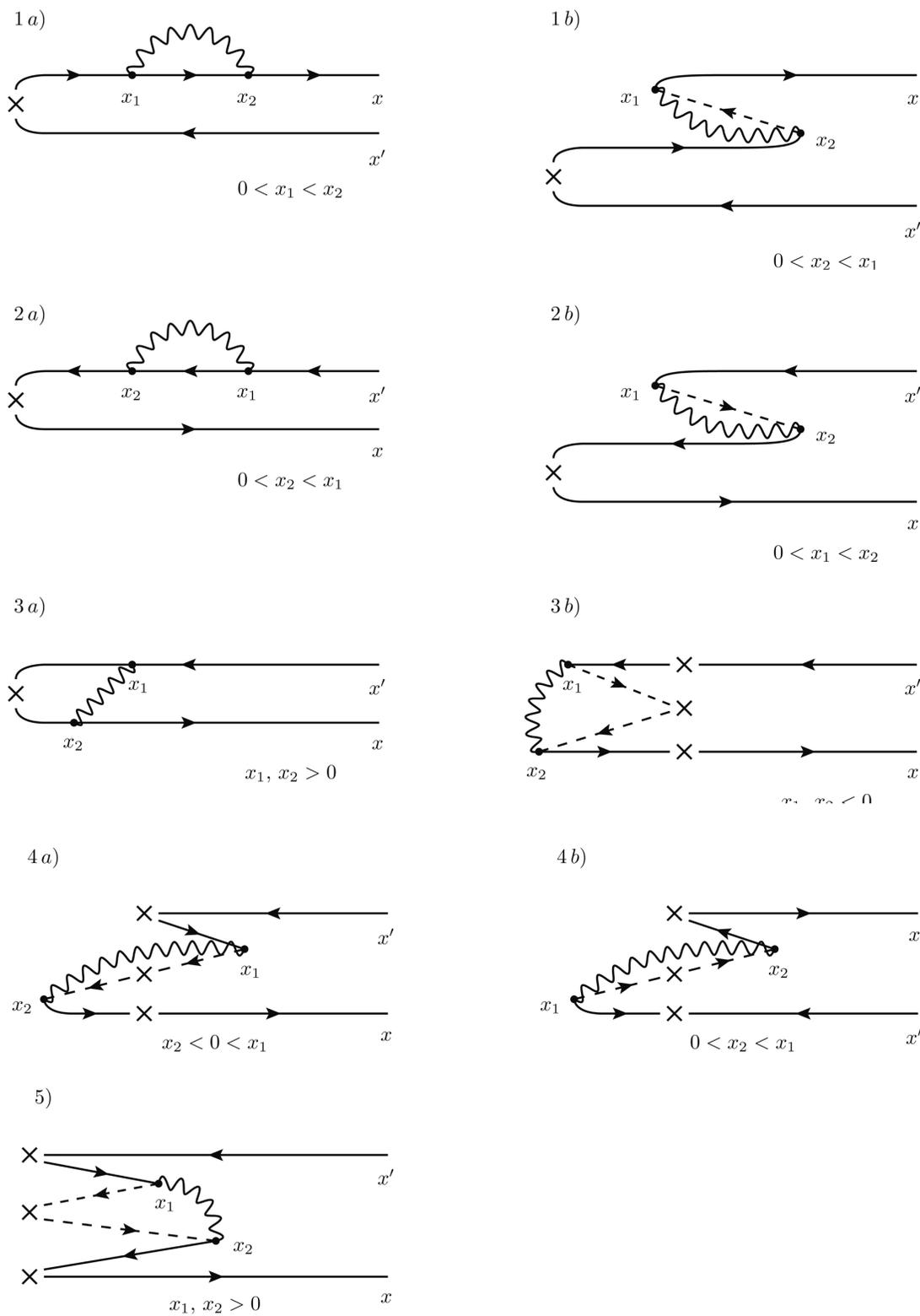
Figure 7.3: First order corrections to the reflection amplitude.

CHAPTER 8

RESONANT IMPURITY

Having considered the properties of electron transport through a single featureless impurity we now turn to the problem of resonant tunnelling in one dimension. Possible realisations of this problem include a setup with two impurities (a double barrier), a single impurity carrying a discrete localised level or a weakly coupled quantum dot with sufficiently large level spacing δ ($\delta \ll T, \Gamma_0$), where Γ_0 is the resonance width. In the model without electron-electron interactions, as well as in the Fermi Liquid in higher dimensions, a resonant transmission is expected for single electrons when their incident energy is close to the energy of the localised state in the double barrier. The resonance is described by the Breit-Wigner formula [63]. However, as we already know, interactions in one dimension lead to the breakdown of the Fermi Liquid theory. Consequently, one may expect that interactions could lead to some qualitative changes in system's behaviour. Indeed, the inclusion of interactions leads to the resonances of non-Lorentzian shape, which has been first found in Refs. [58, 64]. The presence of the resonant transmission effect in Luttinger Liquids is of significant importance for experimentalists. Despite that fact that upon embedding a single impurity into the system with (repulsive) interaction the conductance vanishes as a power-law at low temperatures, it is possible to restore an ideal conductance by creating a double barrier system with a localised energy state. The experimental study of the shape of the resonance can be used as
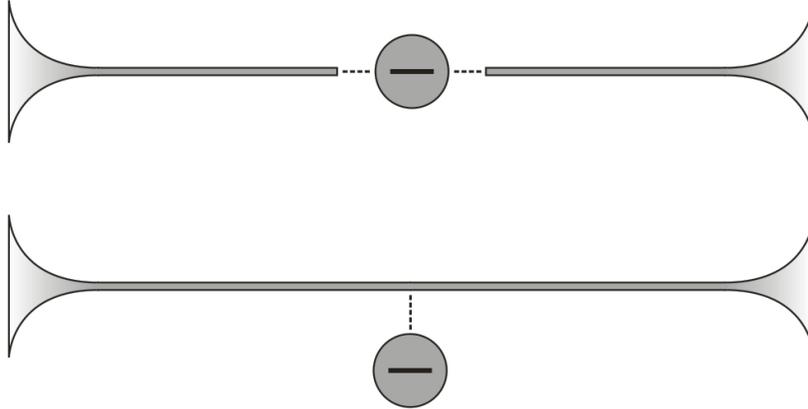
Figure 8.1: Two geometries under consideration: resonant barrier geometry (top) and side-attached geometry (bottom).

an experimental evidence of the Luttinger Liquid behaviour in real one-dimensional systems. In fact, in Ref. [65] the authors have observed the temperature dependence of the effective resonant level width predicted by the Tomonaga-Luttinger model [58, 64].

Following the main spirit of this thesis, the main question we address in this Chapter is the following: what effect does electron-phonon interaction have on the previously studied problem in a Luttinger Liquid without additional coupling? In this Chapter we also consider a dual geometry to the one mentioned above, i.e. where the impurity carrying a discreet level is side-attached to the Luttinger Liquid (see Fig. 8.1). In this case the transmission becomes anti-resonant: now the reflection coefficient is described by the same Breit-Wigner formula. This setup is as realistic and several experimental works have been performed with quantum wires [66, 67, 68, 69] and carbon nanotubes [70, 71, 72]. The consideration of the side-attached geometry was inspired by the work [73]. The results have been published in Ref. [74].

8.1 Resonant level

We begin with description of the resonant barrier geometry (see Fig. 8.1, top) when the resonant level is embedded into the Luttinger Liquid at $x = 0$. The action consists of three terms:

$$S = S_{\mathrm{LL}} + S_{\text{el-ph}} + S_{\mathrm{T}} . \tag{8.1}$$

We assume the usual Luttinger Liquid decoupling of the (spinless) electron field into the sum of right- and left-moving electrons,

$$\psi(x, t) \simeq \psi_{\mathrm{R}}(x, t) e^{i k_{\mathrm{F}} x} + \psi_{\mathrm{L}}(x, t) e^{-i k_{\mathrm{F}} x} , \tag{8.2}$$

with $\psi_{\mathrm{R,L}}$ labelled by $\eta = \pm 1$ below. The standard Luttinger part of the action is

$$S_{\mathrm{LL}} = \sum_{\eta = \pm 1} \int \mathrm{d}x \mathrm{d}t \, \bar{\psi}_\eta(x, t) \, i \partial_\eta \psi_\eta(x, t) - \frac{1}{2} \int \mathrm{d}x \mathrm{d}t \, V_0 \, n^2(x, t) . \tag{8.3}$$

Here $n \equiv (\bar{\psi}_{\mathrm{R}} \psi_{\mathrm{R}} + \bar{\psi}_{\mathrm{L}} \psi_{\mathrm{L}})$ is the electron density, V_0 is the screened Coulomb interaction (we do not differentiate between g_2 and g_4 processes in this Chapter) and $\partial_\eta \equiv \partial_t + \eta v_{\mathrm{F}} \partial_x$. We use here the zero-temperature formalism: the only role played by temperature is providing an alternative low-energy cut-off in the renormalisation group equations below.

The tunnelling action has the form:

$$S_{\mathrm{T}} = \int \mathrm{d}t \left\{ i \bar{d}(t) \left(\partial_t + \varepsilon_0 \right) d(t) + \sum_\mu \left[t_0 \, \bar{d}(t) \psi_\mu(t) + \text{h.c.} \right] \right\} , \tag{8.4}$$

where $\psi_\mu(t) \equiv \psi_\mu(x = 0, t)$, $d(t)$ is the field corresponding to the electron localised at the resonant level with the energy ε_0 counted from the Fermi level. In the resonant-barrier geometry the index μ refers to the left and right electron subsystems separated by the barrier.

