PHD

The child's interpretation of a numerical variable.

Harper, Eon William

Award date:
1979

Awarding institution:
University of Bath

Link to publication

THE CHILD'S INTERPRETATION

OF A NUMERICAL VARIABLE

Submitted by Eon William Harper

for the degree of Ph.D.

of the University of Bath

1979.

Eon W. Harper.

ABSTRACT

The study attempts to outline what are some of the major obstacles to learning the language of algebra. It is in two Parts.

Part 1 is theoretical and is the outcome of a belief that the wrong question might often have been asked about the learning of mathematics, i.e. perhaps the important question to ask is not '<u>How</u> do children learn mathematics'?, but '<u>Why</u> do pupils <u>fail</u> to learn mathematics'?.

Chapters 1 and 2 suggest the reason for this might be due to the fact that mathematics is a system of language systems. As such it demands of the learner a sequence of conceptual adaptations to new meanings for concepts as new languages are introduced.

With particular reference to the algebraic language itself it is suggested that a pupil might be conditioned early in life to think of a letter in arithmetic as an ordered entity with an unique numerical determination, and thus might 'carry over' this understanding into algebra itself.

To comprehend the algebraic language, however, the pupil will need to develop an understanding that the letter is a numeral in its own right - i.e. it is used to convey the symbolic number concept.

Part 2 is empirical. Tasks were specially devised to show that pupils would demonstrate two distinct, logically consistent, usages of a letter, the first matching that of the Mediaeval mathematician, and the second that of the contemporary mathematician.

144 pupils of ages 11-18 years from two grammar schools were interviewed in a structured situation, using the tasks as investigatory material. Responses to each task were arranged into hierarchies of algebraic sophistication, and these were used:

(a) to study the development of the symbolic number concept,

and

(b) to generate three broad levels of algebraic activity.

The results suggest that the symbolic number concept

(i) is available to a small minority of pupils of ages 12-18,

and

(ii) might be associated with dynamic imagery.

ACKNOWLEDGEMENTS

The author wishes to express his gratitude to Professor K. Austwick for his encouragement, support and advice during the course of this study. The author is indebted also to the many pupils and teachers who gave freely of their time and thoughts. Without their help the project would not have been possible.

CONTENTS

Page

TABLES

page

FIGURES WITH DATA

'I do not think x + 1 = 2 is meaningful only when x is replaced by a value. Relations may be expressed by means of variables (in this case number variables). We must have rules for the use of variables as well as rules for the use of numbers.'

Professor R. L. Goodstein

(Personal communication 27.X.75)

INTRODUCTION

"When we come to algebra and have to operate with x and y there is a natural desire to know what x and y really are. That, at least, was my feeling. I always thought the teacher knew what they were but would not tell me".

Bertrand Russell (1927) p.89.

" I know what I remember (about the learning of algebra) from talks by teachers and what struck me in studying textbooks. Many teachers depicted the process of learning this language to me as a mystery. After a shorter or longer period of struggling the pupil finally masters this language though neither he nor the observing teacher knows what has happened. Nobody seems to know what the original obstacles were or how they have been overcome".

Freudenthal (1973) p.340.

The present study is an attempt to gain some insight into how adolescents interpret the symbols a, b, x, y, ... used to convey the concept of a "general" or "symbolic" number. As such it should be regarded as exploratory rather than definitive. In the final event it probably begs more questions than it asks, and certainly many more than it answers.

Freudenthals's observation above suggests that in 1973 relatively little was known about the problems inherent to the learning of the algebraic language. The present study began within this climate of ignorance when little research evidence was available either to suggest a method of approach or a direction in which to turn attention to look for answers.

The particular difficulty associated with a study of algebraic thinking is that of finding a precise boundary between arithmetic and algebra itself. Nunn discussed this bifurcation in some detail during the second decade of the present century(1). His good intentions and important insights appear, however, to have had a marginal, if any, effect. Algebra continues to be a source of much confusion and immense difficulty for many pupils (notwithstanding Bertrand Russell).

Nunn came to the conclusion that arithmetic could not be distinguished from algebra by an appeal to subject matter. What was needed to the contrary was a consideration of the two distinct attitudes which had to be brought to bear upon the same subject matter to give rise on the one hand to an understanding of methods of calculation, and on the other to the processes involved in the calculation:

---

(1) Nunn, T. P., (1919). See Chapter 1.

"But now let the boy's attention shift from the actual manipulation of the figures (used to find the area of a rectangle) to the process which the manipulation follows"; he writes, "and let him observe that the essence of that process is the multiplication of the length of the rectangle by the breadth. At this moment he has crossed the frontier which separates arithmetic from algebra; for it is an important part of the business of algebra, to disengage the essential features of an arithmetical process of given type from the numerical setting which a particular case presents".(2)

Although the present study supports Nunn's observation, it suggests that the distinction between the two subjects is to be found in a more fundamental difference. Namely, in the fact that arithmetic and algebra involve, respectively, two different conceptions of "number". Both the theoretical and empirical aspects of the study aim to provide evidence to support this viewpoint.

Thus for the purposes of the study the frontier between arithmetic and algebra is considered to have been crossed when the learner adopts a usage of the

---

(2) Nunn, T. P., op. cit. p.2.

the letter as a "numerical variable". The main aim of the study is, as suggested above, to attempt an explanation of this statement. Essentially however it means that the criterion once used by textbook authors of the "pre-revolutionary" period[3] for deciding upon what should constitute the content of "Algebra" and "Arithmetic" texts is reinstated here as a criterion that the adolescent is thinking "algebraically".

Textbook authors of that period considered the two subjects to be different by virtue of the fact that Arithmetic did not use letters for "general numbers"[4]. (Nunn objects strongly to the term "generalised number" suggesting it is meaningless[5]).

Today the term is commonly recast in a variety of different ways: e.g. "numerical variable"[6], "arbitrary number"[7], "unknown"[8], "placeholder"[9], "unspecified member of a set"[10], . . . Each of these

---

(3) i.e. prior to the introduction of "modern mathematics" in the 1960's.

(4) See, for example, Channon and McLeish Smith's (1948) algebra course. They write (p.1) "In Algebra numbers are used as in Arithmetic, but letters are also used. Whereas Arithmetic deals with particular numbers, Algebra uses letters for general numbers".

cont. over

terms however, seems to be an attempt to express the same concept.

The existence of an inconsistent terminology throughout classroom resource material suggested at the outset that some clarification of the meaning of the term "algebraic variable" was needed. For that clarification the author turned to a study of the evolution of algebra in the history of mathematics, with the tentative assumption that what has evolved in history must eventually be recreated by the pupil in the classroom.

During this investigation it became increasingly clear that seventeenth century mathematicians had witnessed a revolutionary change in world view, as important, if not more so, than the revolution witnessed by scientists after the introduction of Relativity Theory, and by astronomers after Copernicus. This revolution was the direct outcome of the introduction into mathematics of the language of symbolic formalism by

---

(cont.) This distinction was rigidly applied by some authors. Fawdry's (1931) Arithmetic course (for example) does not include a single letter for numbers. Formulae are introduced but expressed rhetorically throughought.

(5) Nunn, T. P., op. cit., p.6.

(6) See e.g. Skemp, R. R., (1964) p.22.

(7) See e.g. Holt, M., & Marjoram, D.T.E., (1966) p.107.

(8) See e.g. Freudenthal, H., (1973) p.294.

(9) See e.g. SMP Book B (1972) p.6.

(cont. over)

Francois Vieta during the last decade of the sixteenth century, and brought with it the need to adopt a new, symbolic conception of "number".

Chapter 2 discusses the difficulties mathematicians (algebraists) faced in making any real headway in algebra prior to this time, and suggests that pupils will find the same difficulty until the symbolic number concept has been accomodated. The notion that ontogentic development in some sense parallels the historical evolution (phylogenetic development) of mathematical ideas constitutes the essential theoretical framework within which the study takes place. As such this is a departure from traditionally orientated studies of mathematical development.

The major research paradigm for developmental studies is Piagetian, where advances in intellectual capacity are considered to parallel, or be made possible by, a maturation of logical process(11). Thus a pupil's failure to deal successfully with a particular mathematical structure may often be explained as an outcome of a need to acquire new logical operations. For example,

---

(10) See e.g. Skemp, R. R., (1971) p.228.
(In logic the variable is often considered to play the same role as the pronoun 'it' - see Quine, W. V. O., (1962) p.68.)

(11) Inhelder, B., and Piaget, J. J., (1958). See also Piaget, J. J., (1953) and (1972).

Collis suggests that the adolescent at the stage of "concrete generalisations" (13 - 15 years) makes fundamental errors when dealing with algebraic relations because he has not yet developed the operation of reciprocal inversion[12]. Bruner objects strongly to such explanations:

> "Psychological events", he writes, "require explanation in terms of psychological processes and are not fully explicated by translation into . . . .logical terms. Cognitive growth is a series of psychological events. A child does not perform a certain act in a certain way at a certain age because . . . (his) act exhibits a certain underlying logical structure . ... . what is needed for a psychological explanation is a psychological theory. <u>How</u> does a culture . . . . affect his ways of looking at the world? . . . are we any nearer an <u>explanation</u> of a child's solution to a problem to say that the solution presupposed some kind of grasp of the principle of logical implication? Is this not only a more refined and conceivably more useful way of describing the formal properties of the behaviour observed -

---

(12) Collis, K. F., (1974).

much as it would be useful to say that a return by a player in a tennis match indicated that he was able to intersect the ball's trajectory in a fashion that could be described by a particular set of equations?"[13]

Notwithstanding Bruner's objections there is, as Smedlund[14] has aptly pointed out, a circular relation between "logicality" and "understanding" which may have no solution within the Piagetian paradigm, viz: in order to study logical development it is necessary to assume that the subject has understood all the instructions and questions asked of him. To study understanding on the other hand, one must assume that the pupil is thinking logically. The choice as to which direction should be taken appears to be an open one. Thus Donaldson[15] prefers the latter, and accordingly suggests that Piagetian psychology is misguided, both in theory and in practice. Along with Smedlund, however, she is a member of a minority group.

Since the present study attempts to gain an understanding of the meanings pupils give to letters used in algebra it is necessary and, the author believes,

---

(13) Bruner, J. S., et. al., (1966).

(14) Smedlund, J., (1970) and (1977).

(15) Donaldson, M., (1978).

profitable, to presume the pupil to act logically throughout, and to account any misconceptions directly to a mis-match of language. This research strategy was first suggested by Smedlund[16]. The tasks used in the empirical study, and described in Chapter 3, were the outcome of a conscious appreciation that mathematical terms might have distinct meanings in different mathematical systems, i.e. that "logicality" could be brought to bear in more than one language system.

Thus the research strategy used here is not Piagetian. The Piagetian approach has been ably represented by Collis in a number of important studies[17]. Collis claims that mathematical development "parallels" the Piagetian developmental sequence and can be seen to comprise four important stages:

(1) early concrete-operational (7 - 9 years);
(2) middle concrete operational (10-12 years);
(3) concrete generalisations (13-15 years);
(4) formal-operational (16+ years).[18]

The fact that we produce so few mathematicians however, leads to a somewhat puzzling question, i.e. how can it be, if it is true that all (or the majority)

---

(16) Smedlund, J., (1977) op. cit.

(17) Collis, K. F., (1969), (1971), (1972) and (1973).

(18) Collis, K. F., (1974) op. cit.

of adolescents reach the stage of late formal operations by the age of 16+ or 17+, that we are not submerged by mathematicians? Not all pupils develop into capable mathematicians at this age and only a minority, (if folk-lore has any basis in fact) ever comprehend the language of algebra. For this reason it seems that the language itself poses special problems which may be overcome only by the minority.

Thus it may well be that what distinguishes the mathematician from the non-mathematician is that the former develops a meaning for terms such as 'a', 'b', 'x', 'y' . . . not available to the latter. The present study ignores the Piagetian paradigm to explore this possibility.

Alternatively, Chapter 1 suggests that many of the difficulties pupils face in understanding mathematics is due, intrinsically, to the nature of the subject itself, as a system of language systems. Each system harbours its own meaning for key concepts and accordingly its own "reality". In this sense the meaning a child gives to a concept (e.g. "number", "triangle", . .) is related to a particular, but not the only possible, mathematical "reality" which includes that concept. The child's major problem is thus to make successive conceptual adaptions to new meanings for established concepts as new languages are introduced.

The study concludes that there is evidence to believe that a minority of pupils do acquire a new conception of number transcending that required for non-generalised conventional arithmetic. This evidence is gathered from discussions with 144 secondary school pupils from two grammar schools using specially prepared tasks presented in a structured situation. Task development and experimental design is discussed in Chapters 3 and 4.

Pupil responses to each task were analysed and arranged in ascending hierarchies of algebraic sophistication (Appendix II) and this data used to generate what appear to be three levels of algebraic activity.

The first, the "level of fictitious measures" is associated with an interpretation of the letter as an object with a unique, unknown content; the second, the "level of discovered content" is associated with the letter treated as a "pigeon hole" or "box" for numerals; and the third level, that of the "species", is associated with the letter regarded as a symbolic number. Chapters 5, 6, and 7 respectively illustrate these levels using pupil transcripts.

When a letter is used as an object with a unique measure, no "variation" is possible in our contemporary sense of the word. When the letter is used as a a "pigeon hole" or as a "symbolic number" however, "variation"is possible. Chapter 8 discusses the three forms of variation associated with different conceptions of the letter.

The study ends with a discussion of the implications for teaching and research, and includes a commentary upon an important paper, published by Kuchemann[19] during the later stages of the present project, which supports consequentially, if not substantially many of the findings here.

(19) Kuchemann, D., (1978).

CHAPTER 1 : MATHEMATICAL REALITIES

"The problem of Universals ought to be reconsidered in the light of the fact that as knowledge grows languages change".

Lakatos (1976) p.92.

1.1. Abstract

This chapter suggests that mathematics is essentially a piecemeal of languages distinguished by different axiomatic systems.

In the classroom the introduction of a new language system often demands of the learner a fundamental change in interpretation of already familiar concepts. For this reason the teacher needs to anticipate, and so warn pupils of, any need to change prior developed expectations.

This view is used in Chapter 2 to explain how the introduction of a new meaning for letters during the seventeenth century radically changed the course of mathematics, and in Chapter 3 to explain the theoretical ideas which determined the nature of the tasks used in the empirical study.

1.2. Numerical realities

Popper[20] draws attention to the fact that our expectations of reality determine the nature of the world in which we live. His views seem to be shared, in part, by both Polanyi[21] and Kuhn[22] each of whom illustrate with numerous examples why it is that scientists must accept that 'fact-finding' activities are determined by their own tacit assumptions about what constitutes "reality", and does not take place in a void.

'Normal science' is conducted in an established theoretical paradigm where certain habits of mind dictate activity. Thus for Koestler, the creative act is the 'defeat of habit by originality'[23].

Habits of mind are brought to bear in new situations and often lead to misconceptions or misinterpretations of the facts before us. An interesting experiment by Bruner and Postman[24] illustrates this phenomenon in the case of perception.

---

(20) Popper, K., (1972). See in particular the appendix "The bucket and the searchlight: Two theories of knowledge", pp.341-361.

(21) Polanyi, M., (1958).

(22) Kuhn, T., (1970).

(23) Koestler, A., (1969) p.96.

(24) Bruner, J. S. & Postman, L., (1949). Reported in Kuhn, T., (1970) p.63.

Experimental subjects were asked to identify on short and controlled exposure a series of playing cards containing anomalies such as a red six of spades. In each experimental run a single card was displayed to a single subject in a series of gradually increasing exposures. After each exposure the subject was asked what he had seen and the run was terminated by two successive correct identifications.

Even on long exposures, the anomalous cards were nearly always identified as normal (e.g. the red six of spades was often "seen" as the six of hearts). At forty times the average exposure required to recognise normal cards for what they were, nearly ten per cent of the anomalous cards were still not correctly identified. The majority of subjects merely interpreted what they saw in terms of their prior prepared conceptual strategies. A variety of similar studies illustrate precisely the same point, i.e. that the perceived size, colour, shape, etc. of experimentally displayed objects varies with the subjects' previous training.[25]

Kuhn[26] and Koestler[27] draw attention to the parallel phenomenon in scientific thinking with examples

---

(25) Hastorf, A. H., (1950); Bruner, J. S., Postman, L., Rodrigues, J., (1951); Stratton, G. M., (1897).

(26) Kuhn, T., (1970) op. cit.

(27) Koestler, A., (1973).

from man's changing vision of the universe. The experimental subjects in Bruner and Postman's experiment clearly "saw" things differently when the anomalies were recognised. After each successive scientific revolution suggests Kuhn (e.g. the Copernican revolution) scientists appear to operate also in what seems to be a new reality, often seeing things which hitherto had gone unnoticed.(28)

In the pre-Copernican and post-Copernican world the terms "motion", "universe" and "planet" took on different meanings and would thus call to mind different images for scientists working within each paradigm. The word "motion" for the pre-Copernican astronomer would in all probability be associated with an image of a geocentric universe, whereas for the post-Copernican astronomer the term would be interpreted in terms of an heliocentric model.

Today, we can perhaps image each interpretation - but this was not necessarily the case for pre-Copernican astronomers. Very few scientists had any conception of curved space until after the Einsteinian revolution.

---

(28) Kuhn, T., (1970) op. cit. p.116.

The fact that "reality" is not present in the same form for successive generations of scientists leads readily to the question as to whether reality exists in the same form for all men at a particular time in history, and across cultural barriers.

Cross-cultural studies by Whorf[29], Sapir[30], Lee[31], and others have led these researchers to reject the usual view of a constant relationship between language and thought and to suggest that each language embodies and perpetuates a particular world view. The speakers of a language are here considered to be partners to an agreement to see and think of the world in a certain way; the world can be structured in many ways; and the language learned as a child directs the formation of his particular structure, This departs from the common-sense notion in that (a), it holds that the world is differently experienced and conceived in different linguistic communities, and (b) it suggests that language is causally related to these psychological differences.

---

(29) Whorf, B. L., (1940) and (1950).

(30) Sapir, E., (1949).

(31) Lee, D. D., (1938). See also Bruner, J. S., (1974) Chapter 2 for educational implications.

Goodstein has suggested that such infamous unsolved problems of mathematics as 'Fermat's Last Theorm' express the need for a new mathematical language,(32) and Lakatos has pointed out that Descarte's failure to establish Euler's theorem about polyhedra was due to the fact that he had no conception of 'edge' and 'vertex'. Euler introduced these concepts when he first proposed his theorm, indicating that in some sense he had perceived polyhedra differently to Descartes.(33)

The historical evolution of mathematics seems to support Goodstein's view that mathematics develops step by step as new languages are created. Thus mathematicians within each period tend to operate within a dominant theoretical paradigm as in science.

In Antiquity, until the Pythagorean 'discovery' of the incommensurability of $\sqrt{2}$, the dominant theory was arithmetic. According to Pythagoras everything was number (by which he meant 'everything could be expressed in terms of the ratios of counting numbers').

The failure of the theory to account for incommensurable lengths, and the thorny Zeno paradoxes, however, eventually led through Eudoxus and Euclid to the establishing of geometry as the dominant paradigm.

---

(32) Goodstein, R. L., (1965) p.90.

(33) Lakatos, I., (1976) p.6.

Geometrical thinking dominated the mathematicians' activities until the seventeenth century when Descartes reinstated arithmetic and merged the two subjects through the medium of analytic geometry. During the present century the work of Frege, Russell and Whitehead has established logicism as the dominant theory.

Changes in dominant theory seem often to be preceded by a period of crisis in the community; often caused by new discoveries which contradict prior expectations, by paradox, or by a desire for a 'completeness' which gives rise to the introduction of new and more powerful explanatory concepts(34). Thus Eudoxus responded to the Pythagorean dilemma with the theory of proportions, and logicism is underpinned by the concepts of 'set' and 'one to one correspondence'. These concepts have in turn given rise to their own paradoxes and to a variety of attempts to overcome them, despite the fact that these were originally hoped to provide a complete foundation for mathematics(35).

---

(34) See Wilder, R.L., (1968) and Kline, M., (1962) for an analysis of the cultural and psychological forces which affect the evolution of mathematical ideas.

(35) See Wang, H. (1974) and Goodstein, R.L., (1976) for discussions concerning the relationship of set theory to arithmetic.

If mathematical evolution can be explained as an outcome of a desire for 'completeness', i.e. a desire to construct a language system which dispenses with paradox and 'exceptions to the rule', it is likely, by analogy, that this is precisely what causes the child who wants to learn mathematics to develop new mathematical ideas. Equally, if the introduction of new concepts give rise to crises in the community, and to a need to make conceptual adjustments, then again the introduction of new concepts in the classroom may be assumed to create the same need in the child.

Experience of everyday classroom situations and the content of numerous articles in mathematics education journals, suggests that there are certain concepts in mathematics which prove to cause many adolescents great difficulty. In particular the concept of a fraction, an imaginary number, and a negative number, seem to create difficulties which are not yet fully understood.

A consideration of what is entailed in accepting these concepts in relation to concepts previously attained however, suggests a possible reason both for the child's problems and for the reluctance of mathematical communities in the course of history to welcome the ideas with open arms when these were first proposed.

Firstly however, consider some of the problems associated with the learning of geometry and in particular with the learning of non-Euclidean geometries.

Mathematics in schools soon develops in the child important assumptions about the meanings of such terms as "parallel", "triangle", "plane" and "point". Without exception, geometry for the under 16's means Euclidean geometry, and "triangle" means a figure with three 'straight' sides whose angles sum to 180°.

Euclidean geometry grew out of activities with the drawing board, the straight edge and the compass and led to an appropriate interpretation of the nature of space. In Euclidean geometry it is assumed that there is just one line which can be drawn parallel to a second through a point in the plane - the content of the famous "Parallels Axiom".

When non-Euclidean geometries are introduced however, all previous assumptions, interpretations, and expectations, need re-appraisal. Here the Riemann triangle has an angle sum greater than 180°, and the Lobachewsky triangle an angle sum less than 180°. In Riemann geometry no lines can be drawn parallel to a second through a point in the plane, and in Lobachewsky geometry two lines can be drawn parallel to a point in the plane[36].

The non-Euclidean geometries grew out of attempts to provide a complete and secure foundation for Euclidean geometry but, contrary to all expectations, gave rise to totally new self-consistent geometries underpinned by different axiomatic systems.

Neither geometry can be reduced one to the other and each constitutes a different mathematical system, sharing common terms, each of which have different usages in each system. The Euclidean "triangle" is not the Riemann "triangle" and neither of these is the Lobachewsky "triangle". Equally, Riemannian and Lobachewsky geometry give rise to alternative models of space, and any attempt to interpret either in Euclidean terms is doomed to failure.

To "understand" the non-Euclidean geometries a radical change in conceptual outlook is required of us, a change which numerous nineteenth century mathematicians and scientists resisted when these were first introduced. By analogy one would expect a similar resistance from pupils in the classroom who have been brought up in the Euclidean tradition.

---

(36) See Kasner, E. D., and Newman, J., (1965) Chapter IV for an interesting and illuminating discussion.

Riemann and Lobachewsky introduced new meanings for well-established terms such as 'parallel', 'plane', and 'triangle'. In the same way the introduction of new 'types' of arithmetical objects might also require a change in meaning of the established conception of "number".

It was pointed out above that the recognition of the "monster"(37) $\sqrt{2}$, had an important effect upon Greek mathematics. It also totally destroyed Pythagoras' cherished view that the world <u>was</u> 'commensurable number'(38) - which is ironic in view of the fact that the monster emerged from the famous theorem about right-angled triangles which now bears his name.

'$\sqrt{2}$', which contradicted all previous expectations associated with the meaning of the term 'number' was eventually incorporated into the body of mathematics - but not readily by those previously committed to seeing the world in terms of the language of commensurable quantities only.

The concept of a negative number appears to have met a similar resistance. First used by Fibonacci in the Twelfth century to deal with problems of profit and loss(39) it was not readily accepted as a part of

---

(37) This term is borrowed from Lakatos, I., (1976) op. cit.

(38) See Russell's comment p.54-55 (1967).

(39) Quoted in Hooper,A(1961) p.368.

mathematics until the later decades of the sixteenth century.

Diophantus consistently dismissed equations of the form '5x + 20 = 4' as 'absurd'[40] and the majority of mediaeval mathematicians rejected negative roots of quadratic and cubic equations. Tied almost exclusively to thinking in geometrical terms where a line had 'length' but no 'direction' the concept of a negative number could not be entertained. Many pupils appear to have a similar problem to that of Diophantus (although they might not be quite so expressive in their objections), when the negative numbers are met for the first time. The difficulties the concept creates are well known.

Many and varied suggestions have been made as to how the concept might be introduced[41], but the real nature of the difficulty it presents remains something of a mystery. However, an analogy with the difficulties inherent in learning to deal with non-Euclidean geometries might help to point to the source of the problem here.

---

(40) Hooper, A. (1961) op. cit. p.90.

(41) See e.g. the Mathematics Association Report on the teaching of arithmetic (1964); The Schools Council "Mathematics for the Majority" teaching manual "Number Appreciation" (1971); and the Schools Council "Critical Review" report on "Number" (Demter, J.,

cont. over

Non-Euclidean geometries generate difficulties because they demand changes in prior-prepared conceptual understandings of terms such as "triangle", "line", "plane" etc. Here there is a prior commitment to a tacit range of assumptions and expectations associated with Euclidean geometry itself, and developed, perhaps, through exclusive dealings with that system.

The changes in meaning of familiar terms, demanded when non-Euclidean geometries are to be assimilated, are linked to changes made in the axiomatic basis of Euclidean geometry itself i.e. to the very assumptions which made Euclidean geometry possible. Thus the non-Euclidean geometries may be considered to establish a new geometrical "reality".

The reality of Euclidean geometry is the reality of the drawing-board and of lines produced with a straight edge. The reality of the non-Euclidean geometries on the other hand, is a reality of curved lines and curved space. Once conditioned into the Euclidean system the non-Euclidean systems may be difficult to appreciate.

---

and Cundy, M., (1977)). Each draws the teacher's attention to the need to point to the similarity of structure between e.g. the counting number system and the system of directed numbers. The suggestion to be made here is that it is also the <u>differences</u> which are crucial both from the psychological and pedagogical viewpoint.

When arithmetical systems are considered an important parallel emerges. The Ancient Greeks lived in a counting-number reality. Here each counting number was assumed to be a solution of an equation of the form 'x + a = c', where 'x' is an unknown, and 'a' and 'c' counting numbers.

For equations to be meaningful 'a' had to be less than 'c'. The Greek arithmetical reality was therefore underpinned by an assumption that such equations had <u>at most</u> one solution. Some equations had none.

It is therefore understandable that any suggestion that an equation such as

'x + 2 = 1'

has a meaningful solution, will invite immediate confusion - if not disbelief.

Yet in order to feel comfortable with the negative numbers an acceptance that such an equation is meaningful is totally necessary. The directed numbers (dealing only, for illustrative purposes, with the 'whole' numbers), are solutions of equations of the form 'x + c = a', where any one of the relationships $a > c$, $c > a$, $c = a$ is possible.

That is, a commitment to the directed number system is associated with an assumption that equations of this form have <u>exactly one</u> solution.

Thus the introduction of directed number carries with it a demand that the learner makes a fundamental change in assumption about the number of solutions an equation has, and commits himself to the new system.

A desire to universalise the subtract operation to achieve "completeness" eventually led mathematical communities to accept the negative numbers when it was realised that no inconsistencies arose. In turn this meant accepting (-1), (-2), . . .each as a meaningful mathematical entity in its own right, along with the "counterparts" (+1), (+2). . . .

Here a new meaning for "number" is interwoven into the fabric of mathematics. It was only after centuries of hesitation, however, that the new reality associated with this change in meaning was finally accepted by mathematical communities, and only then that the new objects were used with any vestige of confidence.

Continuing the analogy with geometry, it seems clear that the new arithmetical reality is different to that previously experienced by virtue of the changes in the properties of "number" the new system brings with it. Thus, in Euclidean geometry, the angle sum of a triangle is 180°. In the non-Euclidean geometries however the angle sum is either greater or smaller than 180°. That is, important changes occur in the properties of similarly named objects in each system. These changes are associated with new "images" or "models". Thus the Riemann triangle has to be modelled on the sphere, and the Lobachewsky triangle on the pseudosphere.

Equally when it comes to the arithmetical realities differences in the properties of similarly named objects are also evident. In the real number system, it is possible, for example, to combine two elements, under the operations 'x' or '+' and arrive at a result smaller than either of the original elements. This is decidedly not a property of elements of the counting number system.

The parallel discussed above, and the reactions of mathematical communities faced with the problem of making changes in conceptual outlook to accomodate each system are so similar, that this suggests the need to make successive adaptations to new systems may be an important obstacle to learning. To enter a new reality, a "deconditioning" process has to take place.

The argument is supported by the fact that the parallel continues when the imaginary numbers are introduced. Here there is a need to incorporate into the body of mathematics an element 'i' which contradicts again a range of tacit assumptions about the properties of numbers previously experienced - namely that the square of any number is positive.

Again, as in the step from the counting number to the directed number system, important changes in conceptual understanding of the meaning of the term "number" have to be appreciated. In the 'real' number system the ordering of any two elements is well-established and in a sense intuitively supported by counting and measuring activities. The imaginaries however, are not "ordered" in this same sense. Here the idea of a "modulus" has to be entertained.

Controversy over the imaginaries raged for many decades after their initial introduction. Again, mathematical communities, faced with the need to construct a new reality resisted, sometimes with hostility, the conceptual demands made upon them.[42]

---

(42) The seventeenth century English Mathematician John Wallis admonished his timid colleagues in the "Arithmetic" with the following observation: 'These Imaginary Quantities (as they are commonly called) arising from the supposed Root of a negative square (when they happen) are reputed to imply that a case proposed is Impossible. And so indeed it is, as to the first and strict notion of what is proposed. For it is not possible that any Number (Negative or Affirmative) multiplied

cont. over

At any particular time it is necessary to be committed to a particular reality. But such a commitment can both readily lead to error, and to a refusal to entertain new possibilities. Thus the subjects in Bruner and Postman's experiment reported earlier were conditioned into a particular way of seeing playing cards. This way of seeing was 'carried over' into the experimental situation, so that changes made in the make-up of the cards went unnoticed. Bruner reports that the experiment caused some of the subjects a good deal of distress when the anomalies began to be appreciated. Habit, as Koestler[43] has argued dictates activity.

---

by itself can produce (for instance (-4)), since that like signes (whether + or -) will produce + and therefore not -4.

But it is also Impossible that any Quantity (though not a supposed Square) can be <u>Negative</u>, since that it is not possible that any <u>Magnitude</u> can be <u>less than nothing</u> or any <u>Number Fewer than None</u>. Yet is not that supposition (of negative Quantities) either Unuseful or Absurd when rightly Understood. And though, as to the base Algebraic Notation, it impart a Quantity less than nothing yet, when it comes to a physical application it denotes as Real a Quantity as if the sign were +, but to be interpreted in the contrary sense'. Quoted in Smith, D. E. (1958) p. 260.

(43) Koestler, A. (1965)

Precisely the same phenomenon can be experienced in everyday language. Young children, with only singular or particular meanings available for terms often "misinterpret" statements made by the adult (this is not restricted to children alone, of course). The following is an example from the author's own experience.

The author's son, three years old at the time, had been listening to a BBC newscaster reporting an earthquake disaster in Turkey, in which many people had lost their lives and thousands more left homeless.

Rushing breathlessly with the news the three year old gasped out:

"Do you know, ten thousand people have lost their homes in Turkey. Isn't that sad".

The author sympathised, happy that the child appeared to be developing a social conscience.

But then the child added "Do you think they'll ever find them again?"

This "error" is identical in form to the kind of error made by Bruner and Postman's subjects. The child has developed a meaning for a term and interpreted a statement in the only way made possible by prior experience. Again habitual interpretation is "carried over" into an incongruous context.

Precisely the same problems may occur in mathematics learning because of the piecemeal nature of the subject itself.

Thus the common error:

'$\frac{1}{2} + \frac{1}{3} = \frac{2}{5}$'

suggests that the pupil making it remains committed to the counting number reality, and continues to give counting-number meanings to the terms '1', '2', '+', etc. The pupil does not "see" either '$\frac{1}{2}$' or '$\frac{1}{3}$' as a number in its own right. What he appears to "see" to the contrary are the symbols '1', '2' and '3' separated by horizontal lines.

But the symbols '1', '2', '3' belong to the counting-number system with which he is already familiar and the rules for adding counting numbers are well-established.

The error suggests that the pupil has not made the necessary conceptual adjustment required to allow him to accept that equations of the form:

'$2x = 1$' always have a solution, and therefore that 'x' is a number in its own right (i.e. he continues to believe equations of the form '$ax = b$' have at most one solution).

How such a conceptual adjustment takes place and how it can be best helped by teaching seems to be a question of fundamental importance for psychology.

The discussion leads to an important question for teaching. Paradoxically, it would seem that although teaching may be necessary to induct a pupil into a particular mathematical system the experience itself can develop habitual ways of interpreting data which might later prove to be an obstacle to further development.

The need to resolve this dilemma appears to be an important one, and might perhaps only be overcome by the teacher remaining constantly aware that at any time he/she might be conditioning a pupil into a way of seeing which will later need to undergo radical adjustment.

Thus the primary school teacher supports the counting number reality, At some later time the child will need to make an adjustment to the real number system. Here the expectations he has developed during counting activities and operations with the whole numbers will need to undergo review.

It seems that the teacher will need to take these points into account and, when the real number system is contemplated, make every attempt at the outset to explain to the pupil that in the new system now to be introduced some quite <u>unexpected</u> results are to be <u>expected</u>.

Thus it would appear to be as important, if not more so, to point to the difference in the meanings given to "number" in one mathematical system, as it is to suggest there are important similarities.

> "Classical mathematics", writes Goodstein, "is a very complicated collection of systems, the various systems having terms in common, but the usages of the terms being frequently vastly different in the different systems. In particular "number", "proof", "rational", "real", "equation", are used in a multitude of different ways. This would not be a source of confusion were it not for the temptation, that is strongly felt by mathematicians, to think that there is only one criterion of validity. . . Through how many changes has the term 'circle' passed since the time of Archimedes!"[44]

---

(44) Goodstein, R. L., (1965) op. cit. p.84.

1.3. Summary

Mathematical reality changes with mathematics itself as new meanings for established concepts are introduced into the language. The reactions of communities of mathematicians to the introductions of new meanings for concepts suggest that conceptual security often has to be destroyed when a new mathematical system is to be accomodated. Pupils may, therefore, face a similar dilemma. In particular this would be the case if ontogenetic development recapitulates the phylogenetic development of mathematical ideas. The present study assumes that in some sense this is the case.

Both Poincare[45] and Poyla[46] have also proposed that Haeckel's "fundamental biogenetic law" about ontogony recapitulating phylogeny may have important applications in the field of mental development, and in particular in the case of mathematics.

Certainly, the kinds of errors pupils make (e.g.

---

(45) Poincare, H. (1973) - see below;

(46) Polya, G., (1962).
Poincare writes: "Zoologists maintain that the embryonic development of an animal recapitulates in in brief history the whole history of it's ancestors throughout geological time. It seems it is the same for the development of minds". (C. B. Halstead's (1913) authorised translation p.437).

the error '$\frac{1}{2} + \frac{1}{3} = \frac{2}{5}$' reported earlier) and the notation they use, often suggests this might be true. These phenomena would not be such a point of interest were it not for the fact that it is unlikely that the errors and notations have been learned from textbooks or in the classrooms.

Errors of this kind clearly do not originate from the teacher. Neither, however, do such notations as 'm5', 'n4' etc., commonly found in pupils' work. These are positively discouraged in favour of the modern '5m', '4n' notations. They were, however, standard practice for a number of mediaeval algebraists.

Polya uses the "biogenetic principle" to suggest a "genetic" approach to curriculum planning which would aim to help the child recapitulate all the great steps taken by mathematicians throughout the centuries.(47) (One would naturally attempt to help them avoid making the same great errors). The present study uses the "biogenetic principle" to help explain pupil responses to algebraic material, and suggests that two distinct interpretations of letters associated with two algebraic "realities" are to be found in pupils' thinking correlated with distinct meanings given to algebraic terms prior to, and after 1600.

---

(47) Polya, G., (1965) p.132.

Chapter 2 discusses the historical evolution of algebra, and shows that a new meaning for letters was introduced by Vieta in 1591 which gave rise to the possibility of a new means of interpreting both algebraic and geometrical data. This recognition was used to help devise the tasks for the empirical study, described in Chapter 3.

CHAPTER 2 : ALGEBRAIC REALITIES

"The invention of variables was perhaps, the most important event in human evolution. The command of their use remains the most significant achievement in the history of the individual human being. Ordinary algebra simply carries to a higher stage of usefulness in a special field the device which common language employs over the whole range of discourse."

P. Nunn, (1919) p.8.

"A variable is ambiguous in its denotation and accordingly undefined".

Russell and Whitehead, (1927) p.4.

2.1. Abstract

This chapter outlines the implications for our understanding of the term"algebra" of the introduction into mathematics of the language of symbolic formalism by Francois Vieta during the final decade of the sixteenth century.

It is suggested that his introduction of the "species" concept brought with it a new way of interpreting geometrical data and made possible the development of the calculus and function theory. Adopting the concept appears to be consistent with an acceptance that the number '$x = \frac{0}{0}$' is a meaningful mathematical entity.

The Tasks described in Chapter 3 were developed from the ideas expressed here.

2.2. The Vietan Revolution

In Western culture, prior to the dawn of the seventeenth century, algebraists restricted their activities to seeking a variety of means of extracting the true numerical identity of a letter from a given equation. Equations almost invariably contained only one letter and all coefficients were whole numbers. Equations in two or more unknowns were rarely entertained and generally dismissed as 'indeterminate'.

In total, the achievement of algebraists during this period, constituted little more than an 'accumulation of an haphazard collection of rules for solving a variety of equations'.(48) After 1600 however, the rapid influx of new methods of solution and the development of new mathematical topics, suggests that a radical change in view had taken place.

Until 1600 the body of mathematics was largely geometrical with some algebraic and trigonometrical appendages. After 1650 however, algebra became established not only as an effective methodology for dealing with all branches of mathematics - as in co-ordinate geometry and theories of motion - but also as an end in itself. Pure geometry was eclipsed for about a hundred

---

(48) Dantzig, T. (1954) p.86.

years and became at best an interpretation of the algebraic language and a guide to algebraic thinking through the medium of co-ordinate geometry.

Seventeenth century mathematicians were to prove to be some of the most productive in mathematical history. Their activities stretched into many fields, both old and new. They enriched classical topics with new results, cast new light upon ancient fields, and created new topics for mathematical research. Descartes unified algebra and geometry into analytic geometry which was later to stimulate the invention and guide the evolution of the calculus at the hands of Newton and Leibnitz. Fermat dealt confidently with 'indeterminate' equations and, with Pascal, founded the mathematical theory of probabilities. Kepler introduced celestial mechanics, and Galileo the mechanics of freely falling bodies - the beginnings of the theory of elasticity.

Chapter 1 drew attention to the fact that mathematics can be considered to be a piecemeal of language systems each harbouring it's own reality by virtue of the fact that key concepts take on different meanings in each system. It is therefore of particular interest to the purposes of the study that the seventeenth century innovations outlined above followed closely upon the heels of the publication of Francois Vieta's "Introduction to the Analytical Art"(49) (1591), in

(49) See Appendix to Klein, J. (1968) which includes the first English translation.

which is to be found the following important passage:

> "In order that this work be assisted by some art, let the given magnitude be distinguished from the undetermined unknown by a constant, everlasting, and very clear symbol as for example, by designating the unknown magnitude by means of the letter A, or some other vowel E, I, O, U, or Y, and the given magnitude by means of the letters B, G and D, and other consonants".(50)

The innovation suggested here is deceptively simple and might tempt us to dismiss Vieta as a mathematician whose major contribution to his field was to improve algebraic notation(51). But closer scrutiny of the passage indicates that far more than this is involved. Vieta is, in fact, suggesting that what in the past has been considered to be the sole preserve of a "given" or "known" in algebra - the conventional numeral - should now be augmented by a new type of numeral. That is, he is introducing the concept of a "symbolic number".

The important point of interest in view of the discussion in Chapter 1 however, is that he "borrows"

---

(50) See Klein, J. (1968) op. cit.p.340.

(51) See also Struick's comment (1956) p.118.

a familiar term from mathematics - a letter previously used as a temporary replacement for an unknown conventional numeral - and injects it with a new meaning. At the same time he lifts mathematics onto a new, symbolic, plane:

> 'After Vieta, the very nature of the world is governed by the symbolic number concept, a concept which determines the modern idea of science in general. . . The condition for this whole development is the transformation of the ancient conception of arithmos and it's transfer into a new conceptual dimension'.(52)

Vieta gave the name 'species' to his letters, proposed rules and postulates governing their use and transposition,(53) and so essentially set the scene for the first axiomatic of number.

The tasks devised for the present study, and described in Chapter 3, owe their final form as much to the need to understand the nature of the conceptual transformation demanded by an accommodation of the species concept as they do to providing an instrument to detect how such a usage of the letter can be realised.

---

(52) Klein, J. (1968) pp. 184-5.

(53) See 'The Analytical Art', Propositions I, II, and III, Appendix to Klein, J. (1968).

The following discussion outlines the theoretical ideas which gave rise to the tasks.

Kline (to be distinguished from Klein) suggests that the introduction of the 'literal notation' spells an important change in attitude towards mathematics by the mathematician.

Until the seventeenth century, he suggests, the concepts of mathematics were immediate idealisations of, or abstractions from, experience. After 1600 however, the mathematician contributed concepts, so that for the genesis of it's ideas, mathematics in that period turned away from the sensory towards the intellectual faculties.(54)

It is suggested below that this "contribution of concepts" in the form of the species provides both the means of giving a satisfactory solution to one of the unsolved problems of antiquity (the solution of the indeterminate equation), and simultaneously, the potential to construe geometrical data in a way possibly unknown to the Ancients.

The Vietan"species" is a letter which has a 'life of it's own' when regarded as a "given" or "known". That is, it's values do not have to be found.

---

(54) Kline, M. (1972), p.392.

Prior to Vieta, however, letters were used exclusively as "unknowns", and played the role of temporary substitutesfor conventional numerals.

For this reason, any relationship between letters was necessarily a relationship between undetermined constituents. That is, although a relation such as 'x = y' might have been used by the pre-Vietan mathematician to indicate the equality of two conventional numerals, the mathematican was still left with the problem of deciding which two particular numerals might be intended, and this knowledge could be gained only by having available additional knowledge about the relationship of 'x' to 'y' - such as, for example, the knowledge that 'x = 2y', or 'x = 1'.

Letters used in this way might conveniently be termed 'unspecified outcomes' or 'hypothetical judgements' for numerals. In a very real sense these have no "life of their own".

It is a metter of historical fat, however, that within half a century after the publication of "The Analytical Art", Fermat had given specific meaning to such relationships as equations of loci of points in a plane. Here Fermat interpreted each letter not as a temporary replacement for a numeral, but as an entity with an <u>infinite number of guaranteed determinations</u>.

This change in conception of a simple relation, and the obvious desire to model the relation in geometry, suggests that mathematicians after Vieta perceived the relationship of algebra to geometry in a new way.

Prior to Vieta, the letter as an "unknown" could be used to illustrate a geometrical figure and so temporarily stood in place of the numeral which would eventually replace it; but after Vieta geometry was needed to model the algebraic language - a direct reversal of the status of the two languages. After this "switch" in dominant theory there followed a hundred years of unparalleled innovation, including the birth of the calculus, of function theory, and of abstract algebra.

Historical evidence thus suggests not only that the meaning given to letters underwent a radical revision at the hands of Vieta, but also that this change in meaning made possible widespread innovation during the seventeenth century.

To illustrate what this change in usage of the letter might imply, consider the following examples.

Example 1

Firstly, consider the problem of calculating the value of 'x' in Fig. 1.

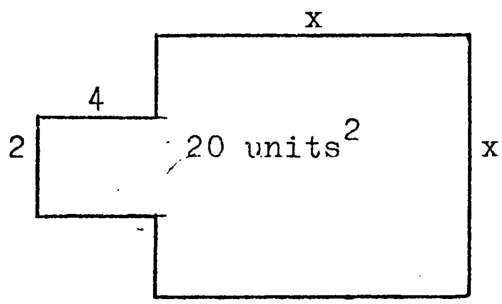


Figure 1.

This problem is easily solved in a language system which does not include the species concept. 'x' can be treated as a classical unknown, and found from the equation

'$x^2 + 8 = 20$'

Thus the majority of Mediaeval mathematicians prior to Vieta would probably have found little difficulty with it (the negative root, however, would have been avoided).

In Chapter 1 it was suggested that the introduction of new meanings for concepts implies a destruction of previous assumptions about the properties of the mathematical objects involved. In the example above, should we allow geometry to dictate our thinking, the assumptions we would probably make about the properties of the letter 'x' (and which, it is postulated, are the assumptions made by the pre-Vietan mathematician) are:

(i) that 'x' has a potential and singular determination;

and (ii) that 'x' has a potential and singular ordering with respect to any numeral. For example, 'x' is either greater than, or less than, or equal to '4'.

Thus it is possible that the pre-Vietan mathematician considered letters to be ciphers representing ordered, unique arithmetical entities (numerals, outcomes of counts, outcomes of measures). The fact that equations in more than one unknown were deemed 'indeterminate' provides testimony to this fact.

Secondly, by way of contrast, consider the problem of calculating the value of 'x' inFig. 2.(55)

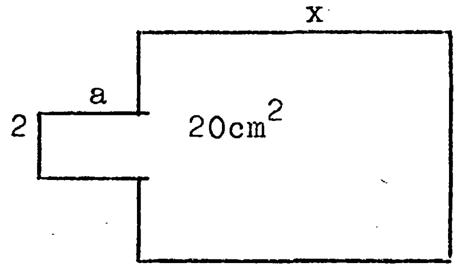


Figure 2.

For the contemporary mathematician this poses no problem. 'x' can be derived from the equation:

'$x^2 + 2a = 20$',

and the 'solution'

'$x = 20 - 2a$,'

considered to be meaningful in it's own right.

---

(55) Here, the 'given', '4', has been replaced by the 'given', 'a'.

The Mediaeval mathematician, however, would have deemed such a problem 'indeterminate'.

Here, the assumptions made about the nature of 'a' (which may be considered to be a Vietan "given") are as follows, and can be compared with the assumptions made above for Figure 1:

(i) 'a', being a "species", has an infinite number of possible determinations;

and (ii) 'a' is non-ordered with respect to both 'x' and the conventional numerals.

That is, any ordering is possible. 'a' and 'x' are non-ordered numerals.

Thus in the languages of pre-Vietan algebra (arithmetic with letter appendages) and symbolic formalism, the assumptions made about the properties of a letter, and the role it plays in algebra, are distinct. In the first language system it has a potential determination and a potential ordering. In the second language system it has guaranteed determinations and is non-ordered. Equally, in the former language system it plays the role of an adjunct to arithmetic. In the latter system however, it plays a central role - here the conventional numerals are the adjuncts.

With a conception of the letters 'a' and 'x' as species, Figure 2, although it is used to suggest the relationship which obtains between the letters, does not determine the values of the letters. To the contrary, the values given to each letter will dictate the final form of the figure.

This draws attention to an important point about the nature of cognition when the species concept is available. Since both 'a' and 'x' each have an infinite number of determinations, Figure 2 can be interpreted either as a representation of an infinite class of figures - some of which are suggested in Figure 3 - or as a dynamically changing system as 'a' and 'x' vary in value.