In this case an electron can leave the left subsystem to the quantum dot and enter it from the quantum dot only as a right- and left-mover, respectively – and conversely for the right subsystem, so that the labels $\mu = \pm 1$ mean

$$\bar{\psi}_- = \bar{\psi}_{\ell,\mathrm{L}}\,, \qquad \psi_- = \psi_{\ell,\mathrm{R}}\,, \qquad \bar{\psi}_+ = \bar{\psi}_{r,\mathrm{R}}\,, \qquad \psi_+ = \psi_{r,\mathrm{L}}\,, \tag{8.5}$$

where L, R refer to the left- and right-movers, as before, and ℓ, r to the left and right subsystems. We have assumed the tunnelling amplitude t_0 to be the same in all channels. This is only the case when it is implemented as a symmetric double-barrier. The resonant properties for $t_1 \neq t_2$ could be different but we leave it aside here. Finally, we model the electron-phonon action with a conventional coupling of the electron density to the gradient of the phonon field, assuming the phonon spectrum to be linear with a cutoff at the Debye frequency, $\omega_{\mathrm{D}} = c\,q_{\mathrm{D}}$. Then the phonon action (neglecting possible electron backscattering justified for $T \ll ck_{\mathrm{F}}$) has the following form:

$$S_{\text{el-ph}} = \int \mathrm{d}x\mathrm{d}t \left[-\frac{1}{2}\phi(x,t)\,\mathcal{D}_0^{-1}\,\phi(x,t) + \phi(x,t)\,n(x,t) \right]\,. \tag{8.6}$$

Here $\phi(x,t)$ is the phonon field (in which the electron-phonon coupling constant is absorbed) and $\mathcal{D}_0(x,t)$ is the free phonon propagator with the Fourier transform given for 1D acoustical phonons by

$$\mathcal{D}_0^R(\omega,q) = \pi v_{\mathrm{F}} \alpha_{\mathrm{ph}} \frac{\omega_q^2}{\omega_+^2 - \omega_q^2}\,, \qquad \omega_q = c\,q\,. \tag{8.7}$$

The action defined by Eqs. (3.41), (8.4) and (8.6) is quadratic in fields d and ϕ, which can thus be integrated out. The integration over the phonon fields results only in substituting V_0 in the action (3.41) by the dynamical coupling,

$$V(x,t) = V_0 + \mathcal{D}_0(x,t)\,. \tag{8.8}$$

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Performing the integration over the field $d(t)$ results in transforming S_T in Eq. (8.4) into

$$\widetilde{S}_T = -\sum_{\mu,\nu} \int dt dt' \, \bar{\psi}_\mu(t) \sum_{\mu,\nu}(t-t') \, \psi_\nu(t') \,, \tag{8.9}$$

$$\hat{\Sigma}(\varepsilon) = \frac{v_F \hat{\Gamma}_0}{\varepsilon - \varepsilon_0 + i\delta \, \mathrm{sgn}\, \varepsilon} \,, \qquad \hat{\Gamma}_0 \equiv \Gamma_0(\hat{1} + \hat{\sigma}_x) \,, \tag{8.10}$$

where $\Sigma(t-t')$ is the Fourier transform of $\Sigma(\varepsilon)$. $\hat{\Gamma}_0$ is the matrix with all elements equal to the tunnelling rate $\Gamma_0 \equiv \pi\nu_0|t_0|^2$, where $\nu_0 = (\pi v_F)^{-1}$ is the density of states of the conduction electrons in the absence of interactions.

If phonons were originated from a three-dimensional environment, their propagator would still have the form of Eq. (8.7) but with $q \to |\boldsymbol{q}|$ where $\boldsymbol{q} \equiv (q, \boldsymbol{q}_\perp)$. Since the Luttinger Liquid electrons are coupled to the phonon propagator along the wire only, one would have to integrate out $\boldsymbol{q}_\perp$. This results in a less singular form of $\mathcal{D}_0(\omega, q)$ which, however, would not qualitatively change the results obtained below, since the effective interaction, Eq. (8.8), still might change sign in various parametric regions. All the specific quantitative expressions below, which depend on the electron-phonon coupling, have been derived for the one-dimensional propagator of Eq. (8.7). Let us stress that this is also the case where the electron-phonon coupling can be particularly strong. It has been demonstrated [75, 76] that in suspended single-wall carbon nanotubes the stretching modes (described as longitudinal phonons with a linear dispersion) are coupled to electrons with the dimensionless coupling strength of the order of 1.

The action given by Eqs. (8.3) (with $V_0 = 0$) and (8.9) describes the resonant transmission through the Fermi gas hybridised with the resonant level. The hybridisation makes the electron Green function to acquire an off-diagonal part, $g_\mu(x - x'; t, t') \to G_{\mu\nu}(x, x'; t, t')$, describing the resonance-induced left-to-right connection. In the mixed position-energy rep-

resentation it has the following matrix form

$$\hat{G}(x, x'; \varepsilon) = \hat{g}(x - x'; \varepsilon) + iv_{\mathrm{F}} \hat{g}(x; \varepsilon) \hat{\mathsf{T}}(\varepsilon) \, \hat{g}(-x'; \varepsilon) \,. \tag{8.11}$$

Here $\hat{g}$ is the matrix with diagonal elements given by Eq. (F.2), and the $\hat{\mathsf{T}}$-matrix has the form

$$\hat{\mathsf{T}}(\varepsilon) = \frac{-i\hat{\Gamma}_0}{\varepsilon - \varepsilon_0 + \frac{i}{2}\hat{\Gamma}_0 \, \mathrm{sgn}\, \varepsilon} \,. \tag{8.12}$$

The transmission is proportional to $G_{12}G_{21}$ This gives the well-known Fermi-gas result with the resonant transmission:

$$\mathcal{T}_0(\varepsilon) = \frac{\Gamma_0^2}{(\varepsilon - \varepsilon_0)^2 + \Gamma_0^2} \,. \tag{8.13}$$

We will now use the effective action represented by the sum of the terms given by Eq. (8.3) with the substitution (8.8) and Eqs. (8.9)–(8.10) to show that the electron-electron and electron-phonon interactions results in $\Gamma_0 \to \Gamma(\varepsilon)$ in the transmission probability (8.13), with the energy dependence of $\Gamma(\varepsilon)$ being qualitatively different from that found in the phononless case [58, 73]. First of all, we use the same functional bosonisation method described in details in Section 3.3. This approach accounts for the non-local interaction V by introducing new bosonic fields $\theta_\eta(x, t)$, as in (3.43), with the corresponding correlation functions (3.47)–(3.49). As a result of this procedure, the self-energy part in resonant-level term (8.9) dresses as:

$$\Sigma(t - t') \to \Sigma_{\mu\nu}(t - t') = \mathrm{e}^{-i\theta_\mu(t)} \, \Sigma(t - t') \, \mathrm{e}^{i\theta_\nu(t')} \,. \tag{8.14}$$

in presence of interactions.

The polaron fields $\theta_\mu(t) \equiv \theta(x = 0, t)$ entering the self-energy (8.14) are defined at the

origin where the quantum dot (or barriers) carrying the resonant level is placed. Integrating out all the fields at $x \neq 0$ results in the zero-dimensional action which governs the renormalisation of the self-energy in Eqs. (8.9)–(8.10) and (8.14), and, thus, the renormalisation of the resonance width in Eqs. (8.13), $\Gamma_0 \rightarrow \Gamma(\varepsilon)$. In the phononless case this action is fully equivalent to that used for describing the resonant transmission (reflection) through the Luttinger Liquid [58, 73]. The only difference due to the electron-phonon coupling is that the correlation function of the local fields $\theta(t)$ is governed by the two-branch polaron modes, Eq. (3.49). Of course, in the resonant-level geometry the fields θ_μ live only on a half-axis, like in the weak link limit of Section 5.2. Therefore, for their correlation function we obtain:

$$\langle \theta_\mu(-\omega) \, \theta_\nu(\omega) \rangle^R = \frac{\pi \delta_{\mu\nu} (\Delta_{\mathrm{WL2}} - 1)}{\omega_+} \, . \tag{8.15}$$

Here Δ_{WL2} was defined in Eq. (5.21). Without electron-phonon coupling one would have $\Delta_{\mathrm{WL2}} = 1/K$, which in the Luttinger Liquid with repulsive interactions gives a scaling exponent $1/K - 1 > 0$. The electron-phonon coupling modifies this exponent in such a way that is may change sign upon changing such parameters of the system as the ration of Fermi and sound velocities and strengths of electron-electron and electron-phonon interactions. Therefore, whereas the phononless exponent is sign-definite, this is not necessarily true for the exponent $\Delta_{\mathrm{WL2}} - 1$. This strongly affects the resonant transmission properties, since the direction of the renormalisation group flow can now be alternated at strong enough coupling to phonons, when the effective interaction becomes attractive.

As in the case of the resonance transport through the phononless Luttinger Liquid [58, 73], we shall write a renormalisation group equation for the tunnelling amplitude t_0. In our formalism we should start with renormalising the self-energy part Σ in the tunnelling action (8.9) with the gauge substitution (8.14).