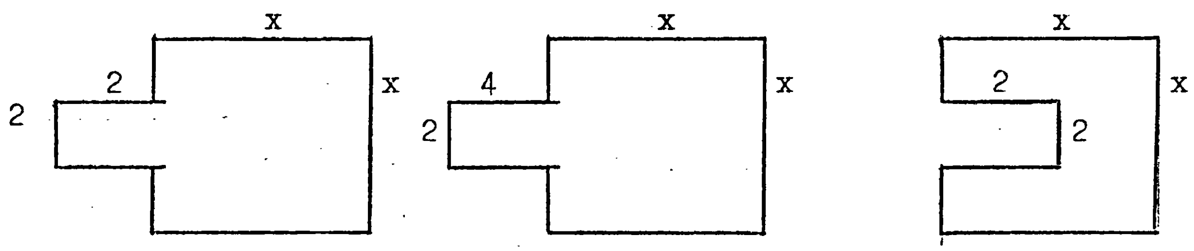


Figure 3.

This latter interpretation suggests that the examples in Figure 3 can be considered to be "snapshots" taken to "stop the action" for particular values of 'a' - which in turn suggests that the pre- and post-Vietan mathematician probably construed the same geometrical figure in quite distinct ways - the former as a static entity, the latter as a dynamic system. It is suggested here that it is precisely this power of the species concept which accounts for the rapid development of dynamics and coordinate geometry soon after the publication of Vieta's work.

Example 2

Consider secondly the problem faced by Diophantus - that of giving a solution to the equations which now bear his name. Although Diophantus was the first mathematician to treat 'indeterminate' equations seriously, he was generally satisfied to give just one solution(56).

Consider the Diophantine equation

'$x^2 + xy = 50$'

Diophantus' method of solution was to propose a relational identity of 'unknowns' and to determine values from the resulting equation in one unknown. For example, for the equation above, suppose '$y = x$'. Then x^2 is 25, and x is 5. 'Inspired guesswork' of this kind has often earned him the reputation of 'conjuror' and non-algebraist.

Diophantus' major problem however, was the want of a language system which included the symbolic number concept. For, equipped with the 'species' concept, x can be assumed to have infinitely many possible determinations where, for each chosen value,

---

(56) Boyer, C. B. (1968) p.202.

$$y = \frac{50 - x^2}{x}$$

The solution of the equation is, therefore,

$$(x, \frac{50 - x^2}{x}).$$

To make this change in usage of the letter more explicit, an alternative letter (say 't') can be introduced as follows:

Put $x = t$, $t \neq 0$.

Then $y = \frac{50 - t^2}{t}$.

The solution is therefore

$$(t, \frac{50 - t^2}{t}).$$

It is suggested here that this second, more powerful approach to the solution of indeterminate equations in more than one 'unknown' is made wholly possible by an acceptance of the letter as a meaningful entity in it's own right, and accounts for the rapid development of co-ordinate geometry and function theory, each introduced during the first half of the seventeenth century. The concept immediately enables the mathematician to regard an equation such as

'$2x + y = 10$'

as a definition of two functions (in this case, $t \rightarrow \frac{10 - t}{2}$ and it's inverse, $t \rightarrow 10 - 2t$) and the way is then open for the development of the calculus.

2.3. The Relevance of '$x = \frac{0}{0}$'

In Chapter 1 it was suggested that new arithmetical entities are accepted into the body of mathematics through a desire for completeness. Thus the negative numbers satisfy the desire to universalise the subtract operation, and the fractions the divide operation.

These steps are however, often resisted by mathematicians because, it seems, each step involves a destruction of tacit assumptions about the number field. Thus an acceptance of the negative numbers coincides with an acceptance that equations of the form '$x + a = b$' always have a solution, that terms such as '$x = -1$' have a meaning in their own right, and with a need to model these as displacements on the number line. In the realm of counting numbers only, these equations have at most one solution and no concept of direction is involved.

The non-Euclidean geometries were constructed also out of a desire to achieve completeness. The 'parallel axiom' enunciated by Euclid appeared to some mathematicians of the nineteenth century to be subject to some doubt, and in the course of attempts to establish the completeness of the axiomatic system, non-Euclidean geometries were constructed. Again the new mathematical systems met resistance from contemporary mathematicians conditioned to the Euclidean outlook.

Vieta's 'Universal Language' introduces the concept of the 'species', a concept formerly not used in mathematics. It would seem that this concept may also be an outcome of a desire for completeness, in the sense that it appears to match an acceptance of the solution of the equation '$o.x = o$';i.e. '$x = \frac{o}{o}$', as a meaningful mathematical entity.

According to Russell and Whitehead (see Introduction to this Chapter) the variable is "ambiguous in it's denotation and accordingly undefined". Equally, the 'number', 'x', which satisfies the relation '$x = \frac{o}{o}$' is also 'ambiguous in its denotation'. Here however, the "number" can be defined either to be identical to any conventional numeral within our range of experience, or to whatever numeral(s) we might consider to be appropriate to any particular problem or mathematical situation. The letter used in this way is no longer an 'unknown' to be determined, and its value does not have to be "found". Its value is guaranteed from the outset. In the same way that the word "chair" refers to chairs, so the word 'x' refers to numerals. Each word is ambiguous in its reference to objects but, unlike the classical "unknown", is not forced upon us by ignorance.

When 'x' is used in this way, it allows for a new interpretation of algebraic statements. Consider, for example the statement:

'x + 1 = 2'.

This may be regarded either as an expression containing an unknown number 'x', or as an expression containing the "number" 'x = $\frac{0}{0}$'.

To distinguish the two interpretations, different forms of quantification might be used. Thus the first interpretation might be expressed:

(a) '(?x) x + 1 = 2'(57)

whilst the second might be expressed:

(b) '($\exists$x) x + 1 = 2'.

(b) is a generalisation i.e. a proposition. Here, 'x' can be defined to be any numeral we may so wish.

Accordingly (b) is true when x is defined to be '1', but false for all other possibilities. 'x' is "full of numerals" and so (b) is immediately a significant statement. On the other hand (a) is an "empty shell" awaiting the true identity of 'x'.

The two different modes of interpretation of 'x + 1 = 2' are correlated with the two distinct attitudes illustrated earlier which can be taken towards geometrical illustrations and indeterminate equations,

---

(57) Freudenthal recommends 'question quantifiers' to distinguish the 'unknown' from the 'arbitrary number'. See his (1973) volume, p.310.

the distinction being due to the inclusion in our repertoire of a new meaning for 'x' which transcends that of the classical "unknown" of pre-Vietan mathematics. For this reason it might be a useful working hypothesis for the teacher and researcher to accredit 'algebraic' thought to a usage of the letter as a "species" and "arithmetical" thought to a usage of the classical unknown.

Thus, if it is true, as Nunn suggests, that the demarcation line between arithmetic and algebra needs to be decided not in particular by reference to the content of each topic, but by reference to distinct attitudes brought to bear upon the same subject matter, it seems that the "algebraic" attitude should be considered to be that conveyed using the letter as a "species".

The present study uses this criterion as the distinguishing feature of "algebraic" activity. The problem for the empirical part of the study was to find a way of showing that some pupils would interpret letters in the classical sense of an "unknown", whilst others would utilise the 'species'. The task turned out not to be an easy one. Many attempts were made to construct questions and task material which would demonstrate the distinction, and many were rejected.

The tasks selected from pilot study material and used here took their final form as an outcome of the ideas discussed in the present chapter, and in particular from the hypothesis that in a language system which does not include the Vietan concept there might be a natural disposition on the user's part to assume that algebraic entities:

(a) have a potential ordering with respect to each conventional numeral;

(b) might also be considered to have a unique identity;

and (c) should (a) and (b) be true, then the letter could not be thought of as a "variable" in the sense that the contemporary mathematician thinks of the "variable". (That is, any "variation" which takes place is necessarily "potential" and not "guaranteed" from the outset).

Essentially this means that the pupil using the letter in the pre-Vietan sense can be expected to treat the letter as an "unknown" with no immediate 'content', whilst the pupil using the letter in the post-Vietan sense will assume the letter to have a 'content' from the outset.

Chapter 3 describes the tasks selected to provide evidence of the existence of two distinct outlooks, and explains how the content of each matches with the ideas expressed in the present chapter.

CHAPTER 3: TASK CONSTRUCTION

3.1. Abstract

This chapter discusses Task Development for the empirical study and explains how the content of each task may be considered to have validity as a translation into experimental terms of the ideas introduced in Chapters 1 and 2.

3.2. Introduction

Klein's suggestion that Vieta introduced a symbolic-number concept into mathematics during the final decades of the sixteenth century suggests that this concept allowed for a new interpretation of algebraic data.

Klein believes that in pre-Vietan mathematics, letters always intend a specific number of units - that is a specific numerical amount, or a specific measure of a concrete entity such as the length of a line, amount of liquid in a container, number of people in a room etc. Here he speaks of the letter having a potential determination. In the Vietan conception however the letter has possible determinations since it owes nothing directly to activities of counting or measuring.

The concept of 'possible' determinations appears to encompass Dantzig's[59] view that what distinguishes mathematics in the two periods is the changed attitude towards what is 'possible' and 'impossible'. With a symbolic conception of number, he suggests, everything appears 'possible'. For this reason the Vietan conception quickly gave rise to an acceptance of both negative and imaginary numbers by allowing the mathematician to recognise that the restrictions previously imposed upon the field of the operand were of his own making - that is, due merely to human tradition. Remove this tradition and solutions of equations such as '$2x = 1$', '$x^2 + 1 = 0$', can be readily accepted as meaningful mathematical entities.

The present author's suggestion (Chapter 2, Section 2.3) is that the 'species' concept is consistent with an acceptance that

$$'x = \frac{0}{0}'$$

is a meaningful and indispensable mathematical object. Due to it's inherent ambiguity it allows 'x' to be freely defined and to be considered either to have a single numerical definition (in which case it is a constant), or to have an infinite number of numerical definitions (in which case it is a variable).

---

(59) Dantzig, T. (1954) p.89

The present chapter explains how this suggestion is embodied in the tasks developed for the empirical study.

Pilot studies were conducted to devise appropriate tasks and to decide upon the most appropriate wording to be used. In particular, attempts were made to devise tasks which would take into account the suggestion made in Chapter 2, Section 2.2., that the species concept might be associated with a new interpretation of geometrical data and of statements involving more than one letter. The four tasks selected for the study are described in detail below.

3.3. The Parallel Lines Task (PLT)

Familiarity with mathematical languages provides us with important means of organising perceptions - that is, of interpreting data. This can be demonstrated by the following simple experiment.

Consider Figure 4 and what it might represent. Some suggestions are given below which may be used by the reader to help him construe his experiences.

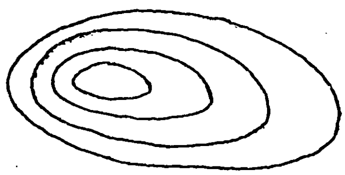


Figure 4.

(a) ripples in a pond; (b) part of an ordinance survey map; (c) a view through an hollow eliptic pipe; (d) the annular rings of a tree.

However, the figure may also be regarded as a number sign, viz. 1111. For Goodstein, 'making a tally' is 'regarding a group as a number sign'. Counting translates the group regarded as a number sign into a conventional numeral(60). In this case the conventional numeral would be '4'.

In this example the cues given below the figure give rise to new ways of interpreting it. As each cue is taken into account attention turns to new features and the figure is 'embedded' in a new context.

Consider now Figure 5 and the question 'Which tree is taller?' Clearly the question cannot be answered since insufficient information is given.

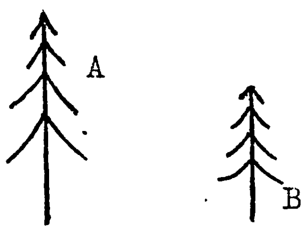


Figure 5.

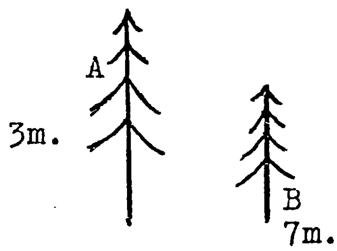


Figure 6.

(60) Goodstein, R.L. (1965) p.58.

When numerals are incorporated as in Figure 6 however, it is 'clear' that tree B is 'arithmetically' taller than tree A. Tree B may be further away from us, or it may be drawn to a different scale. Each of these interpretations of the data is aided by the introduction of numerals.

Figure 7 shows two lines A and B of <u>numerical</u> "length" 5cm and 10cm respectively. Consider the question "Which line is longer?"

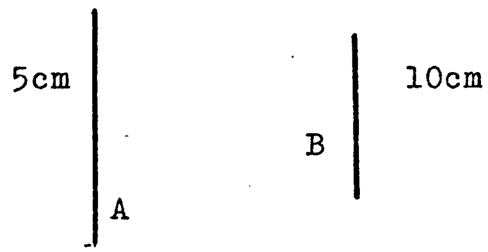


Figure 7.

Ignoring the numerals and assuming each line is in the plane of the paper leads to the answer 'lineA'. By this we mean that should each line be measured then the numerical outcome for A will be greater than the numerical outcome for B.

If the lines are not in the plane of the paper and the numerals ignored, or if line B is curved and we see merely a projection, then it is impossible to say which line is longer.

When the numerals dominate our perception however, then line B is longer and again a variety of reasons can be given to support this conclusion. The words "longer" and "line" have several meanings, correlated with the variety of possible interpretations of Figure 7.

The Parallel Lines Task (PLT) incorporates these ideas. If letters are eventually incorporated into our language as numbers, then these too should be used to organise our perceptions.

Consider Figure 8. Here there are lines of "length" a cm and b cm respectively.

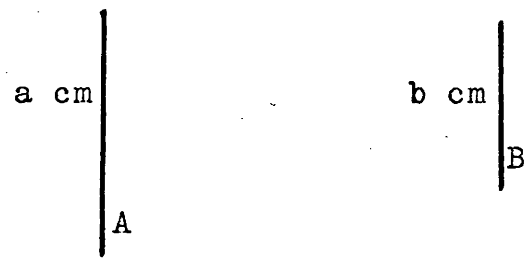


Figure 8.

Consider again the question "Which line is longer, line A or line B?"

As for Figure 7 the question is again ambiguous and any answer depends upon a range of tacit assumptions. In particular however, the answer depends upon what the symbols 'a' and 'b' are considered to symbolise.

'a' cm and 'b' cm may be the <u>outcomes</u> of measuring the line with a ruler. Here we might assume the lines are in the plane of the paper. Should the lines be assumed to be projections of curved lines, to be in perspective, or to be scalar representations, then 'a' and 'b' are necessarily hypothetical judgements standing in place of a measured outcome until that measure can be made, i.e. until further information is given.

In each interpretation above each 'line' is considered to be a 'concrete' line with a <u>particular</u> measurable length.

A second possibility however, is that each letter is a "species", and has infinitely many possible determinations as of right. The lines are then merely attempts to model the species. That is, each line is a 'number line', and has no "length" in the sense implied above.

The question 'Which line is longer, A or B?" thus allows for a variety of interpretations, each correlated with the interpretation given to each letter. Responses to the questions 'When is line B longer than lineA?" "When is line A longer than lineB?" and "When are the lines equal in length?" should give some indication as to how the data is interpreted. For example:

(a) if 'b' is a measured outcome of the concrete line, line B cannot be longer than line A;

(b) if 'b' is considered to be a measured outcome of a line B which is 'further away' from us than a second line A, then line B will be longer only if 'b' is greater than 'a';

(c) if the lines are scale drawings and b cm represents the length of the 'true' line then again the 'line' which line B represents will be longer if 'b' is greater than 'a';

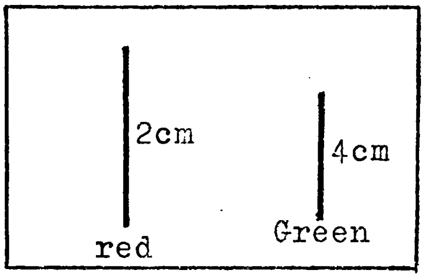
and (d) if the lines are merely representations of 'species' then no decision can be made about respective 'lengths'. However, in this case a theorem can be stated: viz. when 'a' (is defined to be) greater than 'b', line A is longer; when 'b' (is defined to be) greater than 'a', line B is longer; and when 'a' and 'b' are equal then the lines are equal in length.

The parallel Lines Task allows for each of these interpretations and takes the final form given in Figure 9 over the page. It contains three 'subtasks'.

Subtask 1 is introduced both to draw the pupils' attention to the numerical description of "length" and to provide a means of interpreting responses to the 'algebraic' subtasks.

Subtask 1

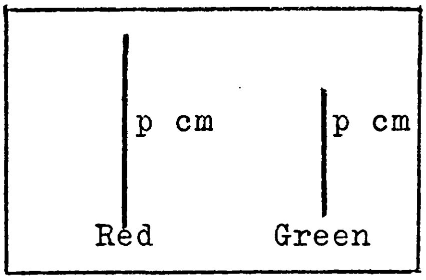
1. Is the red line longer than the green line, the green line longer than the red line, are they equal in length or could any of these be possible?
2. Why?
3. When is the green line longer than the red line?
4. When is the red line longer than the green line?
5. When are they equal in length?



Subtask 1

Subtask 2

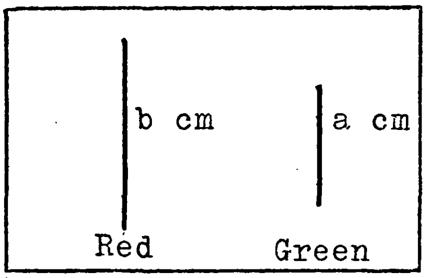
1. Is the red line longer than the green line, the green line longer than the red line, are they equal in length, or could any of these be possible?
2. Why?
3. When is the green line longer than the red line?
4. When is the red line longer than the green line?
5. When are they equal in length?



Subtask 2

Subtask 3

1. Is the red line longer than the green line, the green line longer than the red line, are they equal in length, or could any of these be possible?
2. Why?
3. When is the green line longer than the red line?
4. When is the red line longer than the green line?
5. When are the lines equal in length?



Subtask 3

Figure 9: The Parallel Lines Task.

Subtask 2, introduced next, incorporates two lines with identical algebraic "lengths", since symbolic formalism demands that each letter takes on identically defined values.

Finally, Subtask 3, described in detail above, is introduced. (A full description of the interview situation is given in Section 4.4.)

3.3.1. Relationship of PLT to Theory.

It was suggested both in Chapters 1 and 2 that important expectations about the nature of mathematical entities are developed through experience, and that the introduction of new concepts often disappoints those expectations. Illustrations were given from both Geometry and Arithmetic in Chapter 1.

In a language system which does not utilise the species concept, letters await an outcome by calculation, measuring or counting. In symbolic formalism however, each letter can be defined to be equal to any numeral independently of the concrete context.

Arithmetic with letter appendages (i.e. a language which does not utilise the species) suggests that we should think of mathematical entities, and in particular, letters, as potentially ordered entities with a potential numerical content (see Chapter 2).

The PLT is designed to support each of these expectations. Firstly, two lines which appear to be of unequal 'length' are presented, one line approximately twice the "length" of the second. Secondly, each line is a 'concrete' line drawn on paper, and is of a different colour. This suggests individuality and uniqueness. Thus the notion that 'a' and 'b' and 'p' and 'p', each have unique numerical identities is supported. It can be expected then that the subject who interprets a letter as an unknown outcome of a measure or count will give the letter the content suggested by the figure.

The subject equipped with the species concept however should use this concept to dominate his immediate perceptions for the following reasons.

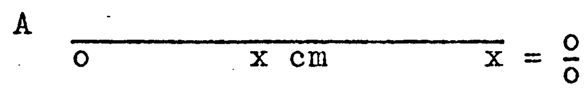


Figure 10.

Consider Figure 10 which incorporates the 'concrete' line A of 'length' x cm.

If 'x' is the outcome of measuring the line then it has a unique value - in this case approximately 6. However, if x is a species then '$x = \frac{0}{0}$' and this can be defined to be identical to any numeral. Thus there is a direct contradiction between the two interpretations of 'x' and a choice must be made.

The figure suggests that x has a unique value. But if this is true then all numerals which can be defined to be $\frac{0}{0}$ must be identical. Arithmetic does not allow these identities. Accordingly either 'x' as a unique outcome of a measure, or 'x' as a species, must be selected.

Since there is nothing in the figure to suggest that 'x' is the actual measure of the line, nor whether scale or perspective is involved, then the species concept must, necessarily, take precedence.

A subject who imposes a relationship between the letters 'a' and 'b' to answer the questions in Subtask 3 may thus be considered to be using the species concept unless he demonstrates in his response that he is thinking also of scale or of perspective. For example, should he respond 'the red line might be nearer to the green line so 'b' might be bigger than 'a' then it is clear that he is thinking of concrete entities and is not using the species.

The interpretation a subject is giving can thus be decided by 'follow up' questions whenever it is not clear what is intended.

Responses to the PLT were used in this study to decide which pupils were using the species to organise perception and so allowing variation in a geometrical setting.

The remaining tasks were devised to study pupils' interpretation of letters, and their readiness to allow "variation", in a non-geometrical setting. These tasks are described below.

3.4. <u>The Equations Task</u>

Prior to Vieta the majority of mathematicians avoided "indeterminate" equations. The exception to the rule was Diophantus who showed an ability to make numerical substitutions for one letter so to conclude a correlated value of the second. Thus it would seem that he was aware that a letter might have a variety of determinations in a particular situation, but he was unable to use the 'species' either to give a "general" solution or to interpret an equation as a definition of a(two) function(s).

Vieta's work was, in a sense, a continuation of that of Diophantus; which suggests that the origin of the species concept is to be found, in particular, in dealings with indeterminate equations. The Equation Task (ET) and the Literal Number Task (LNT)- (Section 3.5.) were devised to investigate how that concept might develop.

The theoretical discussion in Chapters 2 and 3 led to the expectation that the "species" concept was correlated with

(a) an understanding that a letter did <u>not</u> have a unique (unknown) 'content';

and

(b) an understanding that two letters, or a letter and a numeral, did <u>not</u> necessarily have an established ordering.

It was suggested that it was likely, in pre-Vietan mathematics, that the mathematician harboured the opposite of each of these assumptions at a subconscious level of thought. In this sense the meanings given to letters prior to Vieta are different to the meanings used by the contemporary mathematician.

Thus Vieta's introduction of the species implies the possibility that more than one legitimate meaning for algebraic statements exists.

Chomsky[62] has drawn attention to the fact the same sentential form, e.g. "I was sent to Coventry", might have more than one "deep structure" association due to the existence of distinct meanings for terms. Thus "I was sent to Coventry", although it has one "surface structure", clearly has two distinct meanings.

---

(62) Chomsky, N., (1965); (1968) and (1972).

In the same way, if more than one meaning exists for the letter 'x', then algebraic statements might have correlated with them more than one deep structure association. Thus

'x + y = 10'

may have more than one legitimate meaning - a meaning correlated with 'x' and 'y' regarded as species, and a meaning correlated with 'x' and 'y' regarded as classical unknowns.

The questions in the Equations Task, which incorporates the three equations

(a) $x + y = 10$

(b) $2x + y = 9$

(c) $5x = y$

were phrased to "force" any distinct deep structure associations used by pupils "to the surface"(63). The final form of the Task is given in Figure 11, below.

Subtask 1 EQUATION: $x + y = 10$

1. If this is true, is the value of x always, sometimes or never greater than the value of y?
2. Why?
3. When is the value of x greater than the value of y?
4. When is the value of x equal to the value of y?
5. When is the value of x less than the value of y?

---

(63) The PLT depends for it's success upon the same possibility - viz. that two legitimate meanings for the term 'length' exist associated respectively with arithmetical and algebraic deep structures.

Subtask 2 EQUATION: $2x + y = 9$

1. If this is true, is the value of x always, sometimes or never greater than the value of y?
2. Why?
3. When is the value of x greater than the value of y?
4. When is the value of x equal to the value of y?
5. When is the value of x less than the value of y?

Subtask 3 EQUATION: $5x = y$

1. If this is true, is the value of x always, sometimes or never greater than the value of y?
2. Why?
3. When is the value of x greater than the value of y?
4. When is the value of x equal to the value of y?
5. When is the value of x less than the value of y?

Figure 11: The Equations Task.

Questions 3, 4 and 5 demand that a pupil uses a substitution strategy to find possible values for each letter, and then allows each letter to take on these values in turn. Thus the pupil must allow some form of "variation".

Should the pupil not be prepared to allow a letter more than one value however, then "variation" cannot take place. Consequently alternative means of increasing values other than by successive identification with numerals must be found. (Pilot studies showed that some pupils were prone to interpret the letter in this second sense, and used a variety of strategies to increase or decrease "values").

The three equations were selected to demonstrate the influence upon interpretations of algebraic data of varying degrees of suggestions of ordering and uniqueness, i.e. to show that pupils will be influenced by suggestions of ordering inherent to algebraic material itself.

Thus the equation

'x + y = 10'

does not suggest a particular ordering of letters since there is only one 'x' and one 'y' available. Equally '10' can be made up in a variety of ways (1 + 9, 2 + 8, 3 + 7, . . .) each of which provides a potential content for the letters. The equation does not therefore particularly support also an expectation of uniqueness.

On the other hand

'2x + y = 9'

suggests that 'x' might be smaller than 'y', since two 'x's are needed and only one 'y' to make up '9'. Here again however, any uniqueness expectation is not particularly supported. 9 can be made up as 2.1 + 7, 2.2 + 5. 2.3 + 3, etc.

The final equation, '5x = y' however, supports both the ordering expectation and the uniqueness expectation. It both suggests that x is less than y (since $x = \frac{1}{5}y$) <u>and</u> that its value cannot be known since 'y' is not known. Answers to the questions for each Subtask should therefore exhibit distinct response patterns and allow for an attempt to analyse how the species concept might develop.

3.5. The Literal Number Task

The Equations Task incorporates letters into a functional relationship. To answer the questions the pupil needs to consider possible numerical replacements for each letter and then discover the range of possible replacements for one letter which satisfies the required relation.

Here then, the task invites a <u>substitution</u> strategy i.e. it gives some measure of the pupils' readiness, or ability, to find numerical values which satisfy a given relation. The equations are constructed so as to suggest ordering and uniqueness to various degrees as explained in Section 3.4.

In symbolic formalism, however, letters have "meaning in themselves" independently of a stated relation. Thus the letter 'a' is not simply a cipher for an 'unknown number', but is a <u>number</u> having no particular ordering <u>and</u> no particular determination. The letters 'a' and 'b' are non-ordered numerals.

The Literal Number Task was devised to study the extent to which pupils would be prepared to accept a "letter in itself" as a non-ordered entity.

The PLT (Parallel Lines Task) demands that the pupil is prepared to state a relationship between letters to demonstrate a conception of the species. He may be distracted however, by the geometrical drawings accompanying the letter.

In the Literal Number Task the pupil is asked to compare pairs of numbers in turn <u>independently</u> of any conscious attempt to distract. The pairs of "numbers" chosen are:

(a) 't + t, t + 4';

(b) 'm + m, m + k';

(c) 'a + b + 3, a + c + 4';

and the questions asked are replicas of those for the PLT and ET, i.e.

(1) Which is larger, 't + t' or t + 4' (m + m or m + k; a + b + 3 or a + c + 4)?

(2) Why?

(3) When is (t + t, m + m, a + b + 3) larger?

(4) When is (t + 4, m + k, a + c + 4) larger?

(5) When are they equal?

(See Figure 12 below).

The questions here invite a matching strategy ('t' with '4', 'm' with 'k', 'b + 3' with 'c + 4') and include all possible combinations of comparisons possible in symbolic formalism (i.e. two letters which are identical, two letters which are different, and a letter and a numeral).

Subtask 1 LITERAL NUMBERS: t + t, t + 4.

1. Which is larger, t + t or t + 4?
2. Why?
3. When is t + t larger?
4. When is t + 4 larger?
5. When are they equal?

Subtask 2 LITERAL NUMBERS: m + m, m + k.

1. Which is larger, m + m or m + k?
2. Why?
3. When is m + m larger?
4. When is m + k larger?
5. When are they equal?

Subtask 3 LITERAL NUMBERS: $a + b + 3$, $a + c + 4$.

1. Which is larger, $a + b + 3$ or $a + c + 4$?
2. Why?
3. When is $a + b + 3$ larger?
4. When is $a + c + 4$ larger?
5. When are they equal?

Figure 12: The Literal Number Task.

This task should allow for an understanding of what prevents any pupil giving an 'algebraic' response to the PLT, and can be used to detect the extent to which he may desire letters to be ordered independently of explicit and conscious ordering suggestions.

3.6. The Zetetic Task

Klein argues that pre-Vietan algebra is different from post-Vietan algebra by virtue of the differences in intention of the mathematicians during each period(64). Diophantus, although the most eminent of algebraists in the pre-Vietan period, did not direct his attention towards generality when dealing with algebraic expressions. Moreover, for want of a symbolic means of expressing numerical generality he was able only to illustrate his thinking with specific examples, using

(64) Klein, J.(1968)

letters for unknown quantities to be determined.

In the "Analytical Art", Vieta draws attention to the difference between his own approach and that of Diophantus by solving a problem first enunciated and solved by Diophantus in the "Arithmetica". Vieta however uses letters for 'givens' as well as letters for undetermined quantities.

The two solutions are given below, (firstly that of Diophantus and secondly Vieta's) with a 'modern-day' equivalent of each to illustrate the distinctions and the greater power of the Vietan method.

Diophantus' enunciation of the problems is:

"To divide a given number into two numbers with a given difference",

and his solution:

"So let the given number be $\bar{\rho}$ (one hundred), and let the difference be $\mathring{M}\bar{\mu}$ (forty units). To find the numbers, let the less be taken as $\varsigma\bar{\alpha}$ (one unknown). Then the greater will be $\varsigma\bar{\alpha}\mathring{M}\bar{\mu}$ (one unknown and forty units). Then both together become $\varsigma\bar{\beta}\mathring{M}\bar{\mu}$ (two unknowns and forty units). But they have given as $\mathring{M}\bar{\rho}$ (one hundred units). $M\bar{\rho}$ (one hundred units) then, are equal to $\varsigma\bar{\beta}\mathring{M}\bar{\mu}$ (two unknows and forty units). And taking like things from like: I take

$\overset{o}{M}\bar{\mu}$ (forty) units from the $\bar{\rho}$ (one hundred) and likewise $\bar{\mu}$ (forty) from the $\bar{\beta}$ (two) numbers and $\bar{\mu}$ (forty) units. The ϛ$\bar{\beta}$ (two unknowns) are left equal to $\overset{o}{M}\bar{\xi}$ (sixty units). Then each ϛ (unknown) becomes $\overset{o}{M}\bar{\lambda}$ (thirty units).

As to the actual numbers required; the less will be $\overset{o}{M}\bar{\lambda}$ (thirty units) and the greater $\overset{o}{M}\bar{o}$ (seventy units), and the proof is clear".(65)

In our contemporary notation this argument is as follows:

"Suppose the smaller of the numbers which go to make up 100 is x.
Then the larger number is 'x + the difference'
i.e. 'x + 40'.
The sum of these is x + x + 40 = 2x + 40 = 100 (given).
Subtracting 40 from each side of 2x + 40 = 100
we have 2x = 60.
Hence x = 30.
Thus the smaller number is 30, and the larger
30 + 40 = 70.

It will be seen here that Diophantus uses only one letter for an unknown. This was his general method

---

(65) See Klein,J (1968) p.331

of working wherever problems appeared to demand the use of two, i.e. he used the data available to express the second unknown in terms of the first as in the case above: 1st unknown = x; 2nd unknown = x + 40.

A more sophisticated method, using different letters for distinct unknowns would be:

Let x and y be the two numbers, and $x > y$.

The sum, $x + y = 100$.

The difference, $x - y = 40$.

Hence $x = 70$,

and $\quad y = 30$.

Here x and y is each introduced as a tentative substitute for numerals yet to be found. There is, however, no usage of a letter as a species, i.e. as a "number" in it's own right which is not awaiting replacement by conventional numerals. This is where the Vietan method transcends each of the above.

Vieta's enunciation of the problem is;

"Given the difference of two "sides" (i.e. numbers) and their sum, to find the two "sides";

and his solution;

"Let the less "side" be A; then the greater will be A + B. Therefore, the sum of the "sides" will be A2 + B. But the same sum is given as D.

Therefore, A2 + B is equal to D. And, by antithesis, A2 will be equal to D - B, and if they are all halved, A will equal $D\frac{1}{2} - B\frac{1}{2}$.

Or let the greater "side" be E. Then the less will be E - B. Therefore, the sum of the "sides" will be E2 - B. But the same sum is given as D. Therefore, E2 - B will be equal to D, and by antithesis, E2 will be equal to D + B. And if they are halved E will be equal to $D\frac{1}{2} + B\frac{1}{2}$.

Therefore, with the difference of the "sides" given, and their sum, the "sides are found".

Vieta followed this with an illustration in which he replaced D by 100 and B by 40 to show that $D\frac{1}{2} - B\frac{1}{2}$ and $D\frac{1}{2} + B\frac{1}{2}$ gave the numbers required.

In contemporary notation his solution is as follows:

Suppose the difference is b, and the sum a.
Suppose the smaller number is x.
Then the larger number is x + b.
Thus $x + x + b = 2x + b = a$.
Thus $2x = a - b$,
and $x = \frac{a - b}{2}$.

Or, let the larger number be y. Then the smaller number is y - b.

Then the sum, a = 2y - b.

Hence $2y = a + b$,

and $y = \frac{a + b}{2}$.

Synthesising these two separate 'halves' of the solution we arrive at the simpler solution:

Let x and y be the numbers ($x > y$) and 'a' and 'b' the sum and difference respectively.

Then $x + y = a$,

and $x - y = b$.

Hence $2x = a + b$; $x = \frac{a + b}{2}$

and $2y = a - b$; $y = \frac{a - b}{2}$

The obvious advantage of Vieta's solution over that of Diophantus' is that Vieta uses letters, and not conventional numerals, for 'givens'.

The solution is immediately general by virtue of the fact that the "givens" 'a' and 'b' have guaranteed, or possible, determinations, so that it is clear that the argument works for <u>any</u> difference and sum. Diophantus, on the other hand, argues from a particular case.

The introduction of the species clearly aids the expression of generalisations. If we are conversant with this "symbolic" usage of the letter, then it is likely that wherever possible it will be used. The final task given to subjects was a reinterpretation of the problem above as follows (Figure 13):

> "If you are given the sum and the difference of any two numbers, show that you can always find out what the numbers are. Make your answer as general as possible".
>
> Figure 13: The Zetetic Task.

Here it was expected that pupils who demonstrated a usage of the species in the PLT would be more likely than those who did not to utilise the letter as a 'given' in this task and thus that solutions would be more 'general'. That is, the ability to transcend geometrical ordering in the PLT would correlate with a more 'generalised' approach to dealing with mathematical problems.

3.7. Introductory Task

As an introduction to the four main tasks pupils were asked for 'values' of each letter x and y which made each of the following statements in turn, true:

(a) $x + y = 6$.

(b) $2x + y = 6$.

(c) $3x = y$. (See Figure 14, page 86).

This introductory task is used to check that pupils have the necessary arithmetical ability to deal with equations involving more than one letter, and that the term 'value of x' and 'value of y' is understood in this context. Only those pupils capable of giving correct responses to this introductory task were to be considered for interview.

Subtask 1 EQUATION: $x + y = 6$

Can you give me a value for x and a value for y which makes this true?

Subtask 2 EQUATION: $2x + y = 6$

Can you give me a value for x and a value for y which makes this true?

Subtask 3 EQUATION: $3x = y$

Can you give me a value for x and a value for y which makes this true?

Figure 14: Introductory Task.

CHAPTER 4 : EXPERIMENTAL DESIGN

4.1. Data Collection

The process of data collection is exclusively that of the "method clinique" used extensively by Piaget and his co-workers. That is, pupils are presented with the tasks described in Chapter 3, in a clinical situation, and an attempt is made to reconstruct the development of algebraic thought from response patterns.

In the present study the independent variables are those of age, and of mathematical ability as judged by the pupils' mathematics teacher (see Section 4.3.). Piaget generally deals only with age as an independent variable, whilst Krutetskii[(66)], whose work is now well-known in Europe, selects both age and "general mathematical ability". Whereas Piaget is concerned to study the general development sequence of cognitive growth, Krutetskii is more interested in those specific abilities which separate the mathematician from the non-mathematician. The present study is in keeping with Krutetskii's approach and suggests that one of the major advantages the mathematician has over his non-mathematical colleague is his understanding and usage of the letter as a 'species'.

---

(66) Krutetskii, V. A., (1976)

Data collection took place in the three stages described below:

Stage 1:

Eight mathematics teachers from two secondary schools, and five University lecturers, completed the Parallel Lines Task (PLT).

Each of the eight teachers also completed the remaining tasks.

Five teachers, and three University lecturers, were specialist mathematicians. The remaining three teachers taught mathematics as a subsidiary subject, and the remaining two lecturers were social scientists.

The Stage 1 investigation was included for two reasons:

(a) some of the questions asked - in particular the PLT and ET - are not found in school texts and, as far as the author is aware, are rarely (if ever) entertained in the classroom. None of the teachers in the Stage 1 study included such questions in their teaching. It was thus important to determine how the specialist mathematician might respond to the tasks, and in particular the PLT;

and (b) responses from specialist mathematicians could then be used as "models" against which to interpret pupil responses.

The PLT drew the expected form of response from the five school mathematicians and one non-specialist mathematics teacher, and from the three University mathematicians.

The remaining four subjects were distracted by geometrical data and gave "non-algebraic" responses. (Non-algebraic responses are discussed in detail in Chapter 5).

The following transcript is an example "model" response from one of the mathematics teachers to each Subtask of the PLT:

<u>Subtask 1</u> (See diagram overpage)

Q. Is the red line longer than the green line, the green line longer than the red line, are they equal in length, or could any of these be possible?

A. The red looks longer, but if you want an arithmetical response, it's the green. I presume they're drawn to a different scale.

Q. When is the green line longer than the red line?

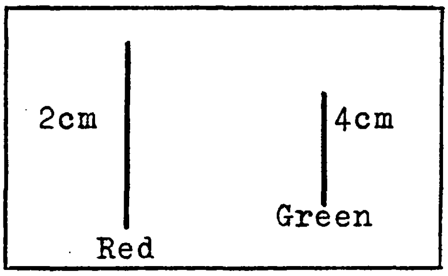
A. Always - arithmetically.

Q. When is the red line longer than the green line?

A. Never - arithmetically speaking.

Q. When are the lines equal in length?

A. Never - again, arithmetically speaking.



PLT Subtask 1.

<u>Subtask 2</u> (See diagram overpage)

Q. Is the red line longer than the green line, the green line longer than the red line, are they equal in length, or could any of these be possible?

A. Algebraically they're equal. Non-algebraically the red line is longer.

Q. When is the green line longer than the red line?

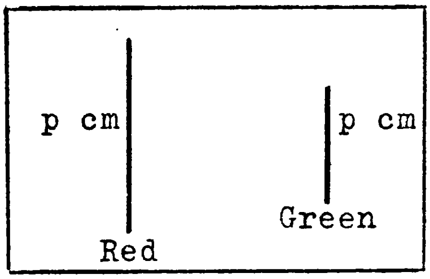
A. Never.

Q. When is the red line longer than the green line?

A. Never.

Q. When are the lines equal in length?

A. Always.



PLT Subtask 2.

Subtask 3

Q. Is the red line longer than the green line, the green line longer than the red line, are they equal in length, or could any of these be possible?

A. The red looks longer, but it needn't be.

Q. Why?

A. Well, it will depend upon the values given to a and b.

Q. When is the green line longer than the red line?

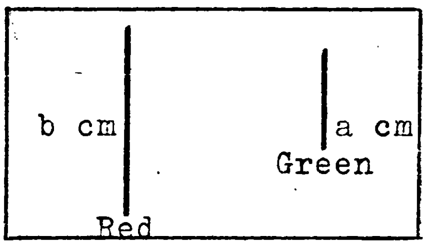
A. When a is greater than b.

Q. When is the red line longer than the green line?

A. When b is greater than a.

Q. When are the lines equal in length?

A. When a equals b.



PLT Subtask 3.

Each mathematics teacher and one non-specialist (science) teacher also gave expected responses to the remaining tasks. The following are examples drawn from each task for the same teacher above.

<u>Equations Task</u>: '5x = y'.

Q. If this true, is the value of x always, sometimes, or never greater than the value of y?

A. Sometimes.

Q. Why?

A. It depends upon the value of x. When x is positive, y is greater. When x is negative, x is greater....And they're equal when x is zero.

<u>Literal Number Task</u>: 'a + b + 3, a + c + 4'.

Q. Which is larger, a + b + 3, or a + c + 4?

A. Either.

Q. Why?

A. Well. . . when b is greater than c + 1 that (a + b + 3) is larger, but when b is less than c + 1 that (a + c + 4) is larger.

Q. And they're equal when?

A. When b equals c + 1.

Zetetic Task (Written)

"$x + y = a$ so $x = \frac{a+b}{2}$

$x - y = b$ $y = \frac{a-b}{2}$

Given a and b, x and y are known".

The remaining two non-specialist mathematics teachers gave solutions at variance with these for '$5x = y$' and for the Zetetic Task. Each gave the response 'When x is greater than one-fifth of y' to the question 'When is the value of x greater than the value of y?' in $5x = y$, and neither used letters for 'givens' in the Zetetic Task. Each used specific numerals for the sum and product and then solved the resulting simultaneous equations.

The responses of the teachers and lecturers gave support to the view that letters might be interpreted in the two distinct senses of an "unknown" and a "species".

Stage 2

The purpose of the second stage was to demonstrate that the distinct usages made of the letter by the teachers and lecturers would be reflected at the learner level.

This stage, which involved intensive interviews with 72 pupils from a boys grammar school, concentrated upon both comprehending the variety of possible responses to each task, and devising categories of response-type. At this stage pupils were often asked to explain particularly "incongruous" responses.

Prior to introducing Subtask 2 of the PLT, each pupil was asked how the "picture" in Subtask 1 (a short line marked 4cm,and a longer line marked 2cm) could be "true in reality". The intention was twofold:

(i) to impress upon the pupils that a rational explanation for the incongruous situation was possible;

and (ii) to study the variety of rationalisations which might be used in a numerical setting and which, should these arise also in Subtasks 2 and 3, could be deemed "non-algebraic" by association.

Wherever a pupil was unable to rationalise the situation (Subtask 1 caused intense confusion, supporting the author's view, expressed elsewhere(67), that confusion in the classroom may often be due to the existence of incompatible meanings for important terms)

---

(67) Harper, Eon , (1978).

the author explained that the lines might be "in perspective", "drawn to scale", or "bent".

The effect of this was to cause pupils to offer similar explanations throughout Subtasks 2 and 3. "Species", or "algebraic" responses were surprisingly rare. It was possible at this stage that the attempt to help the pupil in Subtask 1 had, in fact, had the opposite effect.

Stage 3

The Stage 2 study was repeated one year later with a second group of 72 pupils from a coeducational grammar school to provide an indication of consistency of the tasks across distinct populations.

This second series of interviews was less intensive and Subtask 1 of the PLT was omitted. Although this reduced the number of "scalar", "perspective" and "bent line" explanations to Subtasks 2 and 3, it did not significantly change the incidence of "non-algebraic" responses. The number of pupils who appear to use the letter as an organiser of perception is consistently few.

Responses of pupils at Stage 3 were again classified and the responses at Stage 2 reorganised into the more comprehensive system developed during this third stage.

Response-types to each task are presented in Appendix II.

4.2. The Experimental Groups

The schools selected for Stages 2 and 3 of the study were grammar schools in the south west of England, the first a boys school (School A) and the second co-educational (School B).

Pupils selected for the study may thus be considered to have a high general academic ability relative to the total school population. Grammar schools were selected to attempt to show that problems with algebra are not confined only to those pupils who are generally considered to be less academically able. Whatever problems can be discerned in the grammar school, it is assumed, will be reflected elsewhere.

Each school has an excellect record in mathematics at both 'O' and 'A' level.

Each pupil in School A is required to pass an entrance examination, and each is expected to attain an 'O' level pass in mathematics at the end of his fourth or fifth year in the school.

One third of School A's population take 'O' level mathematics at the end of the fourth year, entering an "accelerated" stream during this year. Successful candidates study additional mathematics in the fifth year. There are three mathematics sets in each year. Pupils in Sets 2 and 3 take 'O' level at the end of the fifth year.

Pupils entering the "accelerated" stream at the 4th year level who are successful at both 'O' and 'Additional Maths' level, and enter the sixth form to take mathematics as a main subject, complete the 'A' level course at the end of their first year in the sixth form.

During his year in the 'Upper Sixth' a successful candidate either studies for Oxbridge examinations, for 'special' papers in Mathematics, or attempts to improve his 'A' level grades.

A pupil entering the sixth form from a non-accelerated stream (Sets 2 and 3) takes the 'Additional Mathematics' examination at the end of his first year in the sixth form and the 'A' level examination after a further year.

There are approximately 90 pupils in each Year 1 - 5, and 18 and 20 members respectively of the lower and upper sixth form studying mathematics. Twelve pupils in the Lower Sixth and fifteen in the Upper Sixth were from previous years' 'express' groups when the present study was undertaken.

The teaching staff comprises three full-time mathematics teachers, and three scientists, each of whom shares his teaching time between the science and mathematics departments.

School B is approximately the same size, although there are only two mathematics sets of 36 and 35 in Year 3 due to a fall in numbers during 1975. Here there are no "accelerated" streams and all pupils sit the 'O' level examination at the end of the fifth year.

Two years are taken for the 'A' level course, and a minority of pupils spend a third year in the sixth form either to improve grades or to sit "special" papers.

There are 16 pupils in the lower sixth and 17 pupils in the upper sixth studying mathematics.

The teaching staff comprises three specialist mathematicians and two non-mathematicians - one a P.E. teacher, and the second a scientist.

4.3. Selection of pupils for the Study

Twelve pupils from each Year-group 1 through 5, and 12 'A' level candidates in each school were selected for the study, using a stratified sampling procedure to produce a representative sample of mathematical abilities across each Year-group.

The Head of Mathematics in each school was asked to rank pupils in each Year-group using the results of the previous year's school mathematics examination.

Consultation with the teachers of individual pupils then helped to re-order the ranking where necessary. Thus, where the teacher felt that a pupil had underachieved in the examination and deserved a higher ranking than in the final list, this was taken into account to achieve a final ordering.

Using the final ranking for each year twelve pupils were selected. The most able mathematician in each Year 2 to 5 was ranked 1, the 8th most able ranked 2, the 15th ranked 3, ...the 85th ranked 12.

A slight modification of this selection procedure was necessary for the 2nd and 3rd year of School B, each of which had less than 85 pupils.

For School A this procedure meant that the 'express' group pupils in Years 4 and 5 were those ranked 1 - 4.

For School B the pupils ranked 1 - 4 came from the top set of three in Years 2, 4 and 5, and from the top set of two in Year 3. (See Tables 13 and 14 Appendix I)

In Year 1 and for the 'A' level candidates, the initial ranking of pupils was attained through consultation with the teachers. Following this, 12 pupils in Year 1 of each school were selected using the procedure described above.

Of the 'A' level candidates the 12 pupils were selected from both the upper and lower sixth forms.

For School A the 'A' level pupil considered to be most able by his teachers was ranked 1, the 4th ranked 2 and so on. In the final ordering pupils ranked 1 - 5 were, fortuitously, those from the upper sixth form, who had completed 'A' level the previous year and had entered the sixth form from "express" groups. Pupils ranked 6 - 9 were those from the previous year's "express" group. Those ranked 10 - 12 were from "non-express" groups.

For School B the final ranking was more variable between years. Pupils ranked 1 and 2 were from the upper sixth, those ranked 3, 4, 5 from the lower sixth.

Tables 13 and 14 (Appendix I) show how the final ranking in each Year-group relates to the teaching sets in each school.

This procedure gives a total of 72 pupils in each school for interview.

Pupils in School B were interviewed one year later than pupils for School A, at approximately the same time of year (December - March).

4.4. Conduct of the Interview

Interviews took place during normal class-lessons, and pupils were asked to be released as requested.

The interview was conducted, in each school, in an annex room to one of the main teaching areas. Each interview was recorded and transcribed.

The results discussed in Chapters 5 - 7 use these transcriptions throughout. The only written work expected of each pupil was the response to the Zetetic Task.

Each interview began with a general discussion to set the pupil at ease. Each pupil was asked about his own feelings towards mathematics, what he believed were his strengths and weaknesses and which topics he enjoyed. He/she was informed that for each task there was no "correct" answer in the sense in which answers may be considered to be "correct" or "incorrect" in classroom work, and the objectives of the study were explained. That is, each pupil was informed that the author was attempting to understand how different people used letters in algebra, and that it was hoped he/she would help with the study. Thus no "score" for general mathematical ability or intelligence was intended, and the results for individuals were not to be disclosed to others.

It was hoped that this procedure would relax pupils, and that each would feel more agreeably inclined to discuss his own responses.

Each pupil was asked not to disclose the questions asked of him to his friends and peers during the investigatory period.

The tasks were presented to each pupil in the following order:

(1) Introductory Task

a request for values of x and y which make each of the following statements, in turn, true:

$x + y = 6$

$3x = y$

$2x + y = 6$ (See Figure 14, Section 3.7, p.86)

Each pupil in the study completed this task successfully.

(2) Parallel Lines Task

Subtask 1, (School A only - see Section 4.1.) using the numerals 2 and 4. (See Figure 9, Section 3.3, p. 65).

(3) Equation Task

$x + y = 10$ (see Figure 11, Section 3.4, p.72).

(4) Parallel Lines Task

Subtask 2, using the letters p and p. (See Figure 9 , Section 3.3, p.65).

(5) Equations Task

$2x + y = 9$ (see Figure 11, Section 3.4, p.73)

(6) Parallel Lines Task

Subtask 3, using the letters 'a' and 'b'. (See Figure 9, Section 3.3, p.66)

(7) Equations Task

5x = y (see Figure 11, Section 3.4, p.73).

(8) Literal Number Task

t + t, t + 4 (see Figure 12, Section 3.5, p.77).

(9) Literal Number Task

m + m, m + k (see Figure 12, Section 3.5, p.77).

(10) Literal Number Task

a + b + 3, a + c + 4 (See Figure 12, Section 3.5, p.78).

(11) Zetetic Task

(see Figure 13, Section 3.6, p.84).

For each Subtask of the PLT the pupil's attention was drawn to the figures and letters using the following introduction.

"Here we have a red line which is a cm ('2cm', 'p cm') long" - pointing to the line - "and a green line which is b cm ('4cm', 'p cm') long. Alright?" When the pupil indicated he understood, this was followed by the relevant questions:

"Now I want you to tell me, 'Is the red line longer. . . ." etc.

The questions were written also on each card for the pupil to read.

A similar procedure was followed for Subtasks of the ET and LNT: "Here we have the equation 'x + y = 10'. Now I want you to tell me, if this is true is. . ." etc. - "Here we have the numbers 't + t' and 't + 4'. Which of the numbers is larger. . ." etc. Again the questions were written on each card for the pupil to read.

Pupils were informed before the tasks were introduced that some questions might appear to contradict an earlier response - for example, should a pupil suggest that 'x' is less than 'y' in '5x = y', the question "When is the value of 'x' greater than the value of 'y'?" contradicts this statement.

If this was the case, the pupil was asked not to consider that the new question indicated that his earlier response was incorrect. The questions had been formulated prior to the interview, written down, and would be asked independently of the nature of earlier responses.

For the Zetetic Task, each pupil was given a card to record his answers. The question was again written on a separate card and the author explained its meaning as follows:

"Suppose you are given the sum and the difference of any two numbers - do you know what "sum" and "difference" means?" If the pupil said "yes" he was asked for an example, and if incorrect ("sum" means "product" to some pupils) was corrected. If he responded "no" the meaning of each was explained. Then:

"Suppose you are given the sum and the difference of any two numbers, but you are not told what the numbers are. Alright? You're given the sum and the difference but not the numbers".

When the pupil indicated he understood, this was followed by:

"I want you to show that given the sum and the difference, you can always find out what the numbers are. Try to make your answer as general as you can".

The pupil was then left to work upon the task with the written question to remind him of the problem. No time limit was set. Some pupils took approximately twenty minutes to complete it.

Interview times varied in School A from between 15 and 20 minutes for the more able sixth-form pupils, to 45 - 50 minutes for younger pupils. In School B these times were reduced by about one third.

4.5. Objective and Hypotheses

The theoretical analysis of Chapters 1 and 2 had an important objective, and the empirical investigation aims to support a number of hypotheses arising from it. These are as follows:

(a) Objective

To illustrate the nature of algebraic thought and to understand what are some of the major obstacles to learning the algebraic language.

As a starting point it was considered that the child's problems in creating, or accomodating new mathematical concepts might parallel the problems mathematical communities faced during history.

Although the mathematician's major aim is to devise an internally consistent and paradox-free universal language, this aim is achieved in a piecemeal fashion as he recognises the inconsistencies and weaknesses of his present language system.

As new systems and languages are introduced, these often require of the mathematician a change in conceptual outlook and a destruction of previous expectations about the meanings of key concepts. Examples were given from geometry and number in Chapter 1. Thus as mathematics develops, each new language demands a new conceptual understanding, and a simultaneous destruction of existing expectations.

This suggestion was applied to the changing meaning given to the terms "unknown" and "number". Vieta introduced the concept of a "symbolic number" or "species" which transcends previous understandings of the terms established by previous generations.

In particular it was suggested that the introduction of the "species" concept implies a destruction of two prior expectations about the role of letters which might be developed and supported through working with arithmetic. These were:

(i) that letters, like conventional numerals, have an established ordering property. Thus any number pair (e.g. 2 and 4) have a unique order relationship ($2<4$). By way of contrast, this is not a property of the letter used as a species (e.g. of 'a' and 'b').

and

(ii) that a letter has a unique numerical content in the same sense that the box in the sentence '2 + □ =7' might be considered to have a unique numerical content.

These possibilities gave rise to the tasks devised for the empirical study, which are used to investigate the following hypotheses:

(a) some pupils will demonstrate a usage of a letter as a means of reorganising immediate perception, in the same sense that conventional numerals can be used to reorganise immediate perception (Chapter 3, Section 3.3).

(b) responses to the Equations Task (ET) and Literal Number Task (LNT) will support the view that pupils bring into their dealings with algebraic material expectations of order and numerical uniqueness of the elements (letters) involved.

and (c) pupils using the letter to re-organise perception in the PLT will demonstrate a greater capacity to think in general terms in the ZT than pupils who do not. That is, the ability to use the letter as an organiser of perception is correlated with an ability to demonstrate general results.

Pupil responses to each task were organised into a variety of categories and a statistical procedure applied to determine three major levels of algebraic activity. The analysis of response - types and the statistical procedure applied are given in Appendix II.

Chapters 5, 6 and 7 illustrate the three levels of activity using pupil transcripts, and Chapter 8 discusses the nature of "variation" at each level. These four Chapters may thus be considered to be an attempt to satisfy Objective (a) above, i.e. to illustrate the nature of algebraic thought, and to understand what are some of the major obstacles to algebraic development.

Evidence to support hypotheses (a) to (c) is presented in Chapter 9.

5.0. Introductory remark to Chapters 5, 6 and 7.

The three "Levels of Activity" illustrated in the following three chapters were derived by the procedure described in Appendix II.

Each level appears to have associated with it a distinct interpretation of "mathematical reality", the latter two levels (Levels II and III) corresponding to 'Diophantine' and 'Vietan' algebra.

The three Levels are, respectively:

Level I : The Level of Fictitious Measures. (Chapter 5)

Level II : The Level of Discovered Content. (Chapter 6)

Level III : The Level of the Species. (Chapter 7)

Each Level is defined by a series of response-types to each task which appear to share a common interpretation of the letter. Some pupils respond consistently at one particular Level, whilst others alternate between Levels according to the task-content. This is to be expected if different meanings for concepts truly exist.

Tables 27(a) & (b), Appendix II, Section II.6.2 indicate the interpretation most commonly used by each pupil across the battery of tasks.

CHAPTER 5 : THE LEVEL OF FICTITIOUS MEASURES

5.1. Introduction

At the first level of activity to be described below the pupil rarely uses a substitution strategy in the ET, and when he does, substitution is "reluctant" - often only one correlated pair of numerals (e.g. x = 1, y = 9) being suggested to satisfy a particular relation (x > y, x < y, x = y).

In the LNT Subtask 2 ('m + m', 'm + k') the pupil does not use a matching strategy, refusing to allow a letter to have another letter as it's 'content'. He is more likely to assume each letter has a predetermined ordering, or to replace the letter by a number.

In the Zetetic Task, the pupil makes little headway. The overall impression given is that the pupil often treats the letter as an object (such as an apple or a pear) which "contains" a fictitious measure. He is prone to think that the letter has a unique (unknown) 'measure' and an established ordering. This first level of activity is accordingly called the "level of fictitious measures".

The most noticeable feature of the pupil working at this level is that he prone to influence by inherent suggestions of ordering or uniqueness.

Where a task does not, in particular, suggest an ordering of entities, (as in 'x + y = 10') the pupil might be willing to make a numerical replacement for a letter .

When suggestions of ordering and uniqueness are introduced however (as in the PLT and '5x = y'), this substitution strategy disappears - but activity continues to be underpinned by the belief that not more than one particular ordering, and not more than one particular replacement, can truly exist. The pupil suffers acutely from the "Russell Syndrome" (see Introduction - monograph I).

5.2. The Level of Fictitious Measures - responses to the PLT

The observations below about the nature of responses made to the PLT at the "level of fictitious measures" applies also to many pupils classified at the "level of discovered content" (Chapter 5).

Pupils working at the latter level are more often distinguished from those working at the former by their ability to use a matching strategy in the ET, and their ability to regard an equation as a system of co-varying numeral-pairs.

In his response to PLT Subtask 1, the pupil responding at the level of fictitious measures will be prepared to use numerals to organise his perceptions, and allow these to determine his interpretation of the meaning of 'length'. He will not, however, be prepared to allow the letter to change the fact that two concrete lines exist, and he does not, in this sense, transcend the arithmetical meaning.

In the following transcript for example, the pupil allows conventional numerals to influence perception (see Appendix II, Section II.2.2. for further examples):

Richard (2:8) 12yrs, 7mths. School A. PLT Subtask 1. (See diagram over

Q. Is the red line longer than the green line, the green line longer than the red line, are they equal in length, or could any of these be possible?

A. Well, erm, well, the green line is longer in truth but. . . .if it's true. . .but if the two centimetres are in different units or something then the red line might be longer.

Q. When is the green line longer than the red line?

A. When they were the same units.

Q. Can you explain what you mean by that?

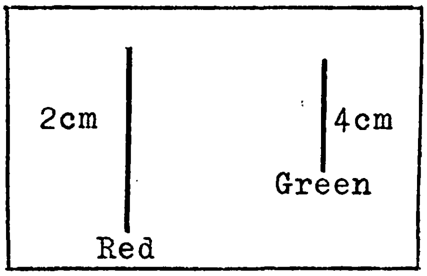
A. Well, the red line might be a scale or something. 4cm is bigger. If it was the same units it would be longer. . .if you drew them the same units the green is longer.

Q. When is the red line longer than the green line?

A. If the units in the red line were say three times as big.(i.e. as those in the green line).

Q. When are the lines equal in length?

A. If the units in that (pointing to the red line) were twice as big as the units in that (the green line). (i.e. if the lines were re-drawn to a scale in which the scale for the red line was twice the scale for the green line).



PLT Subtask 1.

The question has clearly caused the pupil confusion due to his ability to entertain two distinct meanings of the word "length". He is particularly aware, however, of the "numerical" meaning and allows the existence of the numerals to influence his thinking.

Thus he rationalises the figure by introducing scale. His responses to Subtasks 2 and 3 do not exhibit the same confused reaction. (This is a general truth for all pupils). He simply interprets the letter as a cipher whose numerical content is the measure of each concrete line (in Subtask 2 the lines are considered to be in perspective) - (see Appendix II, Section II.2.4. for further examples).

PLT - Subtask 2

Q. Is the red line longer than the green line, the green line longer than the red line, are they equal in length or could any of these be possible?

A. They could be equal.

Q. Why?

A. Well, that one (the green line) could be away from you. . .further away from you.

Q. When is the green line longer than the red line?

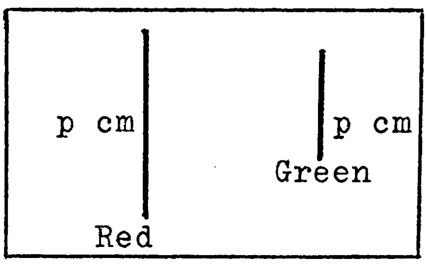
A. When it's brought nearer to you or timesed by any number.

Q. When is the red line longer than the green line?

A. As it is now or again if it's timesed by any number.

Q. When are the lines equal in length?

A. If they're the same height or nearness or p is the same.



PLT Subtask 2

PLT Subtask 3

Q. Is the red line longer than the green line, the green line longer than the red line, are they equal in length, or could any of these be possible?

A. Well, the b line, the red line is longer.

Q. Why?

A. Well, because it looks longer; well, it compares with 'a' to be longer.

Q. When is the green line longer than the red line?

A. Well, if you had like 3a cm. If you timesed the green line 'a'; if you multiplied the actual length by a number.

Q. When is the red line longer than the green line?

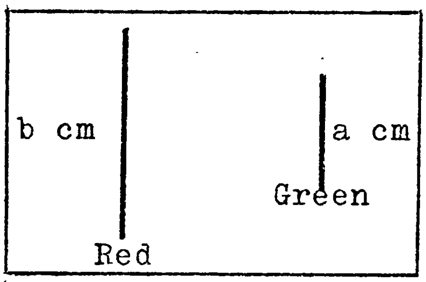
A. Well it is at present ordinarily. As one b it would be longer. If it was half it would be small. But if it was anything more than one b it would be longer.

Q. When are the lines equal in length?

A. Well if you had b equals a.

Q. When is that?

A. Well, if the green was further away it might be. Or if you added some more on as it is now you could get a equals b.



PLT Subtask 3.

In Subtask 2, the lines were considered to be in perspective and each 'p' denoted the measured outcome of a line. In this restricted sense the pupil has used the letter to transcend immediate concrete orderings, but each line (as in Subtask 1) is clearly a single line with a measured outcome 'p'.

These same observations apply to Subtask 3, in which each letter takes it's content from a concrete line. As such, differences in length are achieved either by concrete transformations (pushing one line into the distance) or by numerical operations (multiplying the present length by '3' to give '3a' etc.).

The pupil does not give an impression that what he might "perceive" here is a dynamic system of lines - that is, two 'lines' each of which denotes the relative position of two points moving either towards or away from each other. Should that have been the case then each letter would have been considered to have guaranteed numerical determinations, and the relative length of each line to depend upon the relationship of 'a' to 'b'.

All responses to the PLT defined to be at the level of "fictitious measures' involve similar interpretations of the figure viz. each line is "real", "concrete", and has a fixed measurable length. As such each letter illustrates the line and is a temporary substitute for the numeral which will be obtained when the line is measured. Invariably the red line is concretely longer than the green line.

Some pupils do not mention letters in Subtasks 2 and 3 but rely totally upon spatial perception. Thus they do not apply numerical operations to the letter to increase 'length', but instead restrict themselves either to denying that any ordering other than that suggested is possible, or to suggesting a variety of concrete methods of causing change (e.g. cutting lines in half, extending lines etc.). The following is an illustrative example (see Appendix II, Section II.2.6. for further examples).

Mary (1:12) 11yrs. 3mths. School B. PLT Subtask 3.

Q. Is the red line longer than the green line, the green line longer than the red line, are they equal in length or could any of these be possible?

A. The red line's longer.

Q. Why?

A. Because it looks longer.

Q. When is the green line longer than the red line?

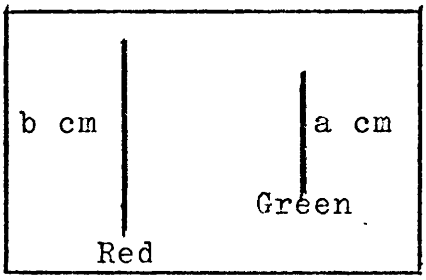
A. When it's doubled.

Q. When is the red line longer than the green line?

A. Now.

Q. When are the lines equal in length?

A. When you cut some off the red line.



PLT Subtask 3.

Effectively here the letter is totally ignored.

5.2.1. Summary

In the PLT, pupils working at the level of fictitious measures either

(a) ignore the existence of letters (but not numerals);

or (b) treat the letter as a temporary replacement for a measured outcome.

The pupil's method of changing the "length" of a line is either:

(i) to apply a concrete transformation (cut the line, draw it longer, push it in to the distance etc)

or (ii) to apply a numerical operation to a letter treated as a temporary replacement for a measured outcome (multiply by 2 etc.).

The pupil gives little evidence that what he "perceives" is anything more than two concrete lines drawn on paper, each of which has a fixed numerical length.

5.3. The Level of Fictitious Measures - responses to the ET

(See Appendix II, Section II.3. for the full range of response-types to the ET).

The three equations in the ET suggest ordering and uniqueness to varying degrees and as such dramatically influence the pupil's thinking processes. Some pupils however, will give a consistent 'fictitious measure' response to each Subtask (see Sections 5.3.1. and 5.3.2. below.)

In such a response the pupil exhibits an inability to "see" the equation as an integrated system (and thus does not give responses of the form 'when $x > 5$', 'when $x > 3$' etc.), and either treats each letter as an object with a fixed, unknown, content, or as a "pigeon-hole" for a numeral.

5.3.1. The "fictitious measure" response to the ET - re-arrangement strategies

In a "fictitious measure" response the pupil makes one or more of the assumptions

(a) each letter has an indeterminate content;

(b) the letters have an ordering dictated by the relation.

The following is an example of a pupil making assumption (a).

Example 1: A. J., (3:;2), 13yrs. 10mths, School A,

'x + y = 10'

Q. If this true, is the value of x always, sometimes or never greater than the value of y?

A. Sometimes.

Q. Why?

(4) A. Well, because you could. . .no. Never. Never because you only know what the values of x and y are together. You don't know what they are individually.

Q. When is the value of x greater than the value of y?

A. If erm. . .well..x would be greater than y if it was ten plus y equals x. (10 + y = x).

Q. Why would that be?

A. Because you need something to add to y to make x.

Q. When is the value of x equal to the value of y?

(10) A. Well. . .well, it could be now. x could be 5 and y could be 5.

Q. When is the value of x less than the value of y?

A. Well. . .if you rearrange it it will be. The value of x will be less than the value of y because x plus ten equals y (x + 10 = y). If you swopped y and ten you would need to add to x to make it equal y. That means x must be smaller.

Here the pupil makes explicit his belief that 'x' and 'y' can't be known and responds accordingly.

(Statement (4)). In Statement (10) he shows that he can find values for x and y which satisfy the required relation - but these values are mere 'fictions'. No-one can be certain that the true measure of x <u>is</u> 5.

A point of interest about this response (and many responses using x and y as objects with fictitious measures) is that the pupil is attempting to make a generalisation. He wants to answer the question 'whatever the true measures of x and y might be, how can we ensure that x is greater than y?'

His solution is to make an illegitimate transposition of the elements in the equation. In a sense, therefore the pupil is thinking in general terms - but the 'generality' of fictitious measures is not the 'generality' we know.

The following are further illustrations of "fictitious content" responses applied to '5x = y' in which each pupil suggests a re-arrangement of the data.

Example 2: Peter (4:12), 14yrs. 11mths; School A,
'5x = y'

Q. If this true, is the value of x always, sometimes or never greater than the value of y?

A. Never.

Q. Why?

A. Because you've only got one y and you've got five x's.

Q. When is the value of x greater than the value of y?

A. Well. . .you'd have to have something like x and 2y.

Q. When is the value of x equal to the value of y?

A. When you remove the five it will be. You'll get x equals y.

Q. When is the value of x less than the value of y?

A. When it's five x equals y.

Example 3: Matthew (3:11), 14 yrs, 1mth. School B.

'5x = y'

Q. If this is true, is the value of x always, sometimes or never greater than the value of y?

A. Never.

Q. Why?

A. Because y is five times x, so x must be smaller.

Q. When is the value of x greater than the value of y?

A. Remove the five. . .or you could divide that (5x) by 5.

Q. When is the value of x equal to the value of y?

A. When you put 5x = 5y. When you put an extra 5 on the y.

Q. When is the value of x less than the value of y?

A. Now.

In each transcript it is clear that the pupil does not "penetrate" the surface-structure of the equation so to identify and compare a numerical value for x and a numerical valuefor y.

This re-arrangement strategy is a feature of pupil activity at the level of fictitious measures. Each letter appears to be "seen" much as a piece on a chess board. Each piece has it's own particular content (value, measure) which is either greater than or less than the measure of the second piece.

In the equation $5x = y$, five pieces 'x' have between them a content equivalent to that of 'y' and hence each piece must have a smaller content than y. This being the case the measure of each piece 'x' can be raised above the measure of 'y' only by making an illegitimate transposition of elements in the equation (there is, however, an alternative - see Section 5.3.2. below).

Each pupil above assumes from the outset that the letters 'x' and 'y' have an ordered content (the measure of $x <$ the measure of y). This assumption might also be applied to '$2x + y = 9$', as in the following example.

Example 3: Gareth (4:11), 15yrs, 3mths. School A,

'2x + y = 9'

Q. If this is true, is the value of x always, sometimes or never greater than the value of y?

A. x is less than y.

Q. Why?

A. Because you need two of x and only one y to make up nine, so x must be smaller.

Q. When is the value of x greater than the value of y?

A. When you have only one x and two y's. When it's x plus 2y.

Q. When is the value of x equal to the value of y?

A. When it's two and two I suppose (2x + 2y = 9).

Q. When is the value of x less than the value of y?

A. In the equation.

and to 'x + y = 10' as follows:

Example 4: Simon (2:5), 13yrs. 1mth, School B,

'x + y = 10'

Q. If this is true, is the value of x always, sometimes or never greater than the value of y?

A. x is less than y.

Q. Why?

A. It's not as far into the alphabet.

Q. When is the value of x greater than the value of y?

A. It can't be.

Q. When is the value of x equal to the value of y?

A. It can't be or they'd be equal.

Q. When is the value of x less than the value of y?

A. It is now.

Examples 3 and 4 indicate how readily the pupil is influenced by an implicit ordering suggestion.

In example 3, the coefficient '2' for x immediately implies that each object 'x' must have a measure less than that of the object, 'y'.

In Example 4, it is alphabetic ordering which influences the pupil's thinking. (Two pupils - pupil (2:5) above in School B and pupil (1:12) in School A interpreted consistently throughout all tasks in terms of alphabetic positioning. Many more pupils working at the present level did so in the LNT (see Section 5.4 below). (This phenomenon questions the wisdom of introducing 'codes' into school mathematics unless care is taken to explain that the letter used in a code does not have the same properties as the letter used in algebra; and at the same time indicates how easily some pupils might be influenced by prior experiences).

Sometimes, the pupil's belief that a letter <u>has</u>

a unique numerical measure leads him to believe that he might have found it, as in the following example:

Example 5: David (2:12), 12yrs. 11mths. School A,

'x + y = 10'

Q. If this is true is the value of x always, sometimes, or never greater than the value of y?

A. It's never greater.

Q. Why is that?

A. Because they're always equal.

Q. When is the value of x greater than the value of y?

A. Never.

Q. When is the value of x equal to the value of y?

A. It is now. There. 5 and 5.

Q. When is the value of x less than the value of y?

A. It can't be. They're always equal.

This transcript exhibits <u>false-content</u> i.e. the pupil believes that the letter has a unique value and is convinced that he has found it.

5.3.2. Fictitious measure responses to the ET - numerical operation strategies

One of the disturbing features of pupil responses at the level of fictitious measures is the strange usage the pupil makes of the term "value of x". In the FLT two strategies were noted for increasing the length of a line:

(i) concrete transformations ("cutting" lines etc.).

(ii) numerical operations (multiplying 'a' by 2 etc.).

The re-arrangement strategy described above may be considered to be analogous to the transformation strategy in the PLT. The pupil simply re-arranges collections of x's and y's to arrive at a situation which satisfies the required relation.

The numerical operation strategy applied in the PLT to the letter considered to be a 'stand in' for the measured outcome of the line, may be applied also to the ET (in particular to '5x = y'). The outcome is a response which is particularly incongruous - but very common at the present level of activity. The following are examples (further examples are given in Appendix II, Section II.3).

Example 1: Matthew (1:10) 11yrs, 8mths, School B,

'5x = y'

Q. If this is true is the value of x always, sometimes, or never greater than the value of y?

A. It can't be bigger.

Q. Why?

A. Because you've got five of x and only one of y. So x has got to be smaller.

Q. When is the value of x greater than the value of y?

A. When it takes less than 5 of them to make y. Say it was $\frac{1}{2}$x. x would be bigger then.

Q. When is the value of x equal to the value of y?

A. When they're both multiplied to make the same.

Q. When is the value of x less than the value of y?

A. When it's less than five of it. When it's four of it or three of it.

Example 2: Jonathan (4:9) 14yrs. 11mths, School B,

'5x = y'

Q. If this is true, is the value of x always, sometimes or never greater than the value of y?

A. y is bigger than x.

Q. Why?

A. Because it takes five x's to make a y.

Q. When is the value of x greater than the value of y?

A. When it's multiplied by say 6. When it's 6x.

Q. When is the value of x equal to the value of y?

A. When it's five x.

Q. When is the value of x less than the value of y?

A. When it's multiplied by something less than 5. Say by 3 or 4. 3x or 4x. That's smaller.

Example 3: Thomas (3:10) 14yrs. 0mths, School A.

'5x = y'

Q. If this is true, is the value of x always, sometimes or never greater than the value of y?

A. Sometimes.

Q. Why?

A. Well, because you don't know the numerical values of x and y.

Q. When is the value of x greater than the value of y?

A. When x is erm. . .when x is greater than y.

Q. And when is that?

A. Well, for example, six x would be larger than y is this statement is true.

Q. When is the value of x equal to the value of y?

A. When it's multiplied by five.

Q. When is the value of x less than the value of y?

A. When x is less than 5x.

Q. Can you explain your ideas to me?

A. Well, five x is equal to y, so we could assume that five x is ten and y is ten. Therefore, anything over ten is greater than y, and anything under ten is less.

Q. What about when x is equal to y?

A. Well, five x equals y. "When is the value of x equal to the value of y?" Five x equals y, ten x equals two y and so on. Any of those.

Q. But isn't the value of x equal to the value of y when x is 0?

A. I don't understand what you mean.

Example 4: Andrew (2:11) 13yrs. 0mths, School A,

'5x = y'

Q. If this is true, is the value of x always, sometimes, or never greater than the value of y?

A. x is greater than y. No. It could be the same because you don't know what it actually equals. So, y could be a far greater number than x.

Q. When is the value of x greater than the value of y?

A. When x equals a lower number. If say that (pointing to y) equals 5, and you found that x equals 2, you would have to times 5 (the coefficient of x) by 2 and get 10. Then you would find that y was less and x could be more.

Q. When is the value of x equal to the value of y?

A. If 5x equalled y it's bound to be the same.

Q. And when would that be?

A. If x equalled 1 and y equalled 5. Their values would be the same then.

Q. When is the value of x less than the value of y?

A. When y is a larger number. Say y equalled. . .erm. . 7 and x equalled 1. Five times 1 would equal 5, and y would be the greater one.

Q. And would five x equal y if that was the case?

A. No. Sorry. . .x couldn't be greater than y if 5x equalled y. It couldn't be.

Q. Can we go back to the earlier questions?

A. Yes, alright.

Q. When is the value of x greater than the value of y?

A. It will always equal it.

Q. Why is that?

A. Well, y has got to equal what x is.

Q. Why?

A. Because of the way you wrote the sum.

Q. So y has got to equal what x is?

A. Yes, well, the sum of x. If x was 2 and y was 10, five twos are ten. They would be the same then.

Q. The value of x would equal the value of y?

A. Yes.

In each example, the pupil treats a letter as an object with a fictitious measure. In particular the measure for 1x is five times less than the measure for 1y. Thus multiplying the measure of 1x by five gives the measure for 1y (i.e. "the value of x is equal to the value of y when it's 5x etc."). Multiplying the measure of 1x by 6, gives a value greater than the measure for 1y. (i.e. "the value of x is greater than the value of y when it's 6x"). This strategy is identical to that used in the PLT.

The pupil is totally convinced during such discussions that his statements are making sense. He remains unaware that the mathematician today does not think of letters in this way but considers each to have guaranteed determinations at all times - so that multiplication is not needed as a strategy to raise the value of one letter above the value of a second letter.

For the mathematician the letter can "vary in itself" over all numerals. Multiplying the letter thus does not necessarily increase it's value. Moreover, whatever value might be obtained by applying such a strategy can be obtained independently of multiplication by allowing "variation in itself". The multiplication strategy used by the pupil is thus immediately obsolete to the mathematician.

Some pupils working at the level of fictitious measures in the ET will use more than one strategy in their attempts to deal with an equation. The following is an illustration:

Debbie (4:9) 14yrs. 11mths. School B, '5x = y'

Q. If this is true, is the value of x always, sometimes, or never greater than the value of y?

A. y is bigger.

Q. Why?

A. Because it's five times something and that (pointing to x) is only one times something.

Q. When is the value of x greater than the value of y?

A. When say it's multiplied by something more than 5.

Q. When is the value of x equal to the value of y?

A. It would be if you removed the five.

Q. When is the value of x less than the value of y?

A. Say if x was 2 and y was 10.

In the final statement Debbie suggests two possible measures - one for x and one for y. At the level of fictitious measures pupils may offer possible values for letters in the ET but the values are almost always singular offerings. Section (c) below discusses the substitution strategy at this level.

5.3.3. Level of fictitious measures - substitution strategies to the ET

Substitution in '5x = y' is rare. The pupil looks upon 'y' as an object with an unknown measure greater than that for x and this gives rise to transposition, or numerical operation, strategies to increase or decrease values.

The equations 'x + y = 10', '2x + y = 9' however do not influence the pupil's thinking to the same extent. The most common response to each question is to offer a pair of whole numbers which satisfy the required relation. This may often be preceded (in particular for '2x + y = 9') by a decision that either 'x' or 'y' has the greater measure (depending upon whether the pupil reads '2x' as "two 'x' objects which have the same measure each" - in which case x is likely to be considered to have a smaller measure than y; or whether '2x' is read as the outcome when the "measure for x has been doubled" - in which case the pupil is likely to assume that x and y originally had the same measure but that the measure for 'x' is now twice the measure for y). The following are examples. (The notation  represents three states of numerical equilibrium and is discussed below).

Example 1: Debbie (4:7) 14yrs. 11mths, School B, 'x + y = 10'

Q. If this is true is the value of x always, sometimes, or never greater than the value of y?

A. They're both the same. (x  y).

Q. Why?

A. Because they must both be five.

Q. When is the value of x greater than the value of y?

A. If x was 8 and y was 2 it would be. (x  y)

Q. When is the value of x equal to the value of y?

A. Both five. (x  y)

Q. When is the value of x less than the value of y?

A. The other way round. If x is 2 and y is 8.
(x  y)

This response exhibits <u>reluctant ordering</u> i.e. the pupil assumes an ordering of values originally (, x = y) but then finds values which contradict the original decision (). (In the example transcripts in sections 5.3.1. and 5.3.2 pupils retained the original ordering and used a variety of alternative strategies to substitution to increase one measure above another.

Such a response does not necessarily imply that the pupil has relinquished the view that x and y are objects with true measures. The lack of willingness to state more than one possible pair of values, and the original ordering decision suggest that each letter continues to be considered to have a unique content. Thus, for example, x <u>might</u> have the measure '5', <u>or</u> '2' - but we simply cannot tell which is true from the equation. Debbie's response to '2x + y = 9' supports this view:

Example 2: Debbie (4:7) 14yrs. 11mths, School B,

'2x + y = 9'

Q. Is the value of x always, sometimes or never greater than the value of y?

A. x is never bigger. ().

Q. Why?

A. Because x is multiplied by 2. No. They're the same. They're both 3. ().

Q. When is the value of x greater than the value of y?

A. When. . .if it was 4 and 1. If x was 4 and that 1, two times 4 is 8, add one is nine. ().

Q. When is the value of x equal to the value of y?

A. When they're both 3. ().

Q. When is the value of x less than the value of y?

A. Two times one is two and seven is nine. ().

Q. So which is larger, the value of x or the value of y?

A. I suppose it might be any of them.

5.3.4. Summary

The impression given by each response in this section is that an indeterminate equation is seen as a collection of individual units - an 'x', a 'y', a '2', a '9', etc. each of which contains numerical information. Each individual unit has a content. As such the content of an individual unit must be either greater than, equal to, or smaller than the content of a second. With this particular view of letters the pupil necessarily assumes one of the equilibrial states x  y, x  y, x  y must be true. (When it comes to the LNT the question "which is larger" seems to confirm this belief - see below).

In the ET pupils working at the level of fictitious measures thus:

(a) treat the letter as an object with an undetermined (unique) content;

(b) do not allow the value of the letter to change in any real sense. Any change in the content of a letter is achieved by numerical operations upon the letter or by embedding the letter in a new context (e.g. 5x = 5y).

(c) sometimes believe they have found the true content;

(d) believe that a true ordering of content exists;

(e) are restricted in their substitution strategy;

(f) vary in the response-type made to each equation, sometimes offering possible numerical values for letters in '$x + y = 10$' and '$2x + y = 9$' but always treating the letter as an object with an unknown content in $5x = y$;

(g) appear to 'see' an indeterminate equation as a collection of individual units, each with a fixed content.

5.4. <u>The Level of fictitious measures - responses to the LNT</u>

In both the PLT and ET pupils have been shown to exhibit false ordering i.e. they will assume a letter has an ordered content given to it by the immediate data, or will use the data to support their own expectations of ordering. This attitude is reflected in the LNT.

Here the pupil assumes an ordering at the outset (e.g. $m + m > m + k$) and, as in the PLT and ET, either maintains that belief throughout, or might reluctantly admit that any ordering can be possible. Where the pupil agrees from the outset that any ordering exists, he demonstrates his assertion by making numerical replacements for letters i.e. he does not 'match' across the numbers to obtain a content for a letter (e.g. '$m = k$', '$t = 4$', '$b + 3 = c + 4$') and then allow for the letter to vary above or below this matched content, but restricts himself to stating a possible number for each letter which satisfies the required relation.

5.4.1. False ordering without correction: (i) maintained ordering

In false ordering without correction the pupil usually assumes the letter to have an ordering by virtue of it's position in the alphabet, and then denies that any alternative is possible. The following are illustrations:

Example 1: Trevor (5:10) 15yrs. 5mths, School A,

'm + m, m + k'

Q. Which is larger, m + m or m + k?

A. m + m. (——•——)

Q. Why?

A. Because m is further on in the alphabet.

Q. When is m + m larger than m + k?

A. It already is. (——•——)

Q. When is m + k larger than m + m?

A. It isn't. (——•——)

Q. When are they equal?

A. They're not. (——•——)

Example 2: A. J. (3:12) 13yrs. 11mths, School A,

't + t, t + 4'

Q. Which is larger, t + t or t + 4?

A. t + t. (——•——)

Q. Why?

A. Because that (pointing to t + t) is t + t and that (pointing to t + 4) is only 4.

Q. When is t + t larger than t + 4?

A. It <u>is</u> larger. ()

Q. When is t + t equal to t + 4?

A. It isn't. ()

Q. When is t + t less than t + 4?

A. It isn't. ()

Example 3: Debbie 14yrs. 11mths, School B, 'a + b + 3, a + c + 4'.

Q. Which is larger a + b + 3 or a + c + 4?

A. a + c + 4. ()

Q. Why?

A. Because b comes before c and 3 is less than 4.

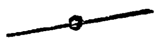
Q. When is a + b + 3 larger?

A. Never. ()

Q. When is a + c + 4 larger?

A. Now. ()

Q. When are they equal?

A. They can't be. ()

These responses are identical in nature to those in the PLT. Here however the 'content' for the letter is not obtained from geometrical data. Whatever 'content' the letter has is 'fictitious' and suggested purely by alphabetic ordering.

Many pupils at this level have an ordering fixation - a fact which has already been demonstrated in the PLT and the ET.

Sometimes the pupil will attempt to give a condition for which one 'number' (e.g. m + k) is greater than the second based upon 'algebraic' reasoning.

Usually such attempts involve torturous usage of the data, as in the following example:

Christine (5:12) 15yrs. 10mths, School B, 't + t, t + 4'

Q. Which is larger, t + t or t + 4?

A. t + 4. ()

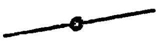
Q. Why?

A. Because t + 4 is 4t[68] and t + t is only 2t. No, it's t^2. No, that's wrong. It's 2t.

Q. When is t + t larger than t + 4?

A. Never. ()

Q. When is t + 4 larger than t + t?

A. Always. ()

Q. When are they equal?

A. Never. ()

---

(68) Many pupils make the same error. 't + 4' appears to mean to them 't added four times' or 't, add four of them'. This, of course, will give rise to totally incorrect strategies for solving equations.

5.4.2. False Ordering Without Correction: (ii) Re-established ordering

In each transcript above the pupil has decided upon an ordering and rejected any possibility of an alternative. The following are examples of re-established ordering in which the pupil makes an original decision, contradicts it, and then returns to his/her original equilibrial position.

Example 1: Debbie (4:7) 14yrs. 11mths, School B,

'$t + t$, $t + 4$'

Q. Which is larger, $t + t$ or $t + 4$?
A. They're both the same. ()
Q. Why?
A. Because you can put 4 and 4 for them.
Q. When is $t + t$ larger than $t + 4$?
A. You could have $t + 8$ and $t + 4$. ()
Q. When is $t + 4$ larger than $t + t$?
A. When it's $t + 2$ and $t + 4$. ()
Q. When are they equal?
A. When it's $t + 4$ and $t + 4$. ()
Q. So which is larger, $t + t$ or $t + 4$?
A. They're both the same. ()

Example 2: Christine (5 :12) 15yrs. 10mths. School A

'$a + b + 3$, $a + c + 4$'

Q. Which is larger, $a + b + 3$ or $a + c + 4$?
A. $a + c + 4$. ()

Q. Why?

A. Because it's the bigger number. c is bigger than b.

Q. When is a + b + 3 larger?

A. When it's put into numbers it could be, Say it's 3 for b and 2. . .No, one for c. ()

Q. When is a + c +4 larger?

A. Say if c was 4 and b was 2. ()

Q. When are they equal?

A. Two for b and one for c. ()

Q. So which is larger, a + b + 3 or a + c + 4?

A. a + c + 4. ()

Q. Why?

A. Because c's usually the bigger number.

Example 3: Daniel (1:12) 11yrs. 6mths. School A,

'm + m, m + k'

Q. Which is larger, m + m or m +k?

A. m + m. ()

Q. Why?

A. m is a bigger number. Say m was 10, k would be 8.

Q. When is m + m larger than m + k?

A. I've just told you that! ()

Q. When is the m + k larger than m + m?

A. Well. . .if you could put m ten and k twelve it would be. ()

Q. When are they equal?

A. They can't be equal. You can't have them the same.

Q. So which is larger, m + m or m +k?

A. m + m. ()

Pupils at this level clearly find immense difficulty accepting that more than one ordering can obtain at any particular instant. As was the case for the pre-Vietan mathematician, their world is a world of concrete objects having an established measure. In the same sense that an apple cannot at one and the same time have more than one particular measure (mass) so pupils working at this level believe also that the letter must have one particular measure.

Despite the fact that he might be able to suggest possibilities which correlate with each "equilibrial position", nevertheless the pupil appears to need to assume just one, and to return to this for "security". His conceptual security thus appears to lie in dealing with an ordered world of static entities with fixed measures.

Not all pupils take such an extreme position. Some are willing to admit (often reluctantly) that any ordering might exist. The following are examples:

5.4.3. False ordering with correction

Geoffrey (3:11) 14yrs. 3mths. School A, 'm + m, m + k'

Q. Which is larger, m + m or m + k?

A. m + m. ()

Q. Why?

A. Because m is further into the alphabet.

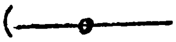
Q. When is m + m larger than m + k?

A. It is now. ()

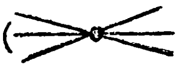
Q. When is m + k larger than m + m?

A. Do you want me to put numbers in?. . .If I can put numbers in it's when k is say 10 and m is say 4. ()

Q. When are they equal?

A. 10 for k and 10 for m say. ()

Q. So which is larger, m + m or m + k?

A. I suppose it could be any couldn't it! ()

Eleanor (4:11) 14yrs. 7mths. School B, LNT, 't +t, t + 4'

Q. Which is larger t + t or t + 4?

A. t + 4. ()

Q. Why?

A. Because you've got two unknowns in t + t and you've got a four in t + 4.

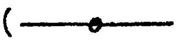
Q. When is t + t larger than t + 4?

A. Can I put numbers in? If I can put numbers in, it's when t + t is 8 plus 8, and t plus 4 is 2 plus 4. ()

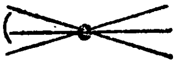
Q. When is t + 4 larger than t + t?

A. When t + t is t + 2, and t + 4 is 4 + 4. ()

Q. When are they equal?

A. When the same numbers are put in. When it's 4 plus 4 and 4 plus 4. ()

Q. So which is larger, t + t or t + 4?

A. I suppose it could be any of them. It depends on what numbers you put in. ()

Katy (4:5) 14yrs. 7mths. School B, 'a + b + 3, a + c + 4'

Q. Which is larger, a + b + 3, or a + c + 4?

A. a + c + 4. ()

Q. Why?

A. Because you've got one more there (pointing to a + c + 4).

Q. When is a + b + 3 larger?

A. Erm. . .when that number (pointing to b) is bigger than that (pointing to c). ()

Q. When is a + c + 4 larger?

A. When c is a bigger number than b. ()

Q. When are they equal?

A. They might be now. ()

Q. So which is larger, a + b + 3 or a + c + 4?

A. It might be any. ()

Sometimes, as pupils worked through the LNT they would change response types after reflecting upon previous "wrong turnings", and by the time they reached 'a + b + 3, a + c + 4' were reluctant to assume an ordering. Their general form of response was then "you can't tell, can you. You don't know what a and b and c are". This was usually followed by the pupil offering possible replacements for each letter to satisfy the required ordering. These replacements often ignored the fact that 'a' was identical to 'a' and seemed to be chosen almost indiscriminately (thus 'a + b + 3' is larger when it's '4, 12, 3' and 'a + c + 4' is '1, 1, 4' etc.).

Other pupils who generally worked at the level of fictitious measures in the ET began the task with this attitude. Thus, although false ordering and false content responses were not evident, suggestions for literal values were often tentative, as though the pupil believed one such value might be the true one but was not particularly certain which it might be. The following is an example from a pupil working consistently as the level of fictitious measures in the ET:

Kay (4:12) 15yrs. 3mths. School B, 't + t, t + 4'

Q. Which is larger, t + t or t + 4?

A. You can't be sure.

Q. Why?

A. Because you don't know what t is.

Q. When is t + t larger?

A. If t was 6 it would be.

Q. When is t + 4 bigger?

A. If t was 2 or 3.

Q. When are they equal?

A. When t is 4.

'm + m, m + k'.

Q. Which is larger, m + m or m + k?

A. You can't tell.

Q. Why?

A. They could be anything at the moment.

Q. When is m + m larger?

A. When m is 7 and k is 3.

Q. When is m + k larger?

A. When k is 5 and m is 2.

Q. When are they equal?

A. When k is 5 and m is 5.

What is most noticeable about such responses is that a direct matching strategy is not used to give one letter as the 'content' for the second (e.g. in'm + m, m + k', m = k). The notion that a letter, having a fixed unknown value, can have another letter as it's content seems to be either inconceivable, or of no value in mathematics. (If m is unknown and k is unknown, what sense is there in saying that m is equal to k, bigger than k, or smaller than k?).

Such responses suggest that the pupil continues to look upon 'letters' as objects (such as oranges and pears) which have a fixed content (analogously a fixed mass) which cannot be known. Whatever "variation" exists here would appear to be an "hypothetical variation". We can make a guess at what might be the relative masses of two pieces of fruit, but we cannot be certain if our guess is "true" until we have carried out the necessary weighing procedures. Analogously, the pupil can make a guess at what might be the case, but he cannot be certain which is true until further information has been given. What he certainly does not appear to do is define each to be possible.

5.4.4. Summary

The pupil who responds at the "level of fictitious measures" to the LNT:

(a) often exhibits a desire for ordered content;

(b) appears to "see" literal numbers as objects with a fixed (undetermined) content;

(c) does not use a matching strategy to give the letter a literal content;

(d) has little respect for the formal syntax of the algebraic language;

(e) is prone to errors of the form "t +4 = 4t; ('t' added 4 times is 4t? - 't' add four of them is 4t? - see footnote (68) p.143).

5.5. The Level of Fictitious Measures - responses to the ZT

Pupils working at the level of fictitious measures are of all ages 11 - 16 (see Tables 27 (a) and (b), Appendix II, Section II.6.1.). Only sixth-forms were free of pupils who consistently interpreted letters as objects with "fictitious measures". (The sixth form pupils in the present study however, were 'established' mathematicians - it is thus more than likely that many sixth form non-mathematicians and many adults interpret letters at the same level.)

This view of the letter as an object with a fixed content corresponds almost invariably with an incapacity to make any real headway with the Zetetic Task - which demands a general argument. The following are illustrative examples of attempts to deal with the ZT, (See Appendix II, Section II.5. for the full range of response-types to the ZT).

Example 1: Eleanor (4:11) 14yrs. 7mths. School B,

"Nos = 4,2

5,1

Nos = 3,2

8,4.

Nos = 6,2"

Example 2: Kay (4:12) 15yrs. 3mths. School B.

"diff. 2 sum 10 = 4 + 6"

Example 3: Katy (4:5) 14yrs. 7mths. School B.
"Ex. If the sum of a number is 12 and the difference is 2, you divide the sum by the difference, which would give you six and the number either side of this are the numbers which were added together".

Example 4: Katharine (4:10) 15yrs. 1mth. School B.
"By putting the numbers in brackets and making an equation."

Example 5: Christine (5:12) 15yrs. 10mths. School B,

"Well - if you've got: 10 = sum
2 = difference
you just have to say two numbers that add up to 10 that have a difference of 2:
e.g. 20 = sum, difference = 4
no = 12 + 8, 12 + 8 = 20, and difference of 4. You have to find no's that add up to that no (x) that have a difference 4. I've said that!" (All this was written down).

5.6. Summary - The Level of Fictitious Measures

At the "Level of Fictitious Measures" the letter is treated as an object with a unique content.

Pupils interpreting the letter consistently in this sense are generally to be found in the lower mathematics sets, and are of all ages 11 - 16 (see Tables 27 (a) and (b), Appendix II, Section II.6.1). Many of them (especially those in the 4th and 5th years) make combinatorial errors (e.g. '$t + 4 = 4t$', '$t + t = t^2$', 'dividing m + k by m gives 1 + k', etc.).

It seems that early in secondary school life the pupil might have been conditioned, through excessive exercise in "finding" true values, to think of letters in the way described above; and that this experience may have done little but confirm prejudices - viz: that mathematical objects <u>are</u> ordered and <u>have</u> unique contents.

The conception of the letter as an object with a fictitious content belongs to pre-Vietan mathematics. Diophantus, in his attempts to deal with indeterminate equations, indicates he transcends this conception; but the way in which many Mediaeval mathematicians interpreted the letter might well have reflected much of what has been reported above. In particular their inability to entertain multiple values and negative quantities, and their dependance upon geometrical illustration as the ultimate proof method suggests an important parallel.

5.6.1. Archetypes

The following transcripts are included here to illustrate general activity at the level of fictitious measures. Each pupil is in the final year of an 'O' level mathematics course and is expected to attain a pass. It is an open question, however, as to what the majority of mathematics might have meant to each during his/her school course. Each pupil can be considered typical of pupils working at the present level.

Andrew (5:6) 15yrs. 9mths. School B.