Having integrated out the fields with $x \neq 0$, all the interaction effects enter via the corre-

lation functions of the bosonic field θ, defined by Eq. (8.15). Therefore, the renormalisation group equation for Σ is obtained by a usual integration over fast components of this field, see Appendix D for a more detailed description of the procedure. We do not integrate over fast components of the fermionic field $\psi(t)$ and do not rescale the time variable since these two procedures exactly cancel each other, which follows from the absence of the renormalisation of the self-energy in the noninteracting case. We assume that the fast Fourier components of the field $\theta_\mu(t)$ have frequencies $E \leq |\omega| \leq E'$, with E being the running cutoff and $E'/E - 1 \ll 1$. Integrating them out leads to the following increment for the self-energy:

$$\delta\Sigma_{\mu\nu}(\varepsilon) = - \int\limits_{E \leq |\omega| \leq E'} \frac{\mathrm{d}\omega}{2|\omega|} \left[(\Delta_{\mathrm{WL2}} - 1)\Sigma_{\mu\nu}(\varepsilon) - \delta_{\mu\nu}(\Delta_{\mathrm{WL2}} - 1)\Sigma_{\mu\nu}(\varepsilon + \omega) \right] . \qquad (8.16)$$

In such a renormalisation group scheme [58] the self-energy $\Sigma_{\mu\nu}$ acquires a dependence on the running cutoff E on top of the dependence on ε. The initial condition for the renormalisation group equations is that at the ultraviolet cutoff, $E = E_0$, ($E_0 \sim \varepsilon_{\mathrm{F}}$ is the bandwidth), $\Sigma_{\mu\nu}$ is independent of E and has all matrix elements equal to $\Sigma(\varepsilon)$ given by Eq. (8.10). Then, as long as $E \gg |\varepsilon - \varepsilon_0|$, one may discard the second term in Eq. (8.16), thus arriving at the following equation:

$$\frac{\mathrm{d}\Sigma_{\mu\nu}(\varepsilon; E)}{\mathrm{d}l} = -(\Delta_{\mathrm{WL2}} - 1)\Sigma_{\mu\nu}(\varepsilon; E) , \qquad (8.17)$$

where $l \equiv \ln E_0/E$. Note that $(\Delta_{\mathrm{WL2}} - 1)$ happens to be the edge electron density of states exponent, equal to $1/K - 1$ in the phononless case [1] as we have already mentioned. In presence of electron-phonon it is still given by Eq. (5.21), which will be shown elsewhere.

Equation (8.17) is solved by substituting $\Sigma_{\mu\nu}(\varepsilon; E)$ in form (8.9), i.e., with all matrix elements equal, but with Γ_0 replaced by $\Gamma(E)$. This leads to the renormalisation group

equation for Γ:

$$\frac{d\Gamma(E)}{dl} = -(\Delta_{\mathrm{WL2}} - 1)\,\Gamma(E)\,, \qquad E \gg |\varepsilon - \varepsilon_0|\,. \tag{8.18}$$

With lowering the running cutoff one eventually reaches the region $E \ll |\varepsilon - \varepsilon_0|$ where the second term in Eq. (8.16) must be taken into account. The self-energy still has the form of Eq. (8.9) but with the substitution

$$\hat{\Gamma}_0 \to \hat{\Gamma}(E) = \Gamma_{\mathrm{diag}}(E)\hat{1} + \Gamma_{\mathrm{off}}(E)\hat{\tau}_1\,. \tag{8.19}$$

When $E \ll |\varepsilon - \varepsilon_0|$ the second term in Eq. (8.16) cancels the first one for $\mu = \nu$, so that Γ_{diag} saturates at $\Gamma(|\varepsilon - \varepsilon_0|)$ obtained by solving Eq. (8.18), while Γ_{off} continues to be renormalised according to the following equation:

$$\frac{d\Gamma_{\mathrm{off}}(E)}{dl} = -(\Delta_{\mathrm{WL2}} - 1)\Gamma_{\mathrm{off}}(E)\,, \qquad E \ll |\varepsilon - \min\{\Gamma_0,\, \varepsilon_0\}|\,. \tag{8.20}$$

Behavior of the renormalised $\Gamma(\varepsilon)$ is shown in Fig. 8.2. Let us stress that the condition of applicability written above does not follow directly from Eq. (8.16), which is perturbative in Γ_0. However, it follows from considerations non-perturbative in tunnelling (but perturbative in the interaction strength) [6, 73] that the inequality in (8.20) practically means the off-resonance condition. The impurity remains off-resonant if the level width, renormalised according to Eq. (8.18), remains narrow, i.e. $\Gamma(\varepsilon_0) \ll \varepsilon_0$. Only in this case Γ_{off} renormalises as in Eq. (8.20). Otherwise, we should put $\varepsilon_0 = 0$ and describe the resonant situation entirely in the frame of Eq. (8.18).

8.1.1 Transmission coefficient

The transmission coefficient, $\mathcal{T}(\varepsilon)$, is obtained by replacing in the $\hat{\mathsf{T}}$-matrix (8.12) the bare tunnelling rates, $\hat{\Gamma}_0$, by the renormalised ones, $\hat{\Gamma}(E = \varepsilon)$, found from Eqs. (8.18) and (8.20).

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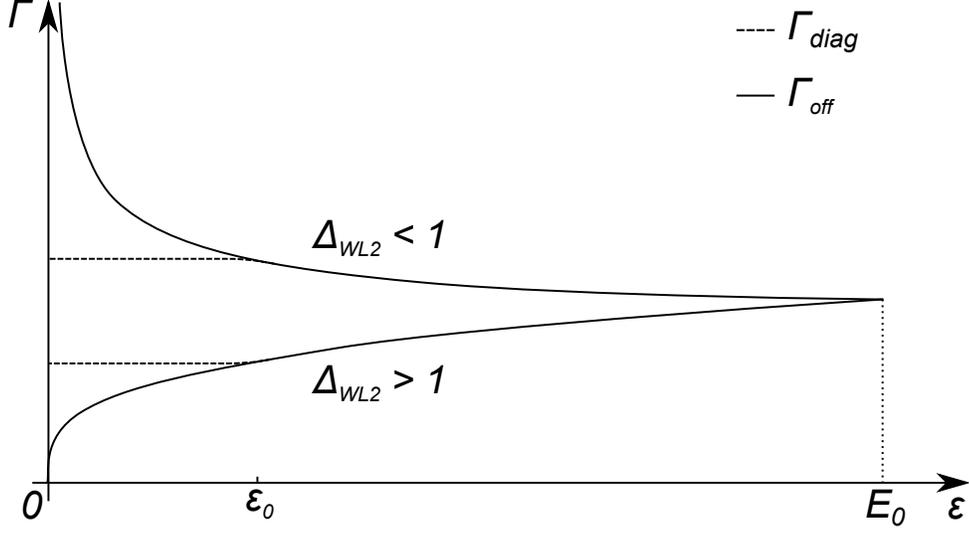
Figure 8.2: Resonant-barrier geometry: $\Gamma_{\mathrm{off}} \propto |\varepsilon/E_0|^{\Delta_{\mathrm{WL2}}-1}$ with the value of Δ_{WL2} depending on the electron-phonon coupling strength. $\Gamma_{\mathrm{diag}} = \Gamma_{\mathrm{off}}$ at $\varepsilon \gtrsim \varepsilon_0$ and saturates at ε_0 (dashed line) for $\varepsilon \lesssim \varepsilon_0$.

The off-diagonal element of the transmission matrix, $\mathsf{T}_{\mathrm{off}}$, is equal to the transmission amplitude in the resonant-barrier geometry, so that

$$\mathcal{T}(\varepsilon) = |\mathsf{T}_{\mathrm{off}}(\varepsilon)|^2 . \tag{8.21}$$

Here $|\mathsf{T}_{\mathrm{off}}(\varepsilon)|^2$ has the following form found from Eq. (8.12) with the substitution (8.19) at $E = \varepsilon$:

$$|\mathsf{T}_{\mathrm{off}}|^2 = \frac{\Gamma_{\mathrm{off}}^2}{(\varepsilon_0 - \varepsilon + \Lambda)^2 + \Gamma_{\mathrm{diag}}^2} , \qquad \Lambda \equiv \frac{\Gamma_{\mathrm{off}}^2 - \Gamma_{\mathrm{diag}}^2}{4\varepsilon_0} . \tag{8.22}$$

For $\varepsilon \gtrsim |\varepsilon_0|$, the diagonal and off-diagonal elements of $\hat{\Gamma}$ are equal and renormalise in the same way, Eq. (8.18). This leads to the transmission coefficients of the Fermi-gas form, Eq. (8.13), but with the renormalised tunnelling rate,

$$\Gamma(\varepsilon) = \Gamma_0 \left(\frac{\varepsilon}{E_0}\right)^{\Delta_{\mathrm{WL2}}-1} , \tag{8.23}$$

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which fully describes the case of resonance. When the impurity level is off-resonance, we substitute ε_0 for $(\varepsilon_0 - \varepsilon)$ in Eq. (8.22) and take into account that Γ_{off} continues to renormalise, Eq. (8.20), while Γ_{diag} saturates:

$$\Gamma_{\text{off}}(\varepsilon) = \Gamma(\varepsilon_0) \left(\frac{\varepsilon}{\varepsilon_0}\right)^{\Delta_{\text{WL2}}-1}, \qquad \Gamma_{\text{diag}} = \Gamma(\varepsilon_0). \qquad (8.24)$$

8.1.2   Resonant conductance

At nonzero but low temperatures T, the two-terminal conductance $G(T)$ is proportional to $\mathcal{T}(\varepsilon)$ with the low-energy cutoff at $\varepsilon \sim T$ (the Fermi energy corresponds to $\varepsilon = 0$). In the off-resonance situation, $|\mathsf{T}_{\text{off}}(\varepsilon)|$ in Eq. (8.22) vanishes with $\varepsilon \to 0$: the resonant level remains decoupled from conduction electrons even when $\Gamma_{\text{off}}(\varepsilon)$ diverges when $\varepsilon \to 0$.