PLT - Subtask 2 (See diagram over)

Q. Is the red line longer than the green line, the green line longer than the red line, are they equal in length or could any of these be possible?

A. The red line's longer.

Q. Why?

A. Because it looks longer.

Q. When is the green line longer than the red line?

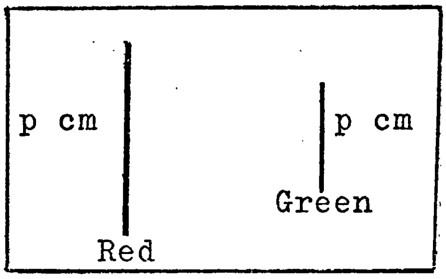
A. Would it be when it's doubled or trebled?

Q. When is the red line longer than the green line?

A. When it's halved.

Q. When are the lines equal in length?

A. When the green line's doubled.



PLT - Subtask 3

Q. Is the red line longer than the green line, the green line longer than the red line, are they equal in length, or could any of these be possible?

A. The red line's longer.

Q. Why?

A. Because it's larger than a.

Q. When is the green line longer than the red line?

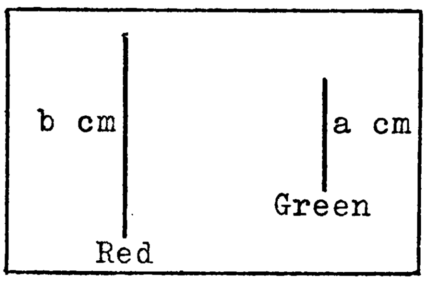
A. Would it be when it's squared? When it's 'a' squared?

Q. When is the red line longer than the green line?

A. When the green line is left as it is.

Q. When are the lines equal in length?

A. When the red line's timesed by a and the green line is timesed by b.



<u>Equations Task</u> (x + y = 10)

Q. If this is true, is the value of x always, sometimes or never greater than the value of y?

A. They're the same.

Q. Why?

A. Because there's nothing to tell you which is which.

Q. When is the value of x greater than the value of y?

A. When the y is taken over to the other side.

Q. When is the value of x equal to the value of y?

A. When they're both divided by ten.

Q. When is the value of x less than the value of y?

A. When just x is divided by 10.

(5x = y)

Q. If this is true, is the value of x always, sometimes, or never greater than the value of y?

A. y is larger.

Q. Why?

A. Because you've got to times x by something which is 5.

Q. When is the value of x greater than the value of y?

A. When y is taken over to the other side (5y = x).

Q. When is the value of x equal to the value of y?

A. When y is divided by 5.

Q. When is the value of x less than the value of y?

A. When y is timesed to 5 times as much as x.

(2x + y = 9)

Q. If this is true is the value of x always, sometimes, or never greater than the value of y?

A. y is a greater value.

Q. Why?

A. Because the x has been timesed by more than one.

Q. When is the value of x greater than the value of y?

A. When y is taken over to the other side.

Q. When is the value of x equal to the value of y?

A. I don't know. I don't think it can, can it? x is always smaller.

Q. When is the value of x less than the value of y?

A. When y is taken over because y is the larger at the beginning. If you took it over it would be 9y and that's bigger than 2x.

Literal Number Task - (t + t, t + 4)

Q. Which is larger, t + t or t + 4?

A. t + 4.

Q. Why?

A. Because you've added plus 4.

Q. When is t + t larger than t + 4?

A. It's not.

Q. When is t + 4 larger than t + t?

A. When it's (t + 4) divided by 4.

Q. When is t + 4 equal to t + t?

A. When you times t + t by 2. t + t is 2t and t + 4 is 4t. So if you times t + t by 2 you'll get 4t.

(m + m, m + k)

Q. Which is larger, m + m or m + k?

A. m + k.

Q. Why?

A. Because m is further into the alphabet than k.

Q. When is m + m larger than m + k?

A. It is now.

Q. When is m + m less than m + k?

A. When that (pointing to m + k) is timesed out by m. You would get m^2 + km.

Q. When is m + m equal to m + k?

A. When they're both divided by km.

Q. So which is larger, m + m or m + k?

A. m + k.

(a + b + 3, a + c + 4)

Q. Which is larger, a + b + 3 or a + c + 4?

A. The second.

Q. Why?

A. Because 4 is larger than 3.

Q. When is a + b + 3 larger than a + c + 4?

A. When it's squared.

Q. When is a + c + 4 larger than a + b + 4?

A. When it's squared.

Q. When is a + c + 4 larger than a + b + 3?

A. I don't see that. I don't think it is.

Q. When are they equal?

A. When one is timesed by 3 and the other by 4.

Q. So which is larger, a + b + 3 or a + c + 4?

A. a + c + 4.

Zetetic Task.

No response.

Example 2: Katherine (4:10) 15 yrs. 1mth. School B.

PLT Subtask 2

Q. Is the red line longer than the green line, the green line longer than the red line, are they equal in length or could any of these be possible?

A. The red line is longer.

Q. Why?

A. Because it's longer. I can see it is.

Q. When is the green line longer than the red line?

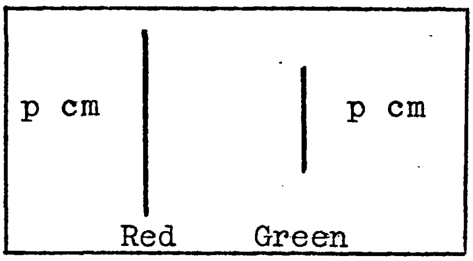
A. Never.

Q. When is the red line longer than the green line?

A. Now.

Q. When are the lines equal in length?

A. They won't be.



PLT Subtask 3

Q. Is the red line longer than the green line, the green line longer than the red line, are they equal in length, or could any of these be possible?

A. The red line is longer.

Q. Why?

A. Because it's longer on the card.

Q. When is the green line longer than the red line?

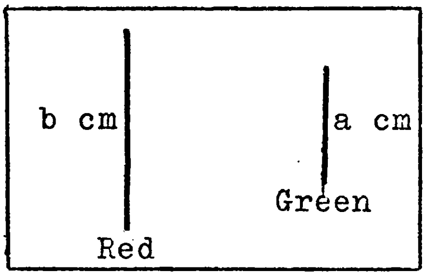
A. When the green line gets longer than the red line.

Q. When is the red line longer than the green line?

A. When you extend it.

Q. When are they equal in length?

A. They'll never be equal.



<u>Equations Task</u> (x + y = 10)

Q. If this is true, is the value of x always, sometimes, or never greater than the value of y?

A. They're both the same.

Q. Why?

A. Because two goes into ten five times.

Q. When is the value of x greater than the value of y?

A. I don't know. Would it be two and eight?

Q. When is the value of x equal to the value of y?

A. When x is 5 and y is 5.

Q. When is the value of x less than the value of y?

A. When x is 2 and y is 8.

(5x = y)

Q. If this is true, is the value of x always, sometimes, or never greater than the value of y?

A. They're both the same. No, I mean y is bigger.

Q. Why?

A. Because you need five x's to make a y.

Q. When is the value of x greater than the value of y?

A. They're both the same. Five x <u>equals</u> y.

Q. When is the value of x equal to the value of y?

A. Now.

Q. When is the value of x less than the value of y?

A. It can't be. They're both the same.

(2x + y = 9)

Q. If this is true, is the value of x always, sometimes, or never greater than the value of y?

A. x is larger than y.

Q. Why?

A. Because x is 2 and y is only 1.

Q. When is the value of x greater than the value of y?

A. When it's changed round - the y for the x.

Q. When is the value of x equal to the value of y?

A. When you get 2x and 2y.

Q. When is the value of x smaller than the value of y?

A. When you put a 2y there (for the y) and an x there (for the 2x).

<u>Literal Number Task</u> (t + t, t + 4)

Q. Which is larger, t + t or t + 4?

A. t + 4.

Q. Why?

A. Because t + t is 2t and t + 4 is 4t.

Q. When is t + t larger than t + 4?

A. Never.

Q. When is t + 4 equal to t + t?

A. They can't be.

Q. When is t + t less than t + 4?

A. It isn't, t + 4 is larger.

(m + m, m + k)

Q. Which is larger, m + m or m + k?

A. You can't tell.

Q. Why?

A. Because they're all letters.

Q. When is m + m larger than m + k?

A. m + m is m^2 so it is bigger than m + k now.

Q. When is m + m equal to m + k?

A. Could it be when you change them into numbers?. . .
I don't know.

Q. When is m + m less than m + k?

A. I don't know.

(a + b + 3, a + c + 4)

Q. Which is larger, a + b + 3 or a + c + 4?

A. a + b + 4 is bigger.

Q. Why?

A. Becuase you've got one more than the other.

Q. When is a + b + 3 larger?

A. It won't be.

Q. When is a + c + 4 larger?

A. Now.

Q. When are they equal?

A. When you add one on to the a + b + 3.

Zetetic Task.

'By putting the numbers in brackets and making an equation'.

CHAPTER 6 : THE LEVEL OF DISCOVERED CONTENT

6.1. Introduction

There is a group of pupils who are relatively free of an interpretation of the letter as an object with a unique measure in the ET and LNT. Pupils in this group do not use "numerical operation" and "re-arrangement strategies" in their dealings with the ET, but consistently "work within the given system". Thus although a pupil might consider 'x' to be less than 'y' in '5x = y', he is likely to resist both re-arranging elements, and applying numerical operations (multiplying x by 6 etc.), to achieve alternative orderings suggested by the questions.

In the LNT this group of pupils do not exhibit "false ordering" responses. They appear, unlike their peers, not to desire any particular equilibrial state, and so accept that any ordering relationship of literal entities <u>might</u> exist.

In a "pure" algebraic setting it appears that these pupils make a reference to collections of <u>numerical</u> identities for letters. However their responses to the PLT and ZT suggest that a concept of the letter as a mathematical entity with guaranteed determinations independently of context has not yet developed. They do not consistently use the letter as an organiser of perception in the PLT, and do not use the letter as a preferred alternative to numerals for "givens" in the ZT.

Although a pupil will allow a letter a variety of numerical identities, he will defer a decision as to which, and how many, until he meets with a given situation. Thus in the PLT and in the ZT a pupil may allow the letter precisely one value, whereas in the ET he may allow several, or an infinite number.

These pupils are accordingly said to respond at the "level of discovered content". The present chapter illustrates the nature of responses at this level.

6.2. The Level of Discovered Content - responses to the ET

Responses to the ET associated with this level fall into three different categories. The first type (multiple substitutions) for '$x + y = 10$' might sometimes be used also by pupils who are influenced by ordering and uniqueness in the remaining ET subtasks at the Level of Fictitious Measures.

The second two types are considered to be those definitive of activity at the Level of Discovered Content.

6.2.1. Multiple Substitution Responses

Here the pupil does not indicate either that he believes x and y to be ordered at the outset, or that the values of x and y cannot be known. He refers 'through' to or discovers, possible numerical replacements for x and y which satisfy the given relation, and states in 'step-by-step' fashion whole number values which satisfy the relation.

The following is an example for 'x + y = 10'.

Nigel (1:2) 12yrs. 5mths, School A. 'x + y = 10'

Q. If this is true, is the value of x always, sometimes or never greater than the value of y?

A. It could sometimes be greater than the value of y.

Q. Why?

A. Because x could be 2 and y eight which equals ten, or x could be 8 and y two which equals ten.

Q. When is the value of x greater than the value of y?

A. When x is 6 or 7 or 8 or 9 and y is 4 or 3 or 2 or 1.

Q. When is the value of x equal to the value of y?

A. When x is 5 and y is 5. When the two numbers are both 5. When the letters represent fives.

Q. When is the value of x less than the value of y?

A. When x is 4, 3, 2, 1.

(The "multiple substitution" response to '2x + y = 9' is identical - see Appendix II, Section II.3.2. for an example. There were no "multiple substitution" responses to '5x = y').

The feature of the multiple substitution response is that the pupil does not commit himself originally to a particular ordering, and later does not restrict himself to stating just one possible number-pair for 'x' and 'y' which satisfies 'x$>$y' and 'x$<$y'. He appears to accept that any ordering might exist.

Possible replacements for letters are stated in a step-by-step fashion and are always whole numbers. There is no indication that any other strategy is involved other than a direct replacement of the letter by a number, much as one might place objects in a box one by one and remove each prior to replacing the second. (Thus some "multiple substitution" responses take the form "when x is 1 and y is 9, or when x is 2 and y is 8, or when x is 3 and y is 7, or. . .", as though the pupil places '1' in the box 'x', removes it, then replaces it by '2' etc.).

In this sense the pupil's thinking appears to be "sporadic" rather than "fluid". The "multiple substitution" response may be considered to be a transitionary response transcending that of the "fictitious measure" but not yet "algebraic".

The second form of response used by pupils is an advance upon this.

6.2.2. Borderline - Algebraic Responses

This, and the "algebraic" response, is the definitive form of response to the ET at the level of discovered content.

In the "borderline-algebraic" response for 'x + y = 10' and '2x + y = 9' the pupil finds the numerical value for which x is equal to y ('x = y = 5' for the former, and 'x = y = 3' for the latter) and then avoids mentioning by name each possible whole-number replacement which satisfies the required relation. (In '5x = y' he uses the terms "Never", "Never", "Always" to respond to questions).

The following is an example of the "borderline-algebraic" response for '2x + y = 9':

Ian (5:11) 15yrs. 3mths. School A. '2x + y = 9'

Q. If this is true, is the value of x always, sometimes or never greater than the value of y?

A. Sometimes.

Q. Why?

A. Because they would be equal if x was 3. But x could be more or less.

Q. When is the value of x greater than the value of y?

A. When x is 4 or more.

Q. When is the value of x equal to the value of y?

A. When they're both 3.

Q. When is the value of x less than the value of y?

A. When it's less than 3. Two or one.

Here it seems that all possible whole number replacements for both 'x' and 'y' might be considered to have an identical 'status'. That is, no one numeral is considered to be the most likely candidate for <u>actual</u> value, Thus false ordering does not feature in the response. Each numeral is welcome as a possible replacement.

The "borderline-algebraic" response to '5x = y' suggests a similar interpretation:

Neil (6:5) 16yrs. 5mths. School B. '5x = y'

Q. If this is true is the value of x always, sometimes, or never greater than the value of y?

A. y is always larger,

Q. Why?

A. Because x is divided by 5. You need five x's to make one y.

Q. When is the value of x greater than the value of y?

A. It can't be greater.

Q. When is the value of x equal to the value of y?

A. When they're both nought.

Q. When is the value of x less than the value of y?

A. All the time.

Here the pupil assumes an ordering of 'x' and 'y', but because he constantly refers "through" to <u>numerical</u> values for letters he neither makes re-arrangements, nor applies a numerical operation to 'x'.

He knows that the <u>numerical</u> replacement for 'x' is smaller than the <u>numerical</u> replacement for 'y'. His attention is thus focussed upon the numerical relationship which obtains between determinations of 'x' and 'y'. For all the numerals within his immediate repetoire (ignoring his "error" with zero) the numerical replacement possible for 'x' is smaller than the numerical replacement possible for 'y'.

This attention to the relationship which obtains between multiple numeral-pairs at the expense of exclusive attention to the <u>number of objects</u> available is what prevents the "re-arrangement" and "numerical operation" strategies being used.

As 'x' increases it's numerical value, so does 'y'. Consequently, 'x' cannot be greater than 'y' and cannot be made greater than 'y'. The pupil thus deals with an "integrated system" of numeral pairs and can hold in mind a numerical relationship, comparing and contrasting relative orderings as the numeral pairs vary within the constraints of the relation.

Pupils at the level of fictitious measures do not demonstrate this same "fluidity" of thought.

(The contrast between pupil responses at this level and at Level I may have important implications for teaching. Pupils at the present level have the cognitive capacity to deal with an "integrated, variable system" of numeral-pairs which satisfy a relation. This capacity is clearly due, in part, to a defeat of the belief that the letter actually "contains" a unique value. Thus the capacity might be best developed by paying a premium to "indeterminate" equations at the expense of "determinate" single-letter equations in the classroom).

6.2.3. Algebraic responses

The two response-types above are more common amongst younger pupils. The older pupils, with more experience at hand, generally give an "algebraic" response.

Here the pupil discovers the "boundary" value for which 'x' is identical to 'y', and uses this knowledge to separate the range of variability of each letter into three constituent 'parts':- that for which the letters are identical (e.g. '$x = y = 5$' for '$x + y = 10$') and the ranges above and below this value (e.g. '$x > 5$', '$x < 5$', for '$x + y = 10$').

The indication is that the pupil recognises, almost at a glance, that x and y covary over sets of numerals. His problem from this point forward is simply to determine how. Thus in '$x + y = 10$' and '$2x + y = 9$', as 'x' increases, 'y' decreases and vice-versa; whereas in '$5x = y$', and 'x' increases/decreases, 'y' increases/decreases respectively.

The following are examples for each equation:

David (5:5) 16yrs. 2mths. School A. '$x + y = 10$'

Q. If this is true, is the value of x always, sometimes, or never greater than the value of y?

A. x could be the same, smaller, or it could be greater than y, depending upon what you want.

Q. When is the value of x greater than the value of y?

A. When x is greater than 5.

Q. When is the value of x equal to the value of y?

A. When x is equal to 5.

Q. When is the value of x less than the value of y?

A. When x is less than 5.

Mark (5:6) : 16 yrs. 2mths. School A. '2x + y = 9'

Q. If this is true, is the value of x always, sometimes or never greater than the value of y?

A. Sometimes.

Q. Why?

A. It depends on the value of x and y.

Q. When is the value of x greater than the value of y?

A. When x is greater than 3.

Q. When is the value of x equal to the value of y?

A. When x is 3.

Q. When is the value of x less than the value of y?

A. When x is less than 3.

Mark (4:2) 15 yrs. 4 mths. School A. '5x = y'

Q. If this is true, is the value of x always, sometimes, or never greater than the value of y?

A. y is always greater.

Q. Why?

A. Because you need 5x for every y.

Q. When is the value of x greater than the value of y?

A. Never.

Q. When is the value of x equal to the value of y?

A. Never.

Q. When is the value of x less than the value of y?

A. Always.

6.2.4. Summary

Responses at the level of discovered content indicate that the pupil is not prone to believe that a particular equilibrial state must exist between two literal "numbers".

He appears either to treat an indeterminate equation as a statement including "boxes" or "pigeon holes" into which numerals can be "posted" to satisfy the relation, or as a co-varying system.

When he does assume an ordering in '5x = y' he does not use "re-arrangement" or "numerical operation" strategies to attain a new relative ordering. He accepts that the ordering must be true for all numerals within his range of experience.

Thus he holds in mind a system of numeral-pairs which co-vary together in the way specified by the relation.

6.3. The Level of Discovered Content - responses to the LNT

The distinguishung feature of a response to the LNT at this level is again that it does not exhibit false-ordering or false-content.

The pupil is prepared to impose a relation between letters, or a letter and a numeral (e.g. 'k = m', 't > 4', . . .), to answer questions about relative orderings.

An expression such as "When k = m" however, can have two distinct meanings. An understanding of this is crucial to an understanding also of why only some pupils who impose a relation between letters in the LNT do so in the PLT.

This distinction lies in the difference between the letter intended as a "classical (Diophantine) unknown" and as a "species".

The relation 'k = m' can be true for some (unspecified) member of a collection of numerals or be defined to be true for all numerals. In the first interpretation, 'k' and 'm' is each an 'hypothetical judgement' for a numeral which is identical to both 'k' and 'm'. This numeral, however, remains uniquely 'unknown' as in the context:

"Suppose 'k' is the height to the nearest metre of the Post Office Tower, and 'm' the height to the nearest metre of the glider, and suppose k = m".

Here 'k' and 'm' each has a specific (unknown) value selected from a range of potential values.

But in the context:

"Let 'k' be the distance travelled from the Post Office Tower of the aeroplane at any time 't', and 'm' the distance travelled from the Post Office Tower of the helicopter at any time 't', and suppose that for all 't', 'k = m'"; 'k = m' is true for all possible values of k (and these may be considered to be infinite in number).

The two contexts above are different because each is embedded in a different calculus. The first context does not involve the species concept; the second context involves the species concept in the form of "all t".

Some pupils who respond "algebraically" (i.e. impose the relations >, <, =) to the LNT transcend orderings in the PLT - others do not. Although it is not possible to identify which pupils are using the species from responses to the LNT alone, the PLT clearly "filters out" the non-species users.

The non-species user intends by an expression such as 'k = m' <u>an</u> unspecified numeral. Equally, the expression 'k > m' ("suppose 'k' is the height of the Post Office Tower and 'm' the height of Paddington Station") may also intend distinct unspecified numerals. This interpretation may be considered to be "caught" by the PLT, and the non-species user naturally gives 'a' and 'b' the unique content suggested by each line. The species user, however, intends <u>all</u> numerals by a letter. Consequently, the expression "when k > m" is, for him, a <u>definition</u> about <u>all</u> numerals which are possible determinations of 'k' and 'm'. The lines in the PLT do not "capture" this concept, and the pupil thus transcends orderings.

Pupils at the level of discovered content have, then, achieved the concept of the Diophantine unknown. But the concept of the species lies beyond this.

The following are illustrative examples of pupils using the Diophantine unknown in the LNT at this level.

Roger (5:2) 16yrs. 1mth. School B. 't + t, t + 4'.

Q. Which is larger, t + t or t + 4?

A. It depends on t.

Q. When is t + t larger?

A. When t is greater than 4.

Q. When is t + 4 larger?

A. When t is less than 4.

Q. When are they equal?

A. When t equals 4.

Mark (5:6) 16yrs. 2mths. School A. 'm + m, m + k'

Q. Which is larger, m + m or m + k?

A. You can't tell.

Q. Why?

A. Because you don't know m and k.

Q. When is m + m larger?

A. When m is larger than k.

Q. When is m + k larger?

A. When k is greater than m.

Q. When are they equal?

A. When m equals k.

In Subtask 3, pupils at this level often fail to give the correct response to one or each of the questions "When is a + b + 3 larger?" "When is a + c + 4 larger?"

The reason for this is not particularly clear. A common response to the former question is "When b is greater than c". This might suggest that the pupil thinks only of whole number determinations, that he is simply "careless", or that the concept of the letter as a classical unknown makes the response "when b is greater than c + 1" difficult for him. To give this response the pupil needs to reform 'a + c + 4' into 'a + (c+1) + 3'. He may be reluctant to do this because conjoining 1 to an unspecified numeral will give rise to a new, unique, numeral. On the other hand the act of reforming 'a + c + 4' might be difficult itself when 'c' is not interpreted as a letter which can "vary in itself" over all numerals. Whichever is the case, this phenomenon would seem to deserve closer investigation in future studies.

The following is an example response to LNT 3 from a pupil classified at the level of discovered content:

Colin (1:3) 12yrs. 2mths. School A. 'a + b + 3, a + c + 4'

Q. Which is larger, a + b + 3 or a + c + 4?

A. You don't know.

Q. Why?

A. Because it depends on b and c.

Q. When is a + b + 3 larger?

A. When b is bigger than c by more than 1.

Q. When is a + c + 4 larger?

A. When b is smaller than c.

Q. When are they equal?

A. When b and c are the same. . .No. . .If b is. . .
If you add one to b. . . If b is bigger by 1 than c.

6.3.1. Summary

Pupils at the "level of discovered content" exhibit a matching strategy "across" literal numbers to obtain a literal or numerical content for the letter. The letter is often interpreted however, as a classical (Diophantine) unknown.

6.4. The Level of Discovered Content - responses to the PLT

It has been reported above that many pupils who give "algebraic" responses to Subtasks 1 and 2 of the LNT do not transcend orderings in the PLT.

Some pupils, however, do so in one Subtask, although the response often lacks conviction. These pupils almost invariably * give "algebraic" or "borderline-algebraic" response to '$x + y = 10$' (see Table 28(b), Appendix III).

This suggests that at the present level statements such as "when $m > k$" in the LNT have more than one meaning (in the sense described in Section 6.3), and that a number of pupils are intuiting the species concept.

Those pupils who recognise that a letter in an

* Pupil (2:3) School B, gave a "multiple substitution" response.

'indeterminate' equation can legitimately be identified with each of a collection of numerals, and afford each identification an equal status as a replacement for the letter, must at some time transfer this understanding into new contexts.

The PLT suggests that each letter has a unique numerical identification. The concept of the letter as an identifier of more than one numeral clashes with this.

The author has shown elsewhere[(69)] that confusion in mathematics is the result of a clash of interpretative frameworks and arises when a person, having the capacity to entertain more than one meaning for a term is

(a) undecided as to which one to select,

and (b) cannot rationalize his interpretations as distinct aspects of the same situation.

In the present context, the distinct interpretations given to the letter in each context above (in ET1 and the LNT) give rise to the need to create a concept of the letter as a unique arithmetical entity with multiple determinations, i.e. a non-ordered numeral. This concept will bring with it the ability to transcend concrete

---

(69) Harper, Eon. (1978)

orderings.

At the intuitive stage, however, one might expect a pupil to exhibit confusion in the face of the PLT, as first one, and then the second, interpretation of the letter is appreciated.

The following are transcripts from two pupils (Pupils (2:3 and (1:1)) in School A which suggest that this stage has been reached.

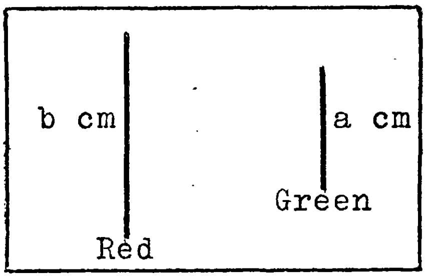
Example 1. Robert (2:3) 13yrs. 0mths. PLT. Subtask 3.

Q. Is the red line longer than the green line, the green line longer than the red line, are they equal in length or could any of these be possible?

A. The red line's longer than the green line (muttering under his breath, "as far as I can <u>see</u>"). . .er. . definitely not equal in length. The red line's longer.

Q. Why?

A. It's longer on paper, although you haven't given any definite measurements for the lines; you've just given a cm and b cm. You've put them in for a reason . . .that (pointing to the green line) being shorter than that (pointing to the red line) haven't you! What I am trying to get at is that the red line is longer than the green line on paper, although you haven't given any definite lengths for them. However. . .if you put 'b' equals 5 and 'a' equals 6, although it doesn't look right, the green line will be longer; but you haven't given any definite indication.



Q. When is the green line longer than the red line?

A. Never. . .unless, as I say, you did replace a and b with a number and a was larger than b.

Q. When is the red line longer than the green line?

A. It is at present and if b is bigger than a.

Q. When are the lines equal in length?

A. When a and b are equal.

Example 2. Timothy (1:1) 11yrs. mths. School A. PLT. Subtask 3.

Q. Is the red line longer than the green line, the green line longer than the red line, are they equal in length, or could any of these be possible?

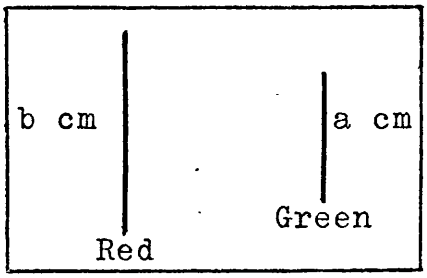
A. The red line's longer.

Q. Why?

A. Well it looks longer. . .but it could be shorter I suppose. . .I suppose it could be shorter.

Q. Why?

A. Well, you don't know what a and b are. So it could be anything really.



Q. When is the green line longer than the red line?

A. Well, . . .if you extend it it could be. Or if you had 'a' a bigger number (than b). If a was, say, 10 and b was 6 it would be longer. But you don't know, do you?

Q. When is the red line longer than the green line?

A. Well. . .well. . .it is now, or if b is a bigger number than a.

Q. When are the lines equal in length?

A. When a equals b.

Each pupil above gave an "algebraic" response to '$x + y = 10$', a "borderline-algebraic" response to '$2x + y = 9$' and '$5x = y$', and "algebraic" responses to Subtasks 1 and 2 of the LNT (see Table 28(b), Appendix III), supporting the suggestion made above that an acceptance that each letter can name a variety of numerals in the same context, and yet be used as a unique entity, is the important combination which gives rise to an ability to transcend geometrical orderings.

The "species" concept therefore appears to be generated by a synthesis of the concept of the letter as a Diophantine unknown and as an identifier of each member of a class of numerals. This inevitably means that although the letter retains the unique referential property assumed by the Diophantine unknown, it is also considered to refer simultaneously to a range of guaranteed identifications of equal 'status'.[70]

---

(70) This reconstruction agrees with Hadamard's (1959) major thesis that new mathematical discoveries are made by combining existing ideas, and Koestler's view that "scientific discovery . . .can be described. . .as the permanent fusion of matrices of thought, previously believed to be incompatible". (1969, p.94).

Piaget arrives at a similar conclusion about the nature of the natural number concept:

". . .the concept of a whole number", he writes, "is a product of logical operations, but it combines operations between them in an organized manner which is irreducible to logic alone. . .The concept of a whole number is therefore psychologically a synthesis of class and of the asymmetric relation, in other words a synthesis of logical operations,

cont. over

The present author knows of no explicit attempts in resource materials to encourage and stimulate the development of the species concept as a letter with a priori determinations[71]. If pupils come upon the concept then it may be through good fortune, having a creative mind, or through good blackboard teaching.

In the absence of explicit teaching, however, Polanyi's theory[72] of 'tacit' knowledge may offer an explanation of how a pupil, working alone, might develop the concept. Polanyi suggests that much of the scientist's success depends upon 'tacit knowledge' i.e. upon knowledge acquired through practice, which cannot be articulated explicitly.

Equation solving, although it may treat the letter at the outset as a classical "unknown" leads at the conclusion to an identification of the letter with a numeral. For example, in the question

"Solve 2x + 1 = 4",

'x' may be interpreted as an "unknown".

---

(70) cont. co-ordinated amongst themselves, however, in a new way, as a result of the elimination of distinctive qualities." (1972) pp.30-31.

(71) Sedivy, however, recognises the importance of this. "In traditional teaching", he writes, "secondary school pupils are confronted with propositional forms with given parameters i.e. with parameters

cont. over.

When the equation is solved, however, viz: '$x = \frac{3}{2}$', the letter is identified with the numeral '$\frac{3}{2}$'. That is, in this final statement 'x' is no longer an unknown but an alternative way of writing '$\frac{3}{2}$'.

Constant practice with equation solving will, in time, lead to numerous identifications of the letter with different numerals.

This tacit knowledge, acquired at the level of subconscious thought may later prove to be of major importance. A turning of attention towards the fact that the letter has, in the past, been identified with a multitude of numerals, synthesised with the concept of the Diophantine unknown provides the ingredients required to create the species concept.

It might be an error however, to underestimate the importance of the indeterminate equation as a catalyst to this development.

---

(71) cont. which have been chosen by authors of texts. In the modern teaching of mathematics the activity of students should be emphasized in all aspects of teaching. Parameters as tools of mathematical language should also be chosen by the pupils themselves". Sedivy, J. (1976)

(72) Polanyi, M. (1958) and (1966)

A need to satisfactorily solve the indeterminate equation gives rise to the need to develop a new concept of "number". Inevitably the letter introduced to provide a 'general' solution to the equation encompasses all possible truth values of one of the original letters (regarded as a Diophantine "unknown"), as in the case:

"x + y = 10,

so the solution is (t, 10 - t)".

Here however, the same principle as that described above is evident. The letter 't' is itself unique, yet simultaneously encompasses all possible identifications of the letter 'x' which gives rise to true arithmetical statements. The "indeterminate equation" and discussions about it's "general solution" may therefore provide the stimulus needed to help many pupils transcend the concept of the letter as a Diophantine unknown.

The transcripts above, and the fact that <u>all</u> pupils who transcended orderings consistently in the PLT gave algebraic responses to both the equation 'x + y = 10' and to Subtasks 1 and 2 of the LNT both lend support to the ideas outlined here and suggest that the switch in dominant theory witnessed in the history of mathematics after Vieta's time might be recapitulated in the classroom.

6.4.1. Summary

The responses to the PLT of a number of pupils who work consistently at the level of discovered content in the LNT and ET indicates that these pupils are beginning to develop the species concept as a means of organising perception. This may indicate that the "Vietan Revolution" is recapitulated in the classroom.

6.5. The Level of Discovered Content - responses to the ZT

The "level of discovered content" is defined to be the level at which responses indicate the pupil

(a) transcends a usage of the letter as an object with a unique content in a Subtask of the ET,

(b) avoids false-ordering and false-content responses in a Subtask of the LNT;

but does not (a) utilise the species consistently to organise perception in the PLT nor (d) utilises the letter as a preferred alternative to numerals for "givens" in the ZT.

The definitive form of response of pupils working at this level in the ZT is a "Rhetorical" or "Diophantine" response type. These responses again suggest that the pupil can deal with a system of numerals, comparing and contrasting sums, differences, and relationships between these successfully to arrive at a general result.

The following are examples of each. It will be seen that these match very closely the type of algebraic activity common to mathematics prior to Vieta's time.

6.5.1. Diophantine Solutions

Example 1. Neil (6:8) 16yrs. 5mths. School B.

"for example

x + y = 10

x - y = 2

Using simultaneous equations

2y = 8

y = 4

and when you substitute, y = 4, x = 6.

Although I have chosen nice easy numbers using this method you can find any numbers given there (sic) sum and there (sic) difference".

Example 2. Philip (5:7) 15yrs. 8mths. School A.

"Let the sum be x and the biggest number be 10

The other number is therefore x - 10

But we are given the difference. And the difference is

10- (x - 10) = 20 - x.

Therefore you can find x and from this you can find the other number.
(Say the difference was 6. 6 = 20 -x, so x is 14. So the other number is 4)".

This was the only solution of this type - in which just one letter was used - given in the study. The majority of pupils used simultaneous equations. The solution above, however, is particularly reminiscent of Diophantus' solution - see Chapter 3, Section 3.6.

6.5.2. Rhetorical Solutions

These solutions are usually given by younger pupils who might not yet have been introduced formally to algebra as a means of dealing with "general" problems.

Example 1: Jane (2:1) L2yrs. 8mths. School B.

"You divide the sum by 2 then divide the difference by 2. Then to get the first number add the sum divided by 2 to the difference divided by 2. To get the second number take the difference divided by 2 away from the sum divided by 2.

e.g. sum = 8 difference = 2

$\frac{8}{2} = 4$ $\frac{2}{2} = 1$

1st number = 4 + 1 = 5

2nd number = 4 - 1 = 3"

Example 2: Philip (2:2) 13yrs. 1mth. School B.

"To find the 2 numbers subtract the difference from the sum and divide the answer by 2.

e.g. Sum 29 difference 3 $29 - 3 = 26$

$26 \div 2 = 13$

One number is 13, the other is 13 + the difference - 16."

In the "Diophantine" solution the pupil uses each letter as a classical unknown as in the LNT. The level of discovered content is therefore the level at which the pupil has, in the least, developed a concept of the letter as an entity with a <u>potential</u> determination. Thus in situations where the context demands it the pupil is ready to identify the letter with a variety of numerals.

6.5.3. <u>Summary</u>

The response type to the ZT at the level of discovered content does not utilise the species as a means of aiding a general argument. However, the pupil exhibits a cognitive capacity to deal with systems of numeral pairs as in the algebraic response to the ET.

Words, such as "sum" and "difference" may be illustrated with a single numeral, but the pupil shows that when he 'solves' the problem he is then allowing the numerals associated with these words to "vary". What he does not do is introduce a formal means of expressing "variation in itself" using the species concept, or "given", of Vieta.

It would appear, however, that the capacity to understand such a usage may be present. This is supported by the fact that some pupils giving this type of response also begin to question the "reality" of the situation in the PLT.

6.6. The Level of Discovered Content - Summary

Pupils at this level treat the letter as a Diophantine unknown and allow the letter, in appropriate contexts, a variety of identifications of equal status.

Thus the pupil demonstrates:

(a) an ability to hold in mind a co-varying system of numeral pairs;

(b) an ability to transcend suggested orderings in a 'pure' algebraic setting;

(c) an awareness that the term "length" may not refer to a measured outcome (or scalar representation of a measured outcome).

Pupils at Level III, that of the species, demonstrate each ability (a) and (b) and often that they have also developed an algebraic conception of "length".

6.6.1. <u>Archetype</u>

The following transcript is presented as an illustration of pupil activity expected at the present level. This may form a basis for future investigations into the nature of algebraic thought, and is reproduced here for that reason.

<u>The Level of discovered content</u>

Barry (3:2) 13yrs. 10mths. School A.

<u>PLT Subtask 2</u> (See diagram over)

Q. Is the red line longer than the green line, the green line longer than the red line, are they equal in length, or could any of these be possible.

A. They're equal in length if you go by the letters, but they don't look it.

Q. When is the green line longer than the red line?

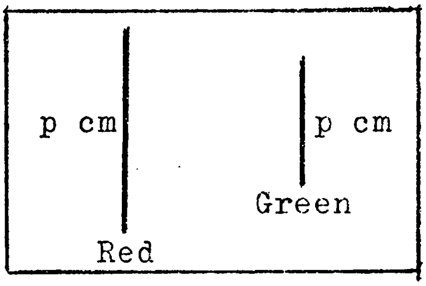
A. Well, . . .If you brought the green line up closer it would be.

Q. When is the red line longer than the green line?

A. As they are now. It's longer now. . .It looks longer now.

Q. When are the lines equal in length?

A. You'd have to bring the green line up till it was near the red line. . .or you could push the red line back.



PLT Subtask 2.

PLT Subtask 3

Q. Is the red line longer than the green line, the green line longer than the red line, are they equal in length, or could any of these be possible?

A. Any could be possible.

Q. Why?

A. Because you don't know what a and b are.

Q. When is the green line longer than the red line?

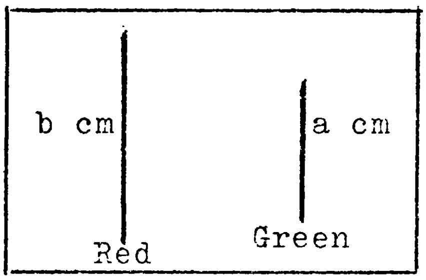
A. Well,. . .well, if a was a bigger number it would be. If a was a bigger number than b. You would say it was longer then even though it didn't look it.

Q. When is the red line longer than the green line?

A. Well, . .well, it is now. Or if you said b was a bigger number than a.

Q. When are the lines equal in length?

A. When a and b are equal.



PLT Subtask 3.

<u>ET</u> 'x + y = 10'

Q. If this is true, is the value of x always, sometimes, or never greater than the value of y?

A. Sometimes.

Q. Why?

A. It depends upon x and y.

Q. When is the value of x greater than the value of y?

A. When, . . .when x is bigger than 5.

Q. When is the value of x equal to the value of y?

A. When they're both 5.

Q. When is the value of x less than the value of y?

A. When x is less than 5.

<u>ET</u> '2x + y = 9'

Q. If this is true, is the value of x always, sometimes, or never greater than the value of y?

A. Sometimes.

Q. Why?

A. Well, it depends on x and y.

Q. When is the value of x greater than the value of y?

A. Erm. . .they're equal if it's 3. . .if it's bigger than 3 it is.

Q. When is the value of x equal to the value of y?

A. When x is 3 and y is 3.

Q. When is the value of x less than the value of y?

A. When x is less than 3.

<u>ET</u> '5x = y'

Q. If this is true, is the value of x always, sometimes, or never greater than the value of y?

A. Never.

Q. Why?

A. Well, because you've got 5x equals y. So x must be smaller. See, if that (x) was 2, y is 10. So x is smaller.

Q. When is the value of x greater than the value of y?

A. Greater? (Yes). Oh. .erm. .It <u>can't</u> be. It's never greater. It's always got to be smaller.

Q. When is the value of x equal to the value of y?

A. Never. .I don't think. .unless it's nought. If it was nought they'd be equal.

Q. When is the value of x less than the value of y?

A. Always. It's always smaller. It's <u>got</u> to be.

<u>LN</u> 't + t, t + 4'

Q. Which is larger, t + t or t + 4?

A. It depends. It depends what t is.

Q. When is t + t larger?

A. If t is bigger than 4 it is.

Q. When is t + 4 larger?

A. When t is under 4.

Q. When are they equal?

A. When t is 4.

<u>LN</u> 'm + m, m + k'

Q. Which is larger, m + m or m + k?

A. It depends.

Q. Why?

A. Well, you don't know what m and k are.

Q. When is m + m larger?

A. When m's a bigger number than k.

Q. When is m + k larger?

A. When m's smaller than k.

Q. When are they equal?

A. If m and k are the same.

LN 'a + b + 3, a + c + 4'

Q. Which is larger, a + b + 3 or a + c + 4?

A. Either could be.

Q. Why?

A. Because you haven't been told what b and c are.

Q. When is a + b + 3 larger than a + c + 4?

A. Erm. .if b is bigger than c. .if b is bigger than c by more than 1. . .I think.

Q. When is a + c + 4 larger?

A. If b is less than c.

Q. When are they equal?

A. b would have to be one bigger than c.

Zetetic Task

"x - y = 2. . .(1)

x + y = 8. . .(2)

(1) + (2) ∴ 2x = 10

x = 5

Substitute with (2)

5 + y = 8

y = 8 - 5

y = 3

You can do this for any numbers.

CHAPTER 7 : THE LEVEL OF THE SPECIES

7.1. Introduction

This chapter is concerned in particular with responses to the ZT and PLT which indicate that the concept of a species is used either to aid a general discussion, or to organise perception.

Unlike the pupil responding at the former level, whose concern is with those numerical denotation(s) of a letter which might satisfy a particular relation, the pupil responding at the present level uses the letter as a preferred alternative to conventional numerals i.e. as a new type of numeral.

7.2. The Level of the Species - responses to the PLT

Chapter 6, Section 6.4. presented transcripts from two pupils in School A (pupils (1:1) and (2:3) which indicated that these pupils had begun to question the "reality" of geometrical orderings in the PLT.

Each pupil explained his concern by suggesting numerical replacements for each letter which would contradict what was immediately available to perception.

After Vieta, geometry was dethroned from it's position of supremacy, and relegated to a servile position as a means of illustrating algebraic relationships.

Chapter 2 suggested that this change in dominant theory might be associated with a possible new means of interpreting geometrical data - i.e. a conception of a figure as a 'dynamic system' in which the concept of 'instantaneous relative displacement' replaced that of 'static measured outcome'.

Both the responses reported to the PLT in Chapter 6, Section 6.4. and responses of pupils reported below suggest that this conceptual change might be a reality experienced by pupils in the classroom when the species concept is available.

The first transcript below is from a sixth-form pupil in School A, the second from a second-year pupil in School A. The sixth-form pupil responded "algebraically" to each task in the LNT and ET, and gave a "Vietan" solution to the ZT. He was considered to be the most able mathematician in the school.

The second-year pupil also gave "algebraic" or "borderline-algebraic" responses to each subtask of the and ET and a Vietan solution to the ZT. This pupil was a particularly able mathematician for his age.

Each transcript below demonstrates that geometrical orderings are subordinated to algebraic relationships i.e. the species concept is used consistently to organise perception: (see Appendix II, Section II.2.5. and II.2.7. for further examples).

Gregory (6:1) 17yrs. 5 mths. School A. PLT Subtask 2

Q. Is the red line longer than the green line, the green line longer than the red line, are they equal in length, or could any of these be possible?

A. They're equal.

Q. Why?

A. Because they're both p cm.

Q. When is the green line longer than the red line?

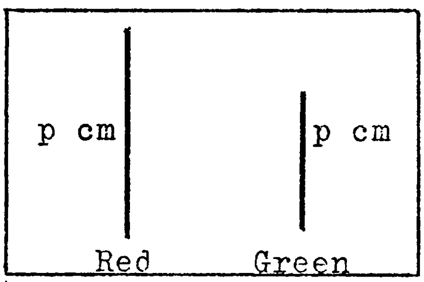
A. Never.

Q. When is the red line longer than the green line?

A. Never.

Q. When are the lines equal in length?

A. Always.

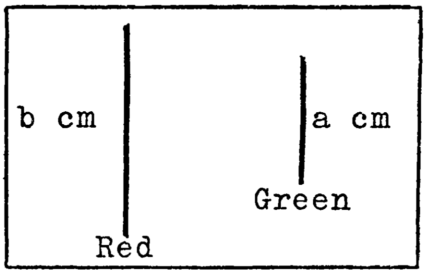


Subtask 2

Subtask 3

Q. Is the red line longer than the green line, the green line longer than the red line, are they equal in length, or could any of these be possible?

A. Is this the only information we're given? - (yes). Then I'd assume a and b are meant to be general numbers. So any could be possible depending upon whether a is greater than b, less than b, or equal to b.



Subtask 3

Robert (2:1) 13yrs. 0mths. School A. Subtask 2.

Q. Is the red line longer than the green line, the green line longer than the red line, are they equal in length or could any of these be possible?

A. The red looks longer, but the figures say they're the same.

Q. When is the green line longer than the red line?

A. Never - I'm going by the figures.

Q. When is the red line longer than the green line?

A. Never - if I'm going by the figures again.

Q. When are the lines equal in length?

A. Always.

Subtask 3

Q. Is the red line longer than the green line, the green line longer than the red line, and are they equal in length, or could any of these be possible?

A. Well, again the red line looks longer, like the last one (Subtask2). But it doesn't have to be. It depends on 'a' and 'b'.

Q. When is the green line longer than the red line?

A. When a is greater than b.

Q. When is the red line longer than the green line?

A. When b is greater than a.

Q. When are the lines equal in length?

A. When a and b are equal.

Each pupil above exhibits a total commitment to an algebraic relation as a determination of "length". Each line is thus clearly a 'sketch' for a letter which cannot capture the totality of numerical entities "internal" to that letter.

Each pupil has transcended any ordering suggestions in the figure and, unlike pupils who work at "non-species" levels, any suggestion that the content for the letter is given by the outcome of applying a measuring process to the line.

This new conception of "reality" was, in the present study, found in a minority of pupils (see Figure 15, Chapter 9, Section 9.2.1.). These pupils perform at a relatively high level across the remaining Subtasks, and are generally considered by their teachers to be comparatively able mathematicians.

As suggested by Stage 1 of the study (Chapter 4), the PLT appears to separate the potential (or actual) mathematician from the non-mathematician when it is used "unseen". (i.e. it may be that once a pupil has been given the "answer" to the task he will respond "algebraically" to it. The extent to which this implies that the pupil has made the necessary "flip" in cognition suggested here cannot be known except by future investigation. After some of the interviews the author returned to the PLT and asked pupils how they would react if told that e.g. "the green line (in Subtask 3) is longer than the red line when 'a' is greater than 'b'". Pupils at the level of fictitious measures generally responded "I wouldn't believe you"; those at the level of discovered content: "I suppose so, yes, if you put a bigger number for 'b' than for 'a'".

What distinguishes the present pupils from those at the second level, is that each appears consistently to question direct geometrical orderings unless sufficient information is given to establish the truth of that ordering - as would be the case for the hypotenuse of an Euclidean triangle and one of its' sides. "Length" for these pupils thus has an algebraic connotation and is, potentially, or actually, correlated with non-ordered, relative, displacements .

7.3. The Level of the Species - responses to the ZT

In the ZT, the 'species' or 'given' of Vieta enters in the form of a letter used for the 'sum' and the 'difference'.

Chapter 6 gave an example of a "Diophantine" solution to the ZT in which the pupil added a note to the effect that although he had chosen particular numerals for the sum and the difference, his method would 'work' for any chosen pair. A number of pupils giving "Diophantine" solutions added similar notes (see examples of pupil responses Chapter6, Section 6.5.). Pupils giving "Rhetorical" solutions also often felt the need to illustrate their argument with a numerical example (see Chapter 6 Section 6.5. for an example).

The advantage the species concept brings is that of an <u>immediate</u> generality. The letter, introduced as a "general" numeral, transcends any need to make reference to particular numerals. This is, in fact, precisely the justification for using it - to avoid mentioning conventional numerals which are guaranteed determinations of it[73].

The following are two illustrations of "Vietan" solutions given to the ZT.