The resonance, $\varepsilon_0 \lesssim \Gamma(\varepsilon_0)$, is described by Eq. (8.22) with $\Lambda = 0$. This corresponds to the Fermi gas expression, Eq. (8.13), with $\varepsilon_0 = 0$ and Γ_0 substituted by its renormalised value, $\Gamma(\varepsilon)$, Eq. (8.23). The critical exponent $(\Delta_{\text{WL2}} - 1)$ is strongly affected by the electron-phonon coupling, as illustrated in Fig. 8.3: without phonons Δ_{WL2} is a monotonically decreasing function reaching 1 at $K = 1$. Crucially, conductance G is either ideal or vanishing at $T = 0$, depending on whether $\Delta_{\text{WL2}} > 2$ or $\Delta_{\text{WL2}} < 2$.

For $\Delta_{\text{WL2}} > 2$, $|\mathsf{T}_{\text{off}}(\varepsilon)|$ in Eq. (8.22) vanishes with $\varepsilon \to 0$, as in the case of a strong electron-electron coupling in the phononless Luttinger Liquid ($K < 1/2$ without pair tunnelling [1]). This happens because $\Gamma(\varepsilon) \to 0$ faster than $\varepsilon \to 0$, i.e. the resonant level remains effectively decoupled from conduction electrons.

On the contrary, for $\Delta_{\text{WL2}} < 2$ we have $|\mathsf{T}_{\text{off}}(\varepsilon)| \to 1$ for $\varepsilon \to 0$ which leads to an ideal resonance, Eq. (8.21).

Thus, the effective decoupling of the resonant electron level from conduction electrons at $\Delta_{\text{WL2}} = 2$ leads to a metal-insulator transition. Fig. 8.3 shows that the electron-phonon coupling shifts the transition towards stronger electron-electron coupling. Note also that
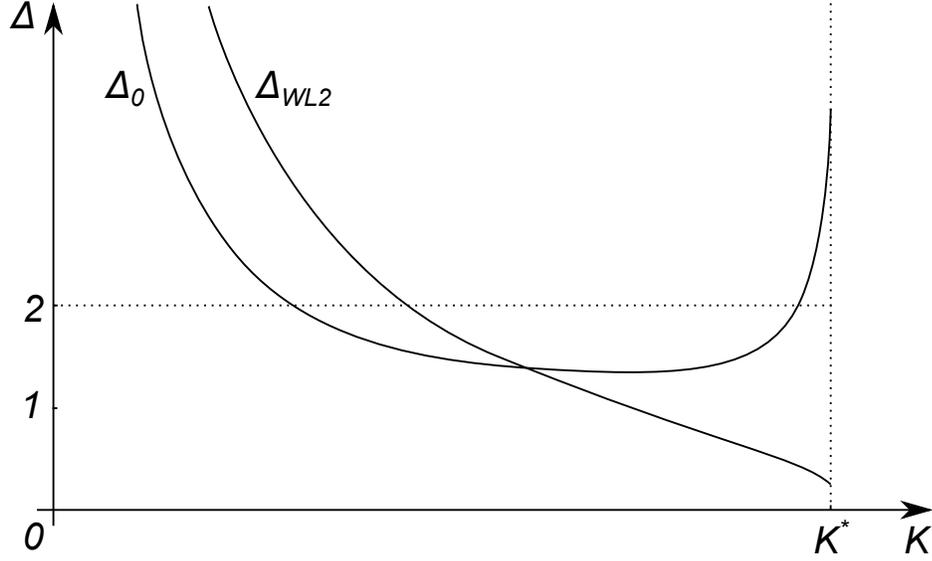
Figure 8.3: The renormalisation group scaling dimensions of the effective resonance width for the resonant barrier geometry, $\Delta_{\mathrm{WL2}} - 1$, and the side-attached geometry, $\Delta_0 \equiv (\Delta_{\mathrm{WS1}} + \Delta_{\mathrm{WL2}})/2$. Here $K^* \equiv \min\{1, v/(v_{\mathrm{F}}\alpha_{\mathrm{ph}})\}$ is the boundary of the applicability region: we demand $\alpha < 1$ to stay away from the Wentzel–Bardeen instability [49].

when $\Delta_{\mathrm{WL2}} < 1$, the effective resonance width diverges with $\varepsilon \to 0$ rather than vanishes as in the phononless Luttinger Liquid.

8.2 Anti-resonant level

In this Section we focus on the side-attached geometry when the resonant level is not embedded into but is rather in direct proximity of the one-dimensional system away from its ends. The description of this case follows closely the material of the previous Section with a few minor changes. The main difference is as follows: in the resonant-barrier geometry we had two electron-phonon subsystems on either side of the resonant level coupled by the tunnelling term (8.4), whereas in the side-attached geometry we have two translation invariant subsystems of chiral (left- and right-moving) electrons coupled by the same term (8.4) but

with the index μ simply denoting one of these two chiral species:

$$\bar{\psi}_- = \bar{\psi}_{\mathrm{L}}\,, \qquad \psi_- = \psi_{\mathrm{L}}\,, \qquad \bar{\psi}_+ = \bar{\psi}_{\mathrm{R}}\,, \qquad \psi_+ = \psi_{\mathrm{R}}\,. \tag{8.25}$$

Note that the tunnelling amplitude t_0 in this case must be the same in both channels, which is, generally, not the case for the resonant-barrier geometry and was only our assumption in the previous Section.

Once the meaning of the index μ has been established for the anti-resonant geometry, the Eqs. (8.6)–(8.12) from Section 8.1 hold true without any changes. The off-diagonal part acquired by Green functions $g_\eta(x,t)$ due to the gauge fields $\theta_\eta(x,t)$ describes resonance-induced backscattering. Therefore, the transmission of electrons past the impurity site without backscattering is proportional to $G_{11}G_{22}$ and one observes resonant reflection, rather than transmission, in this case:

$$\mathcal{R}_0(\varepsilon) = \frac{\Gamma_0^2}{(\varepsilon - \varepsilon_0)^2 + \Gamma_0^2}\,. \tag{8.26}$$

As in Section 8.1, renormalisation group treatment can then be used in order to find the exact scaling behaviour of $\Gamma(\varepsilon)$ at low energies (temperatures). Local correlation functions between fields $\theta_\eta(t)$ yield a different from Eq. (8.15) dimensionless correlation relations, which are, in fact, exactly the same as in Eq. (4.2). This is not at all surprising since the system of interacting electrons and phonons is translation invariant in the side-attached geometry before we couple it to the resonant level.

The right-hand side of the renormalisation equation (8.17) at energies above ε_0 (which corresponds to $E \ll |\varepsilon - \varepsilon_0|$) has a different coefficient instead of $(\Delta_{\mathrm{WL2}} - 1)$. In fact, it turns out that the coefficient is $(\Delta_0 - 1)$, which is the bulk electrons density of states exponent
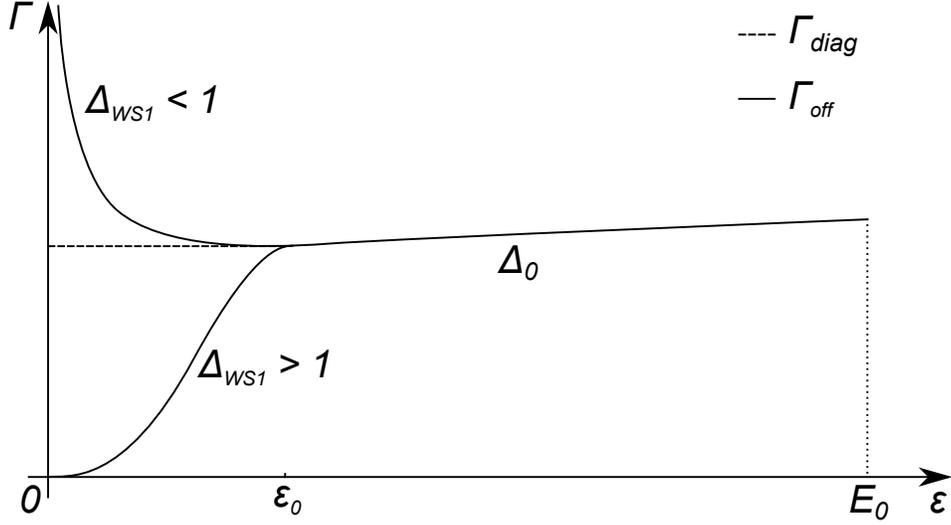
Figure 8.4: Side-attached geometry: $\Gamma_{\mathrm{diag}} = \Gamma_{\mathrm{off}} \propto |\varepsilon/E_0|^{\Delta_0-1}$ for $\varepsilon \gtrsim \varepsilon_0$; Γ_{diag} saturates and $\Gamma_{\mathrm{off}} \propto |\varepsilon/\varepsilon_0|^{\Delta_{\mathrm{WS1}}-1}$ for $\varepsilon \lesssim \varepsilon_0$. The sign of $(\Delta_{\mathrm{WS1}} - 1)$ depends on the electron-phonon coupling strength, while $\Delta_0 > 1$.