The former is that of pupil (6:1) in School A (see also Section 7.2.) and the latter is that of a sixth-form pupil in School B.

Example 1. Gregory (6:1) 17yrs. 5mths. School A.

"$x + y = a$

$x - y = b$

$2x = a + b$, $x = \frac{a + b}{2}$; $y = \frac{a - b}{2}$".

---

(73) 'In learning geometry' writes Bertrand Russell, 'one acquires the habit of avoiding mentioning special interpretations of such a word as "triangle". . . This is essentially the process of learning to associate with the word what is associated with <u>all</u> triangles; when we have learnt this, we understand the word "triangle". Consequently there is no need to suppose that we ever apprehend universals, although we use general terms correctly'. (1927, p.57)

The suggestion here is that we apply this same principle to "number". The "general number" or "species" is essentially a word used to avoid special interpretations of the word "numeral".

Example 2. Ann (6:2) 16yrs. 2mths. School B.

"If $x + y = a$

$x - y = b$

Solving simultaneously

$$2y = a - b \qquad x = a - \frac{(a - b)}{2}$$

$$y = \frac{a - b}{2} \qquad x = \frac{2a - a + b}{2}$$

$$x = \frac{a - b}{2}$$ "

The fact that a pupil will use the letter to organise perception in the PLT does not necessarily mean that he will use it as a preferred alternative to conventional numerals in the ZT and vice-versa. (The relationship between the two tasks is discussed fully in Chapter 9, Section 9.2.3.).

However, pupils who did utilise the species in this context were more likely to transcend suggested orderings in the PLT than their peers and again were comparatively able mathematicians, both from their own teacher's point of view and as judged by their relative performance across the remaining tasks.

Whether the 'species' will be used in the present task or not may depend to a large extent upon classroom experience. Thus the younger pupils who used the letter to organise perception in the PLT tended to give either Diophantine or Rhetorical solutions to the ZT.

7.4. The Level of the Species - responses to the ET and LNT

Definitive responses to the ET and LNT at this level are confined to Subtask 3 of each task.

The placing of the 'D'-type response to '5x = y' (the 'D'-type response is one in which the pupil introduces negative numbers) at this level may be questionable.

Pupils who use letters to organise perception in one Subtask only of the PLT are almost as likely to give the negative number response as those who consistently utilise the species in the PLT. (Chapter 9, Section 9.2.3.).

Whether negative numbers are mentioned or not may again depend to a great extent upon classroom experience.

Since those pupils who consistently utilise the 'species' in the PLT however, are more likely to mention negative numbers, and certainly, when they do, deal with them more confidently than do the former group, the response-type was placed at the present level.

In Subtask 3 of the LNT (a + b+ 3, a + c + 4) the pupil responding at the present level answers each question correctly with statements of the form 'When b is less than c + 1".

As mentioned earlier (Chapter 6, Section 6.3.) it is difficult to decide precisely why the "species" user has more success with this item than the "classical unknown" responder - a phenomenon which deserves further investigation. The answer to it may shed some light upon the problems pupils face when "factorizing" algebraic expressions.

The following transcripts illustrate response-types classified at this level to the ET3 and LNT 3.

Gregory (6:1) 17yrs. 5mths. School A. '5x = y'

Q. If this is true, is the value of x always, sometimes or never greater than the value of y?

A. Sometimes.

Q. Why?

A. It depends upon x. If x is negative, x is larger, If x is positive, y is larger. If x is zero, they're equal.

'a + b + 3, a + c + 4' (same pupil).

Q. Which is larger, a + b + 3 or a + c + 4?

A. Neither.

Q. Why?

A. Again it depends upon b and c. When b is greater than c + 1, a + b + 3 is larger; when it's less, a + c + 4 is larger.

Q. And they're equal when. . .

A. b equals c + 1.

7.4.1. Summary

Pupils who respond at the Level of the Species in the ET and LNT indicate

(a) a capacity to deal with a system of co-varying numeral pairs;

(b) a natural usage of negative numbers;

(c) an ability to transform and re-model literal expressions.

7.5. The Level of the Species - Summary

Pupils exhibiting a usage of the species in the PLT and/or ZT can be expected (almost constantly) to avoid fictitious measure responses in both the ET and the LNT. Their responses to these tasks are generally "algebraic".

The most noticeable feature which distinguishes the pupil at the present level from the majority of pupils at the level of discovered content in his approach to the ET and LNT, is his almost dismissive attitude to the questions. Often the follow-up questions are not necessary (see Gregory, Section 7.4.).

On being asked for example, "which is larger. . ." in the LNT, the pupil will declare that the relationship depends upon the value of the letter, and then might proceed immediately to give conditions for each relative ordering. The same is often true in the ET.

By way of contrast many pupils at the second level tend to restrict themselves to the statement that any ordering is possible and then await the remaining questions, answering each one in turn as it arises.

It is difficult not to conclude that pupils at the present level consistently "see" algebraic relations of the form '$x + y = 10$', '$2x + y = 9$', and '$5x = y$' as dynamic systems of co-variation. Although this is true also of a number of pupils at the previous level, the present pupils have taken an additional step. They have transcended the world of arithmetic by accepting the letter as a preferred alternative to conventional numerals i.e. they have developed the symbolic number concept.

7.5.1. Archetype

As for the previous two levels, the following transcript is included to aid any future investigations. The pupil is working consistently at the level of the species, and may be considered to be representative of all pupils working at that level:

Mark (6:7) 16yrs. 9mths. School B.

<u>PLT Subtask 2</u>

Q. Is the red line longer than the green line, the green line longer than the red line, are they equal in length, or could any of these be possible?

A. They're equal.

Q. Why?

A. Because they're both p cm.

Q. When is the green line longer than the red line?

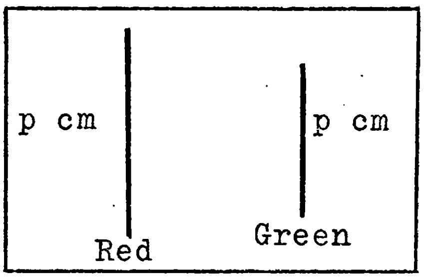
A. Never.

Q. When is the red line longer than the green line?

A. Never.

Q. When are they equal in length?

A. Always.



PLT Subtask 2

<u>PLT Subtask 3</u>

Q. Is the red line longer than the green line, the green line longer than the red line, are they equal in length, or could any of these be possible?

A. Either could be possible.

Q. Why?

A. Because a and b are undefined.

Q. When is the green line longer than the red line?

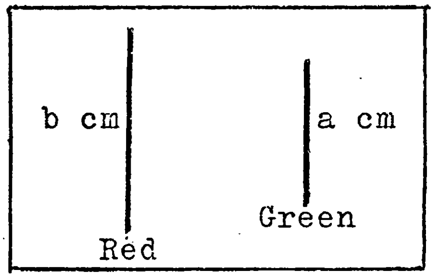
A. When a is greater than b.

Q. When is the red line longer than the green line?

A. When b is greater than a.

Q. When are the lines equal in length?

A. When a equals b.



PLT Subtask 3

<u>Equations Task</u> : 'x + y = 10'

Q. If this is true, is the value of x always, sometimes or never greater than the value of y?

A. It depends upon x and y. Sometimes.

Q. When is the value of x greater than the value of y?

A. When x is greater than 5.

Q. When is the value of x equal to the value of y?

A. When x equals 5.

Q. When is the value of x less than the value of y?

A. When x is less than 5.

'5x = y'

Q. If this is true, is the value of x always, sometimes, or never greater than the value of y?

A. It depends upon x and y. If x is negative, x is greater than y; if x is zero, x equals y and if x is positive, x is less than y.

'2x + y = 9'

Q. If this is true, is the value of x always, sometimes or never greater than the value of y?

A. It depends again. If x is less than 3, y is greater. If x is three, they're equal. And if x is greater than 3, x is bigger.

Literal Number Task : 't + t, t + 4'

Q. Which is larger, t + t or t + 4?

A. If t is larger than 4, t + t is bigger, but if it's less than 4, t + 4 is larger.

Q. And when are they equal?

A. When t equals 4.

'm + m, m + k'

Q. Which is larger, m + m or m + k?

A. Either. When m is greater than k, m + m is greater, when m is less than k, m + k is greater, and they're equal when m equals k.

'a + b + 3, a + c + 4'

Q. Which is larger, a + b + 3 or a + c + 4?

A. It depends, let's see. . .if b is larger than c + 1, that's larger (a + b + 3). If b is less than c plus 1 that's larger (a + c + 4) and if b is c plus 1 they're equal.

Zetetic Task

"Let numbers be x and y

Given values $x + y = m$

$x - y = n$

$m + n = (x + y) + (x - y)$

$= 2x$

hence $x = \frac{m + n}{2}$

$m - n = (x + y) - (x - y)$

$= 2y$

hence $y = \frac{m - n}{2}$".

CHAPTER 8 : THREE ASPECTS OF VARIATION

8.1. Introduction

The present chapter discusses how the concept of "variation" changes with the three interpretations of the letter described in Chapters 5, 6 and 7 i.e. with the letter as an object with a fictitious content, as a Diophantine unknown, and as a species (a non-ordered numeral).

If changes in the meanings given to letters in mathematics occur, then these changes must necessarily influence the concept of variation. That is, one should expect the concept of variation in the post-Vietan world to be different to that in the pre-Vietan world.

The possibility that distinct meanings for algebraic terms exist in the mathematician's make-up was indicated by Hilbert during the early decades of the present century.

The statement

'a + 1 = 1 + a'

he suggests, can have two different interpretations:

(a) it is an 'hypothetical judgement' which 'comes to assert something when a numeral is given';

and (b) it is a "general" statement to the effect that all counting numbers commute with 1.(74)

Steiner, although he believes Hilbert's insight might be of the utmost importance for mathematics, refers to the 'hypothetical judgement' as a "queer fish"(75). The present author has shown elsewhere that that bifurcation has, in fact, important important implications for mathematical education and in particular for the teaching of 'proof'(76).

Equally, Russell has suggested a distinction between the "propositional function" and the "proposition".(77)

The statement

$$'(x + y)^2 = x^2 + 2xy + y^2'$$

he says, is a "propositional function", since it asserts nothing definite unless we are told, or are led to believe, that x and y are to have all possible values or such and such a value. The "propositional function" contains one or more undetermined constituents. Only when values are assigned to those constituents does the expression become a "proposition".

---

(74) See Steiner, M. (1975)

(75) Steiner, M. (1975) op. cit. p.144.

(76) Harper, Eon, (1976)

(77) Russell, B. (1910) p.155

Each of Hilbert's and Russell's discussions take place at the philosophical level. Each however, is concerned with the meanings that can be given to algebraic statements by virtue of different conceptions of the letter which seem to be available, and each draws attention to the fact that an algebraic statement might be intended either as an "hypothetical judgement" or as a universal truth. Intending a statement in the former sense implies that we must then make replacements for letters to arrive at true or false propositions. When the statement is intended in the latter sense, however, the statement is immediately significant. This latter usage, as Russell observes, is the one most difficult to achieve:

> "To 'understand' even the simplest formula in algebra, say $(x + y)^2 = x^2 + 2xy + y^2$, is to be able to react to two sets of symbols in virtue of the form which they express, and to perceive that the form is the same in both cases. This is a very elaborate business, and it is no wonder boys and girls find algebra a bugbear".(78)

The present author suggests that the bifurcation sought by both Hilbert and Russell is that introduced into mathematics by the change in meaning given to

---

(78) Russell, B. (1929) p.89.

letters by Vieta in his "Analytical Art" i.e. the introduction of the letter as a "given". This innovation gave rise to two distinct language systems - the first in which the "givens" are conventional numerals, and where letters are reserved for use as "classical unknowns", and the second in which the "givens" are non-ordered numerals i.e. species. Thus to intend an algebraic proposition is to use the letter not as a classical unknown but as a 'species'.(79)

This usage, as shown in Chapters 5 to 7, coincides with a perception of a letter as a non-ordered, significant, numerical entity, and with a different interpretation of geometrical material. When a mathematician is totally committed to this language, wherein each letter used intends a non-ordered numeral with "possible" (in contrast to "potential") determinations,

---

(79) This bifurcation may be relevant to contemporary discussions concerning the nature of 'understanding' in mathematics. Skemp (1976) proposes two forms: "relational" and "instrumental", whilst Byers and Herscovitz (1978) propose four: "relational", "instrumental", "intuitive" and "formal". 'Understanding' in algebra necessarily deals with the meaning given to terms. "Instrumental" understanding here would be closely linked to the letter used as a classical unknown, "relational" understanding to that when the letter is a species.

each statement he makes necessarily intends a proposition. As such, his activity is not one directed towards achieving generalisations from multiple exemplars, but one of making a direct conjecture, perhaps from a singular instance, and then seeking to refute the conjecture. That is, his activity is analytic.(80)

The bifurcation in the meaning given to the letter, however, as an entity with "potential", and with "possible" (or "guaranteed") determinations, implies also that "variation" takes more than one form. Sections 8.2, 8.3 & 8.4. below use the evidence gathered from pupil responses to explain how the conception of "variation" changes with each usage made of the letter. The first two sections may be considered to deal with two different aspects of variation possible in the pre-Vietan world, and the third section with the concept of variation after Vieta.

8.2. The letter as an object - "concrete" (or "null") variation

Russell's complaint (p.1.) to the effect that he always thought the teacher knew what x and y were but would not tell him, is both a personal confession and, perhaps, a criticism of teaching during his own childhood.

---

(80) See also Nunn, T. P. (1919) p.4.

The statement reflects precisely the psychological state of many pupils in the present study, and at the same time suggests that even the most scintillating mind can be misled by the language of algebra.

Russell eventually transcended this interpretation. The evidence here however, is that many pupils do not do so consistently prior to completing their mathematical studies.

Pupils who use the letter as an object with a fictitious measure (Chapter 5) appear to associate the letter with the world of "static", "concrete" entities, such as apples and stones, each of which has a <u>predetermined</u>, unique numerical "content". (It's mass, it's volume, etc.).

In the physical world of concrete objects, the 'numerical content' of an object can be found only by counting or measuring procedures. Pupils responding at the level of Fictitious Measures appear to believe, temporarily or permanently, that a letter is a "man-made addition" to this world. Letters, however, cannot be "measured". They can only be counted.

Chapter 1, p.31 presented an example of an "incongruous" usage of the word "lost" by a young child.

The meaning the child had developed was, contextually, that associated with "misplacing" objects such as toys. He had not learned to associate the term with "destruction of property". The result is humorous.

In mathematics precisely the same 'erroneous' usage of words can occur when inappropriate associations are made. This may give rise to a belief that the person using terms wrongly is speaking "illogically". Nothing, however, may be further from the truth.

When responses (in particular to the ET Subtask '5x = y') are analysed, it is clear that pupils at the Level of Fictitious Measures are, given their own premises, acting logically.

The fact that pupils working at this level often have difficulty separating the object (e.g. 't') from the numerical content of the object (as, for example, when combinational errors such as 't + 2 = 2t' are made) indicates that the child's premise is that letters behave in much the same way as do objects in the physical world. Thus the response above suggests either a collecting together of the objects 't' and '2', or an interpretation of 't + 2' as "t added twice" or "t, add two of them". Consequently, when "variation" has to be considered, which is the central demand of each task, one should expect the pupil to devise his own strategies appropriate to his own interpretations.

In the classroom pupils are often encouraged to solve problems in algebra from such premises as "let m be the mass of the apple", "let p be the length of side of the square". Such problems, used in excess, have an important pedagogical weakness.

Each apple has just one mass, and each side of a square a specific length. Consequently, many pupils may turn attention to the world of concrete, static objects, to support their thinking. In this world, however, each object has a pre-determined numerical content. Each object is "impregnated with a number" from the outset.

Pupils at the level of fictitious measures appear both to image concrete objects, and to translate sentences like those above literally; e.g. "m already contains the mass of the apple"; "p already contains the length of the side", when dealing with algebraic expressions. As a result, the content of the letter cannot vary in "real" terms. "The value of m" is literally the "mass of the apple". That this is the case comes out most clearly in responses to the ET Subtask '5x = y'.

The reader is asked to image a balance with five apples (of equal mass) in one pan balancing a melon in the second pan, and to consider the question:

"When is the mass of one of the apples greater than the mass of the melon?"

with this image in mind.

The reader is now in precisely the same quandary as the pupil interpreting the letter as an object with a fictitious measure. In this context the question is incongruous. Each apple, and the melon, has a unique mass which is unknown.

The "commonsense" response to the question above with this image in mind is probably, "It can't be" (Type B response - Appendix II, Section II.3.2.). There are, however, alternatives.

Although it is not possible for the mass of one of the apples ('the value of an 'x'') to be greater than the mass of the melon ('the value of y'), were the situation different (e.g. should 5 melons balance an apple), then the mass of the apple would be the greater than the mass of a melon. ("If it was x = 5y it would be" - Type A response, Appendix II, Section II.3.1.)

The third alternative is more sophisticated. Here we might recognise that five times the mass of an apple equals the mass of the melon. Thus six times the mass of an apple will be greater than the mass of the melon, and four times the mass of an apple will be less than the mass of the melon. (The value of 'x' equals the value of 'y', "When it's 5x"; is greater than the value of 'y', "When it's 6x"; and is less than the value of 'y', "When it's 4x" - Type A response: Appendix II, Section II.3.1.).

Pupil responses at this level therefore suggest that the meaning given to a letter is associated with, or derived from, dealings with the world of concrete objects with invariant measures. Each letter is considered to be a unique object with a unique content. As a result "variation" cannot take place, and alternative means of increasing "values" have to be found.

This interpretation of the letter therefore implies "null" or "concrete" variation (variation by increasing/decreasing the number of objects available, or the amount of material available - such as cutting down the length of a line in the PLT).

Pupil responses to the LNT suggest the same association. Here pupils often assume an ordering of the literal entities (e.g. "m + m is larger than k + m") and refuse to use the letter to specify relations (i.e. do not use the word-series "when k is greater than m" etc.).

Questions such as "when is m+k larger than m + m?" for these pupils are analogous to questions such as "when is the cricket ball heavier than the stone?" (where "k" is a cricket ball and "m" is a stone). Since neither a cricket ball nor a stone can vary in mass responses such as "when k > m" are unlikely.

Again, therefore, the pupil is forced back upon alternatives. It is interesting to note that multiplication strategies are not used here - the reason, should one assume the pupil to be responding consistently, is clear. The LNT does not indicate a relationship between letters - which is the opposite case in the ET (i.e. 'y' is '5 times 'x"). Thus, since the relationship between 'k' and 'm' is not given, so that the pupil is not informed whether 'k' is twice 'm' or three times 'm' etc., then the multiplication strategy cannot lead to a conclusive answer. Multiplying the content of 'k' by, say, ten, might <u>not</u> produce a value greater than that for 'm'.

The pupil is therefore reduced to an assumption that one object contains a greater measure than the second at the outset (false-ordering responses), or to proposing tentative numerical possibilities ("Can I put numbers in for them? If I can it's if k is 4 and m is 2").

(The fact that the pupil sometimes proposes different numerical values for the same letter is consistent with the view that letters are merely objects with fictitious contents. There are many objects called apples - but each apple does not necessarily have the same mass).

At each and every turn it is thus possible to commend the pupil for his "commonsense" approach. He intentionally resists using the multiplication strategy in the LNT because he knows this will not lead him to a 'general', conclusive, answer. In the ET however, he may tenaciously cling to the multiplication strategy because he knows the strategy works (zero and negative quantities do not exist in the world of objects). His answers are totally consistent with all the facts and lead, for him, to irrefutable conclusions. There is, therefore, little wonder that algebra is difficult to teach and to understand. Whilst the teacher is convinced that his own thinking is sensible the pupil is thinking otherwise.

The sole reason for condemning such a usage of letters - "object algebra" - must lie with the constraints and restrictions it imposes upon mathematical activity. The pupil at the level of fictitious measures is apparently unable to deal successfully with general arguments because for him everything is specific. The letter cannot be used as an "unknown" in the classical sense because it is an object "impregnated with a number". Thus the pupil can never be sure, should he introduce the letter, that he has chosen the correct one. To write 'x + y = 8' to represent the sum of two numbers in the ZT may turn out to be a disastrous error. x and y might each have the content '5'.

For pupils responding at the level of fictitious measures, therefore, "variation" cannot take place. They achieve differences in the 'content' of letters by concrete transformations, modelling the algebraic language in the reality of a world populated with objects. Their algebraic "reality" is a reality of unique objects with unique contents. If, therefore, there *is* any association between ontogenetic development and the phylogenetic development of mathematical ideas, the discussion above suggests that mathematicians prior to Vieta turned attention almost exclusively to the world of concrete objects with immutable measures and counts. That is, as Kline observes, the objects of mathematics are direct idealizations of "external" objects in the concrete world.(81)

---

(81) See Chapter 2, p.43

8.3. The letter as a classical unknown - potential variation

The nature of variation at the second level of activity is best introduced by an illustrative example of a change in response-type to a Subtask of the ET.

The following transcript, although lengthy, is worth reproducing in full since the pupil explains his thinking processes at the former level when the letter is an object with a unique content.

Example 1 : Thomas (3:10) 14yrs. 0mths. School A.

'x + y = 10'

Q. If this is true, is the value of x always, sometimes or never greater than the value of y?

(2)A. Well, it's impossible to tell from the equation. You don't know exactly what x and y are. It would be greater if there were 2x or 3x probably. Or at any rate more than 10x it would be greater.

Q. When is the value of x greater than the value of y?

A. Er. . .when x's value is greater than y's.

Q. And when is that?

(6)A. When there are more x than y (see below), or when x is in itself larger than y when put into numbers, when it changes in itself. For example, from that equation x could be 6 and y could be 4; or x could be 3 and y could be 7. You can't tell from the equation.

Q. So can you answer my question now. When is the value of x greater than the value of y?

A. No. I can't think how it could be possible

. . . .

Q. When is the value of x equal to the value of y?

A. Oh! Yes! I've just realised how you can answer it.

Q. Have you?

A. Yes, well, especially for this one. It would be for when x is five. And the other is for when x is greater than five.

Q. Well, what was it you were thinking of earlier? Can you explain it to me?

(14)A. I was thinking of the fact that as neither of these are given numbers I'm trying to think how to work it out rather than in figures in which x is larger than y. I was trying to think of the answer rather than possibilities, but it seemed more or less impossible to get an answer.

Q. I see. At one time you said 'when there are more x's than y's', didn't you?

A. Yes.

Q. Can you explain what you meant?

(18)A. Well, x will definitely be larger than y if there were more than 10x, if x is a whole number. If x was a whole number, 1, 2, 3, 4,. . .and there were more than 10x, it would definitely be larger than y.

Q. What do you mean, 'If there were more than 10x?'

(20)A. You can have 1x. You might have 10x. If 10x was a whole number it would be larger than y.

Q. Because the sum of x and y is ten?

(22)A. No, nothing to do with the answer, basically. It's just that if x for example was one, if we imagine that it has always to be a whole number, and we have ten x's which is ten in all, that would mean that for the equation y would have to be nine. Therefore the value of x would be larger than y.

Q. Can we choose something else in front of the x?

A. Yes, alright.

Q. Say eight in front of the x. Would that be alright?

A. Well, it could be, but it might not be because if it was eight, eight x wouldn't necessarily greater than y.

Q. And why is that?

A. x could be one, in which case it would add up to eight, and that would mean y would be nine, and nine is greater than eight.

Q. So if we put an eleven in front of the x, that would mean that y must be less than. . .

A. Yes, that would make the y less than the x wouldn't it? If it was nine, unless you multiplied the y by eleven as well.

Q. Isee. Thank you. Let's go on to the next one. When is the value of x less than the value of y?

A. When x is less than five. I've just seen how to do it.

Here the pupil begins his statement with an interpretation of each letter 'x' and 'y' as an object with a unique content. (Statement (2)).

At this stage it is useful to assume that he "sees" the expression as a combination of three different "spots" of ink: an 'x' (which has an unknown content) a 'y' (which has an unknown content) and a '10' which is familiar and can be made-up in a variety of ways. This means that whatever 'x' and 'y' contain is 'indeterminate' (Statement (14)).

Thus to "cover" himself, he suggests multiplying whatever x is by 3 (Statement (2)). 3 times the value of x might make that value greater than y. Certainly, multiplying x by ten will generate a value greater than y - because , if x were 1, ten times one is 10 and y would only be 9 (Statements (18) - (22)). Equally should x turn out to be 2, multiplying x by ten will give 20, and y will only be 8.

What he has not yet considered, of course, are numbers less than 1. His argument works, as he explains in Statements (18) - (22), only for whole numbers 1 - 9. Should 'x' turn out to be '$\frac{1}{2}$' then the argument breaks down. For 10x is then '5' and 'y' is '$9\frac{1}{2}$'.

During his deliberations he has been generating possible numerical identities for each letter. But his own explanations, and reflection upon them has led him to the conclusion that his argument is not conclusive. It works only for whole-number replacements for 'x' and for 'y'.

Here there is now a problem. No matter which multiplier he chooses for 'x', x might be identified with a numeral so small that it is possible that the multiplier will not be large enough.

It is at this point that his attention turns to the numerals identified with 'x' and with 'y', and here that he recognises the term "value" refers to this class of numerical determinations. His recognition that the value of x can be equal to the value of y independently of multiplication (when x is 5) convinces him that this is what is meant by the term 'value', and he responds accordingly. (It is interesting to note that the pupil returned to the "fictitious measure" interpretation in '5x = y'- his response is given in Chapter 5, Section 5.3.2. Conceptual security for the pupil might well rest in the world of objects with unique measures).

This transcript demonstrates that pupils at the level of discovered content, who give "algebraic" responses to the ET deal not with a world of objects with unique contents, but with classes of numerals. The term "value" to these pupils means "numerical truth values of expressions".

The nature of "variation" to pupils who generally use the letter as a classical unknown, is thus a potential variation over discovered truth values. That is, the range of variation of a letter depends upon the context in which it is used, and upon the probablility that the pupil will discover a numerical content for the letter. This in 'x + y = 10' for example, the fact that the pupil manages to find possible replacements for x and for y, allows him to think of 'x' varying over the range he has discovered. In the PLT however, the pupil often does not allow variation since the content discovered for the letter is, uniquely, the unmeasured length of the line.

Equally, in the LNT, Subtask 1, the pupil discovers the content '4' for 't' which makes 't + t' equal to 't + 4', and then allows the letter to refer to an unspecified numeral greater than this content. The same is true of the LNT Subtask 2.

This "finding" activity is, however, sometimes confused with an interpretation of the letter as an object with a unique content, as in the following transcript:-

Toby (2:9) 12yrs. 9mths. School A. '$5x = y$'

Q. If this is true, is the value of x always, sometimes or never greater than the value of y?
A. Never.
Q. Why?
(4)A. Because if 5x equals y then 5x must <u>equal</u> y.
Q. When is the value of x greater than the value of y?
(6)A. When x is larger than a fifth of y.
Q. When is the value of x equal to the value of y?
A. When y equals x.
Q. And when is that?
A. In this. When x equals a fifth of y.
Q. And when is that?
A. When 5x equals y.
Q. When is the value of x less than the value of y?
A. When y is greater than 5x.
Q. And when is that?
(16)A. When x is smaller than a fifth of y.
Q. Can you explain that to me?
(18)A. Well, if 5x equals y in this context, and if y had a greater value in another context, x, if it was timesed enough would still equal y. But 5x wouldn't equal y.
Q. So x is just one number, is it?
A. Yes.

Q. And y is just one number?

A. Yes.

Q. And five times that number equals y?

A. Yes.

Here the pupil regards 'y' to the right of the equation as an object with a unique content (Statement (4)).

However, his "finding" activity applied to 'x' leads him to the content '$\frac{1}{5}$y', (Statement (6)), and having derived this content, he is then happy to allow x to refer to unspecified numerals above it (Statement (6)) and below it (Statement (16)).

In this transcript, therefore, the pupil appears to be using two interpretations of the letter. Initially, 'x' and 'y' is each an object with a unique content, but when a content for x has been identified the letter is treated as classical unknown viz. it refers to one of a range of numerals greater than and less than '$\frac{1}{5}$y'. It is interesting to note at this point that the letter in the present context refers to just <u>one</u> possible numeral, (Statement (18)). The transcript thus provides evidence for the <u>psychological</u> reality of Hilbert's "hypothetical judgement" i.e. a letter which denotes an unknown (unspecified) numeral.

Each pupil above shows that he is ready to allow a letter to take on more than one identity. But this readiness depends, in each case, upon the pupil actually <u>deriving</u> an <u>identity</u> for x from the situation (in the first case 'x = 5', and in the second '$x = \frac{1}{5}y$') which he recognises is not necessarily unique. That is, each pupil shows evidence that he overcomes the "Russell Syndrome".

Collis[82] introduces the important concept of Acceptance of Lack of Closure (A.L.C.) to explain the development of mathematical ideas.

Thus the pupil who resists the temptation to replace the question mark in the item:

$a + b = 3$

$a + b + c = ?$

by a conventional numeral may be said to "accept lack of closure" in this context. The essential feature of material which requires A.L.C. is that the information provided does not admit of an unambiguous inference about a variable[83].

The transcripts above each provide an illustration of two pupils moving from a position of non-A.L.C. to a position of A.L.C. over a particularly short period

---

(82) Collis, K. F. (1972)

(83) See Lunzer, E. (1973) and Peel, E. A. (1967).

of time, in the sense that each pupil eventually accepts that a letter can refer to more than one numerical possibility.

Thus the step from 'fictitious measure' responses to 'discovered content' responses involves A.L.C. i.e. involves treating the letter as an entity which refers ambiguously to a class of potential numerical determinations. This step, however, is not sufficient to guarantee the species concept has been developed. (This is explained in Section 8.4.)

However, what it does guarantee, when the pupil accustoms himself fully to the view that the letter might refer to more than one numeral is that the letter now has a potential variation over a range of numerical values.

The concept of potential variation will be sufficient to allow the pupil to use the letter as a classical unknown to represent an as yet unidentified numeral, and to interpose relationships such as 'x = y' into a given relationship (e.g. xy + 2x = 10) in order to arrive at a particular identity of x (as Diophantus did). This concept of variation might thus be associated with Diophantine algebra. Here the pupil uses the letter to refer to <u>one</u> of a class of possible identifications, in much the same sense that we might use the term 'Mr. X' to refer to one of a group of people whose true identity we do not know, or whose name we do not wish to disclose. In this context too 'Mr. X' has a 'potential' variation over the range of possible identifications. What it is probably imagined <u>not</u> to be however, is the <u>true name of each</u> member of the group, which is the usage given to the letter at the final level, and which gives rise to a new form of "variation".

8.4. <u>The letter as a species - "variation in itself"</u>

In Chapter 6, Section 6.4., it was suggested that the species concept might be created by a fusion of two incompatible roles given to the letter, viz: that of a numeral identifier (e.g. the pupil accepts that x names <u>each</u> numeral in his repetoire in the same sense that 'Mr. X' can be the name of <u>each</u> person in the group), and the letter as a classical unknown i.e. as a word used to refer to an <u>unspecified</u> member of a group.

Pupils who use the letter as a preferred alternative to numerals in the ZT (Vietan solution) and as an organiser of perception in the PLT demonstrate they consider the letter to be a unique non-ordered entity which is, from the outset, "full of numerals".

To explain how this is possible, consider the relation:

'x $\in$ (1, 2, 3, 4).'

Two alternative interpretations of '$\in$' are possible in this statement, viz.

(a) 'x' refers to an unspecified member of the set;

and (b) 'x' is a word used to avoid mentioning by name each element of the set.

In the first interpretation, x is used in the usual sense in which 'Mr. X' is used. In the second interpretation, however, 'x' is used in a peculiarly transformed sense. It is, at the same time, an unique mathematical object, and an alternative name for each numeral. As such, it is necessarily non-ordered with respect to any numeral.

To accept this usage of the letter is to "acquire" a different form of A.L.C. to that described in Section 8.2. There A.L.C. referred to situations in which an unambiguous inference about the identity of a variable could not be made. This however, says something only about the way in which a pupil responds to or interprets a given variable. The important problem for mathematics teaching is, however, to understand what it means for pupils to acquire the concept of a variable. The answer to this question lies in the fact that "closure" has to be accepted also in so far as numerical orderings are concerned. That is, the pupil has to learn to use the letter as a non-ordered entity.

This form of A.L.C. can arise, it seems, only through the medium of "variation in itself" wherein each letter is considered to identify all possible numerals simultaneously. If 'x' identifies both 3 and 5, and 'y' both 2 and 8 then 'x' and 'y' is each a non-ordered entity. But, for the sake of our own sanity, this usage of the letter can be accepted only when we recognise that what we in fact intend by the relation '$x \in \ldots$' is that we are to use 'x' to avoid mentioning all those entities to the right of the relation. When this usage of the letter is accepted it is clear that we have transcended the world of measures and counts of concrete collections. From this point forward the letter can be used to generate new types of numerals previously unimagined, in the following sense.

Consider the pre-Vietan mathematician who for the first time recognises that 'x' can be used in the sense described above. We can imagine that until this time his numerical repetoire has been restricted to the counting numbers only. Thus the equation

$$x + 2 = 1$$

was previously for him "not possible", since each replacement for x leads to a false arithmetical statement with no modelling in the concrete world. However, now that 'x' has been recognised to be an entity used intentionally to avoid the labour of mentioning all known numerals, this recognition itself leaves open the possibility that previously unimagined numerical entities exist. The 'solution' 'x = -1' to the equation simply opens up the possibility of the existence of a new identification for 'x' and the mathematician has entered the post-Vietan world of symbolic algebra. Here, almost anything appears "possible",(84) and attention thus turns towards discovering models in the "real" world to provide consistency for the concepts generated.

In so far as the "negative" numbers are concerned, this consistency is found in displacements along a number-line i.e. in the concept of "direction". For the imaginary numbers, the Argand diagram provides the consistency model, and for the concept of the species itself consistency is found in a world of dynamic change where the concept of "measured outcome" is replaced by that of "instantaneous displacement from".

8.5. Summary

This Chapter has outlined three interpretations of the letter and three related forms of variation:

---

(84) Dantzig, T. (1954) p.86.

The letter as an object with a unique content	The letter as a Classical unknown	The letter as a species
"null" or "concrete" variation	"potential" variation	"variation in itself"

The first interpretation corresponds to a world populated with objects, the second to a world populated with conventional numerals, and the third to a world in which entities can change their numerical descriptions continuously i.e. with a world of dynamic action.

The chapter completes the attempt to satisfy Objective (a), Chapter 4, Section 4.5.-i.e. to investigate and illustrate the problems involved in the learning of the algebraic language.

What comes out most clearly from the investigation is that letters used in mathematics have distinct meanings to pupils of the same age throughout the secondary school years 1 - 5. As such, "mismatches" of meaning may be a normal condition of many classrooms.

Equally it seems clear that pupils use different "models" of reality to support algebraic thought.

There is therefore a need to devise appropriate classroom models to help pupils accommodate the species concept. In particular attention needs to be paid to two important characteristics of the letter used in this way:

(a) it is introduced intentionally to save the labour of mentioning numerals;

and (b) the letter does not "stand in place of" a numeral or a measured outcome.

By devising appropriate learning schemes it might be possible to help pupils acquire a non-ordered conception of "numerals". In particular it will be important to stress to pupils that letters are numerals in the sense devised by Vieta, and that it is necessary to look upon them at all times as non-ordered entities which "vary in themselves".

CHAPTER 9 : OBSERVATIONS FROM DATA

9.1. Abstract

The Chapter relates the results from the interviews to Hypotheses (a), (b) and (c), Chapter 4, Section 4.5.

9.2. The Species as an Organiser of Perception

The first expected outcome of the study (Chapter 4 Section 4.5.) is that some pupils will demonstrate they have developed a usage of the letter as an organiser of perception i.e. to transcend geometrically suggested orderings. This has been shown to be the case by pupil transcripts in Chapters 6 and 7. The following sections discuss the relative abilities, and distributions, of pupils grouped by response-type to the PLT (see Appendix II, Section II.2.).

9.2.1. Numbers of 'F', 'T' and 'A' pupils

A total of 71 'B' - type responses (responses using the letter to organise perceptions) were made to the PLT. 39 of these were by pupils in School A, and 32 by pupils in School B. 217 responses (105 in School A and 112 in School B) were "non-species" responses. (See Table 18, Appendix II, Section II.2.8.).

20 pupils (11 in School A and 9 in School B) gave 'B' - type responses to each Subtask. These pupils were grouped as "F" responders.

31 pupils (17 in School A and 14 in School B) gave 'B' - type responses to one Subtask. These pupils were grouped as "T" responders.

The remaining 93 pupils (44 from School A and 49 from School B) gave consistent 'A' type responses. These pupils were grouped as "A" responders. Table 1 below shows the proportion of pupils in each year group (for School A and School B) classified as "F", "T" and "A" responders.

Figure 15 is a graphical interpretation of the table, giving the respective number of pupils.

SCHOOL A + SCHOOL B

	1	2	3	4	5	6
F	0	0.08	0.04	0.08	0.12	0.50
T	0.08	0.13	0.21	0.29	0.25	0.33
A	0.92	0.79	0.75	0.63	0.63	0.17

Table 1:

Proportion of pupils in Groups 'F', 'T', and 'A' in each year group - School A and School B.

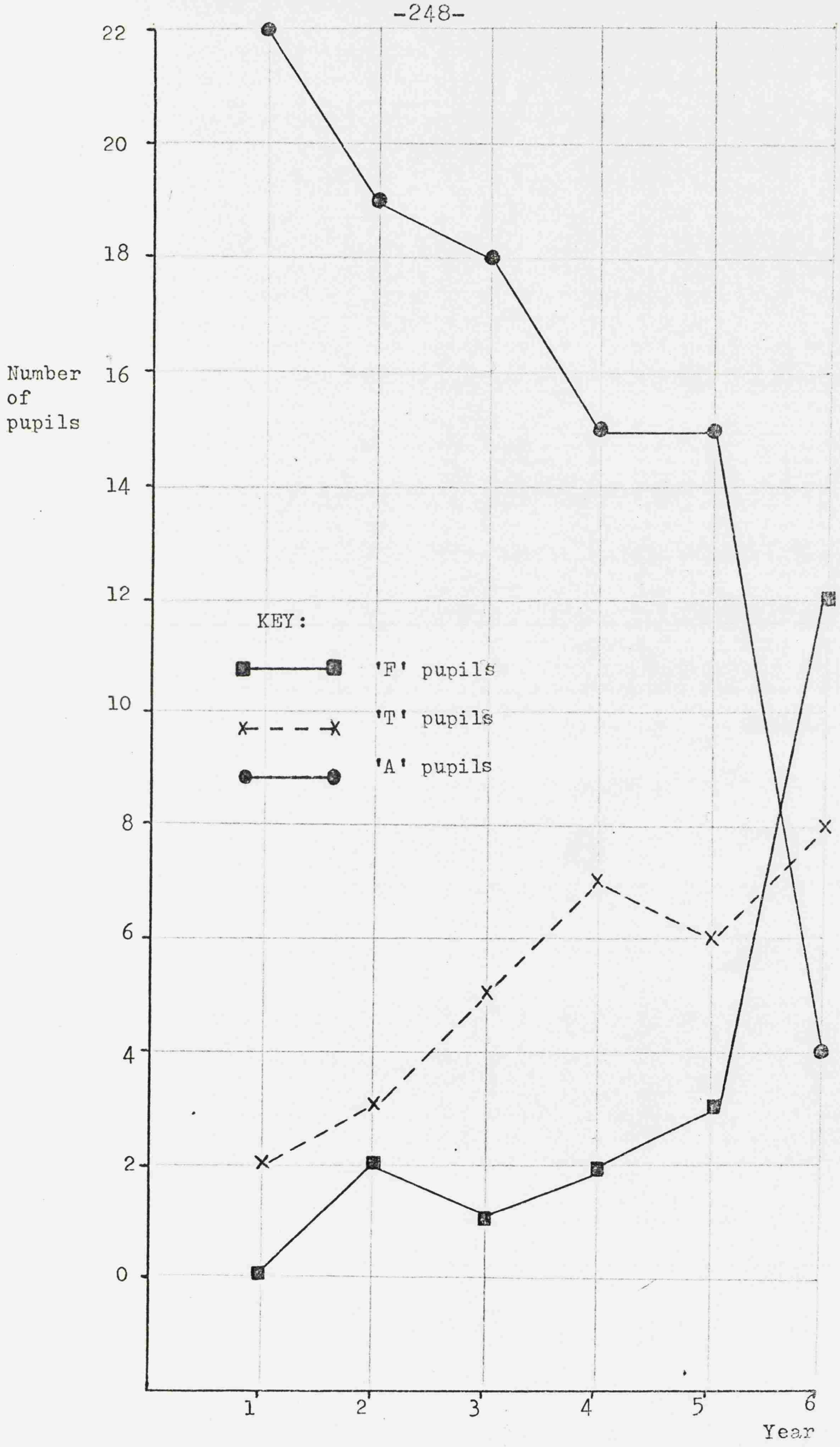


Fig.15: Number of pupils in each Year-group classified in the PLT groups 'F', 'T' and 'A'

The figures indicate that only a small minority of pupils in Years 1 through 5 (a maximum of 3 in Year 5) naturally bring the species concept with them into the PLT as the dominant means of interpreting data. The rise in proportion at the sixth year level is misleading as a general indication of concept acquisition in view of the fact that only 'A' level mathematicians were interviewed.

Although Table 1 indicates a steady fall in the number of pupils classified in group 'A' throughout the years, the indications are that the majority of pupils end their mathematical studies at the fifth year level without adopting the concept as a natural means of interpreting data.

There are however, from Year 1 onwards, pupils who either use the concept consistently in the task, or show a potential to do so.

9.2.2. Mean rankingss of 'F', 'T', and 'A' pupils

Table 2 shows the mean rankings of pupils in groups 'F', 'T', and 'A' in each year group.

SCHOOL A + SCHOOL B

	1	2	3	4	5	6	Total no. of pupils
F	0	1	1	2.5	2.7	4.8	20
T	1.5	5	3.2	3.6	3.8	7.3	31
A	7.5	7.3	7.7	8.4	8.2	10	93

Table 2:

Mean ranks of 'F', 'T', and 'A' pupils in each year-group - School A and B.

The mean rankings of the 'A' group of pupils is greater than that of the 'T' group and, in turn, this is greater than that of the 'F' group, for each year. Here the table indicates that the 'F' classified pupil is considered by his teacher to be amongst the more able mathematicians in his Year-group.

This apparent superiority of the 'F' group is reflected in the remaining tasks. This is discussed below:

9.2.3. <u>Relative abilities of 'F', 'T' and 'A' group pupils across the remaining tasks</u>.

Table 3 shows the proportions of 'F', 'T' and 'A' group pupils classified in each Group FM, FM/D, D, D/S, S (see Appendix II, Section II.6.1.).

	FM	FM/D	D	D/S	S	Total no. of pupils
F	0	0	0	0.3	0.7	20
T	0	0.06	0.39	0.45	0.1	31
A	0.46	0.34	0.14	0.05	0	93

Table 3: Cross-classification of pupil performance to the PLT against performance across the remaining tasks...

The table suggests that whereas the 'T' group pupil often makes 'excursions' into Level III - type responses (so being classified in group D/S), this is rare for the 'A' group pupil.

Equally, whereas some 'T' group pupils make a sufficient number of Level III - type responses to be classified in the 'S' group, this is not the case for the 'A' group pupil. The majority of his responses across the remaining Subtasks belong to Levels I or II.

On the other hand, the majority of the 'F' group work consistently at Level III.

Tables 4(a) - (g) show the proportions of each group 'F', 'T' and 'A' giving each type of response to each Subtask (e.g. in Table 4(a), 0.95 of the 'F' group - 19 out of 20 pupils in the group - gave a 'D' - type response to 'x + y = 10').

Response Type / Group	A	B	C	D
F	0	0	0.05	0.95
T	0	0.06	0.16	0.77
A	0.14	0.46	0.14	0.26

$\chi^2 = 51.79$ (sig. $p<0.01$)

Table 4(a); Proportions of each group 'F', 'T', 'A' giving each type of response to ET1.

Response Type / Group	A	B	C	D
F	0.05	0	0.05	0.9
T	0	0.06	0.32	0.61
A	0.28	0.44	0.12	0.16

$\chi^2 = 84.876$ (sig. $p < 0.01$)

Table 4(b): Proportions of each group 'F', 'T', and 'A' giving each type of response to ET2.

Response Type / Group	A	B	C	D
F	0	0.1	0.45	0.45
T	0.13	0.16	0.39	0.32
A	0.48	0.32	0.18	0.01

$\chi^2 = 60.284$ (sig. $p < 0.01$)

Table 4(c); Proportions of each group 'F', 'T', and 'A' giving each type of response to ET3.

Response Type / Class	A	B	C	D
F	0	0	0	1.0
T	0	0	0	1.0
A	0.22	0.17	0.19	0.39

$\chi^2 = 55.69$
(sig. $p < 0.01$)

Table 4(d): Proportions of each group 'F', 'T' and 'A' giving each type of response to LNT1.

Response Type / Class	A	B	C	D
F	0	0	0	1.0
T	0	0	0	1.0
A	0.22	0.15	0.27	0.37

$\chi^2 = 54.81$
(sig. $p < 0.01$)

Table 4(e): Proportions of each group 'F', 'T', 'A', giving each type of response to LNT2.

Response Type / Class	A	B	C	Di	Dii
F	0	0	0	0.1	0.9
T	0	0	0.03	0.58	0.39
A	0.29	0.09	0.34	0.25	0.03

$\chi^2 = 104.477$
(sig. $p < 0.01$)

Table 4 (f): Proportions of each group 'F', 'T' and 'A' giving each type of response to LNT3.

Response Type / Class	A	B	C	D
F	0	0.05	0.15	0.8
T	0.26	0.13	0.32	0.29
A	0.77	0.13	0.06	0.03

$\chi^2 = 88.528$ (sig. $p < 0.01$)

Table 4(g): Proportions of each group 'F', 'T' and 'A' giving each type of response to the ZT.

Other than the one anomaly in Table 4(b) (due to one 'F' group pupil in School B - pupil (6:2) - giving the following response to '2x + y = 9':

"Q. When is the value of x greater than the value of y?

A. When x is greater than $\frac{9 - y}{2}$",

which is reminiscent of the response to '5x = y', reported in Chapter 8, Section 8.3., involving a confusion of two different forms of variation), each table indicates a superiority or equality of the 'F' group over the 'T' group, and a superiority of the 'T' group over the 'A' group.

Analysing the data in each table using the χ^2 test, whose significance depends only upon the degrees of freedom in the table(84) gives the values of presented by the side of each table. The null hypothesis that there is no relationship between performance in the FLT and performance on another task would be rejected at

(84) Garrett, H. E. (1966); Guilford, J. P. and Fruchter, B., (1973).

the 0.01 level for each contingency table. However, in terms of statistical significance, these figures need to be treated with caution, since the expected frequency in each cell does not always exceed 5[85]. They do, however, provide some support to the subjective assessment above. This limitation is pertinent also to the discussions below.

Using the data in Table 4(g) to form 2 x 2 contingency tables for an 'F' - 'T' group comparison, by choosing each boundary level A/BCD, AB/CD, ABC/D in turn as a "cell boundary", leads to the χ^2* values given in Table 5. Where expected frequencies for all four cells of a contingency table are greater than 5, the value of χ^2 has been underlined. These values can be accepted with more confidence than unmarked figures.

Boundary Level	χ^2	d.f.	Sig./non-sig. at 0.01 level
A/BCD	4.56	1	N.S
AB/CD	<u>5.96</u>	1	N.S
ABC/D	<u>12.75</u>	1	S

Table 5: χ^2 values for the ZT - 'F' and 'T' group comparison.

(85) Lewis, D.C. (1973).

* computed after applying Yate's correction /cont. over

χ^2 is significant at the ABC/D boundary. A null hypothesis that there is no relationship between classification by response to the FLT and an ability to generalise using "givens" in the ZT would be rejected here at the 0.01 level.