(will be proved elsewhere), not the edge one as in the resonant-barrier case:

$$\frac{\mathrm{d}\Sigma_{\mu\nu}(\varepsilon; E)}{\mathrm{d}l} = -(\Delta_0 - 1)\Sigma_{\mu\nu}(\varepsilon; E), \tag{8.27}$$

which can be solved as in the previous Section and gives:

$$\frac{\mathrm{d}\Gamma(E)}{\mathrm{d}l} = -(\Delta_0 - 1)\,\Gamma(E), \qquad E \gg |\varepsilon - \varepsilon_0|. \tag{8.28}$$

8.2.1 Transmission coefficient

Since in the side-attached geometry the off-diagonal element of the transmission matrix, $\mathsf{T}_{\mathrm{off}}$, is equal to the reflection amplitude,

$$\mathcal{T}(\varepsilon) = 1 - |\mathsf{T}_{\mathrm{off}}(\varepsilon)|^2, \tag{8.29}$$

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with $\mathsf{T}_{\mathrm{off}}(\varepsilon)$ defined in the same way as in the resonant-barrier geometry, Eq. (8.22).

For $\varepsilon \gtrsim |\varepsilon_0|$ Eq. (8.28) leads to the following renormalisation equation for $\Gamma(E) \equiv \Gamma_{\mathrm{diag}} = \Gamma_{\mathrm{off}}$:

$$\Gamma(\varepsilon) = \Gamma_0 \left(\frac{\varepsilon}{E_0} \right)^{\Delta_0} , \qquad \Delta_0 \equiv \frac{\Delta_{\mathrm{WS1}} + \Delta_{\mathrm{WL2}}}{2} . \tag{8.30}$$

When the impurity level is off-resonance, the diagonal component Γ_{diag} again saturates at $\Gamma(\varepsilon_0)$, while Γ_{off} continues to renormalise:

$$\Gamma_{\mathrm{off}}(\varepsilon) = \Gamma(\varepsilon_0) \left(\frac{\varepsilon}{\varepsilon_0} \right)^{\Delta_{\mathrm{WS1}} - 1} , \qquad \Gamma_{\mathrm{diag}} = \Gamma(\varepsilon_0) , \tag{8.31}$$

see Fig. 8.4.

8.2.2 Resonant conductance

Similarly to the resonant-barrier geometry, a metal-insulator transition occurs when due to the decoupling of the resonant level at $\Delta_0 = 2$. However, as can be seen from Fig. 8.3, there could be two phase transitions in this case: one as in the phononless Luttinger Liquid with strong repulsive interacting, and an additional one, induced purely by the presence of phonons, at sufficiently strong electron-phonon coupling and relatively weak repulsive electron-electron interaction (K close to 1).

CHAPTER 9

CONCLUSIONS

This thesis has been devoted to the study of the single impurity problem in Luttinger Liquids. The main motivations was to investigate the effects of retarded electron-electron interaction in the system by introducing coupling to inevitable in real systems acoustic phonons. We have focused on the spinless (spin-polarised) case, whereas the inclusion of electron spin degree of freedom would serve as a natural extension of the problem. Throughout the thesis various approaches to calculating low-temperature electron transport through the impurity have been used, such as renormalisation group theory, functional bosonisation method and perturbation theory.

The thesis began with a general introduction into the field of interacting low-dimensional electronic systems. We have outlined the main differences and similarities between physics of interacting electrons in one and higher dimensions by considering the Fermi Liquid theory and is one-dimensional counterpart, Luttinger Liquid theory. This discussion led to the bosonisation method which is commonly used for dealing with strongly correlated systems in one dimension. In the subsequent chapter we have described the basics of the Keldysh technique and introduced the notations used throughout the thesis. We have then presented the functional bosonisation method, which was our tool of choice in this work. It is based on the functional integral approach and uses the Hubbard-Stratonovich transformation in

order to reduce the action describing the system to quadratic form. In this chapter we have rigorously derived the Luttinger Liquid model and shown a general method of calculating various correlation functions. It has been shown that the random phase approximation is exact in one dimension.

The purpose of the following chapter was to provide the reader with a detailed description of the problem. First, we have explicitly derived basic correlation functions, including two-point Green functions for translation invariant system. We have introduced a Luttinger parameter K, which describes the effective strength of electron-electron interactions, and v, the velocity of plasmonic excitations in the system. This was followed by the discussion of the problem of a single impurity embedded in a Luttinger Liquid. We have outlined the main results obtained in Refs. [1] and [64] and, thereby, built a foundation for our main calculations. We have proved the duality relation between the weak and strong impurity limits in the Luttinger Liquid. Also in this chapter, with the help of functional bosonisation, we have developed the description of a Luttinger Liquid coupled to one-dimensional acoustic modes. The presence of phonons strongly modifies the system and results in a Luttinger Liquid two polaronic modes, slow and fast, with the velocities determined by the strength of electron-electron and electron-phonon interactions. One also discovers an intrinsic instability of a system at high values of electron-phonon coupling, called Wentzel-Bardeen instability. When introducing an impurity into the system one should carefully consider its effect on propagation of phonons. Depending on a particular model of the impurity there are several possibilities. Since the renormalisation group-relevant logarithmic corrections come from low-energy contributions, we are only interested in how the impurity affects low-energy phonons. Here only two situations are possible, either the impurity breaks translational invariance of the system or not. Both are addressed separately in the following two chapters of the thesis. The case of a phonon-transparent impurity also includes certain realisations of electron local scattering potential which do not affect phonons at all (e.g., electron density depletion by an

AFM tip).

The aim of the next two chapters was to calculate scaling dimensions of the problem in all previously discussed limits. For arbitrary strong interactions in the system we were only able to do this in the limits of weak and strong electron scattering by the impurity. We have used two approaches for tackling the problem. The first one is the earlier developed functional bosonisation method which allows for a more elegant solution. It, however, could only be applied without major modifications in two situations out of four. For this reason we have also employed a brute force method of direct calculation of the screened polarisation operator in the random phase approximation, from which the scaling dimension can be easily extracted. This approach is somewhat more bulky and requires a solution of an integral equation. In fact, it came as a surprise to us that in it was always possible to solve it in al four cases. This was due to the fact that in all four cases the kernel was either simple or would at least turn out to be separable even for non-trivial retarded phonon-mediated electron-electron interaction. Having calculated the scaling dimensions we could then write down renormalisation group scaling equations for the corresponding coupling constants. Those equations determine electron transport properties through the impurity at low energies (temperatures). We have found the duality between the weak and strong scattering limits, previously only observed for systems with local electron-electron interaction, hold even in presence of phonons. This unexpected relation is one of the main results of this work. Regardless of the strength of interactions, phonon scattering properties of the impurity and other parameters of the system (e.g., ratio of Fermi and sound velocities), upon introducing a weak or strong electron scattering potential one of them is always relevant and the other one irrelevant. The physical meaning is that the system is always either in metallic or insulating state once we have fixed all the system parameters listed above.

Having established the duality relation between two opposite limits in impurity strength, the goal of the next chapter was to investigate how the relation can be violated in real systems

and what effects this would lead to. We have suggested to consider a system in which electron and phonon scattering properties of the introduced impurity are correlated. In particular, we have focused on the situation where the external perturbation controlling the impurity strength acts in the same way on electron and phonon properties. This means that in the limit of weak electron backscattering phonons in the system are completely free in are not affected. On the other hand, strong electron backscattering impurity results also in significant reflection of phonons at the impurity site. Because in the crossover region between the two limits phonon scattering properties change, the duality relation naturally does not hold any longer. Calculations show that in this case in addition to metallic and insulating behaviour one finds a region of parameters which corresponds to some intermediate regime in which an introduces weak impurity is irrelevant, however, a strong scattering potential is relevant. This signals the existence of an unstable fixed point at some intermediate impurity strength. One can study a metal-insulator transition in such a system by controlling externally the correlated electron and phonon scattering strengths.

As we have seen, finding a solution of the single impurity problem for both arbitrary scattering and interaction strengths poses great difficulties. Despite that, major progress has been made when considering in the limit of weak and strong impurity strengths - the approach we had also used so far. However, the next chapter of the thesis has been devoted to another type of perturbative treatment of the problem. We have considered a weakly interacting system with no restriction of the impurity strength. We have succeeded in calculating first order corrections to electron transmission and reflection amplitudes for a system with arbitrary electron and phonon impurity scattering amplitudes. These calculations allow us to obtain the temperature dependence of electron conductance through a barrier of arbitrary strength. They agree with the results obtained in Ref. [6] in the absence of phonons (at $\alpha_{\mathrm{ph}} = 0$). Moreover, a perfect agreement is found when comparing with our results from previous chapters if one goes to the limit where both approaches are valid, i.e. with both

weak interactions and a weak or strong impurity.

In the final chapter we focused on the problem of resonant and anti-resonant tunneling in one-dimensional systems or strongly correlated electrons. We have considered a Luttinger Liquid with electron-phonon coupling, which is hybridised with an impurity carrying a resonant energy level. Two different geometries, an embedded and side-attached impurities, have been studied. Both cases have been previously considered in literature in the context of a Luttinger Liquid, therefore, we our aim has been to investigate the effect of coupling to phonons in such systems. Unlike in higher-dimensions where the resonances are described by the Breit-Wigner formula with a constant resonance width, in one dimension one finds that the effective resonant level width scales as a power-law with a non-universal factor which depends of the strength of interactions in the system. We have calculated the energy-dependencies, both off- and in-resonance, and discovered an additional metal-insulator transition regime at strong electron-phonon coupling for the side-attached geometry.