When χ^2 values are derived by the same method for 'T' - 'A' group comparisons for the same task, the indication is here that the pupil who uses the letter to reorganise perception is more capable of "general" activity no matter which boundary level is chosen (Table 6):

Boundary Level	χ^2	d.f.	Sig./Non-Sig. at 0.01 level
A/BCD	27.06	1	S
AB/CD	35.43	1	S
ABC/D	17.71	1	S

Table 6: χ^2 values for the ZT - 'T' and 'A' group comparison.

These figures lend support to Hypothesis (b) (Chapter 4, Section 4.5.) that the capacity to utilise the "species" as a organiser of perception is correlated with an ability to achieve a general solution to the ZT.

---

* cont.; for continuity on a 2 x 2 contingency table.

whereas 80% of the 'F' group of pupils gave a "Vietan" solution to this task, only 29% of the 'T' group did so, and 3% of the 'A' group. Each 'F' group pupil gave a general solution, whereas 26% of the 'T' group and 77% of the 'A' group failed to do so. This additional capacity to deal with the "general" thus appears to be correlated with an intuited or conscious usage of the letter as a non-ordered numeral i.e. with the "species" concept as an organiser of perception.

With respect to the remaining tasks, the group of 'F' responders show a statistically significant superiority over the group of 'T' responders only in one respect. The value of χ^2 for the boundary D (i)/D (ii) for LNT3 is significant at the 0.01 level (Table 7, over-page).

This phenomenon is discussed in Chapters 5 and 6, and might be due to the 'F' pupil's commitment to "variation in itself", wherein he knows that the "value" of a letter is not destroyed by numerical operations.

Task	Boundary Level	χ^2	d.f.	Sig./Non-Sig. at 0.01 level
ET1	A/BCD	0	1	N.S.
ET1	AB/CD	1.343	1	N.S.
ET1	ABC/D	2.841	1	N.S.
ET2	A/BCD	1.5811	1	N.S.
ET2	AB/CD	0.0436	1	N.S.
ET2	ABC/D	5.0239	1	N.S.
ET3	A/BCD	2.800	1	N.S.
ET3	AB/CD	2.612	1	N.S.
ET3	ABC/D	0.844	1	N.S.
LNT1	A/BCD	0	1	N.S.
LNT1	AB/CD	0	1	N.S.
LNT1	ABC/D	0	1	N.S.
LNT2	A/BCD	0	1	N.S.
LNT2	AB/CD	0	1	N.S.
LNT2	ABC/D	0	1	N.S.
LNT3	A/BCDiDii	0	1	N.S.
LNT3	AB/CDiDii	0	1	N.S.
LNT3	ABC/DiDii	0.6579	1	N.S.
LNT3	ABCDi/Dii	13.203	1	S

Table 7: Values of χ^2 for subtasks of the ET and LNT - 'F' and 'T' group comparison.

Although the 'A' group performs significantly better than the 'T' group only over a small number of Subtasks, the 'T' group perform significantly better than the 'A' group across all remaining Subtasks (Table 8 below).

Task	Boundary Level	χ^2	d.f.	Sig./Non-Sig. at 0.01 level
ET1	A/BCD	9.5204	1	S
ET1	AB/CD	26.504	1	S
ET1	ABC/D	26.105	1	S
ET2	A/BCD	10.9658	1	S
ET2	AB/CD	65.022	1	S
ET2	ABC/D	23.8273	1	S
ET3	A/BCD	12.248	1	S
ET3	AB/CD	28.343	1	S
ET3	ABC/D	27.966	1	S
LNT1	A/BCD	9.4124	1	S
LNT1	AB/CD	18.9646	1	S
LNT1	ABC/D	35.164	1	S
LNT2	A/BCD	7.948	1	S
LNT2	AB/CD	15.6146	1	S
LNT2	ABC/D	37.517	1	S
LNT3	A/BCDiDii	11.488	1	S
LNT3	AB/CDiDii	16.255	1	S
LNT3	ABC/DiDii	44.458	1	S
LNT3	ABCDi/Dii	27.53	1	S

Table 8 : Values of χ^2 for Subtasks of the ET and LNT: 'T' and 'A' group comparison.

These figures thus suggest that the 'T' group has a particular advantage over the 'A' group when dealing with letters in a 'pure' algebraic setting.

The differences in achievement of the 'F', 'T', and 'A' groups across the remaining Subtasks suggested by the data is consistent with the view expressed in Chapters 5 - 8 that a pupil who naturally uses a letter as an organiser of perception in the PLT has an "algebraic" attitude of mind. An ability to deal with "general mathematical arguments" might thus correspond with an ability to construe geometrical data as a dynamic system, and with a conception of a letter as a non-ordered numeral.

9.2.4. Switches of interpretation

For some pupils a usage of the letter as "species" is clearly unstable and the probability it will be used depends upon the task content.

Thus the 'T' group of pupils switched between Type 'A' and Type 'B' responses to the PLT, and not all the 'F' group gave a "Vietan" solution to the ZT (3 of the 20 pupils gave a "Diophantine" solution, and one - pupil (2:1) School B - a "Rhetorical" solution).

There are probably two interrelated reasons for this lack of consistency - firstly, the nature of the task content; and, secondly, psychological reasons.

In the PLT the letters are presented on cards. Some pupils, when they see this use the letter to organise perception, immediately allowing the letter a numerical value which contradicts what is immediately suggested.

Other pupils take the same attitude in one Subtask but not in the other.

These same pupils however, may not use the letter as a preferred alternative for numerals in the ZT. Here the letter has to be introduced by the pupil. The older members of both the 'F' group and the 'T' group generally gave "Vietan" solutions, the younger "Diophantine" or "Rhetorical" solutions.

Although some pupils may thus have a natural potential to utilise the letter as an organiser of perception they may not have had sufficient classroom experience to convince them of its benefits in dealing with general arguments. It seems clear that by the time the pupil has reached the sixth form however that for the majority the two usages of the letter are recognised to be equivalent.

In the absence of further evidence it may be reasonable to speculate here that it may be possible to teach many pupils to use a letter as a preferred alternative to numerals for general argument, but that of the group of pupils who respond to this experience not all may learn to "see" the world of geometrical entities in the way in which the 'T' and 'F' group of pupils potentially see it. Thus it may be that where classroom experience has restricted the use of a letter to an "unknown", a younger pupil, although he might have a natural potential to use it to organise perception, might not yet be aware of its potential for expressing general mathematical results.

On the other hand, those pupils who do use the letter in the ZT to give a Vietan solution may clearly be persuaded by the nature of the PLT to abandon its usage in that context. This might question the "stability" of the PLT as a diagnostic test. However, the theoretical considerations which gave rise to its construction made this "instability" necessary from the outset.

The PLT was originally devised to demonstrate that the letter can be given two distinct meanings correlated with interpretations prior to and after the Vietan "revolution". That is, it was considered that sentential forms involving the letter could have two distinct "deep structure" associations according to whether a meaning was given to letters consonent with pre-Vietan or post-Vietan usage.

It has been suggested earlier (Chapter 8, Section 8.1.) that the two meanings may continue to reside simultaneously in the mathematician's mind, and that this is precisely what gave rise to both Hilbert's and Russell's attempts to establish the distinction between "hypothetical judgements" and "propositions".

Here Chomsky's[86] work, which draws attention to the distinction between "surface" and "deep" structure associations with sentential forms was translated into mathematical terms.

The present author has shown elsewhere[87] that any confusion caused by the logical paradox is due to our ability to entertain two independent deep-structures associated with the same sentential form.

Thus, for example, the statement:

"The Barber shaves all the people in the village" in the paradox of the Barber, and the question

---

(86) Chomsky, N. (1972)

(87) Harper, Eon (1978)

"Who wins the race, Achilles or the tortoise"? in Zeno's famous paradox, can each be interpreted in two distinct logical frameworks wherein key words (in this case respectively "all" and "motion") demand different conceptual understandings.

Our ability to work within each logical framework and so to entertain each meaning of the key term leads to our undoing.

For the mind finds itself flickering uncontrollably between the two frameworks as first one, and then a second, meaning of a statement is appreciated. The outcome is intense confusion.

The parallel in everyday language is suggested by such word series as:

"I was <u>sent</u> to <u>Coventry</u>";

"10,000 people have <u>lost</u> their homes in Turkey";

"There is a green hill far away
<u>Without</u> a city wall", etc.

and in perception is experienced with a "gestalt" figure such as that of a vase and two profiles. Here again the mind appreciates two distinct interpretations, neither of which can be said to be "true" image. The parallel is obvious for the word "length" in Subtask 1 of the PLT.

When Vieta introduced the species he introduced along with it a need to appreciate a new usage for letters, and so bequeathed to the mathematician a difficult problem - namely, that of deciding which interpretation of the letter to use - that as an "unknown" in the classical sense of the word, whose value was to be found by solving a numerical equation (which had been its common usage in mediaeval times and which is reflected here at the Level of Discovered Content); or that of the species in which the letter has guaranteed determinations from the outset.[88]

This bifurcation in the meaning of letters, and the recognition that the problems inherent to logical paradoxes could be explained as an outcome of an attempt to accommodate simultaneous deep structure associations, helped give rise to the PLT. It incorporates all the features described above.

Should it be true that we are capable of entertaining more than one deep structure associated with a sentential form then the questions asked in the PLT should tap one (or more) of these. [illegible] is position.

For the majority of pupils the deep-structure it taps is that of the letter as a classical unknown.

---

(88) See also Whitehead, A. N. (1911)

For some pupils each Subtask consistently taps the "species" interpretation. For the 'T' group of pupils it may tap each interpretation in turn.

Of the 31 'T' responders, 25 gave "Type-B" responses ('species' responses) to Subtask 3 (red line 'a' cm; green line 'b' cm), and 6 gave "Type-B" responses to Subtask 2 (red line 'p' cam; green line 'p' cm) - see Tables 15(a) and (b), Appendix II, Section II.2.8.

25 Type A (non-species) responses were made to Subtask 2 by these pupils of which 20 were due to an "abuse of symbolic formalism" i.e. the pupil suggested that each letter 'p' might have a different value - (see Tables 15(a) and (b). Appendix II, Section II,2.8.)

Each 'T' responder in the sixth form (8 pupils in all)abused symbolic formalism in this Subtask, and the same was true of all but 5 others (pupils (1:1), (2:3), (2:9) and (3:8), School A and (5:8), School B. - See Tables 15(a) and (b), Appendix II, Section II.2.8.)

It is possible that the pupils who'abuse symbolic formalism' in Subtask 2 continue to consider that the letter 'varies in itself' in that Subtask. It is the five remaining members of the 'T' group who show more explicit indication that two distinct deep-structures associated with each letter are being tapped.

In Subtask 3 these pupils appear to use the letter as a "species" and so allow it to "vary in itself". In Subtask 2 however, they suggest making spatial transformations of lines, indicating that the letter is considered to have a fixed, unknown content.

These pupils are either young or are lowly ranked relative to the majority of the 'T' group, and may not yet have reached the stage where "variation in itself" is used consistently.

The fact that pupils (1:1) and (2:3) in School A each showed a good deal of hesitation at Subtask 3, and eventually suggested making numerical replacements for each letter which contradicted immediate perception supports this possibility (see Chapter 6, Section 6 for each transcript).

9.2.5. Summary

Although some 'F' group pupils will utilise the letter as an organiser of perception in the PLT but not as a preferred alternative to numerals in the ZT, these pupils collectively represent a group who can deal significantly better with a general problem than can a 'T' or 'A' group of pupils.

Equally, pupils who indicate they are beginning to question what is immediately presented to perception in the PLT are more capable of giving general solutions to the ZT than are pupils in the 'A' group.

The letter as a non-ordered numeral thus appears to be an important means of expressing general mathematical ideas, and to be correlated with a potential or actual, dynamic interpretation of geometrical data.

Pupils in the 'F' group do not demonstrate an ability to deal with the ET and LNT which is significantly greater than that of the 'T' group. It would thus appear that the capacity to perceive an indeterminate equation as a co-varying system of numerals, and a readiness to apply matching strategies to literal numbers to obtain both a numerical and literal content, is a precursor to a development of the letter as a non-ordered numeral.

Each suggestion:

(a) that some pupils develop dynamic imagery;

and (b) that matching and substitution strategies are precursors to a development of a concept of a non-ordered numeral, would appear to

deserve close attention in future investigations.

The former suggests that a mis-match of language and imagery may prevail in the classroom between the teacher and the pupil, and the latter that pupils gradually learn to accept that a letter has a range of numerical identifications of equal status, prior to developing a concept of the letter as a non-ordered numeral (see also Chapter 6, Section 6.4. and Chapter 8, Section 8.3.).

9.3. The Role of Expectation

Hypothesis (c) (Chapter 4, Section 4.5.) is that the ET and LNT will demonstrate that pupils are influenced by suggestions of ordering and uniqueness in algebraic material.

Responses to the LNT reported in Chapter 5 indicate that a number of pupils naturally assume an ordering of letters, and will maintain, or re-establish, this ordering during the task.

The proportional number of all pupils giving A and B Type responses ("False-ordering without correction" and "False-ordering with correction" - see Appendix II, Section II.4) in each Subtask to the LNT is relatively constant (Table 11):

LNT1	LNT2	LNT3
0.27	0.23	0.24

Table 11: Proportion of all pupils giving A and/or B type responses to each Subtask of the LNT.

By way of contrast, the proportional number of all pupils giving A-Type responses to the ET ("Concrete Variation" responses) is more variable between Subtasks (Table 12):

ET1	ET2	ET3
0.09	0.19	0.33

Table 12: Proportion of all pupils giving Type A responses to each Subtask of the ET.

Thus whereas only 9% of pupils interpret the letter as an object with a unique content in ET1, 19% do so in ET2, and 33% in ET3. ET3 thus appears to cause pupils to interpret the letter at a lower level than does ET2, and, respectively ET2 than ET1, which supports the basis upon which the Equations Task was constructed.

Collis has suggested that a pupil's success with a particular item will depend upon the <u>number of</u>

operations involved in the item[89]; and Brown and Kuchemann that the type of operation is crucial[90].

When 'success' is defined in terms of the pupil's ability to transcend a fictitious measure interpretation of letters it seems that neither model fully explains the pupil's difficulties.

The equation '5x = y', in fact, involves only one operation; yet it attracts a greater number of fictitious measure interpretations than does '2x + y = 9' which involves two.

Equally, '2x + y = 9' involves both an 'add' and a 'multiply' and '5x = y' just a 'multiply'. On the surface of things, therefore, the former equation appears to be more 'complex'. Yet the pupils are less likely to impute ordering and uniqueness to letters in the former equation than in the latter. In the same way, LNT3 involves two operations and LNT1 and 2 only one; yet because ordering suggestions in each item are relatively constant, the number of responses which involve false-ordering are, too, relatively constant. The probability that a pupil will deal successfully with algebraic items therefore appears to depend not

---

(89) Collis, R. F., (1975).

(90) Brown, M. and Kuchemann, D. (1976).

in particular upon "external" factors such as the number and type of operation involved in an item but more so upon the interpretation of the letter the pupil has available to him.

9.3.1. Conclusion

Responses to the ET and LNT indicate that the greater is any suggestion of ordering and uniqueness in "pure" algebraic material, the more likely it is that a pupil will regard a letter as an object with a fictitious measure. This would appear to need to be taken into account both in teaching and in resource production.

Here, attempts might be made to arrange material into an hierarchical system beginning, for example, with equations in two variables which do not particularly support prior-developed expectations (e.g. 'x + y = 10). In particular, however emphasis seems to need to be given in the early stages to attempts to explain how a letter is used in algebra, at the expense of teaching methods of equation solving, and construction/simplification of algebraic expressions. Unfortunately few authors appear to see this as a priority. Skemp's(91) attempt to explain the nature of the algebraic variable from the standpoint of set theoretic

---

(91) Skemp, R. R. (1964)

notations is an exception. Some texts actively misguide pupils by introducing letters as abbreviations for everyday objects: 3b 'stands for' 3 boys(92) - an extreme example of what Galvin and Bell have called 'Fruit-salad algebra'(93). This might suggest that only a minority of authors are aware that simple expressions such as 'the mass of the object', translated into the sentential form 'x', can have three quite distinct meanings to the pupil; that as an object with a fictitious measure that as a classical unknown and that as a species.

9.4. General Observations

Chapter 3, Section 3 specified three important hypotheses about the way in which pupils would respond to algebraic data:

(a) some pupils would use letters to organise perceptions i.e. use the letter as a non-ordered numeral;

(b) some pupils would be influenced by inherent suggestions of ordering and uniqueness because these are intrinsic properties of "number" in non-generalised arithmetic;

and (c) pupils who used the letter as a non-ordered numeral in the PLT would demonstrate a greater

---

(92) See e.g. Avon Resources for Learning - algebra material.

(93) Galvin, W. P. and Bell, A. W. (1977).

ability to deal with a problem requiring a "general" argument.

Each hypothesis has been discussed earlier and shown to be given support by the results of the discussions with pupils.

Generally, it would seem that some pupils are conditioned by counting and measuring procedures to think of mathematical entities as necessarily ordered and numerically unique.

These pupils probably "carry over" this (mis)conception into algebraic material.

The "switches" in interpretation reported here are natural outcomes of the nature of task content, and of the intention to show that distinct, but logical, usages can be made of a letter. As such, tasks had to be devised which would be capable of capturing distinct deep-structure inputs. It is hoped that in the future it will prove possible to devise improved test material which will demonstrate more clearly any differences suggested here. The author believes this not only to be necessary, but also an important area for future research.

CHAPTER 10 : CONCLUSIONS

1. There is evidence to suggest that some pupils develop a symbolic number concept which is used to organise perceptions. This concept transcends that first studied in depth by Piaget(94) i.e. the whole number concept, and belongs to the language of symbolic formalism first introduced into mathematics by Francois Vieta during the sixteenth/seventeenth centuries.

The new, symbolic conception of number, is conveyed using letters in contrast to conventional numerals. The letters are considered to be non-ordered numerical entities with guaranteed (or "possible") determinations.

2. An accommodation of the symbolic number concept might correspond with an accommodation of the concept "instantaneous relative displacement" as a preferred alternative for the concept "measured outcome". Pupils might thus exist in distinct "realities" using different language systems and different imagery. The majority of pupils interpret the letter in geometrical settings as an object 'standing for' the measured outcome of a line.

3. It is probable that the majority of pupils complete secondary school mathematical studies devoid of

(94) Piaget, J. J., (1952)

the symbolic conception of number. These pupils work at one of two alternative levels to that of the "species":

(a) the "level of fictitious measures", in which the letter is used as an object with a unique numerical content;

and (b) the "level of discovered content" in which the letter is used as a "classical unknown".

Pupils working consistently at level (a) believe algebraic elements bear exactly one of three relations to each other. This equilibrium is also a state of mind. Despite the fact that the pupil will contradict his own assertions about the particular state of equilibrium he then returns to it. Conceptual security appears to lie in a world of numerically ordered objects

Pupils working consistently at level (b) think it possible that any one of three equilibrial states <u>might</u> exist. But each state is not necessarily a guaranteed state. Pupils at the level of the species use letters to <u>define</u> the ordering which is to exist.

5. The development of the species concept involves two aspects of A.L.C. (Acceptance of Lack of Closure) viz.

(i) in respect to accepting that a letter is needed when an unambiguous inference about a particular variable is not possible;

and (ii) in respect to accepting that literal entities are non-ordered mathematical objects.

CHAPTER 11 : IMPLICATIONS

11.1. Introduction

The study has attempted to outline some of the mathematical and psychological differences associated respectively with an acceptance of the letter as an element in the language of arithmetic with letter appendages, and in the language of symbolic formalism.

Adolescents often find algebra, as Russell puts it, a"bugbear". The reason for this, the present study suggests, is that to "understand" algebra the learner needs to have accommodated a symbolic conception of "number".

It is clear that the study is only a beginning upon a problem area which affects the lives of all pupils and mathematics teachers. Much more work is needed, with refined investigatory material, to clarify any ideas introduced here which might be thought worthwhile pursuing, and to establish beyond reasonable doubt that some of the reported differences are psychologically real.

The present chapter is an overview. It discusses the outcomes of the study and their implications for teaching and research.

11.2. The need for consistency

Kuchemann[94] identifies six "levels of understanding" of the numerical variable, viz: the letter 'evaluated', 'ignored', as 'object', as a 'specific unknown', as a 'generalised number' and as a 'variable'.

The test item which taps the 'variable' level is the following:

(a) "Which is larger, 2n or n + 2? Explain".

Of approximately 1000 third year secondary school pupils only 6% answered this item correctly.

On the other hand, 25% and 30% respectively answered the following items correctly:

(b) "m + n + q = m + p + q is true (a) Always, (b) Never, (c) Sometimes when. . ."

(c) "If c + d = 10, and c is less than d, what can you say about c?"

These items are considered to tap the "generalised number" level.

Kuchemann's test was devised from an original report by Collis[95], who described three important ways in

(94) Kuchemann, D., (1978)

(95) Collis, K. F., (1975)a

which pupils used letters whilst searching for algebraic relationships:

(i) some pupils replaced a letter by a single number and if this strategy failed to give a result, gave up the task;

(ii) some pupils were prepared to try several numbers and so appeared to have extracted a concept of a "generalised number" which was recognised as an entity in its own right sharing the property common to all numbers within the child's experience. These pupils were 14 - 15 year-olds;

and (iii) some pupils had reached a new "level of abstraction", that of the 'variable'.

The item Collis considered to differentiate the 'generalised number' pupils from the 'variable' level pupils was item (b) above, which Kuchemann himself places at the 'generalised number' level (he does not, however, say why).

Collis considers (b) to tap understanding at the 'variable' level because, he observes, to <u>have</u> the variable concept is to recognise that two letters varying over the same large range of numbers can meet in any particular number. Pupils at the 'generalised number' level, he says, find this possibility "inconceivable".(96)

An alternative answer to item (b) however, which <u>might</u> suggest the pupil is aware that letters are 'ambiguous in their denotation and accordingly undefined'(97) is:

"Always, when n = p".

Collis fails to distinguish the letter used as a Diophantine unknown and as a species. At the 'variable' level we need to <u>define</u> a relationship between letters since each is recognised to be a non-ordered entity.

To answer each item (b) and (c) above however, a pupil needs only a concept of "potential variation" i.e. to think of a number (or numbers), which satisfies a relation. Kuchemann is thus correct to place each item at the same level.

Such mismatches between researchers' ideas of what does, and does not constitute a 'variable' points readily to the state of our present understanding of what the algebraic language entails. It also draws attention

(a) to the problems the pupil must face in the classroom;

---

(96) Collis, K. F. (1975)a op. cit. p.48.

(97) Russell, B., and Whitehead, A. N. (1927) op. cit. p.4.

and (b) to the need to arrive at some consistent viewpoint in the future.

The "definition" of the 'variable' or 'species' in the present study has added yet another dimension. Here it has been considered to be a non-ordered numeral with guaranteed determinations - a concept which transcends that of the classical unknown, which demands a radical change in conceptual understanding of the properties of arithmetical objects, and which involves two different forms of A.L.C. (Chapter 8). The fact that very few pupils in the present study utilised the letter as a species is testimony to the fact that the concept might be a difficult one to achieve.

An attempt has been made (Chapter 6, Section 6.4. and Chapter 8, Section 8.4.) to explain how the concept is developed. Here any appeal to the widely accepted explanatory concept of ""abstraction"[98] was intentionally avoided for the following reasons. An appeal to the process of abstraction cannot explain how a mathematical object, which is understood to be a non-ordered entity, can acquire this property from <u>ordered</u> entities themselves. It seems to the present author that exclusive attention to 'abstractions' leaves out of consideration the fact that the role played by a symbol sometimes undergoes a radical transformation. In the process of

(98) See Dieniez, Z. P. (1961) and (1963)

creating the symbolic meaning for "number" for example, it is true that there are some similarities in the role given to the letter, and in the role given to the conventional numeral and the "unknown". Thus the new "numbers" obey the same laws as do the conventional numerals and letters for "unknowns". But the properties of the new numerals are not the same. Numerals and letters in conventional arithmetic are understood to be ordered, or potentially ordered, entities. The numerals in algebra are not.

To return to item (a) above, ("Which is larger, 2n or n + 2?"), a successful response will be guaranteed when '2n' and 'n + 2' are recognised immediately to be non-ordered entities. The item is, therefore, similar in cognitive demand to the PLT. Suggestions of ordering and uniqueness ('n = n') have to be transcended to achieve a correct result. Equally, the fact that all pupils who answered item (a) correctly also answered items (b) and (c) correctly, (personal communication with Dietmar Kuchemann, 1978) indicates again that the species concept is a synthesis of the Diophantine unknown and of the letter used as an identifier of a range of numerals (see Chapter 6, Section 6.4.). Kuchemann's work here suggests that in algebra we understand letter assemblages to be non-ordered mathematical entities, in the same way that the PLT suggests we understand letters to be non-ordered entities. This can only be the case if we know that each letter we perceive has 'internal' to it the totality of numerals known by us i.e. it is identified simultaneously with each numeral and so 'varies in itself'.

Clearly there is much more work needed before a clear and concise answer can be given to Freudenthal (p.1.) but it is hoped that the present study, considered along with Kuchemann's work will help spur future investigations towards a solution.

In particular two of the major areas of interest are, firstly that of knowing how to help pupils establish a non-ordered conception of 'numeral', and secondly how to convert the finding activity predominant at the level of 'discovered content' into the propositional activity at the level of the 'species'. The two are clearly interdependent. The PLT, however, suggests that a psychological distinction lies in the fact that <u>distinct</u> deep-structures associated with the meaning given to algebraic statements are available to us, and that the 'algebraic' deep structure is associated with that meaning given to the letter by Vieta at the conclusion of the sixteenth century. Section 11. 3. below suggests that this meaning is associated with an <u>analytical</u> activity.

11.3. Mathematical ability and Symbolic Formalism

(a) Analytical abilities

Chomsky observes that 'somehow our brief and personal and limited contacts with the world suffice for us to determine what words mean'(99). Such an ability, to pick up the meanings of words quickly, is however, necessarily relative. The mathematician probably learns the meanings of new words relevant to mathematics more quickly than does the chemist, and vice-versa.

In general terms, however, Chomsky appears to be correct. A clue to the reason for this is offered by Goodstein's observation that in learning to use a general term correctly we do not learn to associate the word with that which is common to exemplars of the word in 'concrete reality', but that we ignore any differences there might be. "Overlooking some differences in objects but not overlooking others," he writes, "is the fundamental operation in language".(100) He supports his argument with the observation that should it be true that a general term is associated with what is common to objects of experience (e.g. "sugar" associated with sugar-lumps) then the too discriminating child might never learn to speak.(101)

---

(99) Chomsky, N., (1972) p.22.

(100) Goodstein, R. L. (1965) p.26.

(101) see over.

Husserl[102] takes a similar stance. Empiricists who base their explanations of 'abstract', 'general' ideas on the relation of similarity, he observes, do so without realising the importance of "objectifying intentional acts". If by saying this figure is a triangle we mean only to say that the figure resembles the shape of other figures we will, he points out, find ourselves involved in an infinite regress. All we will be able to say is that something is a triangle because something else is a triangle, and so on ad infinitum. But it is clear that a recourse to similarly shaped things, so far from explaining the term 'triangle', presupposes in fact that we know how to use it. We must be able to make a distinction between triangularity and the sensation of a triangle.

There is, he says, something more fundamental than the relation of similarity - namely the <u>intentional</u> distinction between "appearance" and what <u>appears</u> in appearance. Thus we need to distinguish between, for example, "globe-appearance" and the "appearing-globe",

---

(101) cont. 'We regard a child's ability to learn languages quickly as a mark of intelligence, yet a too subtle and discerning child might never learn to speak his mother tongue'. Ibid. p.26.

(102) See Pivcevic, E., (1970) Chapter 5. See also Poole, R. (1977).

the latter being the intentional object. The intentional object is not to be confused with the physical object. We can, he says, perceive a 'general object' in a single empirical datum <u>directly</u> without needing to deal with similarity.

The process suggested by each writer above, by which the meaning of a general term acquired, is that of analysis. From a particular instance an attempt is made to attain an immediate generality.

When dealing with 'number' or 'numeral', the problem faced by the mathematician is that of having available a symbolic means of conveying an idea of the "general object". This symbolic apparatus, however, was introduced by Vieta when he used the <u>letter</u> to help him avoid mentioning particular conventional numerals. Vieta thus <u>intended</u> the general when he used the species.

Nunn points out that the mental movement associated with 'analysis' is one in which we "bring to light the essential process concealed in a particular or accidental garb"[103]. This can be achieved by "generalising from a particular instance".

When the PLT is considered, the relationship of Nunn's observations to those of Husserl's above is clear.

---

(103) Nunn, T. P., (1919) p.4.

In the PLT there are two ways of interpreting the data - that associated with the letter used as a "stand in" for the measured length of the line, and that associated with the "species".

When the letter is interpreted as a "stand-in", the line associated with it is a physical object. It is one line and can be measured.

When the letter is interpreted as a "species" however, the picture is transformed, for now we intend a "general" line. This line has no particular measure.

In the former instance we are dealing with "line appearance", and in the latter with the "appearing-line" i.e. with the intentional object. Thus, switching the meanings given to the letters in the PLT is equivalent to seeing the lines firstly as individual objects, and secondly as "general objects". That is, an analytical act is involved, which appears to be associated with the acquisition of the new meaning for letters introduced by Vieta.

Pupils in the present study who naturally transcended orderings in the PLT clearly intended each letter as a 'general' object in the sense above, and each line as a 'general line'. This perception of the figure was attained from one exposure to the illustration. Pupils who did not transcend orderings, clearly 'saw' the physical object before them. The former pupils were, almost invariably, in the "top" sets in their Year Group. In this sense, therefore, there is evidence to suggest that these pupils exhibited an 'analytical ability' in the PLT. Krutetskii[(104)] isolates such an aptitude as one of the important characteristics of the more able school mathematician.

The fact that some of the pupils in the present study who transcended orderings were in the lower forms, whilst many who did not were completing their final year of mathematics (5th form pupils) suggests that a specialised mathematical ability, which is related to an ability to generate the species concept at an early age, might exist. Kuchemann's work which shows that approximately the same small percentage of pupils (between 6% and 8%) in each year 2, 3, and 4 answered the item "Which is larger, 2n or n + 2?" correctly, indicates the same. Future research might well focus upon this possibility. In particular, it is important to know the role the species plays in general analytical activity. Can a pupil intend a "general triangle" or "general line" whilst devoid of the species concept, or are the two concepts inseparable? Is the ability to attain the 'general' with respect to mathematical objects an ability restricted to the few? Each is a question of fundamental

---

(104) Krutetskii, V. A., (1976)

importance to mathematical education.

One thing, however, seems clear, viz: this intentional act, which is associated with a change in usage of the letter, converts the letter directly into a non-ordered entity[105]. Section (b) below continues the discussion and suggests that this conversion might be associated also with a change from static to dynamic imagery.

(b) Language and Imagery

It has been suggested that an accommodation of the 'species' concept is associated with a potential, or actual, ability to perceive geometrical data as a dynamic construct. Little substantive support has been given to this conjecture, other than the obvious differences in response of the 'F' and 'A' groups to the FLT.

---

(105) Consider for example, the proof schema:
For some n,

$$S_n = 2 + 4 + 6 + \ldots + 2n;$$
$$2S_n = (2 + 2n) + (2 + 2n) + \ldots + (2 + 2n) = 2n(n + 1);$$
$$S_n = n(n + 1).$$

In line 1 of the 'proof' 'n' necessarily stands for an unknown ordered numeral. The proof schema is therefore not general. The result has been 'proved' for one value of 'n' only. To look upon "$S_n = n(n + 1)$" as a general statement, we have first to convert 'n', by an analytical act, into the species 'n'. Thus in this final statement 'n' is accepted to be a non-ordered entity with guaranteed determinations. In other words there is a 'logical gap' between arithmetic with letters used as 'unknowns', or 'stand-ins', for a number, and symbolic formalism. See Harper, Eon, (1976) for a full discussion.

It is possible, however, that should existent research reports be re-evaluated some of these might suggest also that some adolescents have acquired a dynamic conception of geometrical data.

One such possibility, to take an example, is work on "false conversations" reported by Lunzer(106). Lunzer is interested in Multiple Interacting Systems (MIS). In one experiment pupils are asked to consider how the perimeter and area of a rectangle changes when deformations are made which

(a) retain perimeter length;

and /or

(b) retain area.

It is only at the age of approximately 14 - 15 years that pupils begin to accept that a constant perimeter can be associated with a changing area, and a constant area with a changing perimeter.

Lunzer believes this experiment demonstrates clearly the role of "abstraction"(107). Pupils who

---

(106) Lunzer, E. A. (1968) and (1973)

(107) "The insistance on false conversation derives from failure to consider area and perimeter in abstraction from the figures from which they are derived." - Lunzer, (1973)

exhibit false conversation are "tied to the figure" immediately available to perception. Older pupils can 'stand apart' from the figure and consider the effect changing of one variable has upon the system as a whole. The figure in question is made up of a pin-board with string, so that deformations of an original rectangle can easily be made.

However the distinction between the different responses might here indicate the emergence of a dynamic interpretation associated with an ability to perform analytical mental acts. What the "non-false conserver" might see in the figure is merely a "snapshot" of a dynamic system (see Chapter 2, Section 2, Example 1). The false conserver might see the static image.

In this sense the "reality" faced by the non-conserver is not the "reality" of the false-conserver - in the same sense that the "reality" of the pre-Copernican astronomer, or the Euclidean geometer, is not the "reality" of the post-Copernican astronomer, or of the non-Euclidean geometer. In the present context such a change in view might be made possible by an accommodation of the species concept as a part and parcel of cognitive make-up.

To illustrate the point further, consider the following figures and the questions associated with them:

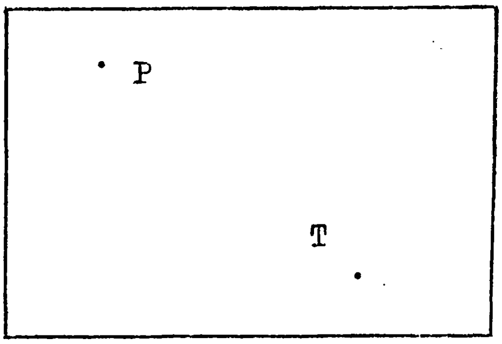


Figure 16.

<u>Question A</u>: P is fixed and T can move. PT = x cm. One possible position for T has been marked. Mark all possible positions for T.

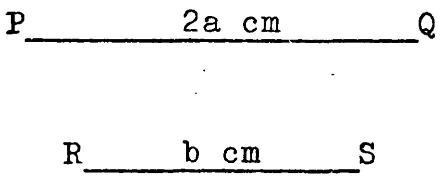


Figure 17.

<u>Question B</u>: "Which is the greater distance, PQ or RS?"

Should a circle, centre 'P', be drawn for Question A then 'x' has been interpreted as a numerical constant. Here the value of 'x' cannot be increased or decreased except by doubling, adding, etc.

An alternative response is to shade the whole rectangle, in which case 'x' might have been interpreted as a "species".

Each interpretation of the letter can be applied to Figure 17. If 'b' and 'a' are "unknowns" awaiting the outcome of a measuring process then PQ is "greater" than RS (assuming no consideration of perspective or scale has interfered with interpretation - if it has, then no conclusive answer can be given).

If 'a' and 'b' are "species", however, then the "highest level" interpretation may be that RS is greater than PQ when $|b| > |2a|$.

This second interpretation suggests a dynamic interpretation of the data in which the concept "measured distance" has been replaced by "instantaneous relative displacement". The former, accordingly, suggests a static interpretation of the data.

In the classroom situation it is possible that there are "dynamic" thinkers and "static" thinkers. Almost invariably, it would seem, the former are the potential mathematicians.

The possibility of two "realities" existing simultaneously in the classroom clearly has important implications for teaching and for resource production.

Classroom teaching and texts usually present figures (either in geometry, or as an aid to algebraic demonstration) as static entities. The medium of chalk and pen make this entirely necessary. Yet there must be ways and means of explaining to pupils that the figure drawn is not necessarily the figure intended.

A "general" figure is, potentially, a dynamic construct, and at all times we must consider that the finishing point for the chalk (or pen) is purely an instanteous position of rest. Yet many pupils will argue, once a theorem or result has been demonstrated, that an original drawing is "incorrect" - an indication that the drawing is to them a static entity and not a "snapshot" of action.

In the final event the distinction suggested here between dynamic and static imagery correlated with analytical and non-analytical activity has an important bearing upon the teaching and demonstration of geometry, trigonometry and analytic geometry, and opens up the all important question of the relationship of language to reality.

11.4. Logical process and the learning of algebra.

A number of reviews of the nature and progress of adolescent thought have concluded that a search for unity is misguided[108]. Thus these reports suggest that

---

(108) Brown, G. & Desforges, C. (1977); Lunzer, E.A. (1973); Neimark, E. D., (1975); Blasi, A. & Hoeffel, E.C. (1974); Wason, P. C. & Johnson-Laird, P. M. (1972)

the Piagetian belief that the INRC group of operations performs the enabling role of all advanced thinking is ill-founded. In particular it is suggested that curriculum planning needs to take into account the special problems associated with a particular subject area, the pupil's personal history of experience, and particular special abilities which give rise to, for example, mathematical thought on the one hand, and literary criticism on the other.

The present study made no attempt to relate distinct forms of cognitive activity to logical process. Here subjects were considered at all times to be acting logically given their own premises, and an attempt was made to account any "errors" to a mismatch in meaning of key terms - in particular to terms such as "x", "value of x", "length", and "variation".

The extent to which this is considered to have been successful cannot be known outside future discussion. However, what it does appear to suggest (to the present author) is that what is important is not in particular that certain pupils may be devoid of certain logical operations, but more so that what is often a meaningful and logically consistent statement to the pupil may not be a meaningful statement to the teacher (and vice-versa).

Thus, for example, in pilot studies, pupils (and teachers) who gave responses to '5x = y' such as:

'x is greater than y when x is greater than one fifth of y'

'x is less than y when there are 4x';

'x is equal to y when you remove the 5';

'x is equal to y now';

and

'x is equal to y when it's 10x equals 2y',

were often astonished to learn that these statements confused the author. Each subject was convinced that his/her statement was consistent with common usage of such terms as "value of x" and "variation".

Of equal importance is that when one accepts the subject's own premises (e.g. that 'x' and 'y' are objects with fictitious positive measures) then such statements <u>do</u> have their own internal logic. It simply appears to be the case that meanings in the language of fictitious measures are different to those in the language of the species.

In the teaching situation this suggests that the teacher needs to be aware that the meanings of the words he uses may not match with the meanings given to the words by many of his pupils, and so must constantly attempt to bridge the "space between the words". Such "mis-matches" in meaning and action may be a normal condition of many classroom situations when algebra is the immediate area of concern.

Some of the tasks used in the present study might be of value here as both a diagnostic and teaching aid. They might, in the least, help the teacher appreciate his pupils' algebraic understandings, help to point to the source of some difficulties, and serve as a means of demonstrating alternative meanings of key terms.

In particular, during this process there is little need to assume an existential status of logical operations. Direct remedial action can be taken to help correct a pupil's understanding by concentrating upon differences in word meanings and interpretations of data immediately available to perception. In this sense the teacher should feel that he is dealing with problems which are real, immediately present, and correctable.

11.5. A crisis for school algebra?

Folk-lore suggests that for many intelligent adults school algebra was a meaningless jargon. The majority of these adults may never have transcended the usage of a letter as an object with a fictitious measure, and so suffered acutely throughout school life from the "Russell Syndrome".

The present study, in conjunction with Kuchemann's work, suggests that this state of affairs might continue to be true. Few of even the most capable secondary school pupils appear to develop the usage of a letter either as a preferred alternative to conventional numerals in general argument or as a means of organising perception.

With this in mind, these results suggest that approximately 85% of pupils conclude their mathematical studies at the fifth year level devoid of either of these usages of the letter.

The figure is of some concern in view of the fact that it cannot be disputed that the species is the vehicle of all important mathematical ideas and that even the simplest formula, expressed symbolically, requires the concept for a clear understanding of the message it conveys. Thus '$x + y = 10$' can be understood as a formula only when an analytical act converts the letters into species, in which case the statement may be written: $(t \rightarrow 10 - t)$

From this point of view there appears to be an important need for an awareness that pupils <u>will</u> "carry over" expectations developed through working with the conventional numerals into algebra itself, and that therefore conscious action needs to be taken to change tacit assumptions which might be developed through exclusive dealings with that language.

The algebraic language would therefore seem to need to be introduced in three steps:

(a) pupils must firstly be convinced that a letter does not harbour a true measure;

(b) an attempt needs to be made to help pupils deal adequately with "indeterminate" equations; that is, pupils will need to develop the ability to hold in mind a varying system of numeral pairs which satisfy a given relation;

and

(c) the 'species' concept needs to be introduced as a letter used intentionally to avoid mentioning particular ranges of numerals which are assumed determinations of it; thus the expression $x \in \{1, 2, 3, 4, 5\}$ might be explained to mean that x is now to be introduced to save the labour of mentioning each numeral in the collection by name. In this sense any statement which then includes 'x' is either universally true or false.

In particular if the apparent 'crisis' in algebra is to be overcome, it would seem that an attempt might be made to make some change in direction. A continuous reference to the letter as an "unknown" can do little but help support prior developed expectations.

11.6. Final Remark

If the present study asks for anything, it makes an appeal for a continued and concerted attempt to understand the problems inherent to learning the algebraic language.

In particular, it would wish to stress the importance of helping pupils and teachers share a common language by ensuring that each is familiar with the concept of the Vietan species; and it asks that attempts should be made to develop programmes for teaching which bear in mind the psychological and the mathematical differences which combine together to distinguish conceptual activity in the language of arithmetic with letter appendages, from that required by the "logistica speciosa".

APPENDIX I

Name	Year:Rank	Age	Math. Set 1,2,3
Timothy	1:1	12-3	- - - -
Nigel	1:2	12-5	- - - -
Colin	1:3	12-2	- - - -
Donald	1:4	11-10	- - - -
Graham	1:5	11-6	- - - -
Alan	1:6	12-5	- - - -
Angus	1:7	11-6	- - - -
Nigel	1:8	11-7	- - - -
Patrick	1:9	12-5	- - - -
Mark	1:10	11-11	- - - -
Andrew	1:11	12-5	- - - -
Daniel	1:12	11-6	- - - -

YEAR 1
(Ranking by teacher recommendation)

Name	Year:Rank	Age	Math. Set 1,2,3
Robert	2:1	13-0	1
Jeremy	2:2	13-4	1
Patrick	2:3	13-2	1
John	2:4	13-5	1
Stephen	2:5	13-2	2
Michael	2:6	12-6	2
William	2:7	12-7	2
Richard	2:8	12-7	2
Matthew	2:9	12-9	3
Brian	2:10	13-1	3
Andrew	2:11	13-0	3
David	2:12	12-11	3

YEAR 2
(Ranking by teacher recommendation and examination results)

Table 13 : Experimental-group Structure - School A (cont. over)

Name	Year:Rank	Age	Math. Set 1,2,3
Graham	3:1	14-3	1
Barry	3:2	13-10	1
Matthew	3:3	14-6	1
Colin	3:4	13-9	1
Tony	3:5	13-6	2
Andrew	3:6	14-1	2
John	3:7	13-11	2
William	3:8	13-9	2
Michael	3:9	13-6	3
Thomas	3:10	14-0	3
Geoffrey	3:11	14-3	3
A.J.	3:12	13-10	3

YEAR 3
(Ranking by teacher recommendation and examination results)

Name	Year:Rank	Age	Math. Set 1,2,3
Michael	4:1	14-7	1
Mark	4:2	15-4	1
Malcolm	4:3	14-9	1
Tony	4:4	14-11	1
Barney	4:5	15-0	2
Ian	4:6	15-2	2
Barney	4:7	15-0	2
Mark	4:8	14-8	2
Brendan	4:9	15-4	3
Daniel	4:10	14-7	3
Gareth	4:11	15-3	3
Peter	4:12	14-11	3

YEAR 4
(Ranking by teacher recommendation and examination results)

/cont.

Name	Year:Rank	Age	Math. Set 1,2,3
Chris	5:1	15-5	1
Paul	5:2	15-6	1
Andrew	5:3	16-3	1
Philip	5:4	16-3	1
David	5:5	16-2	2
Mark	5:6	16-2	2
Philip	5:7	15-8	2
Colin	5:8	15-11	2
Martin	5:9	15-11	3
Trevor	5:10	15-5	3
Ian	5:11	15-3	3
Alan	5:12	15-4	3

YEAR 5
(Ranking by teacher recommendation and examination results)

Name	Year:Rank	Age	U=Upper 6 L=Lower 6
Gregory	6:1	17-5	U
Alan	6:2	17-6	U
Mark	6:3	17-6	U
Jonathan	6:4	17-10	U
Philip	6:5	18-1	U
Peter	6:6	16-7	L
David	6:7	16-7	L
Andrew	6:8	17-11	L
Colin	6:9	16-11	L
Malcolm	6:10	17-5	U
Daren	6:11	17-3	U
Brian	6:12	16-11	L

YEAR 6
(Ranking by teacher recommendation)

/Table 13 end.

Name	Year:Rank	Age	Math. Set 1,2,3
Matthew	1:1	11-8	- - - -
Susan	1:2	11-6	- - - -
Alison	1:3	11-6	- - - -
Jonathan	1:4	11-7	- - - -
Cheryl	1:5	11-6	- - - -
Peter	1:6	11-9	- - - -
David	1:7	11-5	- - - -
Matthew	1:8	11-8	- - - -
Giles	1:9	11-4	- - - -
Susan	1:10	11-9	- - - -
Peter	1:11	11-6	- - - -
Mary	1:12	11-3	- - - -

YEAR 1
(Ranking by teacher recommendation)

Name	Year:Rank	Age	Math. Set 1,2,3
Jane	2:1	12-8	1
Philip	2:2	13-1	1
David	2:3	12-7	1
Peter	2:4	12-3	1
Simon	2:5	13-1	2
Jonathan	2:6	12-9	2
Richard	2:7	13-3	2
Malcolm	2:8	12-7	2
Julie	2:9	13-1	3
Anna	2:10	12-9	3
Samantha	2:11	12-11	3
Alison	2:12	12-8	3

YEAR 2
(Ranking by teacher recommendation and examination results)

Table 14 : Experimental-group Structure - School B (cont. over)

Name	Year:Rank	Age	Math. Set 1,2,-
Richard	3:1	14-3	1
Douglas	3:2	13-5	1
Stewart	3:3	13-11	1
Vanessa	3:4	13-7	1
Ben	3:5	13-8	1
Sarah	3:6	13-10	1
Polly	3:7	13-5	2
Gareth	3:8	13-7	2
Murray	3:9	13-7	2
Timothy	3:10	13-2	2
Matthew	3:11	14-1	2
Conrad	3:12	13:3	2

YEAR 3
(Ranking by teacher recommendation and examination results)

Name	Year:Rank	Age	Math. Set 1,2,3
Nicholas	4:1	15-6	1
Stephen	4:2	14-9	1
Becky	4:3	15-3	1
David	4:4	14-7	1
Katy	4:5	14-7	2
Caroline	4:6	14-11	2
Debbie	4:7	14-11	2
Paul	4:8	15-3	2
Jonathan	4:9	14-11	2
Katherine	4:10	15-1	2
Eleanor	4:11	14-7	3
Kay	4:12	15-3	3

YEAR 4
(Ranking by teacher recommendation and examination results)

/cont.