This thesis is the result of my own work, including the already published and submitted for publication results [48, 74, 57].

APPENDIX A

POLARISATION OPERATOR

Here we derive the polarisation operator in one dimension using Keldysh technique. The definition of a polarisation operator reads for right- and left-moving electrons reads:

$$i\pi_\eta(x - x', t - t') = g_\eta(x - x', t - t')g_\eta(x' - x, t' - t).$$

(A.1)

On a Keldysh time contour the relation between lesser/greater components is:

$$i\pi_\eta^{</>}(x - x'; t, t') = g_\eta^{</>}(x - x'; t, t')g_\eta^{</>}(x' - x; t', t).$$

(A.2)

Using the identity

$$\Theta(t) = -\frac{i}{2\pi} \int \frac{d\omega'}{\omega' - i0} \, e^{i\omega' t}$$

(A.3)

one can derive the following expressions for retarded components of the polarisation operator:

$$\pi_\eta^R(x - x'; t, t') = \Theta(t - t') \left[\pi^>(x - x'; t, t') - \pi^<(x - x'; t, t') \right] , \tag{A.4}$$

$$\pi_\eta^R(\omega, q) = -\frac{1}{2\pi} \int \frac{d\omega'}{\omega' - \omega_+} \left[\pi^>(\omega', q) - \pi^<(\omega', q) \right]$$

$$= -\frac{1}{2\pi} \int \frac{d\omega'}{\omega' - \omega_+} \int \frac{d\varepsilon}{2\pi} \int \frac{dk}{2\pi} \left[g_\eta^>(\varepsilon, k) g_\eta^>(\varepsilon - \omega', k - q) - g_\eta^<(\varepsilon, k) g_\eta^<(\varepsilon - \omega', k - q) \right]$$

$$\tag{A.5}$$

Using that

$$g_\eta^<(\varepsilon, k) = -n_F(\varepsilon) \left[g_\eta^R(\varepsilon, k) - g_\eta^A(\varepsilon, k) \right] = 2\pi i \, n_F(\varepsilon) \delta(\varepsilon - \eta v_F q) , \tag{A.6}$$

$$g_\eta^>(\varepsilon, k) = (1 - n_F(\varepsilon)) \left[g_\eta^R(\varepsilon, k) - g_\eta^A(\varepsilon, k) \right] = -2\pi i (1 - n_F(\varepsilon)) \delta(\varepsilon - \eta v_F q) \tag{A.7}$$

and taking the limit $T \to 0$, we arrive at

$$\pi_R^R(\omega, q) = \frac{1}{2\pi} \frac{q}{\omega_+ - v_F q} , \qquad \pi_L^R(\omega, q) = -\frac{1}{2\pi} \frac{q}{\omega_+ + v_F q} . \tag{A.8}$$

The total polarisation operator $\pi = \pi_R + \pi_L$ yields:

$$\pi^R(\omega, q) = \frac{1}{\pi} \frac{v_F q^2}{\omega_+^2 - v_F^2 q^2} . \tag{A.9}$$

APPENDIX B

JACOBIAN OF THE GAUGE TRANSFORMATION

Several times in the text we have performed the gauge transformation under the interacting electron fields:

$$\psi_\eta(x,t) = \psi_{0\,\eta}(x,t)\, \mathrm{e}^{\mathrm{i}\theta_\eta(x,t)}\,, \qquad \mathrm{i}\partial_\eta\theta_\eta(x,t) = \phi_\eta(x,t)\,, \tag{B.1}$$

where ϕ_η are some bosonic fields. This procedure is sometimes referred to as partial bosonisation. Its purpose is to separate the effects of interactions from the non-interacting electrons described by the wave function $\boldsymbol{\psi}_0(x,t)$. After such transformation all information about interactions is encoded in the correlation function of the bosonic fields $\boldsymbol{\theta}(x,t)$. The Jacobian of this transformation can be written as

$$J[\boldsymbol{\phi}] = \frac{\int \mathcal{D}[\bar{\boldsymbol{\psi}},\boldsymbol{\psi}]\,\exp\left\{\mathrm{i}\sum\limits_{\eta=\pm1}\int \mathrm{d}x\mathrm{d}t\,\bar{\psi}_\eta(x,t)\left[\mathrm{i}\partial_\eta - \mathrm{i}\phi_\eta\right]\psi_\eta(x,t)\right\}}{\int \mathcal{D}[\bar{\boldsymbol{\psi}},\boldsymbol{\psi}]\,\exp\left\{\mathrm{i}\sum\limits_{\eta=\pm1}\int \mathrm{d}x\mathrm{d}t\,\bar{\psi}_\eta(x,t)\mathrm{i}\partial_\eta\psi_\eta(x,t)\right\}}$$

$$= \exp\left\{-\sum_{\eta=\pm1}\sum_{n=1}^{\infty}\frac{\mathrm{i}^n}{n}\mathrm{Tr}\left[g_\eta(x,t)\phi_\eta(x,t)\right]^n\right\}\,, \tag{B.2}$$

where

$$\mathrm{Tr}\left[g_\eta(x,t)\phi_\eta(x,t)\right]^n = \int \prod_{i=1}^{n}\mathrm{d}x_i\mathrm{d}t_i\,\Gamma_\eta^n(x_1,t_1;\ldots;x_n,t_n)\prod_{j=1}^{n}\phi_\eta(x_j,t_j)\,. \tag{B.3}$$

The term Γ_η^n is nothing else than a fermionic loop of order n:

$$\Gamma_\eta^n(x_1, t_1; \ldots; x_n, t_n) = \prod_{k=1}^{n} g_\eta(x_k - x_{k+1}, t_k - t_{k+1}), \qquad (x_{n+1}, t_{n+1}) \equiv (x_1, t_1). \qquad \text{(B.4)}$$

In what follows we prefer to do calculations in (ε, p)-representation, where the chiral free electron Green functions can be written as:

$$g_\eta(\varepsilon, p) = \frac{1}{\varepsilon - \eta v_{\mathrm{F}} p}. \qquad \text{(B.5)}$$

The loop Γ_η^n is a function of $(n-1)$ frequencies and momenta:

$$\Gamma_\eta^n(\omega_1, q_1; \ldots; \omega_{n-1}, q_{n-1}) = \int \frac{\mathrm{d}\varepsilon \mathrm{d}p}{(2\pi)^2} \, g_\eta(\varepsilon, p) g_\eta(\varepsilon + \omega_1, p + q_1) \ldots$$

$$\times \, g_\eta(\varepsilon + \omega_1 + \cdots + \omega_{n-1}, p + q_1 + \cdots + q_{n-1}). \qquad \text{(B.6)}$$

To simplify the notations, we introduce

$$\xi = \varepsilon - \eta v_{\mathrm{F}} p, \qquad \xi_i = \omega_i - \eta v_{\mathrm{F}} q_i, \qquad \text{(B.7)}$$

after which, by using the following identity:

$$\frac{1}{\lambda_1 \ldots \lambda_n} = (n-1)! \int\limits_{0 \leq y_1 \leq \cdots \leq y_{n-1} \leq 1} \frac{\mathrm{d}y_1 \ldots \mathrm{d}y_{n-1}}{[\lambda_1(1 - y_{n-1}) + \lambda_2(y_{n-1} - y_{n-2}) + \cdots + \lambda_n y_1]^n}, \qquad \text{(B.8)}$$

we finally obtain for following expression for the loop Γ_η^n:

$$\Gamma_\eta^n(\omega_1, q_1; \ldots; \omega_{n-1}, q_{n-1}) = \int \frac{\mathrm{d}\varepsilon \mathrm{d}p}{(2\pi)^2} (n-1)! \int\limits_{0 \leq y_1 \leq \cdots \leq y_{n-1} \leq 1} \frac{\mathrm{d}y_1 \ldots \mathrm{d}y_{n-1}}{[\xi + \xi_1 y_1 + \cdots + \xi_{n-1} y_{n-1}]^n}. \qquad \text{(B.9)}$$

The final step is to notice that, according to Eq. (B.3), Γ_η^n is always multiplied by the product

of n ϕ_η-fields, which is symmetric with respect to all possible permutations of (ω_i, q_i), and, hence, permutations of ξ_i. Now, suppose we have swapped ξ_k and ξ_l. Then, by interchanging y_k and y_l, we restore the same integrand in Eq. (B.9). By doing so we have only exchanged these two coordinates in the limits of the integral.

By adding together the expressions

$$\Gamma_\eta^n(\omega_1, q_1; \ldots; \omega_n, q_n) \prod_{j=1}^n \phi_\eta(\omega_j, q_j) \tag{B.10}$$

with all $(n-1)!$ possible permutations of arguments $(\omega_i; q_i)$, we obtain:

$$\sum_{\text{all perm.}} \Gamma_\eta^n(\omega_1, q_1; \ldots; \omega_n, q_n) \prod_{j=1}^n \phi_\eta(\omega_j, q_j) = \int \frac{\mathrm{d}\varepsilon \mathrm{d}p}{(2\pi)^2} (n-1)! \int_0^1 \cdots \int_0^1 \frac{\mathrm{d}y_1 \ldots \mathrm{d}y_{n-1}}{[\xi + \xi_1 y_1 + \cdots + \xi_{n-1} y_{n-1}]^n} . \tag{B.11}$$

Since eventually we integrate over all pairs $(\omega_i; q_i)$ in Eq. (B.3), contributions from each permutation are exactly equal. Therefore, we can substitute Γ_η^n in Eq. (B.3) by its 'symmetrised' version:

$$\widetilde{\Gamma}_\eta^k(\omega_1, q_1; \ldots; \omega_{n-1}, q_{n-1}) = \int \frac{\mathrm{d}\varepsilon \mathrm{d}p}{(2\pi)^2} \int_0^1 \cdots \int_0^1 \frac{\mathrm{d}y_1 \ldots \mathrm{d}y_{n-1}}{[\xi + \xi_1 y_1 + \cdots + \xi_{n-1} y_{n-1}]^n} . \tag{B.12}$$

However, it turns out that for $n > 2$ the above expression is zero. To see that one can integrate over y_{n-1} and then make the shift of variables $(\xi + \xi_{n-1}) \to \xi$ in one of the two remaining integrals, which will cancel them exactly.

Having proved that the only nonzero Γ_η^n comes from $n = 2$, the Jacobian becomes

$$J[\phi] = \exp\left\{ \frac{\mathrm{i}}{2} \sum_{\eta=\pm 1} \int \mathrm{d}x \mathrm{d}x' \mathrm{d}t \mathrm{d}t' \, \phi_\eta(x, t) \pi_\eta(x - x'; t, t') \phi_\eta(x', t') \right\}, \tag{B.13}$$

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which results in substituting $(\hat{V}_0^{-1} - \hat{\pi})$ for $\hat{V}_0^{-1}$ in the corresponding interaction terms, e.g., see Eq. (2.31).

The derivation provided above closely follows the one from Ref. [23]. An alternative derivation has been suggested in Ref. [46, 47].

APPENDIX C

INVERSION

Several times in the text we had to perform an inversion of an operator. The operation is defined as follows. If we have an operator which in q-space has a kernel of the form

$$\hat{K}(q, q') = \hat{a}_q 2\pi\delta(q - q') + \hat{b}_q 2\pi\delta(q + q'),\tag{C.1}$$

where $\hat{a}_q$ and $\hat{b}_q$ are matrices in R/L-space, then the inverted operator has a kernel

$$\hat{K}^{-1}(q, q') = \hat{A}_q 2\pi\delta(q - q') + \hat{B}_q 2\pi\delta(q + q'),\tag{C.2}$$

where

$$\hat{A}_q = (\hat{a}_q - \hat{b}_q \hat{a}_{-q}^{-1} \hat{b}_{-q})^{-1}, \qquad \hat{B}_q = (\hat{b}_{-q} - \hat{a}_{-q} \hat{b}_q^{-1} \hat{a}_q)^{-1}.\tag{C.3}$$

This corresponds to following identity:

$$\int \frac{d\tilde{q}}{2\pi} K(q, \tilde{q}) K^{-1}(\tilde{q}, q') = 2\pi\delta(q - q').\tag{C.4}$$

APPENDIX D

RENORMALISATION GROUP

The general idea behind the renormalisation group analysis is the following. One has a system described by the action S and asks a question how relevant a certain perturbation, described by the action term δS, is at low energies. In order to do so one first needs to introduce a high-energy cut-off, E_0, which sets the maximum energy scale in the system. The cut-off is then lowered from E_0 to some value E by integrating out all the fields with energies in between the two. After this, all fields entering the total action, as well as the time (energy) variables, are rescaled in such a way that the initial unperturbed action S returns to exactly the same form, i.e., stays invariant (at a 'fixed point') under the renormalisation group transformation. The stability of the fixed point is then analysed by looking at how the transformation modifies the coupling constant in δS. The analysis is usually done perturbatively, with the coupling constant being a small parameter, as in our case. One can write a differential (flow) equation describing how the coupling constant changes upon lowering the cut-off, which is then solved in the limit $E \to 0$ to obtain the effective low-energy behaviour. If the perturbation δS grows (decays) under the renormalisation group transformation, then it is classified as relevant (irrelevant). In cases when it is not modified by the transformation it is called marginal. A comprehensive review of the method with many examples of its practical application, including to Luttinger Liquids, can be found in Refs. [77, 78].

high-momentum cut-off.

Here we provide the derivation of the renormalisation group treatment used in Section 4.1.1. The action we analyse is

$$S = S_{\text{e-ph}} + S_{\text{imp}} , \tag{D.1}$$

$$S_{\text{e-ph}} = \sum_{\eta=\pm1} \left[\int \mathrm{d}x \mathrm{d}t \, \bar{\psi}_\eta(x,t) \, g_\eta^{-1}(x,t) \, \psi_\eta(x,t) \right.$$

$$\left. - \frac{1}{2} \int \mathrm{d}x \mathrm{d}t \mathrm{d}x' \mathrm{d}t' \, \theta_\eta(x,t) \, U_{\eta\eta'}^{-1}(x,x';t,t') \, \theta_{\eta'}(x',t') \right] , \tag{D.2}$$

$$S_{\text{ws}} = \lambda_0 \int \mathrm{d}t \, \bar{\psi}_{\text{R}}(0,t) \psi_{\text{L}}(0,t) \mathrm{e}^{-\mathrm{i}[\theta_{\text{R}}(0,t) - \theta_{\text{L}}(0,t)]} + \text{c.c.} , \qquad \theta_\eta(t) \equiv \theta_\eta(0,t) . \tag{D.3}$$

The bare impurity strength λ_0 is assumed to be small and we consider the impurity action S_{ws} as a perturbation. Once we have specifies two energy intervals, $0 \leq |\omega| < E$ and $E \leq |\omega| < E_0$, the fields $\theta_\eta(t)$ can be split into 'slow' (s) and 'fast' (f) modes:

$$\theta(t) = \theta_\eta^{\text{s}}(t) + \theta_\eta^{\text{f}}(t) , \tag{D.4}$$

$$\theta_\eta^{\text{s}}(t) = \int\limits_{0 \leq |\omega| < E} \frac{\mathrm{d}\omega}{2\pi} \mathrm{e}^{\mathrm{i}\omega t} \theta_\eta(\omega) , \tag{D.5}$$

$$\theta_\eta^{\text{f}}(t) = \int\limits_{E \leq |\omega| < E_0} \frac{\mathrm{d}\omega}{2\pi} \mathrm{e}^{\mathrm{i}\omega t} \theta_\eta(\omega) . \tag{D.6}$$

Since the renormalisation group transformation rescales the fields ψ_η and the time variable such that the action $S_{\text{e-ph}}$ stays invariant (these two procedures exactly cancel each other), it is sufficient to only rescale the fields $\theta\eta$, which are responsible for interactions in the system.

We consider the partition function

$$Z = \int \mathcal{D}[\boldsymbol{\psi}(\bar{x},t), \boldsymbol{\psi}(x,t), \boldsymbol{\theta}(x,t)] \, \mathrm{e}^{\mathrm{i}\left(S_{\text{e-ph}} + S_{\text{ws}}\right)} . \tag{D.7}$$

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Since we are studying the action term S_{ws} which contains only local fields at $x = 0$, we can integrate out all other fields:

$$Z \propto \int \mathcal{D}[\boldsymbol{\theta}(t)] \exp\Bigg\{ -\frac{\mathrm{i}}{2} \sum_{\eta=\pm 1} \int \mathrm{d}t\mathrm{d}t'\, \theta_\eta(t) U_{\eta\eta'}^{-1}(t,t') \theta_{\eta'}(t')$$
$$+\, \mathrm{i} \left[\lambda_0 \int \mathrm{d}t\, \bar{\psi}_{\text{R}}(0,t) \psi_{\text{L}}(0,t) \mathrm{e}^{-\mathrm{i}[\theta_{\text{R}}(t) - \theta_{\text{L}}(t)]} + \text{c.c.} \right] \Bigg\}, \qquad (\text{D.8})$$

where

$$\mathrm{i}U_{\eta\eta'}(t,t') = \langle \theta_\eta(t) \theta_{\eta'}(t') \rangle, \qquad \theta_\eta(t) \equiv \theta_\eta(x=0,t). \qquad (\text{D.9})$$

By splitting now the field $\theta_\eta(t)$ according to Eq. (D.4) and integrating out the fast fields, we obtain:

$$Z \propto \int \mathcal{D}[\boldsymbol{\theta}(t)^{\text{s}}] \exp\Bigg\{ -\frac{\mathrm{i}}{2} \sum_{\eta=\pm 1} \int \mathrm{d}t\mathrm{d}t'\, \theta_\eta^{\text{s}}(t) U_{\eta\eta'}^{-1}(t,t') \theta_{\eta'}^{\text{s}}(t')$$
$$+\, \mathrm{i} \left[\lambda(E_0) \int \mathrm{d}t\, \bar{\psi}_{\text{R}}(0,t) \psi_{\text{L}}(0,t) \mathrm{e}^{-\mathrm{i}\left[\theta_{\text{R}}^{\text{s}}(t) - \theta_{\text{L}}^{\text{s}}(t)\right]} \langle \mathrm{e}^{-\mathrm{i}\left[\theta_{\text{R}}^{\text{f}}(t) - \theta_{\text{L}}^{\text{f}}(t)\right]} \rangle_U + \text{c.c.} \right] \Bigg\},$$
$$(\text{D.10})$$

with

$$\langle \mathrm{e}^{-\mathrm{i}\left[\theta_{\text{R}}^{\text{f}}(t) - \theta_{\text{L}}^{\text{f}}(t)\right]} \rangle_U = \mathrm{e}^{-\langle \theta_{\text{R}}^{\text{f}} \theta_{\text{R}}^{\text{f}} \rangle_U + \langle \theta_{\text{R}}^{\text{f}} \theta_{\text{L}}^{\text{f}} \rangle_U}. \qquad (\text{D.11})$$