Name	Year:Rank	Age	Math. Set. 1,2,3
Mark	5:1	15-5	1
Roger	5:2	16-1	1
Fiona	5:3	15-5	1
Philip	5:4	15-8	1
Karen	5:5	15-6	2
Andrew	5:6	15-9	2
Nigel	5:7	16-1	2
Christine	5:8	15-4	3
Sarah	5:9	15-11	3
Rosemary	5:10	15-6	3
Stephen	5:11	15-8	3
Christine	5:12	15-10	3

YEAR 5
(Ranking by teacher recommendation and examination results)

Name	Year:Rank	Age	U=Upper 6 L=Lower 6
David	6:1	17-7	U
Ann	6:2	18-2	U
Michael	6:3	16-5	L
Stephen	6:4	16-4	L
Julian	6:5	16-7	U
Nigel	6:6	17-10	U
Mark	6:7	16-9	L
Neil	6:8	16-5	L
Robert	6:9	17-0	L
Nicholas	6:10	17-1	L
Joanne	6:11	18-2	U
Judith	6:12	17-3	L

YEAR 6
(Ranking by teacher recommendation)

/Table 14 end.

APPENDIX II

APPENDIX II

II.1. Classification of Response-Types

This appendix explains how the responses to each task are classified and presents the data from each school in the form of tables and figures.

II.2. Classification in terms of levels of algebraic sophistication

II.2.1.

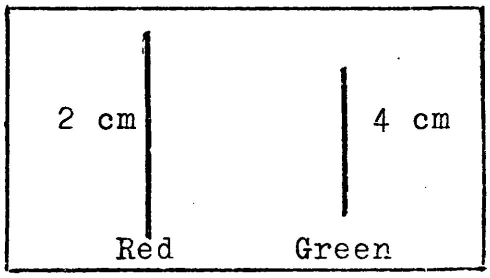
The Parallel Lines Task

Classification of pupil by response for the PLT was decided in terms of a "level of commitment" to symbolic formalism to produce three distinct categories. (That is, to the extent across Subtasks 2 and 3 to which each pupil subordinated immediate perception of concrete orderings to the symbolic language).

II.2.2. Response-types to Subtask 1

In Subtask 1 all pupils showed a readiness to subordinate concrete order to numerical intuition, giving a variety of reasons for this readiness. By comparing responses to Subtasks 2 and 3 with these responses it is possible to suggest which responses may be due to the pupil giving content to the letter from the figure, and which pupils consider the letter to have content independently of the figure.

The following are the four types of response to Subtask 1 which show that the language of arithmetic is being used to organise perceptions.



PLT Subtask 1.

(i) <u>"Length" as a relative numerical construct only</u>

The pupil considers the relationship between numerals to define the meaning of "length".

Gregory (6:1) 17yrs. 5mths. School A.

Q. Is the green line longer than the red line, the red line longer than the green line, are they equal in length, or could any of these be possible?

A. The green line is longer than the red line.

Q. Why?

A. Because it's 4cm in length.

Q. When is the green line longer than the red line?

A. It <u>is</u> longer.

Q. When is the red line longer than the green line?

A. It isn't.

Q. When are the lines equal in length?

A. They're not.

(ii) "Length" as a scalar construct. The pupil regards the lines on the card to be scalar representations.

Mark (4:8) 14yrs. 8mths. School A.

Q. Is the green line longer than the red line, the red line longer than the green line, are they equal in length, or could any of these be possible?

A. Well, . . .erm. . .if you take the units into account the green line's longer but as drawn the red line's longer.

Q. When is the green line longer than the red line?

A. When they're both drawn to the same scale.

Q. When is the red line longer than the green line?

A. As it is now, approximately half the scale.

Q. When are the lines equal in length?

A. The er. . .when the red line is twice the scale as the green line they will be equal.

(iii) "Length" as a spatial judgement in 3D. The pupil suggests the drawing may be in perspective. "Length"then refers to what "appears" to be the case.

Brendan (4:9) 15yrs. 4mths. School A.

Q. Is the green line longer than the red line, the red line longer than the green line, are they equal in length, or could any of these be possible?

A. Well, if you looked at them in the plane. . .erm. . . that (pointing to the green line) could be just shoved back in the distance, so when it comes out it could be equal. . .er. . .yes. Yes. Yes. Oh! I'm not thinking. Erm.. . .the <u>green</u> line is longer. <u>Obviously</u>. Because it's got 4cm by it.

Q. When is the green line longer than the red line?

A. When you bring it up closer to you it will be bigger.

Q. When is the red line longer than the green line?

A. When it's like it is now, in front of the green line.

Q. When are the lines equal in length?

A. When you've got the green line some way behind the 2cm line. Erm. . .when will they be equal? (Yes). When the distance is brought up so that they will be equal.

(iv) <u>"Length" as a spatial judgement in 2D</u>. The pupil suggests the numerals could be ignored.

Michael (3:9) 13yrs. 6mths. School A.

Q. Is the red line longer than the green line, the green line longer than the red line, are they equal in length, or could any of these be possible?

A. The red line's longer. Oh. Well. . .the red line's longer but it's shorter in length. That's a bit puzzling that, isn't it. They can either be possible.

Q. When is the green line longer than the red line?

A. When that (pointing to the green line) is 4cm in length and that (the red line) is 2cm in length.

Q. When is the red line longer than the green line?

A. When they're drawn like that.

Q. When are the lines equal in length?

A. They won't be.

(Some pupils suggested "extending" the green line or "cutting down" the red line - 'folding in half', etc. - others that the green line may be "bent" away from us and only a projection visible. One pupil suggested that the green line might be a 'side projection' of a circle).

Each pupil takes the numerals into account and searches for an explanation for the incompatibility of numerical and spatial meanings of "length". Thus each uses the numerals as a "pivot" around which his responses are organised. Clearly these are a number of meanings of "length" with which pupils are familiar.

II.2.3. Response Types to Subtasks 2 and 3

Subtasks 2 and 3 introduced the new possibility of "algebraic" length. Responses to these subtasks were divided into two major categories.

(a) spatio-numerical interpretations;

and

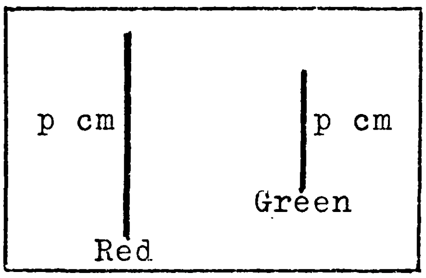
(b) algebraic interpretations.

Algebraic interpretations are considered to utilise the species concept as a means of organising perceptions; spatio-numerical interpretations to treat the letter as having a content given by the figure. Here the letter is an "hypothetical judgement" awaiting the outcome of a measuring activity.

The types of spatio-numerical interpretations correspond roughly with the four types of responses given above for Subtask 1. The following are illustrations for Subtask 2 and 3.

II.2.4. <u>Type A interpretations - Subtask 2</u>

(i) "<u>Length" as a 2D spatial judgement</u>. Each letter 'p' takes a different content according to the perceived differences in length of each line.



PLT Subtask 2.

Nigel (5:7) 16yrs. 1mth. School B.

Q. Is the red line longer than the green line, the green line longer than the red line, are they equal in length, or could any of these be possible?

A. The red line's longer.

Q. Why?

A. That (the red line) has more units in it.

Q. When is the green line longer than the red line?

A. when the green line covers more units.

Q. When is the red line longer than the green line?

A. When you work out there's more units.

Q. When are the lines equal in length?

A. When there are the same units in them.

(pupils may also argue e.g. 'when you double the (green) 'p' etc.).

(ii) <u>"Length" as a 3D spatial judgement</u>. The pupil may accept an identity of each letter, but suggests spatial transformations to attain a difference in "size".

Philip (5:7) 15yrs. 8mths. School A.

Q. Is the red line longer than the green line, the green line longer than the red line, are they equal in length or could any of these be possible?

A. The red line is longer than the green line.

Q. Why?

A. Because p is larger than. . .no. Because the red line is longer in length than the green line.

Q. When is the green line longer than the red line?

A. If it was looked at nearer your eye it would be.

Q. When is the red line longer than the green line?

A. When it's drawn as it is now.

Q. When are they equal in length?

A. When you look at them from the right distance.

(iii) "<u>Length" as a "multiple" construct</u>. The pupil introduces all "numerical" possibilities to explain relative size.

Colin (5:8) 15yrs. 11mths. School A.

Q. Is the red line longer than the green line, the green line longer than the red line, are they equal in length, or could any of these be possible?

A. Any of those could be possible.

Q. Why?

A. Well, from the drawing the red line appears longer than the green line. They could be equal in length because a different scale was being used. And the green line could definitely be longer because of the three dimensional problem. It could also be a curved line going into the desk.

Q. When is the green line longer than the red line?

A. If in fact it was a curved line and it was straightened out.

Q. When is the red line longer than the green line?

A. If in fact it was an optic problem and it was nearer.

Q. When are the lines equal in length?

A. If in fact it was a different scale.

(iv) <u>Abuse of symbolic formalism</u>. The pupil accepts the lines are equal in length, but suggest the 'p's' may represent different numerical entities.

Becky (4:3) 15yrs. 3mths. School B.

Q. Is the red line longer than the green line, the green line longer than the red line, are they equal in length, or could any of these be possible.

A. They're the same.

Q. Why?

A. Because they're both p cm long.

Q. When is the green line longer than the red line?

A. When the p's are not equal. When the green p is more than the red p.

Q. When is the red line longer than the green line?

A. When the p's are not equal.

Q. When are the lines equal in length?

A. When the p's are equal.

(Some pupils suggest that the lines are a different length when p varies - when p is a'variable').

Each response above is considered to be a "non-formal" response. The "formal" response, utilising the species is as follows:

II.2.5. Type B interpretations - Subtask 2

Stephen (4:2) 14yrs. 9mths. School B.

Q. Is the red line longer than the green line, the green line longer than the red line, are they equal in length, or could any of these be possible?

A. They're equal in length.

Q. Why?

A. Because they're both p cm.

Q. When is the green line longer than the red line?

A. Never.

Q. When is the red line longer than the green line?

A. Never.

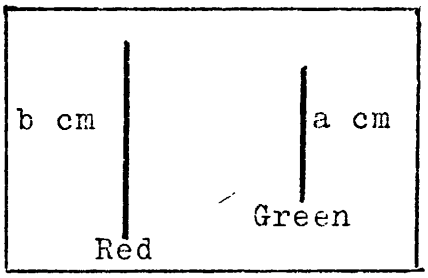
Q. When are they equal in length?

A. Always.

(Some pupils use the terms "for all values", or "All the time").

II.2.6. Type A interpretations - Subtask 3

Interpretations for Subtask 3 are similar to those for Subtasks 1 and 2. The following are examples of A-type interpretations which do not involve the species concept.



PLT Subtask 3

(i) "Length" as a fictitious measure. The pupils consider 'a' to be smaller than 'b' and suggest numerical operations upon each to increase the length of the line.

Nigel (5:7) 16yrs. 1mth. School B.

Q. Is the red line longer than the green line, the green line longer than the red line, are they equal in length or could any of these be possible?

A. The red line is longer.

Q. Why?

A. Because you've drawn it longer.

Q. When is the green line longer than the red line?

A. When there are more a's than b's. When there's two a's or three a's.

Q. When is the red line longer than the green line?

A. When the unit of a is less so that it balances out.

Q. When are the lines equal in length?

A. When the lines are equal.

(ii) "Length" as a 2D spatial judgement. Letters are totally ignored.

Jonathan (4:9) 14yrs. 11mths. School B.

Q. Is the red line longer than the green line, the green line longer than the red line, are they equal in length, or could any of these be possible?

A. The red line is longer.

Q. Why?

A. Because it goes up more centimetres than the other.

Q. When is the green line longer than the red line?

A. It can't be.

Q. When is the red line longer than the green line?

A. All the time.

Q. When are the lines equal in length?

A. They're not.

(iii) "Length" as a 3D spatial judgement. The pupil considers the lines to be in perspective.

Nigel (1:8) 11yrs. 7mths. School A.

Q. Is the red line longer than the green line, the green line longer than the red line, are they equal in length, or could any of these be possible?

A. Any of them could be possible.

Q. Why?

A. Because the red line (he bends down and squints along the card, one eye closed) could be where someone is looking from and if the green line was further away it could be small. And if it was nearer it would look bigger.

Q. When is the green line longer than the red line?

A. When you're looking from the green side.

Q. When is the red line longer than the green line?

A. When the green line is further away.

Q. When are the lines equal in length?

A. I don't know.

(iv) "<u>Length" as a scalar construct</u>. The lines are considered to be scale drawings.

Mark C. (5:6) 16yrs. 2mths. School A.

Q. Is the red line longer than the green line, the green line longer than the red line, are they equal in length, or could any of these be possible?

A. It depends if they're drawn in different scales or not.

Q. When is the green line longer than the red line?

A. When the scale for that (the green line) is smaller than the scale for that (the red line).

Q. When is the red line longer than the green line?

A. Either as they're drawn now, or if that (the red line's) scale is smaller than that (the green line's) scale. But it doesn't even need to be that does it? It could still be drawn so that b was longer than a.

Q. When are the lines equal in length?

A. Well, they could be drawn on different scales representing the lengths that they are now, but they could turn out to be different lengths.

Each of the above are examples of "<u>non-formal</u>" responses for Subtask 3. The following is an example of a "formal" response, utilising the species.

II.2.7. Type B interpretations - Subtask 3

Chris (5:1) 15yrs. 5mths. School A.

Q. Is the red line longer than the green line, the green line longer than the red line, are they equal in length, or could any of these be possible?

A. Any of those could be possible.

Q. Why?

A. Well, because a and b are general numbers, so a can be greater than b, or less than b or equal to b.

Q. When is the green line longer than the red line?

A. When a is greater than b.

Q. When is the red line longer than the green line?

A. When b is greater than a.

Q. When are the lines equal in length?

A. When a equals b.

II.2.8. Classification of pupils by response-type to the PLT subtasks.

Using responses to Subtasks 2 and 3 each pupil was classified into one of three groups as follows:

Group F: Pupils giving Type B responses to each Subtask;

Group T: Pupils giving at least one Type B response to a Subtask;

Group A: Pupils giving a Type A response to each Subtask.

Tables 15(a) and 15(b) indicate the classification of each pupil in Schools A and B. Tables 16(a) and 16(b) show the number of 'B' and 'A' responses in each year group in each school.

Pupil	Response to Subtask 2	Response to Subtask 3	Group F,T,A
1:1	A	B	T
1:2	A	B	T
1:3	A	A	A
1:4	A	A	A
1:5	A	A	A
1:6	A	A	A
1:7	A	A	A
1:8	A	A	A
1:9	A	A	A
1:10	A	A	A
1:11	A	A	A
1:12	A	A	A

YEAR 1

Pupil	Response to Subtask 2	Response to Subtask 3	Group F,T,A
2:1	B	B	F
2:2	A	A	A
2:3	A	B	T
2:4	A	A	A
2:5	A	A	A
2:6	A	A	A
2:7	A	A	A
2:8	A	A	A
2:9	A	B	T
2:10	A	A	A
2:11	A	A	A
2:12	A	A	A

YEAR 2

Pupil	Response to Subtask 2	Response to Subtask 3	Group F,T,A
3:1	A	B	T
3:2	A	B	T
3:3	A	B	T
3:4	A	A	A
3:5	A	A	A
3:6	A	A	A
3:7	A	A	A
3:8	A	B	T
3:9	A	A	A
3:10	A	A	A
3:11	A	A	A
3:12	A	A	A

YEAR 3

Pupil	Response to Subtask 2	Response to Subtask 3	Group F,T,A
4:1	A	B	T
4:2	B	A	T
4:3	B	B	F
4:4	A	A	A
4:5	A	B	T
4:6	A	A	A
4:7	A	A	A
4:8	A	A	A
4:9	A	A	A
4:10	A	A	A
4:11	A	A	A
4:12	A	A	A

YEAR 4

Pupil	Response to Subtask 2	Response to Subtask 3	Group F,T,A
5:1	B	B	F
5:2	B	B	F
5:3	A	B	T
5:4	A	B	T
5:5	A	A	A
5:6	A	A	A
5:7	A	A	A
5:8	A	A	A
5:9	A	A	A
5:10	A	A	A
5:11	A	A	A
5:12	A	A	A

YEAR 5

Pupil	Response to Subtask 2	Response to Subtask 3	Group F,T,A
6:1	B	B	F
6:2	B	B	F
6:3	A	B	T
6:4	B	B	F
6:5	B	B	F
6:6	B	B	F
6:7	A	B	T
6:8	B	B	F
6:9	B	B	F
6:10	A	B	T
6:11	A	A	A
6:12	A	B	T

YEAR 6

Table 15(a): Response-type of each pupil to the PLT- School A

Pupil	Response to Subtask 2	Response to Subtask 3	Group F,T,A
1:1	A	A	A
1:2	A	A	A
1:3	A	A	A
1:4	A	A	A
1:5	A	A	A
1:6	A	A	A
1:7	A	A	A
1:8	A	A	A
1:9	A	A	A
1:10	A	A	A
1:11	A	A	A
1:12	A	A	A

YEAR 1

Pupil	Response to Subtask 2	Response to Subtask 3	Group F,T,A
2:1	B	B	F
2:2	A	A	A
2:3	A	B	T
2:4	A	A	A
2:5	A	A	A
2:6	A	A	A
2:7	A	A	A
2:8	A	A	A
2:9	A	A	A
2:10	A	A	A
2:11	A	A	A
2:12	A	A	A

YEAR 2

Pupil	Response to Subtask 2	Response to Subtask 3	Group F,T,A
3:1	B	B	F
3:2	A	B	T
3:3	A	A	A
3:4	A	A	A
3:5	A	A	A
3:6	A	A	A
3:7	A	A	A
3:8	A	A	A
3:9	A	A	A
3:10	A	A	A
3:11	A	A	A
3:12	A	A	A

YEAR 3

Pupil	Response to Subtask 2	Response to Subtask 3	Group F,T,A
4:1	A	B	T
4:2	B	B	F
4:3	A	B	T
4:4	B	A	T
4:5	A	A	A
4:6	A	A	A
4:7	A	A	A
4:8	A	A	A
4:9	B	A	T
4:10	A	A	A
4:11	A	A	A
4:12	A	A	A

YEAR 4

Pupil	Response to Subtask 2	Response to Subtask 3	Group F,T,A
5:1	A	B	T
5:2	A	A	A
5:3	A	B	T
5:4	B	A	T
5:5	B	B	F
5:6	A	A	A
5:7	A	A	A
5:8	A	B	T
5:9	A	A	A
5:10	A	A	A
5:11	A	A	A
5:12	A	A	A

YEAR 5

Pupil	Response to Subtask 2	Response to Subtask 3	Group F,T,A
6:1	A	B	T
6:2	B	B	F
6:3	B	B	F
6:4	A	B	T
6:5	B	B	F
6:6	B	B	F
6:7	B	B	F
6:8	A	A	A
6:9	A	A	A
6:10	A	B	T
6:11	A	B	T
6:12	A	A	A

YEAR 6

Table 15(b): Response-type of each pupil to the PLT -School B

Response Type	Year	1	2	3	4	5	6	Totals	
A	Subtask 2	12	11	12	10	9	5	59	
	Subtask 3	10	9	8	9	9	1	46	
	Total No. of 'A' Responses	22	20	20	19	18	6		105
B	Subtask 2	0	1	0	2	2	7	12	
	Subtask 3	2	3	4	3	4	11	27	
	Total No. of 'B' Responses	2	4	4	5	6	18		39

Table 16 (a); Number of 'A' and 'B' type responses to Subtasks 2 and 3 of the PLT in each year group - School A.

Response Type	Year	1	2	3	4	5	6	Totals	
A	Subtask 2	12	11	11	10	10	7	61	
	Subtask 3	12	10	10	8	8	3	51	
	Total No. of 'A' Responses	24	21	21	18	18	10		112
B	Subtask 2	0	1	1	2	2	5	11	
	Subtask 3	0	2	3	4	4	9	21	
	Total No. of 'B' Responses	0	3	3	6	6	14		32

Table 16 (b); Number 'A' and 'B' type responses to Subtasks 2 and 3 of the PLT in each year group - School B.

II.3. The Equations Task.

Responses to each equation are divided into four categories in an ascending hierarchy of algebraic sophistication.

These are:

II.3.1. Type A Responses: Fictitious Measure.

The pupil assumes false-ordering and /or false-content and then may use "concrete variation" or suggest "transposing terms" to attain equality or inequality i.e. he does not work within the algebraic constraints set up by the equation.

Examples for each equation are given below.

Example 1: (False content)

Angus (1:7) 11yrs. 6mths. School A. '$x + y = 10$'

Q. If this is true, is the value of x always, sometimes, or never greater than the value of y?

A. Well, it depends on what you want really. You can want it larger or smaller.

Q. When is the value of x greater than the value of y?

A. When you. . .take the values according to their place in the alphabet. Oh no, sorry. When you just say are counting from x to a, if then you take the number of their positions as their value, then 'z' would be one and 'a' twenty-six.

Q. When is the value of x equal to the value of y?

A. When it's necessary to work the problem out.

Q. And when is that?

A. Say when y is five, y plus x equals ten; then they would have to be the same.

Q. When is the value of x less than the value of y?

A. When you take their positions in the alphabet from 'a' to 'z'.

<u>Example 2</u> (False ordering - transposing terms)

Angus (1:7) 11yrs. 6mths. School A. '5x = y'.

Q. If this is true, is the value of x always, sometimes, or never greater than the value of y?

A. Never.

Q. Why?

A. Because five x's equal y.

Q. When is the value of x greater than the value of y?

A. When you put a 5 on the y. When you change the 5 over to the other side.

Q. When is the value of x equal to the value of y?

A. When you remove the 5, or put an extra 5 by the y.

Q. When is the value of x less than the value of y?

A. Like it is at the moment.

Example 3 (False ordering - concrete variation)

William (3:8) 13yrs. 9mths. School A. '5x = y'.

Q. If this is true is the value of x always, sometimes or never greater than the value of y?

A. Never.

Q. Why?

A. Because 5x equals y.

Q. When is the value of x greater than the value of y?

A. When it's multiplied by more than 5. Say when it's 6x or 7x.

Q. When is the value of x equal to the value of y?

A. When it's 5x.

Q. When is the value of x less than the value of y?

A. When it's multiplied by 4, or 3.

Example 4 (False ordering - transposing terms)

A. J. (3:12) 13yrs. 10mths. School A. '2x + y = 9'

Q. If this is true, is the value of x always, sometimes, or never greater than the value of y?

A. It's never greater.

Q. Why?

A. Because you need two x's and only one y to make nine.

Q. When is the value of x greater than the value of y?

A. Well, if you had x plus 2y it would be.

Q. When is the value of x equal to the value of y?

A. Would it be if you put a 2 on the y?

Q. When is the value of x less than the value of y?

A. It is now.

II.3.2. Type B Responses : Placeholder

(i) For x + y = 10 (Multiple subtraction)

The pupil discovers numerical identities for each letter and lists possibilities (one or more) for each question asked.

Example 1

Cheryl 11yrs. 9mths. (1:5) School B. 'x + y = 10'

Q. If this is true, is the value of x always, sometimes, or never greater than the value of y?

A. They're both the same.

Q. Why?

A. They must be five for that to work out.

Q. When is the value of x greater than the value of y?

A. When it's like 8 plus 2.

Q. When is the value of x equal to the value of y?

A. When they're both 5.

Q. When is the value of x less than the value of y?

A. If y was a larger number, like 6.

Example 2

Richard (2:8) 12yrs. 7mths. School A. 'x + y = 10'

Q. If this is true, is the value of x always, sometimes, or never greater than the value of y?

A. x is less than y.

Q. Why?

A. Because you need five x's to make a y.

Q. When is the value of x greater than the value of y?

A. It isn't.

Q. When is the value of x equal to the value of y?

A. It isn't.

Q. When is the value of x less than the value of y?

A. In the equation.

(iii) For '2x + y = 10' the response is similar to that for 'x + y = 10'.

Example 4

Debbie (4:7) 14yrs. 11mths. School B. '2x + y = 9'

Q. If this is true, is the value of x always, sometimes, or never greater than the value of y?

A. x is never bigger.

Q. Why?

A. Because x is multiplied by 2. No, they're the same. Both 3.

Q. When is the value of x greater than the value of y?

A. When. . .if it was 4 and 1. If x was 4 that's two times 4 is 8, add 1 is 9.

Q. When is the value of x equal to the value of y?

A. When they're both three.

Q. When is the value of x less than the value of y?

A. Two times one is two and seven is nine.

II.3.3. Type C Responses : Borderline-Algebraic

(i) x + y = 10 and 2x + y = 9. The pupil shows evidence that he has an overview of the equation as a system, but that he may deal with whole number values only. He does not feel the need to state each possibile identity of x in a step-by-step fashion as in the Type B response.

Example 1

Philip (2:2) 13yrs. 1mth. School B. 'x + y = 10'

Q. If this is true, is the value of x always, sometimes or never greater than the value of y?

A. It could be larger or smaller.

Q. Why?

A. x could be 8 and y, 2; or y, 8 and x, 2.

Q. When is the value of x greater than the value of y?

A. When it's six or more.

Q. When is the value of x equal to the value of y?

A. When they're both five.

Q. When is the value of x less than the value of y?

A. When x is four or less.

Example 2

Patrick (2:3) 13yrs. 2mths. School A. '2x + y = 9'

Q. If this is true, is the value of x always, sometimes, or never greater than the value of y?

A. It could be both.

Q. Why?

A. Because they're equal if x is 3 and y is 3.

Q. When is the value of x greater than the value of y?

A. When it's 4 and more.

Q. When is the value of x equal to the value of y?

A. When it's 2 and below.

Type C Response ('5x = y')

The pupil works within the system set up by the equation, may mention zero, but does not utilise negative numbers.

Example 1

Jane (2:1) 12yrs. 8mths. School B. '5x = y'

Q. If this is true, is the value of x always, sometimes, or never greater than the value of y?

A. Never.

Q. Why?

A. Because five x is y. So x is a fifth of y.

Q. When is the value of x greater than the value of y?

A. Never.

Q. When is the value of x equal to the value of y?

A. When they're both nought.

Q. When is the value of x less than the value of y?

A. Always.

Example 2

Mark (4:2) 15yrs. 4mths. School A. '5x = y'

Q. If this is true, is the value of x always, sometimes, or never greater than the value of y?

A. y is always greater.

Q. Why?

A. Because you need 5x for every y.

Q. When is the value of x greater than the value of y?

A. Never.

Q. When is the value of x equal to the value of y?

A. Never.

Q. When is the value of x less than the value of y?

A. Always.

II.3.4. Type D Responses : Algebraic

The pupil views the equation as an integrated system, and specifies an algebraic relation ($>$, $<$, $=$) between the letter and a number to limit the range of variability of the letter for each question. In '5x = y' he mentions zero and the negative numbers.

Example 1

Jane (2:1) 12yrs. 8mths. School B. 'x + y = 10'

Q. If this is true, is the value of x always, sometimes, or never greater than the value of y?

A. It depends upon x. Sometimes.

Q. When is the value of x greater than the value of y?

A. When x is greater than 5.

Q. When is the value of x equal to the value of y?

A. When x equals 5.

Q. When is the value of x less than the value of y?

A. When x is less than 5.

Example 2

Jane (2:1) 12yrs. 8mths. School B. '2x + y = 9'

Q. If this is true, is the value of x always, sometimes, or never greater than the value of y?

A. It depends upon x. Sometimes greater.

Q. When is the value of x greater than the value of y?

A. When x is greater than 3.

Q. When is the value of x equal to the value of y?

A. When x equals 3.

Q. When is the value of x less than the value of y?

A. When x is less than 3.

Example 3

Paul (5:2) 15yrs. 6mths. School A. '5x = y'

Q. If this is true, is the value of x always, sometimes, or never greater than the value of y?

A. It must be less because y is a multiple of x.

Q. When is the value of x greater than the value of y?

A. It could be if x were negative. Then y will be a greater negative number.

Q. When is the value of x equal to the value of y?

A. When x and y are zero.

Q. When is the value of x less than the value of y?

A. When x is positive.

Tables 17(a) and 17(b) give each pupils' response-type to each equation. Tables 18(a) and 18(b) show the number of each type of response in each year group for each school.

Pupil	Response to ET1	ET2	ET3
1:1	D	C	C
1:2	C	C	C
1:3	D	D	B
1:4	C	C	B
1:5	B	B	B
1:6	C	B	B
1:7	A	A	A
1:8	B	B	A
1:9	B	A	A
1:10	B	B	B
1:11	B	A	A
1:12	A	A	A

YEAR 1

Pupil	Response to ET1	ET2	ET3
2:1	D	D	C
2:2	D	D	C
2:3	D	C	C
2:4	C	C	B
2:5	C	C	C
2:6	C	B	A
2:7	B	B	B
2:8	B	B	B
2:9	D	C	A
2:10	B	A	B
2:11	B	A	A
2:12	A	A	A

YEAR 2

Pupil	Response to ET1	ET2	ET3
3:1	D	D	C
3:2	D	D	C
3:3	D	C	C
3:4	B	B	B
3:5	C	C	B
3:6	B	B	C
3:7	B	B	A
3:8	C	C	A
3:9	A	A	B
3:10	D	B	A
3:11	B	A	A
3:12	A	A	A

YEAR 3

Pupil	Response to ET1	ET2	ET3
4:1	D	D	D
4:2	B	C	C
4:3	D	C	C
4:4	B	B	A
4:5	D	D	B
4:6	B	B	B
4:7	A	A	A
4:8	B	A	A
4:9	D	D	B
4:10	B	B	B
4:11	A	A	A
4:12	A	B	A

YEAR 4

Pupil	Response to ET1	ET2	ET3
5:1	D	D	D
5:2	D	D	D
5:3	D	D	A
5:4	D	D	D
5:5	D	C	C
5:6	D	D	C
5:7	D	D	B
5:8	D	B	B
5:9	B	B	B
5:10	B	B	A
5:11	D	C	C
5:12	B	A	A

YEAR 5

Pupil	Response to ET1	ET2	ET3
6:1	D	D	D
6:2	D	D	D
6:3	D	D	D
6:4	D	D	D
6:5	D	D	C
6:6	D	D	B
6:7	D	D	D
6:8	D	D	C
6:9	C	D	C
6:10	D	C	C
6:11	D	B	C
6:12	D	D	D

YEAR 6

Table 17(a): Response-types of each pupil to the Equations Task - School A

Pupil	Response to ET1	ET2	ET3
1:1	D	D	C
1:2	B	B	A
1:3	C	B	A
1:4	D	C	C
1:5	B	B	B
1:6	C	B	B
1:7	B	B	B
1:8	B	A	A
1:9	B	B	A
1:10	B	A	A
1:11	A	B	A
1:12	B	A	A

YEAR 1

Pupil	Response to ET1	ET2	ET3
2:1	D	D	C
2:2	C	C	C
2:3	B	D	B
2:4	B	B	B
2:5	A	A	A
2:6	C	B	B
2:7	D	B	C
2:8	D	C	A
2:9	B	B	A
2:10	B	B	B
2:11	B	A	A
2:12	A	A	A

YEAR 2

Pupil	Response to ET1	ET2	ET3
3:1	D	D	C
3:2	D	D	C
3:3	D	D	B
3:4	D	D	C
3:5	C	C	A
3:6	B	B	B
3:7	B	A	A
3:8	B	B	B
3:9	D	B	A
3:10	C	C	A
3:11	B	A	A
3:12	B	B	A

YEAR 3

Pupil	Response to ET1	ET2	ET3
4:1	D	D	C
4:2	D	D	D
4:3	D	D	B
4:4	C	C	D
4:5	B	B	A
4:6	D	D	A
4:7	B	B	B
4:8	D	D	A
4:9	D	C	A
4:10	B	A	A
4:11	B	B	B
4:12	B	B	A

YEAR 4

Pupil	Response to ET1	ET2	ET3
5:1	D	D	D
5:2	D	D	D
5:3	C	C	D
5:4	D	D	C
5:5	D	D	C
5:6	A	A	A
5:7	B	B	B
5:8	D	B	B
5:9	B	A	A
5:10	D	D	A
5:11	D	B	C
5:12	B	D	C

YEAR 5

Pupil	Response to ET1	ET2	ET3
6:1	D	B	B
6:2	D	A	C
6:3	D	D	D
6:4	D	D	C
6:5	D	D	D
6:6	D	D	C
6:7	D	D	D
6:8	B	B	C
6:9	D	D	A
6:10	C	D	D
6:11	D	D	D
6:12	D	D	C

YEAR 6

Table 17(b): Response-types of each pupil to the Equations Task - School B

Resp Type	YEAR EQUATION	1	2	3	4	5	6	TOTALS	
A	x+y=10	2	1	2	3	0	0	8	
	2x+y=9	4	3	3	3	1	0	14	
	5x=y	5	4	5	5	3	0	22	
	TOTAL No OF 'A' RESPONSES	11	8	10	11	4	0		43
B	x+y=10	5	4	4	5	3	0	21	
	2x+y=9	4	3	4	4	3	1	19	
	5x=y	5	4	3	4	3	1	20	
	TOTAL No OF 'B' RESPONSES	14	11	11	13	9	2		60
C	x+y=10	3	3	2	0	0	1	9	
	2x+y=9	3	4	3	2	2	0	14	
	5x=y	2	4	4	2	3	5	20	
	TOTAL No OF 'C' RESPONSES	8	11	9	4	5	6		43
D	x+y=10	2	4	4	4	9	11	33	
	2x+y=9	1	2	2	3	6	11	24	
	5x=y	0	0	0	1	3	6	7	
	TOTAL No OF 'D' RESPONSES	3	6	6	8	18	28		69

Table 18 (a): Number of each type of response to Subtasks of the ET in each year group - School A.

Resp. Type	EQUATION	1	2	3	4	5	6	TOTALS	
A	x+y=10	1	2	0	0	1	0	4	
	2x+y=9	3	3	2	1	2	1	12	
	5x=y	7	5	6	6	3	1	28	
	TOTAL No OF 'A' Responses	11	10	8	7	6	2		44
B	x+y=10	7	5	5	5	3	1	26	
	2x+y=9	7	5	4	4	3	2	25	
	5x=y	3	4	3	3	2	1	16	
	TOTAL No OF 'B' Responses	17	14	12	12	8	4		67
C	x+y=10	2	2	2	1	1	1	9	
	2x+y=9	1	2	2	2	1	0	8	
	5x=y	2	3	3	1	4	5	18	
	TOTAL No OF 'C' Responses	5	7	7	4	6	6		35
D	x+y=10	2	3	5	6	7	10	33	
	2x+y=9	1	2	4	5	6	9	27	
	5x=y	0	0	0	2	3	5	10	
	TOTAL No OF 'D' Responses	3	5	9	13	16	24		70

Table 18 (b): Number of each type of response to Subtasks of the ET in each year group - School B.

II.4. The Literal Number Task

Responses to each Subtask of the LNT were divided into four categories in an ascending order of algebraic sophistication:

A	False ordering responses without correction;
B	False ordering responses with correction;
C	Numerical replacement responses;
D	(For Subtasks 1 and 2 only): Algebraic response;
Di	(For Subtask 3): Algebraic response with an incorrect relation stated;
Dii	(For Subtask 3): Algebraic response;

Each level is defined below by illustration.

II.4.1. Type A Responses: False ordering without correction

The pupil suggests that one number is greater than another and retains this expectation throughout, despite suggesting possible identities for letters which contradict this statement.

Example 1

Gareth (3:8) 13 yrs. 7 mnths. School B. 'm + m, m + k'.

Q. Which is larger, m + m or m + k?

A. m + k.

Q. Why?

A. Say m equals 1 and k equals 2 then 2m is 2 and m + k is 3. So m + k is larger.

Q. When is m + m larger than m + k?

A. If m was more than 2 it would be larger.

Q. When is m + k larger than m + m?

A. When m is less than 2.

Q. When are they equal?

A. Well, it can't be because m can't be equal to k or it would be itself.

Q. So which is larger, m + k or m + m?

A. m + k.

Q. And why is that?

A. Because there's more numbers for m + k to be larger.

Example 2

The pupil assumes an ordering and retains it throughout.

Katherine (4:10) 15yrs. 1mth. School B. 't + t, t + 4'

Q. Which is larger, t + 4 or t + t?

A. t + 4.

Q. Why?

A. Because t + t is 2t and t + 4 is 4t.

Q. When is t + t larger than t + 4?

A. It can't be.

Q. When is t + t less than t + 4?

A. It is now.

Q. When are they equal?

A. They can't be.

Example 3

Peter (1:1) 11yrs. 6mths. School B. 'a + b + 3, a + c + 4'

Q. Which is larger, a + b + 3 or a + c + 4?

A. They're both the same.

Q. Why?

A. Because c is after b and that gives an extra one to add on.

Q. When is a + b + 3 larger?

A. When b is 6 and c is 4.

Q. When is a + c + 4 larger?

A. When c is 4 and b is 2.

Q. When are they equal?

A. When c is 4 and b is 2.

Q. So which is larger, a + b + 3 or a + c + 4?

A. They're the same. No! That (a + c + 4) because that's 4 and that's 3.

II.4.2. Type B : False ordering with correction

The pupil assumes an ordering at the outset, but reluctantly accepts any ordering may be possible after answering each question.

Example 1

Katy (4:5) 14yrs. 7mths. School B. 't + t, t + 4'.

Q. Which is larger, t + t or t + 4?

A. t + t.

Q. Why is that?

A. I just guessed.

Q. Can you think when t + t will be bigger?

A. No, not really.

Q. When is t + t larger than t + 4?

A. Well, you're just taking a guess on t, aren't you. I suppose it might be 6 + 6 and 6 + 4.

Q. When is t + 4 larger than t + t?

A. I suppose it's just t. When you remove one of the t's.

Q. When is t + 4 equal to t + t?

A. I suppose until I know what t is. . .you can't really tell until you know what t is.

Q. Which is larger, t + t or t + 4?

A. I suppose you don't know. It might be any.

Example 2

Alan (1:6) 12yrs. 5mths. School A. 't + t, t + 4'.

Q. Which is larger, t + t or t + 4?

A. 2t, No. t + 4.

Q. Why?

A. Well; it is when t is 2.

Q. When is t + t greater than t + 4?

A. When t is over 4.

Q. When is t + t less than t + 4?

A. When it's under 4.

Q. When is t + t equal to t + 4?

A. When it's 4.

Q. So which is larger, t + t or t + 4?

A. It depends on what t is, doesn't it.

II.4.3. Type C Responses : Numerical replacement

The pupil accepts that any ordering might be possible, but treats the letters as "empty boxes" for numerals. He often does not appreciate the formal rules of syntax i.e. 't' always takes a value identical to itself.

Example 1

Eleanor (4:11) 14yrs. 7mths. School B. 'a + b + 3, a + c + 4'.

Q. Which is larger, a + b + 3 or a + c + 4?

A. They could be the same.

Q. Why?

A. Because you could put numbers in.

Q. When is a + b + 3 larger?

A. When it's 3, 2, 3 and that (a + c + 4) is 1, 2, 4.

Q. When is a + c + 4 larger?

A. When it's 4, 5, 6 (for a + c + 4) and 3, 2, 1 (for a + b + 3)

Q. When are they equal?

A. When the three numbers add up to the same.

Example 2

Kay (4:12) 15yrs. 3mths. School B. 'm + m, m + k'.

Q. Which is larger m + m or m + k?

A. They could be anything at the moment.

Q. When is m + k larger than m + m?

A. If k is 5 and m is 2.

Q. When is m + m larger than m + k?

A. If k is 3 and m is 7 they would be.

Q. When are they equal?

A. If k is 5 and m is 5.

II.4.4. Type D Responses : (D(i) for 'a + b + 3, a + c + 4') : Algebraic

The pupil accepts any ordering is possible (but does not give the correct relationship obtaining between the letters in 'a + b + 3, a + c + 4').

Example 1 (D(i)-type)

Jonathan (4:9) 14yrs. 11mths. School B. LNT.

'a + b + 3, a + c + 4'

Q. Which is larger, a + b + 3 or a + c + 4?

A. Any of them could be.

Q. When is a + b + 3 larger than a + c + 4?

A. When b is greater than c by 1.

Q. When is a + c + 4 greater than a + b + 3?

A. When b is equal or less than c.

Q. When are they equal?

A. When b is 1 more than c.

<u>Example 2</u>

Stephen (4:2) 14yrs. 9mths. School B. 't + t, t + 4'.

Q. Which is larger, t + t or t + 4?

A. Any is possible. It depends upon t.

Q. When is t + t larger?

A. When t is larger than 4.

Q. When is t + 4 larger?

A. When t is less than 4.

Q. When are they equal?

A. When t equals 4.

<u>Example 3</u>

Michael (4:1) 14yrs. 7mths. School A. 'm + m, m + k'.

Q. Which is larger, m + m or m + k?

A. Any is possible. It depends upon m and k.

Q. When is m + m larger?

A. When m is greater than k.

Q. When is m + k larger?

A. When m is less than k.

Q. When are they equal?

A. When m equals k.

<u>D(ii) Type (for 'a + b + 3, a + c + 4')</u>. The pupil gives the correct relationships throughout.

<u>Example</u>

Michael (4:1) 14yrs. 7mths. School A. 'a + b + 3, a + c + 4'

Q. Which is larger, a + b + 3 or a + c + 4?

A. It depends upon b and c.

Q. When is a + b + 3 larger?

A. When b is more than 1 more than c.

Q. When is a + c + 4 larger?

A. When b + 3 is less than c + 4.

Q. When are they equal?

A. When b plus one equals c.

Tables 19(a) and 19(b) indicate each pupils' response type for each Subtask. Table 20(a) and 20(b) give the number of responses of each type in each year group for each school.

Pupil	Response to LNT 1	LNT 2	LNT 3
1:1	D	D	Dii
1:2	D	D	Di
1:3	D	D	Di
1:4	C	C	C
1:5	C	C	C
1:6	B	B	C
1:7	A	B	B
1:8	C	C	C
1:9	A	B	A
1:10	B	B	C
1:11	A	A	A
1:12	A	A	A

YEAR 1

Pupil	Response to LNT 1	LNT 2	LNT 3
2:1	D	D	Dii
2:2	D	D	Di
2:3	D	D	Di
2:4	C	C	C
2:5	D	D	C
2:6	D	D	Di
2:7	C	C	C
2:8	C	C	B
2:9	D	D	Di
2:10	C	C	B
2:11	A	A	A
2:12	B	B	A

YEAR 2

Pupil	Response to LNT 1	LNT 2	LNT 3
3:1	D	D	Dii
3:2	D	D	Di
3:3	D	D	Di
3:4	D	D	Di
3:5	D	C	C
3:6	D	D	C
3:7	B	B	C
3:8	C	C	C
3:9	D	D	B
3:10	A	A	A
3:11	B	B	B
3:12	A	A	A

YEAR 3

Pupil	Response to LNT 1	LNT 2	LNT 3
4:1	D	D	Dii
4:2	D	D	Di
4:3	D	D	Dii
4:4	D	D	Di
4:5	C	C	C
4:6	C	C	C
4:7	B	B	A
4:8	C	C	C
4:9	B	C	C
4:10	A	A	A
4:11	C	C	C
4:12	A	B	A

YEAR 4

Pupil	Response to LNT 1	LNT 2	LNT 3
5:1	D	D	Dii
5:2	D	D	Dii
5:3	D	D	Di
5:4	D	D	Dii
5:5	D	D	Dii
5:6	D	D	Di
5:7	D	D	C
5:8	B	A	A
5:9	D	D	Dii
5:10	A	A	B
5:11	D	D	Di
5:12	A	A	A

YEAR 5

Pupil	Response to LNT 1	LNT 2	LNT 3
6:1	D	D	Dii
6:2	D	D	Dii
6:3	D	D	Dii
6:4	D	D	Dii
6:5	D	D	Dii
6:6	D	D	Dii
6:7	D	D	Dii
6:8	D	D	Dii
6:9	D	D	Dii
6:10	D	D	Dii
6:11	D	D	Di
6:12	D	D	Di

YEAR 6

Table 19(a): Response-types of each pupil to the Literal Number Task - School A

Pupil	LNT 1	LNT 2	LNT 3
1:1	D	D	Di
1:2	C	C	C
1:3	D	D	Di
1:4	D	D	Di
1:5	C	C	C
1:6	C	C	C
1:7	C	C	C
1:8	B	B	C
1:9	A	B	B
1:10	A	A	A
1:11	A	A	A
1:12	A	A	A

YEAR 1

Pupil	LNT 1	LNT 2	LNT 3
2:1	D	D	Dii
2:2	D	D	Di
2:3	D	D	Di
2:4	D	D	Di
2:5	A	A	A
2:6	D	D	Di
2:7	D	D	Di
2:8	B	D	C
2:9	C	C	B
2:10	D	B	C
2:11	B	B	A
2:12	A	A	A

YEAR 2

Pupil	LNT 1	LNT 2	LNT 3
3:1	D	D	Di
3:2	D	D	Di
3:3	D	D	Di
3:4	D	D	Di
3:5	D	C	Di
3:6	D	C	C
3:7	D	D	C
3:8	A	A	A
3:9	D	D	Di
3:10	D	C	C
3:11	B	B	C
3:12	A	A	A

YEAR 3

Pupil	LNT 1	LNT 2	LNT 3
4:1	D	D	Di
4:2	D	D	Dii
4:3	D	D	Di
4:4	D	D	Dii
4:5	B	B	B
4:6	D	D	Di
4:7	B	A	A
4:8	C	C	C
4:9	D	D	Di
4:10	A	C	A
4:11	B	C	C
4:12	C	C	C

YEAR 4

Pupil	LNT 1	LNT 2	LNT 3
5:1	D	D	Dii
5:2	D	D	Dii
5:3	D	D	Di
5:4	D	D	Dii
5:5	D	D	Dii
5:6	A	A	A
5:7	C	D	C
5:8	D	D	Di
5:9	D	D	Di
5:10	A	A	A
5:11	B	C	C
5:12	A	A	A

YEAR 5

Pupil	LNT 1	LNT 2	LNT 3
6:1	D	D	Dii
6:2	D	D	Dii
6:3	D	D	Dii
6:4	D	D	Dii
6:5	D	D	Dii
6:6	D	D	Dii
6:7	D	D	Dii
6:8	D	D	Di
6:9	D	D	Dii
6:10	D	D	Dii
6:11	D	D	Di
6:12	D	D	C

YEAR 6

Table 19(b): Response-types of each pupil to the Literal Number Task - School B

Response Type	Subtask	1	2	3	4	5	6	TOTALS	
A	t+t, t+4	4	1	2	2	2	0	11	
	m+m, m+k	2	1	2	1	3	0	9	
	a+b, a+c+4	3	2	2	3	2	0	12	
	TOTAL No OF 'A' RESPONSES	9	4	6	6	7	0		32
B	t+t, t+4	2	1	2	2	1	0	8	
	m+m, m+k	4	1	2	2	0	0	9	
	a+b+3, a+c+4	1	2	2	0	1	0	6	
	TOTAL No OF 'B' RESPONSES	7	4	6	4	2	0		23
C	t+t, t+4	3	4	0	3	0	0	10	
	m+m, m+k	3	4	1	4	0	0	12	
	a+b+3, a+c+4	5	3	4	4	1	0	17	
	TOTAL No OF 'C' RESPONSES	11	11	5	11	1	0		39
D	t+t, t+4	3	6	8	5	9	12	43	
	m+m, m+k	3	6	7	5	9	12	42	
	a+b+3 a+c+4	2	4	3	3	4	2	18	
	TOTAL No OF 'D' RESPONSES	8	16	18	13	22	26		103
E	t+t, t+4	-	--	--	--	--	--	--	
	m+m, m+k	-	--	--	--	--	--	--	
	a+b+3, a+c+4	1	1	1	2	4	10	19	
	TOTAL No OF 'E' RESPONSES	1	1	1	2	4	10		19

Table 20 (a): Number of each type of response to Subtasks of the L N T in each year group - School A.