After the renormalisation group transformation the partition function, Eq. D.10, can be rewritten as

$$Z \propto \int \mathcal{D}[\boldsymbol{\theta}(t)^{\text{s}}] \exp\Bigg\{ -\frac{\mathrm{i}}{2} \sum_{\eta=\pm 1} \int \mathrm{d}t\mathrm{d}t'\, \theta_\eta(t) U_{\eta\eta'}^{-1}(t,t') \theta_{\eta'}(t')$$
$$+\, \mathrm{i} \left[\lambda(E) \int \mathrm{d}t\, \bar{\psi}_{\text{R}}(0,t) \psi_{\text{L}}(0,t) \mathrm{e}^{-\mathrm{i}[\theta_{\text{R}}(t) - \theta_{\text{L}}(t)]} + \text{c.c.} \right] \Bigg\}, \qquad (\text{D.12})$$

where

$$\lambda(E_0) = \lambda(E) \exp\left\{ -\mathrm{i} \int\limits_{E \leq |\omega| < E_0} \frac{\mathrm{d}\omega}{2\pi} \int \frac{\mathrm{d}q}{2\pi} \left[U_{\mathrm{RR}}(\omega, q) - U_{\mathrm{RL}}(\omega, q) \right] \right\}$$

$$= \lambda(E)\mathrm{e}^{-(1-\Delta)\ln(E_0/E)} , \tag{D.13}$$

where we used Eqs. (4.2) and (4.5). If one continues to perform the same transformation over and over again, i.e. integrate out a narrow high-energy band and rescale the fields to bring the action back to its initial form, the following differential equation can be derived for the effective impurity strength $\lambda(E)$:

$$\partial_l \lambda(E) = (1 - \Delta)\lambda(E) , \qquad l \equiv \ln(E_0/E) , \tag{D.14}$$

which is exactly the one we used in Section 4.1, Eq. (4.8).

APPENDIX E

'UNFOLDING' OF THE WEAK LINK GEOMETRY

When solving the problem of electron transmission through a single impurity in the limit of strong scattering (weak link) one comes across the following situation. Before the tunneling term is introduced, the system consists of two absolutely independent half-infinite fermionic subsystems which occupy the space to either side from the impurity ($x = 0$). Each of these subsystems has left- and right-moving electrons which gets reflected once they reach the impurity site. In the most general case, electron wave function acquires an arbitrary phase shift from edge scattering:

$$\psi_{\mathrm{R}}(x = 0, t) = \mathrm{e}^{\mathrm{i}\chi}\psi_{\mathrm{L}}(x = 0, t)\,. \tag{E.1}$$

It is most convenient for us to set $\chi = 0$. The phase of reflection amplitude could be chosen to be different but it would not affect the results since they are not sensitive to electron scattering phases at all.

The above argument allows us to define the 'unfolded' fermionic fields (denoted by the

superscript 'wl') as follows:

$$\psi_{\mathrm{R}}^{\mathrm{wl}}(x,t) = \begin{cases} \psi_{\mathrm{R}}(x,t), & x < 0; \\ \psi_{\mathrm{L}}(-x,t), & x > 0, \end{cases} \qquad \psi_{\mathrm{L}}^{\mathrm{wl}}(x,t) = \begin{cases} \psi_{\mathrm{R}}(-x,t), & x > 0; \\ \psi_{\mathrm{L}}(x,t), & x < 0. \end{cases} \tag{E.2}$$

This procedure is geometrically equivalent to the unfolding of original subsystems of electrons living on a half-line into a system of chiral electrons living on the whole axis, as shown in Fig. E.1.
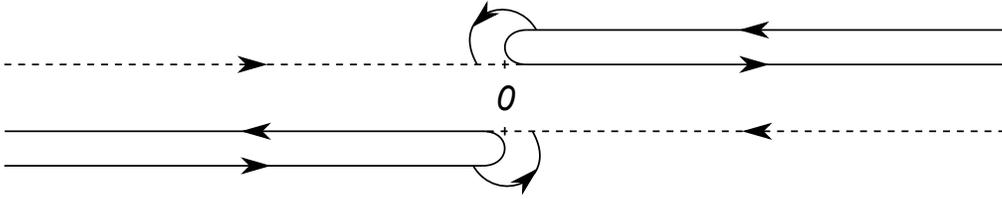

Figure E.1: 'Unfolding' of the weak link geometry.

APPENDIX F

ZERO-TEMPERATURE GREEN FUNCTIONS

Here we provide a list of zero-temperature Green functions. As usual, $\varepsilon \pm i0$ denotes $\varepsilon_\pm$.

$$g_{\mathrm{R}}^{R/A}(\varepsilon, p) = \frac{1}{\varepsilon_\pm - v_{\mathrm{F}} p}, \qquad g_{\mathrm{L}}^{R/A}(\varepsilon, p) = \frac{1}{\varepsilon_\pm + v_{\mathrm{F}} p}. \tag{F.1}$$

In (ε, x)-representation they can be written as matrices in R/L-space:

$$\hat{g}^R(\varepsilon, x - x') = \frac{e^{i\frac{\varepsilon_\pm}{v_{\mathrm{F}}}|x - x'|}}{iv_{\mathrm{F}}} \begin{pmatrix} \Theta(x - x') & 0 \\ 0 & \Theta(x' - x) \end{pmatrix}, \tag{F.2}$$

$$\hat{g}^A(\varepsilon, x - x') = -\frac{e^{-i\frac{\varepsilon_-}{v_{\mathrm{F}}}|x - x'|}}{iv_{\mathrm{F}}} \begin{pmatrix} \Theta(x' - x) & 0 \\ 0 & \Theta(x - x') \end{pmatrix}, \tag{F.3}$$

$$\tag{F.4}$$

A single featureless scatterer can be modelled by a scattering matrix $\hat{S}_0 = \hat{\mathrm{T}}_0 - \hat{1}$. Here $\hat{\mathrm{T}}_0$ is the transmission matrix:

$$\hat{\mathrm{T}}_0 = \begin{pmatrix} t_0 - 1 & r_0 \\ r_0' & t_0 - 1 \end{pmatrix}, \tag{F.5}$$

t_0 and $r_0(r'_0)$ are electron transmission and reflection when coming from the left (right) amplitudes, correspondingly. Since $\hat{S}$ is a unitary matrix they must satisfy $t_0\bar{r}_0 + \bar{t}_0 r'_0 = 1$.

Electron Green function in presence of an elastic scatterer at $x = 0$ is:

$$\hat{G}_0^R(\varepsilon; x, x') = \hat{g}^R(\varepsilon; x - x') + iv_{\mathrm{F}}\hat{g}^R(\varepsilon; x, 0)\hat{\mathrm{T}}_0\hat{g}^R(\varepsilon; 0, x'), \qquad (\mathrm{F}.6)$$

$$\hat{G}_0^A(\varepsilon; x, x') = \hat{g}^A(\varepsilon; x - x') - iv_{\mathrm{F}}\hat{g}^A(\varepsilon; x, 0)\hat{\bar{\mathrm{T}}}_0\hat{g}^A(\varepsilon; 0, x'), \qquad (\mathrm{F}.7)$$

which yields the following expressions:

$$\hat{G}_0^R(\varepsilon; x, x') = \frac{e^{i\frac{\varepsilon_+}{v_{\mathrm{F}}}|x-x'|}}{iv_{\mathrm{F}}}\begin{pmatrix} \Theta(x\,x')\Theta(x - x') + \Theta(x)\Theta(-x')t_0 & \Theta(x)\Theta(x')r_0 \\ \Theta(-x)\Theta(-x')r'_0 & \Theta(x\,x')\Theta(x' - x) + \Theta(-x)\Theta(x')t_0 \end{pmatrix},$$

$$(\mathrm{F}.8)$$

$$\hat{G}_0^A(\varepsilon; x, x') = \frac{e^{-i\frac{\varepsilon_-}{v_{\mathrm{F}}}|x-x'|}}{iv_{\mathrm{F}}}\begin{pmatrix} \Theta(x\,x')\Theta(x' - x) + \Theta(-x)\Theta(x')\bar{t}_0 & \Theta(-x)\Theta(-x')\bar{r}_0 \\ \Theta(x)\Theta(x')\bar{r}'_0 & \Theta(x\,x')\Theta(x - x') + \Theta(x)\Theta(-x')\bar{t}_0 \end{pmatrix}.$$

$$(\mathrm{F}.9)$$

When $x = x'$ this gives:

$$\hat{G}_0^R(\varepsilon; x, x) = \frac{1}{iv_{\mathrm{F}}}\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} + \frac{e^{i\frac{2\varepsilon_+}{v_{\mathrm{F}}}|x|}}{iv_{\mathrm{F}}}\begin{pmatrix} 0 & \Theta(x)r_0 \\ \Theta(-x)r'_0 & 0 \end{pmatrix}, \qquad (\mathrm{F}.10)$$

$$\hat{G}_0^A(\varepsilon; x, x) = -\frac{1}{iv_{\mathrm{F}}}\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} - \frac{e^{-i\frac{2\varepsilon_-}{v_{\mathrm{F}}}|x|}}{iv_{\mathrm{F}}}\begin{pmatrix} 0 & \Theta(-x)\bar{r}_0 \\ \Theta(x)\bar{r}'_0 & 0 \end{pmatrix}. \qquad (\mathrm{F}.11)$$

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