Response Type	Subtask	1	2	3	4	5	6	TOTALS	
A	t+t, t+4	4	2	2	1	3	0	12	
	m+m, m+k	3	2	2	1	3	0	11	
	a+b+3, a+c+4	3	3	2	2	3	0	13	
	TOTAL No OF 'A' RESPONSES	10	7	6	4	9	0		36
B	t+t, t+4	1	2	1	3	1	0	8	
	m+m, m+k	2	1	1	1	0	0	5	
	a+b+3, a+c+4	1	1	0	1	0	0	3	
	TOTAL No OF 'B' RESPONSES	4	4	2	5	1	0		16
C	t+t, t+4	4	1	0	2	1	0	8	
	m+m, m+k	4	1	3	4	1	0	13	
	a+b+3, a+c+4	5	2	4	3	2	0	17	
	TOTAL No OF 'C' RESPONSES	13	4	7	9	4	1		38
D	t+t, t+4	3	7	9	6	7	12	44	
	m+m, m+k	3	8	6	6	8	12	43	
	a+b+3, a+c+4	3	5	6	4	4	2	24	
	TOTAL No OF 'D' RESPONSES	9	20	21	16	19	26		111
E	t+t, t+4	-	--	--	--	--	--		
	m+m, m+k	-	--	--	--	--	--		
	a+b+3, a+c+4	0	1	0	2	3	9	15	
	TOTAL No OF 'E' RESPONSES	0	1	0	2	3	9		15

Table 20 (b): Number of each type of response to Subtasks of the LNT in each year group - School B.

II.5. The Zetetic Task

Responses to this task were divided into 4 distinct types, of which Type A comprises four different sub-levels. They are presented below in an increasing hierarchy of algebraic sophistication.

II.5.1. Type A(i) Response : Misinterpretation/No headway

The pupil did not make any headway and said so, or proclaimed the problem "impossible" (generally because the sum and difference could not be known). His answer card was generally left blank, or he tentatively wrote down a number and then stopped. Some pupils misinterpreted the question, despite repeated attempts to explain what was required. The following is an example:

Example of A(i) type solution to the ZT

Andrew (2:11) 13yrs. 0mths. School A.

What is the difference between 10 and 13?

10 + 3 = 13 ∴ the difference between 10 and 13 = 3

The sum of 10 and 13 = 23

II.5.2. Type A(ii): Responses : spurious generality

The pupil wrote down an example of two numbers, their difference and their sum. This special case was then used to arrive at a "general" result which was, however, spurious for other choices of the sum and difference:

Example of A(ii) type solution to the ZT

Michael (2:6) 12yrs. 6mths. School A.

10, 5	Difference 5	sum - difference = one number
	Sum 15	From this you can find the other number.

II.5.3. Type A(iii) Responses : Numerical statements

The pupil wrote down two numerical expressions, one for the sum and one for the difference but did not draw a conclusion.

Example of A(iii) type solution to the ZT.

Jeremy (2:2) 13yrs. 4mths. School A.

7 - 3 = 4	8 - 2 = 6
7 + 3 = 10	8 + 2 = 10

II.5.4. Type A(iv) Responses : Trial and Error

The pupil wrote down two numerical expressions, one for the sum and one for the difference, and then suggested a process of trial and error.

Example of Type A(iv) solution to the Zetetic Task

John (2:4) 13yrs. 5mths. School A.

6 - 4 = 2

6 + 4 = 10

You can go through all the possibilities until you find the numbers which give the difference of 2.

2	What two numbers add to ten?
10	Work through and you will
2 - 8	find the exact difference of
1 - 9	2.
3 - 7	
4 - 6	

Types B, C, and D are distinguished from Type A by their legitimate attempts at a generalisation.

II.5.5. Type B Responses : Rhetorical

The pupil states (rhetorically) a means of obtaining each number (e.g. "add the sum and difference together and divide by 2. Then subtract this from the sum to give you the other number") and may support this with a particular example.

<u>Example of Type B solution to the ZT</u>

Matthew (3:3) 14yrs. 6mths. School A.

The sum of two numbers e.g. 5, 3 is 8;

The difference is 2;

8 ÷ 2 = 4. If you halve the difference and so get 1, you add 1 to 4 = 5 and subtract 1 from 4 to give 3.

e.g. 24 is the sum, 6 is the difference;

24 ÷ 2 = 12 12 - 3 = 9 12 + 3 = 15

The two numbers are 9 and 15.

II.5.6. <u>Type C Responses : Diophantine</u>

The pupil uses letters for unknowns and writes down two simultaneous equations representing a particular sum and a particular difference. He may then suggest the method is general.

<u>Example of Type C solution to the ZT</u>

Barry (3:2) 13yrs. 10mths. School A.

$x - y = 2. . . .(1)$

$x + y = 8. . . .(2)$

(1) + (2) $2x = 10$

$\underline{x = 5}$

Substitute into (2)

$5 + y = 8$

$y = 8 - 5$

$\underline{y = 3}$

You can do this for any numbers.

II.5.7. Type D Responses: Vietan

The pupil utilises letters for the 'sum', 'difference', and each number.

Example of Type D solution to the ZT

Paul (5:2) 15yrs. 6mths. School A.

Let nos = x and y

n = sum of x and y

m = difference of x and y

General equations $n = x + y$

$m = x - y$

Add together

$m + n = 2x$

∴ find x and substitute back for y.

Tables 21 (a) and 21(b) give each pupil's response type. Tables 22(a) and 22(b) give the total number of each type of response in each year for each school.

Section II.5.8 below describes the statistical analysis applied to the data to generate the three 'levels of algebraic activity' illustrated in Chapters 5, 6, and 7.

Pupil	Response type
1:1	B
1:2	Aiii
1:3	B
1:4	Ai
1:5	Ai
1:6	Ai
1:7	Ai
1:8	Aiii
1:9	Ai
1:10	Aii
1:11	Aii
1:12	Ai

YEAR 1

Pupil	Response type
2:1	D
2:2	Aiii
2:3	C
2:4	Aiv
2:5	B
2:6	Aii
2:7	Aiii
2:8	Ai
2:9	Ai
2:10	Ai
2:11	Ai
2:12	Aiii

YEAR 2

Pupil	Response type
3:1	C
3:2	C
3:3	B
3:4	Aiii
3:5	Aiii
3:6	Ai
3:7	Aiv
3:8	Ai
3:9	Aiii
3:10	Ai
3:11	Ai
3:12	Ai

YEAR 3

Pupil	Response type
4:1	C
4:2	Aiv
4:3	C
4:4	C
4:5	Aiii
4:6	Aiv
4:7	Ai
4:8	Aiii
4:9	Aii
4:10	Ai
4:11	Aiv
4:12	Ai

YEAR 4

Pupil	Response type
5:1	D
5:2	D
5:3	D
5:4	Aiii
5:5	Ai
5:6	Ai
5:7	C
5:8	Ai
5:9	C
5:10	Aiii
5:11	C
5:12	Ai

YEAR 5

Pupil	Response type
6:1	D
6:2	D
6:3	D
6:4	D
6:5	D
6:6	D
6:7	C
6:8	D
6:9	C
6:10	D
6:11	D
6:12	D

YEAR 6

Table 21(a): Response-type of each pupil to the Zetetic Task - School A

Pupil	Response type
1:1	B
1:2	Aiv
1:3	Ai
1:4	B
1:5	Ai
1:6	Ai
1:7	Ai
1:8	Ai
1:9	Ai
1:10	Ai
1:11	Ai
1:12	Ai

YEAR 1

Pupil	Response type
2:1	B
2:2	B
2:3	Aiv
2:4	Aiii
2:5	Ai
2:6	B
2:7	Aiv
2:8	Aii
2:9	Aii
2:10	Ai
2:11	Aiv
2:12	Ai

YEAR 2

Pupil	Response type
3.1	C
3:2	B
3:3	B
3:4	Aiv
3:5	Ai
3:6	Ai
3:7	B
3:8	Aiv
3:9	Aii
3:10	Aiii
3:11	Aiv
3:12	Aiv

YEAR 3

Pupil	Response type
4:1	Ai
4:2	D
4:3	C
4:4	C
4:5	Aii
4:6	Aiv
4:7	Aiii
4:8	Aiv
4:9	B
4:10	Ai
4:11	Ai
4:12	Ai

YEAR 4

Pupil	Response type
5:1	D
5:2	D
5:3	D
5:4	C
5:5	D
5:6	Ai
5:7	Ai
5:8	C
5:9	B
5:10	Aii
5:11	B
5:12	Aii

YEAR 5

Pupil	Response type
6:1	D
6:2	D
6:3	D
6:4	D
6:5	D
6:6	D
6:7	D
6:8	C
6:9	D
6:10	D
6:11	C
6:12	D

YEAR 6

Table 21(b): Response-type of each pupil to the Zetetic Task - School B

Response Type	1	2	3	4	5	6	Totals
A	10	9	9	9	6	0	43
B	2	1	1	0	0	0	4
C	0	1	2	3	3	2	11
D	0	1	0	0	3	10	14

Table 22(a) Number of each type of response to the ZT in each year group - School A.

Response Type	1	2	3	4	5	6	Totals
A	10	9	8	8	4	0	39
B	2	3	3	1	3	0	12
C	0	0	1	2	2	2	7
D	0	0	0	1	3	10	14

Table 22(b) Number of each type of response to the ZT in each year group - School B.

II. 5. 8. Correlations between Task-pairs

For the PLT there are three categorical variables "F", "T" and "A"; for the ET and ZT there are four: "A", "B", "C" and "D", and for the LNT there are five: "A", "B", "C", "D(i)" and "D(ii)".

The hypothesis to be tested here is that these variables are related across tasks in the same way in each school - that is, if a "F" classification to the PLT demands an equivalent cognitive functioning to a "D" response to "5x = y" for pupils in School A, then this is also true in School B. In this sense a cross-comparison of equivalent levels of response will indicate the consistency of the battery of tasks independently of the population tested. (This method also serves to indicate three levels of algebraic activity, as described below.)

The statistic chosen to achieve this is the phi (ϕ) fourfold point correlation coefficient, usually applied to data categorised in terms of a pass/fail dichotomy.

If information consists of numbers of passes and fails for the same pupils in two tests, the data can be set out as a fourfold contingency table as follows:

		Test 1		
		Pass	Fail	Total
Test 2	Pass	a	b	a + b
	Fail	c	d	c + d
	Total	a + c	b + d	

is defined by

$$= \frac{ad - bc}{\sqrt{(a+b)\ (c+d)\ (a+c)\ (b+d)}}$$

and is related to χ^2 on a 2x2 contingency table by $\chi^2 = N\phi^2$

The greater the numbers 'a' and 'd' (those with the same result ('pass' or 'fail') on each test) relative to 'b' and 'c' (those who pass on one test but fail on the other) the greater is ϕ.

Choosing for example, the PLT and ET as two tests, "pass" and "fail" levels can be set arbitrarily for each.

For example, the "pass" level for Subtask 1 of the ET can be set at response-type 'A' (in which case all pupils would "pass"); response type 'B' (in which case only pupils giving a type 'A' response would fail), response type C; or response type D (in which case pupils giving 'A', 'B' or 'C' type responses would fail). Equally, for the PLT the "pass" level can be set at "F", "T", or "A".

For each choice of "pass" level (e.g. "T" on the PLT and "B" on Subtask 1 of the ET), ϕ can be calculated. For this particular pair of tasks, this process will generate six non-zero coefficients, one for each of the following contingency tables:

ET1		"Pass"	"Fail"
		B, C or D	A
"Pass"	T or F		
"Fail"	A		

		"Pass"	"Fail"
		C, D	A, B
"Pass"	TF		
"Fail"	A		

		"Pass"	"Fail"
		D	A, B, or C
"Pass	TF		
"Fail"	A		

	"Pass" B, C, or D	"Fail" A
"Pass" F		
"Fail" T or A		

	"Pass" C, D.	"Fail" A, B
"Pass" F		
"Fail" T, A		

	"Pass" D	"Fail" A, B, C
"Pass" F		
"Fail" T, A		

The <u>maximum</u> of the six values of ϕ will indicate that the associated "pass" level for each task makes an equivalent cognitive demand since this value will be correlated with the maximum value of 'ad' relative to 'bc' i.e. with the maximum number of pupils passing or failing each task relative to those passing one and failing the other.

This procedure can be followed for each pairing of categorical variables - e.g. the PLT and Subtask 2 of the LNT, Subtask 1 of the ET and Subtask 3 of the

LNT etc. This gives 7 + 6 + 5 + 4 + 3 + 2 + 1 = 28 task pairs and 251 ϕ coefficients for each school.

The coefficients are given in Table 23: 1 - 28 (School A); Table 24: 29 - 56 (School B), and Table 25: 56 - 84 (School A + School B).

	Task-pair	ϕmax*	Level at which ϕmax occurs		
1.	PLT/ET1	0.615071	T(PLT)	-	D(ET1)
2.	PLT/ET2	0.733799	T(PLT)	-	C(ET2)
3.	PLT/ET3	0.597148	T(PLT)	-	C(ET3)
4.	PLT/LNT1	0.655115	T(PLT)	-	D(LNT1)
5.	PLT/LNT2	0.674200	T(PLT)	-	D(LNT2)
6.	PLT/LNT3	0.718862	F(PLT)	-	Dii(LNT3)
7.	PLT/ZT	0.669249	F(PLT)	-	D(ZT)
8.	ET1/ET2	0.835937	C(ET1)	-	C(ET2)
9.	ET1/ET3	0.611333	D(ET1)	-	C(ET3)
10.	ET1/LNT1	0.653569	C(ET1)	-	D(LNT1)
11.	ET1/LNT2	0.630144	D(ET1)	-	D(LNT2)
12.	ET1/LNT3	0.697333	D(ET1)	-	Di(LNT3)
13.	ET1/ZT	0.611333	D(ET1)	-	B(ZT)
14.	ET2/ET3	0.664348	C(ET2)	-	C(ET3)
15.	ET2/LNT1	0.722268	C(ET2)	-	D(LNT1)
16.	ET2/LNT2	0.692618	C(ET2)	-	D(LNT2)
17.	ET2/LNT3	0.688875	B(ET2)	-	C(LNT3)
18.	ET2/ZT	0.632340	D(ET2)	-	C(ZT)
19.	ET3/LNT1	0.694066	C(ET3)	-	D(LNT1)
20.	ET3/LNT2	0.714286	C(ET3)	-	D(LNT2)
21.	ET3/LNT3	0.652899	C(ET3)	-	Di(LNT3)
22.	ET3/ZT	0.600000	C(ET3)	-	B(ZT)
23.	LNT1/LNT2	0.971692	D(LNT1)	-	D(LNT2)
24.	LNT1/LNT3	0.844368	D(LNT1)	-	Di(LNT3)
25.	LNT1/ZT	0.636626	D(LNT1)	-	B(ZT)
26.	LNT2/LNT3	0.868966	D(LNT2)	-	Di(LNT3)
27.	LNT2/ZT	0.657143	D(LNT2)	-	B(ZT)
28.	LNT3/ZT	0.652899	Di(LNT3)	-	B(ZT)

Table 23: Maximum ϕ coefficients for each task-pair - School A.

* Extracted from Column 1 of the listing of all coefficients, pp 369-372

	Task-pair	ϕmax*	Level at which ϕ max occurs
29.	PLT/ET1	0.505669	T(PLT) - D(ET1)
30.	PLT/ET2	0.525626	T(PLT) - C(ET2)
31.	PLT/ET3	0.553325	T(PLT) - C(ET3)
32.	PLT/LNT1	0.546536	T(PLT) - D(LNT1)
33.	PLT/LNT2	0.562640	T(PLT) - D(LNT2)
34.	PLT/LNT3	0.630218	T(PLT) - Di(LNT3)
35.	PLT/ZT	0.709527	T(PLT) - C(ZT)
36.	ET1/ET2	0.685409	C(ET1) - C(ET2)
37.	ET1/ET3	0.597026	C(ET1) - C(ET3)
38.	ET1/LNT1	0.680932	C(ET1) - D(LNT1)
39.	ET1/LNT2	0.653569	C(ET1) - D(LNT2)
40.	ET1/LNT3	0.722268	C(ET1) - Di(LNT3)
41.	ET1/ZT	0.606633	C(ET1) - B(ZT)
42.	ET2/ET3	0.592190	C(ET2) - C(ET3)
43.	ET2/LNT1	0.547855	C(ET2) - D(LNT1)
44.	ET2/LNT2	0.515434	C(ET2) - D(LNT2)
45.	ET2/LNT3	0.688875	B(ET2) - C(LNT3)
46.	ET2/ZT	0.471589	C(ET2) - B(ZT)
47.	ET3/LNT1	0.519481	C(ET3) - D(LNT1)
48.	ET3/LNT2	0.538937	C(ET3) - D(LNT2)
49.	ET3/LNT3	0.585101	D(ET3) - Dii(LNT3)
50.	ET3/ZT	0.615071	C(ET3) - C(ZT)
51.	LNT1/LNT2	0.861892	C(LNT1) - C(LNT2)
52.	LNT1/LNT3	0.867217	D(LNT1) - Di(LNT3)
53.	LNT1/ZT	0.640435	D(LNT1) - B(ZT)
54.	LNT2/LNT3	0.864334	B(LNT2) - B(LNT3)
55.	LNT2/ZT	0.663357	D(LNT2) - B(ZT)
56.	LNT3/ZT	0.698493	Dii(LNT3)- D(ZT)

Table 24: (cont.): Maximum ϕ coefficients for each task-pair - School B.

* Extracted from Column 2 of the listing of all coefficients, pp 369-372

	Task-pair	ϕ max*	Level at which ϕ max occurs
57.	PLT/ET1	0.560988	T(PLT) - D(ET1)
58.	PLT/ET2	0.633084	T(PLT) - C(ET2)
59.	PLT/ET3	0.576074	T(PLT) - C(ET3)
60.	PLT/LNT1	0.599406	T(PLT) - D(LNT1)
61.	PLT/LNT2	0.616964	T(PLT) - D(LNT2)
62.	PLT/LNT3	0.671387	T(PLT) - Di(LNT3)
63.	PLT/ZT	0.661277	T(PLT) - C(ZT)
64.	ET1/ET2	0.759380	C(ET1) - C(ET2)
65.	ET1/ET3	0.587811	C(ET1) - C(ET3)
66.	ET1/LNT1	0.667139	C(ET1) - D(LNT1)
67.	ET1/LNT2	0.640210	C(ET1) - D(LNT2)
68.	ET1/LNT3	0.669660	C(ET1) - Di(LNT3)
69.	ET1/ZT	0.566596	D(ET1) - B(ZT)
70.	ET2/ET3	0.628751	C(ET2) - C(ET3)
71.	ET2/LNT1	0.633349	C(ET2) - D(LNT1)
72.	ET2/LNT2	0.602335	C(ET2) - D(LNT2)
73.	ET2/LNT3	0.610746	C(ET2) - Di(LNT3)
74.	ET2/ZT	0.525037	D(ET2) - C(ZT)
75.	ET3/LNT1	0.606819	C(ET3) - D(LNT1)
76.	ET3/LNT2	0.626614	C(ET3) - D(LNT2)
77.	ET3/LNT3	0.606633	C(ET3) - Di(LNT3)
78.	ET3/ZT	0.604721	C(ET3) - B(ZT)
79.	LNT1/LNT2	0.913790	D(LNT1) - D(LNT2)
80.	LNT1/LNT3	0.855717	D(LNT1) - Di(LNT3)
81.	LNT1/ZT	0.638240	D(LNT1) - B(ZT)
82.	LNT2/LNT3	0.836017	B(LNT2) - B(LNT3)
83.	LNT2/ZT	0.659927	D(LNT2) - B(ZT)
84.	LNT3/ZT	0.650085	Di(LNT3)- B(ZT)

Table 25:(cont.): Maximum ϕ coefficients for each task-pair - School A + School B.

* Extracted from Column 3 of the listing of all coefficients, pp 369-372

Note: For 'B', 'C' -PLT- read 'T', 'F' respectively

PLT	ET1	B	B	0.282038	0.187160	0.233281
PLT	ET1	B	C	0.597026	0.501905	0.548936
PLT	ET1	B	D	0.615071	0.505669	0.560988
PLT	ET1	C	B	0.150137	0.103252	0.126514
PLT	ET1	C	C	0.348736	0.310396	0.329814
PLT	ET1	C	D	0.371607	0.410891	0.390280
PLT	ET2	B	B	0.391925	0.244155	0.318538
PLT	ET2	B	C	0.733799	0.525626	0.633084
PLT	ET2	B	D	0.615046	0.515308	0.561777
PLT	ET2	C	B	0.208632	0.068238	0.141480
PLT	ET2	C	C	0.390621	0.304589	0.350428
PLT	ET2	C	D	0.501164	0.401203	0,450578
PLT	ET3	B	B	0.343604	0.469161	0.409254
PLT	ET3	B	C	0.597148	0.553325	0.576074
PLT	ET3	B	D	0.503444	0.500056	0.500340
PLT	ET3	C	B	0.281681	0.292770	0.288430
PLT	ET3	C	C	0.345844	0.473804	0.407150
PLT	ET3	C	D	0.387599	0.333947	0.361290
PLT	LNT1	B	B	0.338754	0.306394	0.322860
PLT	LNT1	B	C	0.477630	0.424893	0.451317
PLT	LNT1	B	D	0.655115	0.546536	0.599406
PLT	LNT1	C	B	0.180328	0.169031	0.175096
PLT	LNT1	C	C	0.254255	0.234404	0.244761
PLT	LNT1	C	D	0.348736	0.301511	0.325074
PLT	LNT2	B	B	0.301511	0.290936	0.297405
PLT	LNT2	B	C	0.460566	0.366211	0.411706
PLT	LNT2	B	D	0.674200	0.562640	0.616964
PLT	LNT2	C	B	0.160503	0.160503	0.161290
PLT	LNT2	C	C	0.245172	0.202030	0.223279
PLT	LNT2	C	D	0.358895	0.310396	0.334596
PLT	LNT3	B	B	0.374454	0.336601	0.355740
PLT	LNT3	B	C	0.460566	0.380898	0.419628
PLT	LNT3	B	D	0.718862	0.630218	0.671387
PLT	LNT3	B	E	0.657952	0.602064	0.632652
PLT	LNT3	C	B	0.199332	0.185695	0.192927
PLT	LNT3	C	C	0.245172	0.210133	0.227575
PLT	LNT3	C	D	0.413014	0.347677	0.379885
PLT	LNT3	C	E	0.735516	0.529971	0.641007
PLT	ZT	B	B	0.597148	0.604964	0.594168
PLT	ZT	B	C	0.615046	0.709527	0.661277
PLT	ZT	B	D	0.543896	0.566577	0.553382
PLT	ZT	C	B	0.502453	0.399580	0.449013
PLT	ZT	C	C	0.582251	0.478634	0.534128
PLT	ZT	C	D	0.669249	0.557086	0.614487
ET1	ET2	B	B	0.607957	0.439931	0.531595
ET1	ET2	B	C	0.384353	0.265693	0.323894
ET1	ET2	B	D	0.257855	0.211604	0.236834
ET1	ET2	C	B	0.598253	0.497964	0.548681
ET1	ET2	C	C	0.835937	0.685409	0.759380
ET1	ET2	C	D	0.598943	0.519134	0.558452
ET1	ET2	D	B	0.464727	0.359320	0.412436
ET1	ET2	D	C	0.702599	0.611157	0.656545
ET1	ET2	D	D	0.712598	0.669342	0.690021
ET1	ET3	B	B	0.437061	0.239818	0.336343
ET1	ET3	B	C	0.298807	0.217922	0.258703
ET1	ET3	B	D	0.141990	0.109711	0.126514
ET1	ET3	C	B	0.500343	0.358276	0.426263
ET1	ET3	C	C	0.579186	0.597026	0.587811
ET1	ET3	C	D	0.329814	0.329814	0.329814
ET1	ET3	D	B	0.446256	0.309481	0.376082
ET1	ET3	D	C	0.611333	0.524142	0.568130
ET1	ET3	D	D	0.424577	0.194789	0.309764
ET1	LNT1	B	B	0.341219	0.464244	0.391819
ET1	LNT1	B	C	0.389949	0.318507	0.353303
				School A	School B	School A+School B

ET1	LNT1	B	D	0.340409	0.342448	0.339635
ET1	LNT1	C	B	0.438379	0.468580	0.453594
ET1	LNT1	C	C	0.472071	0.565504	0.519084
ET1	LNT1	C	D	0.653569	0.680932	0.667139
ET1	LNT1	D	B	0.324350	0.336581	0.330683
ET1	LNT1	D	C	0.376984	0.383775	0.380522
ET1	LNT1	D	D	0.606633	0.505083	0.555752
ET1	LNT2	B	B	0.133631	0.491442	0.293903
ET1	LNT2	B	C	0.408248	0.379653	0.395446
ET1	LNT2	B	D	0.328688	0.332646	0.328837
ET1	LNT2	C	B	0.288989	0.438379	0.366207
ET1	LNT2	C	C	0.506836	0.582765	0.543625
ET1	LNT2	C	D	0.627053	0.653569	0.640210
ET1	LNT2	D	B	0.189275	0.313143	0.253850
ET1	LNT2	D	C	0.417620	0.424641	0.420255
ET1	LNT2	D	D	0.630144	0.584919	0.607295
ET1	LNT3	B	B	0.408514	0.417981	0.407427
ET1	LNT3	B	C	0.510310	0.491365	0.499429
ET1	LNT3	B	D	0.363515	0.296977	0.333034
ET1	LNT3	B	E	0.204124	0.140138	0.171764
ET1	LNT3	C	B	0.424343	0.526702	0.476128
ET1	LNT3	C	C	0.572235	0.610304	0.590980
ET1	LNT3	C	D	0.617734	0.722268	0.669660
ET1	LNT3	C	E	0.474137	0.421282	0.447775
ET1	LNT3	D	B	0.299356	0.381502	0.341096
ET1	LNT3	D	C	0.417620	0.445773	0.431206
ET1	LNT3	D	D	0.697333	0.566434	0.631355
ET1	LNT3	D	E	0.546119	0.420403	0.485149
ET1	ZT	B	B	0.298807	0.258402	0.281761
ET1	ZT	B	C	0.257855	0.181207	0.219280
ET1	ZT	B	D	0.173702	0.134214	0.154770
ET1	ZT	C	B	0.521746	0.606633	0.563562
ET1	ZT	C	C	0.479981	0.483267	0.481056
ET1	ZT	C	D	0.403473	0.403473	0.403473
ET1	ZT	D	B	0.611333	0.525786	0.566596
ET1	ZT	D	C	0.595725	0.418547	0.508688
ET1	ZT	D	D	0.519400	0.393241	0.456337
ET2	ET3	B	B	0.588302	0.456828	0.518681
ET2	ET3	B	C	0.415227	0.300386	0.358230
ET2	ET3	B	D	0.197312	0.188517	0.192927
ET2	ET3	C	B	0.539572	0.294173	0.415845
ET2	ET3	C	C	0.664348	0.592190	0.628751
ET2	ET3	C	D	0.369427	0.412925	0.390605
ET2	ET3	D	B	0.420448	0.362963	0.387365
ET2	ET3	D	C	0.567081	0.559065	0.561727
ET2	ET3	D	D	0.550659	0.352564	0.450578
ET2	LNT1	B	B	0.474164	0.371403	0.421916
ET2	LNT1	B	C	0.502069	0.434428	0.467932
ET2	LNT1	B	D	0.526702	0.366224	0.447958
ET2	LNT1	C	B	0.461644	0.285830	0.372416
ET2	LNT1	C	C	0.587655	0.417055	0.501557
ET2	LNT1	C	D	0.722268	0.547855	0.633349
ET2	LNT1	D	B	0.309708	0.192450	0.248848
ET2	LNT1	D	C	0.370486	0.352282	0.360587
ET2	LNT1	D	D	0.539462	0.441367	0.490277
ET2	LNT2	B	B	0.344863	0.302483	0.321545
ET2	LNT2	B	C	0.526787	0.443911	0.487018
ET2	LNT2	B	D	0.510136	0.350722	0.431882
ET2	LNT2	C	B	0.410891	0.258536	0.332572
ET2	LNT2	C	C	0.627646	0.386193	0.506191
ET2	LNT2	C	D	0.692618	0.515434	0.602335
ET2	LNT2	D	B	0.275659	0.169450	0.218321
ET2	LNT2	D	C	0.421076	0.276026	0.349389
ET2	LNT2	D	D	0.557218	0.460640	0.508765
ET2	LNT3	B	B	0.499253	0.408019	0.452991
ET2	LNT3	B	C	0.688875	0.419203	0.557390
				School A	School B	School A + School B

ET2	LNT3	B	D	0.505146	0.365359	0.436579
ET2	LNT3	B	E	0.283654	0.151888	0.219597
ET2	LNT3	C	B	0.510295	0.337414	0.422724
ET2	LNT3	C	C	0.627646	0.409860	0.517837
ET2	LNT3	C	D	0.666928	0.560034	0.610746
ET2	LNT3	C	E	0.531085	0.390592	0.464192
ET2	LNT3	D	B	0.342347	0.235586	0.287074
ET2	LNT3	D	C	0.421076	0.295540	0.358605
ET2	LNT3	D	D	0.592603	0.482214	0.537378
ET2	LNT3	D	E	0.589506	0.450341	0.518855
ET2	ZT	B	B	0.415227	0.227028	0.322252
ET2	ZT	B	C	0.358320	0.232981	0.296444
ET2	ZT	B	D	0.241379	0.139385	0.191059
ET2	ZT	C	B	0.551268	0.471589	0.506416
ET2	ZT	C	C	0.495232	0.380386	0.439955
ET2	ZT	C	D	0.381502	0.364719	0.372521
ET2	ZT	D	B	0.567081	0.416631	0.491398
ET2	ZT	D	C	0.632340	0.420389	0.525037
ET2	ZT	D	D	0.599930	0.416806	0.507372
ET3	LNT1	B	B	0.556378	0.423390	0.487041
ET3	LNT1	B	C	0.560584	0.480384	0.518778
ET3	LNT1	B	D	0.500343	0.441367	0.467677
ET3	LNT1	C	B	0.358895	0.280306	0.319394
ET3	LNT1	C	C	0.506028	0.367512	0.436774
ET3	LNT1	C	D	0.694066	0.519481	0.606819
ET3	LNT1	D	B	0.170544	0.179605	0.175096
ET3	LNT1	D	C	0.240460	0.249068	0.244761
ET3	LNT1	D	D	0.329814	0.320374	0.325074
ET3	LNT2	B	B	0.387466	0.308997	0.347293
ET3	LNT2	B	C	0.591864	0.483046	0.532535
ET3	LNT2	B	D	0.479068	0.475264	0.474579
ET3	LNT2	C	B	0.319438	0.259565	0.288871
ET3	LNT2	C	C	0.487950	0.357873	0.423231
ET3	LNT2	C	D	0.714286	0.538937	0.626614
ET3	LNT2	D	B	0.151794	0.170544	0.161290
ET3	LNT2	D	C	0.231869	0.214669	0.223279
ET3	LNT2	D	D	0.339422	0.329814	0.334596
ET3	LNT3	B	B	0.550879	0.416806	0.481129
ET3	LNT3	B	C	0.522233	0.447532	0.481476
ET3	LNT3	B	D	0.440712	0.439031	0.436307
ET3	LNT3	B	E	0.382971	0.326718	0.356718
ET3	LNT3	C	B	0.396718	0.319939	0.358230
ET3	LNT3	C	C	0.487950	0.376418	0.432346
ET3	LNT3	C	D	0.652899	0.562262	0.606633
ET3	LNT3	C	E	0.553010	0.502761	0.529189
ET3	LNT3	D	B	0.188517	0.197312	0.192927
ET3	LNT3	D	C	0.231869	0.223279	0.227575
ET3	LNT3	D	D	0.390605	0.369427	0.379885
ET3	LNT3	D	E	0.602861	0.585101	0.593230
ET3	ZT	B	B	0.377139	0.445364	0.406394
ET3	ZT	B	C	0.357117	0.389249	0.374885
ET3	ZT	B	D	0.249711	0.308074	0.278780
ET3	ZT	C	B	0.600000	0.615071	0.604721
ET3	ZT	C	C	0.567081	0.522278	0.545644
ET3	ZT	C	D	0.438955	0.471909	0.455150
ET3	ZT	D	B	0.393730	0.424577	0.408602
ET3	ZT	D	C	0.466303	0.518265	0.491303
ET3	ZT	D	D	0.513012	0.513012	0.513012
LNT1	LNT2	B	B	0.773331	0.845960	0.811530
LNT1	LNT2	B	C	0.735516	0.747018	0.739568
LNT1	LNT2	B	D	0.502453	0.544566	0.523305
LNT1	LNT2	C	B	0.631266	0.684728	0.658971
LNT1	LNT2	C	C	0.964274	0.861892	0.912232
LNT1	LNT2	C	D	0.708440	0.691953	0.699735
LNT1	LNT2	D	B	0.460242	0.532327	0.496165
LNT1	LNT2	D	C	0.703031	0.670059	0.686855
				School A	School B	School A + School B

				School A	School B	School A + School B
LNT1	LNT2	D	D	0.971692	0.855199	0.913790
LNT1	LNT3	B	B	0.804298	0.816094	0.810443
LNT1	LNT3	B	C	0.735516	0.804400	0.769397
LNT1	LNT3	B	D	0.436615	0.486172	0.460918
LNT1	LNT3	B	E	0.245172	0.229416	0.237721
LNT1	LNT3	C	B	0.783983	0.713854	0.748190
LNT1	LNT3	C	C	0.745948	0.750428	0.747624
LNT1	LNT3	C	D	0.615610	0.674200	0.644303
LNT1	LNT3	C	E	0.345683	0.318142	0.332302
LNT1	LNT3	D	B	0.571585	0.615882	0.593487
LNT1	LNT3	D	C	0.637633	0.696932	0.666965
LNT1	LNT3	D	D	0.844368	0.867217	0.855717
LNT1	LNT3	D	E	0.474137	0.409224	0.441340
LNT1	ZT	B	B	0.280591	0.423022	0.351809
LNT1	ZT	B	C	0.309708	0.296648	0.303483
LNT1	ZT	B	D	0.208632	0.219718	0.214201
LNT1	ZT	C	B	0.442109	0.524512	0.482211
LNT1	ZT	C	C	0.436677	0.411377	0.424229
LNT1	ZT	C	D	0.294164	0.304694	0.299425
LNT1	ZT	D	B	0.636626	0.640435	0.638240
LNT1	ZT	D	C	0.598943	0.467302	0.533146
LNT1	ZT	D	D	0.403473	0.391925	0.397675
LNT2	LNT3	B	B	0.805203	0.864334	0.836017
LNT2	LNT3	B	C	0.654654	0.763815	0.708734
LNT2	LNT3	B	D	0.388613	0.461644	0.424577
LNT2	LNT3	B	E	0.218218	0.217841	0.218978
LNT2	LNT3	C	B	0.813029	0.750322	0.780276
LNT2	LNT3	C	C	0.777778	0.804115	0.790509
LNT2	LNT3	C	D	0.593617	0.581087	0.587754
LNT2	LNT3	C	E	0.333333	0.274204	0.303137
LNT2	LNT3	D	B	0.555405	0.598253	0.576597
LNT2	LNT3	D	C	0.618070	0.676984	0.647228
LNT2	LNT3	D	D	0.868966	0.835937	0.852497
LNT2	LNT3	D	E	0.487950	0.421282	0.454268
LNT2	ZT	B	B	0.234255	0.401679	0.318799
LNT2	ZT	B	C	0.275659	0.281681	0.279555
LNT2	ZT	B	D	0.185695	0.208632	0.197312
LNT2	ZT	C	B	0.422890	0.505608	0.464358
LNT2	ZT	C	C	0.421076	0.354562	0.386996
LNT2	ZT	C	D	0.283654	0.262613	0.273145
LNT2	ZT	D	B	0.657143	0.663357	0.659927
LNT2	ZT	D	C	0.616392	0.483267	0.549820
LNT2	ZT	D	D	0.415227	0.403473	0.409323
LNT3	ZT	B	B	0.323478	0.394432	0.358057
LNT3	ZT	B	C	0.342347	0.249711	0.296444
LNT3	ZT	B	D	0.230619	0.241379	0.236015
LNT3	ZT	C	B	0.422890	0.460377	0.441670
LNT3	ZT	C	C	0.421076	0.297786	0.359914
LNT3	ZT	C	D	0.283654	0.273145	0.278401
LNT3	ZT	D	B	0.652899	0.646764	0.650085
LNT3	ZT	D	C	0.650971	0.489145	0.569432
LNT3	ZT	D	D	0.477841	0.381502	0.429579
LNT3	ZT	E	B	0.553010	0.542326	0.543105
LNT3	ZT	E	C	0.589506	0.624875	0.607119
LNT3	ZT	E	D	0.526787	0.698493	0.608825

STOP --

II.6. Discussion

Figures 18 and 19 below show which response-types are connected by maximum phi-coefficients in each school. Tables 26 (a) and (b), (overpage), reflect these interconnections.

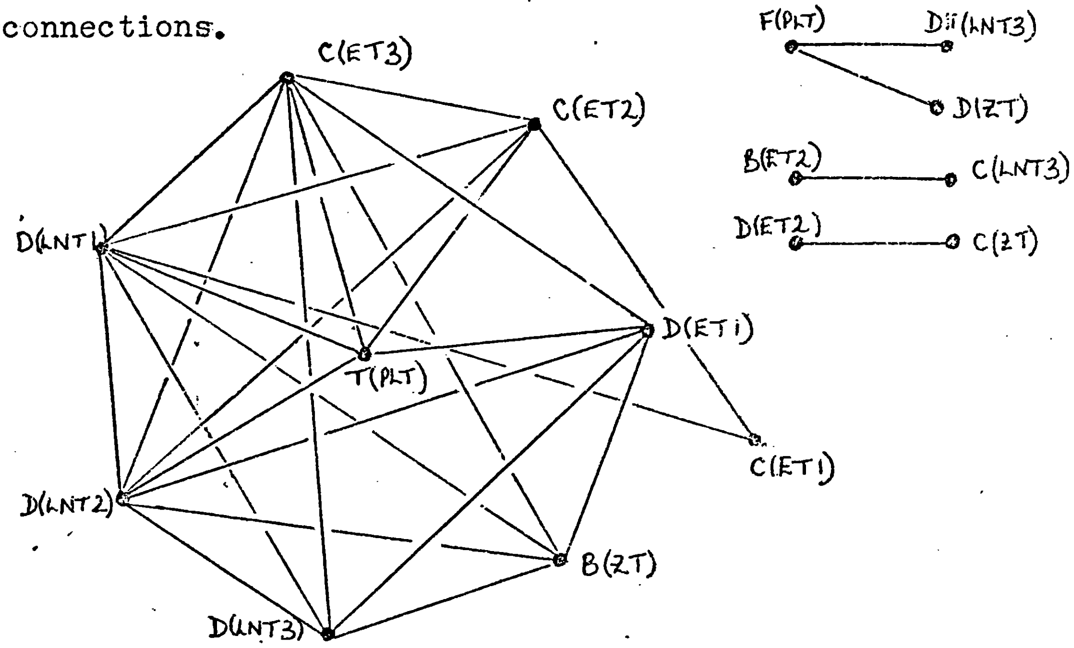


Fig. 18: Subtasks related by maximum phi-coefficients - School A.

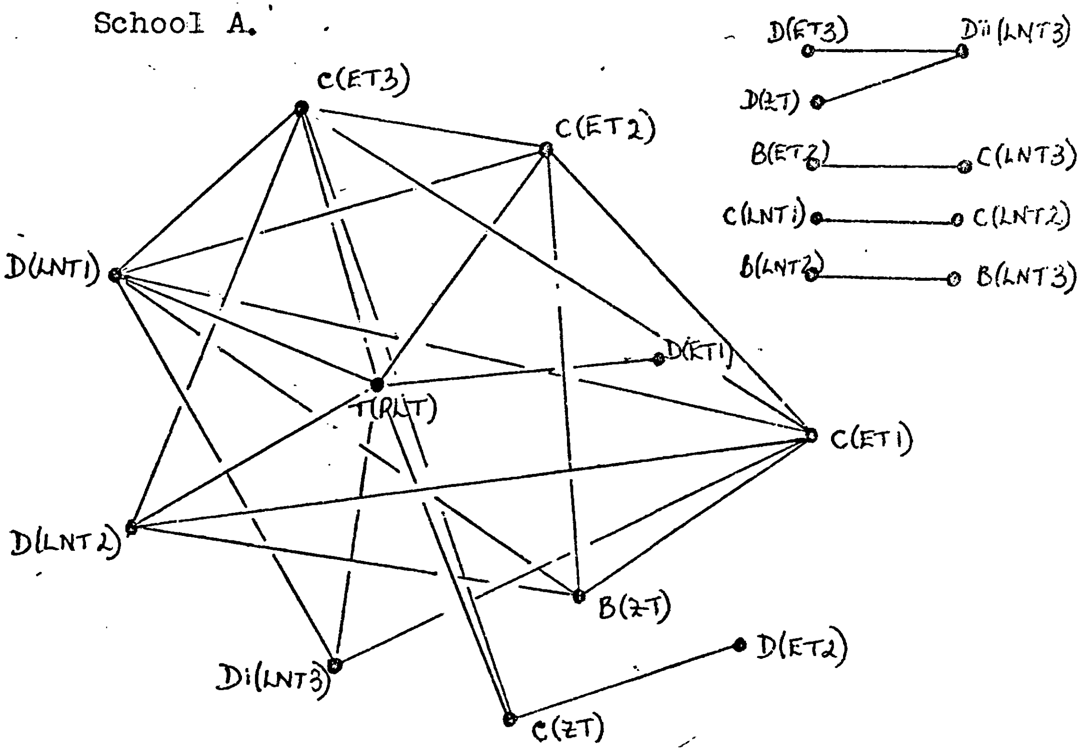


Fig. 19: Subtasks related by maximum phi-coefficients - School B.

:B(ET2): :C(LNT3):
T(PLT):C,D(ET1):C(ET2):C(ET3):D(LNT1):D(LNT2):Di(LNT3):B(ZT) :D(ET2): :C(ZT)
F(PLT): :Dii(LNT3);D(ZT)

Table 26(a): Response-types connected by maximum phi coefficients - School A.

:B(LNT2):B(LNT3): :C(LNT1):C(LNT2): :B(ET2): :C(LNT3):
T(PLT):C,D(ET1):C,D(ET2):C(ET3):D(LNT1):D(LNT2):Di(LNT3):B,C(ZT)
:D(ET3): :Dii(LNT3): D(ZT)

Table 26(b); Response-types connected by maximum phi coefficients - School B.

:B(LNT2):B(LNT3):
T(PLT:C,D(ET1):C,D(ET2):C(ET3):D(LNT1):D(LNT2):Di(LNT3):B,C(ZT)

Table 26(c) Response types connected by maximum phi coefficients - School A + School B.

Figures 18 and 19 show that discrepancies arise in two areas where maximum phi's are exhibited:

(a) in each school the role of D(ET 1) is shared by C(ET 1): and

(b) in school B the role of B(ZT) is shared by C(ZT).

This means that D(ET 1) is mediationally related to C(ET 1) in each school, and B(ZT) mediationally related to C(ZT) in School B, and that these response-types need to be treated as equivalent if total consistency is required. When these are treated as demanding an equivalent cognitive understanding then no inconsistency arises across the two populations. However, this decision involves accepting also that D(ET 2) is equivalent to C(ET 2), so giving rise to the following group of equivalent response-types:

C,D(ET 1) - C,D(ET 2) - C(ET 3) -
D(LNT 1) - D(LNT 2) - D(LNT 3) -
B,C(ZT)

This set of equivalences is exhibited when maximum phi's are considered for the total population:

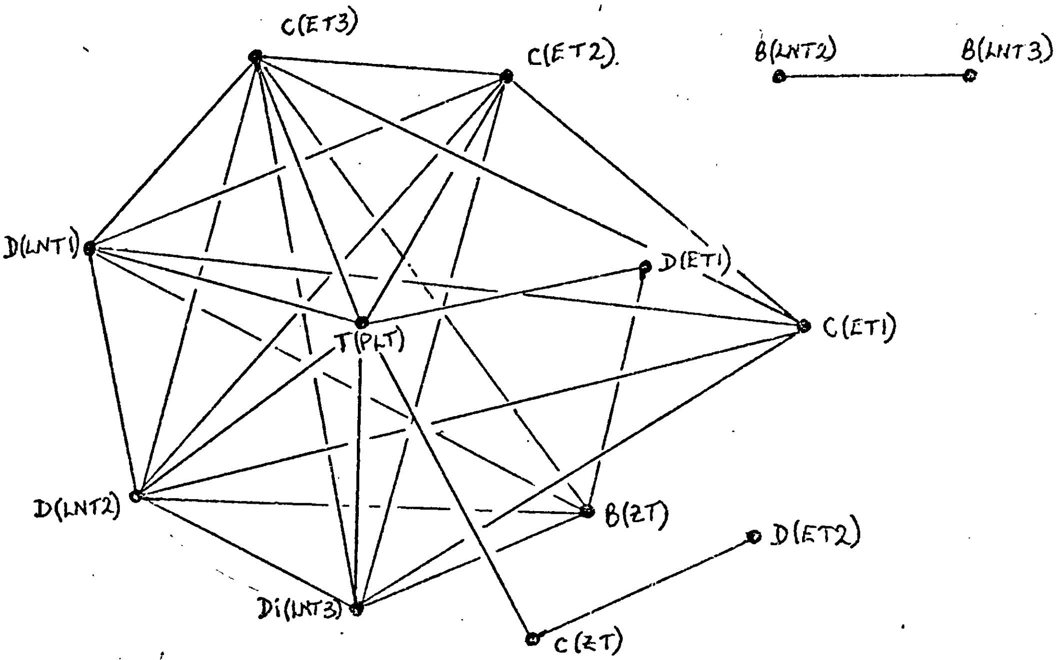


Fig. 20: Subtasks related by maximum phi-coefficients - School A + School B.

Here again D(ET 1) and C(ET 1) are mediationally connected, along with B(ZT) and D(ZT), and D(ET 2) and C(ET 2) - see Table 26(c), p.374.

Each response-type listed appears to be underpinned by a particular shared understanding, viz: in the ET and LNT it is impractical to name all possible indentifications for a letter which would satisfy a requested relation, and so the letter is used to refer to unspecified possibilities, (e.g. ': t is greater than 4', in LNT 1, or 'x is 6 or more' in ET 1). That is, there is no evidence of concrete variation in ET 1, ET 2, and in the LNT; and there is no indication of concrete variation or that the letter is considered to contain a unique content in '5x = y'. The pupil indicates he considers a variety of possible determinations for a letter. In the ZT the pupil demonstrates that he can make a meaningful attempt at a generalisation - but he does not introduce the letter as a 'given'.

This range of response-types is thus chosen to be definitive of the Level of Discovered Content (Chapter 6).

A similar process gives rise to two further levels of activity. The maximum phi-equivalences for School A identify F(PLT), D(ii)(LNT 3) and D(ZT) as mediationally connected, and for School B, D(ZT), D(ET 3) and D(ii)(LNT 3).

This produces:

F(PLT) _ D(ET 3) _ D(ii)(LNT 3) _
D(ZT)

as definitive of the Level of the Species (Chapter 7).

The remaining response-types are taken to be definitive of the Level of Fictitious Measures (Chapter 5):

A,B(ET 1) _ A,B(ET 2) _ A,B(ET 3) _
A,B,C(LNT 1) _ A,B,C(LNT 2) _
A,B,C(LNT 3) _ A(ZT) _ A(PLT).

II.6.1. Classification of Pupils

The nature of the tasks cause pupils to switch between different interpretations of the letter. Thus not all pupils respond consistently at a particular level.

Thus a pupil who responds at Level 11 in '$x + y = 10$' might be influenced by ordering and uniqueness suggestions in '$5x = y$' and so respond at Level I.

To indicate the Level at which each pupil most commonly responds and so to indicate how competence relates to age, Tables 27 (a) and (b) were produced by taking into account the number of 'excursions' made by each pupil into each group of response-types as follows:

The response-types for pupil (1:7) School A are:

A(PLT) _ A(ZT) _ A(ET 2) _ A(ET 3) _
A(LNT 1) _ B(LNT 2) _ B(LNT 3).
(See Appendix III).

This pupil responded consistently at Level I and is classified in Group FM (see Table 27(a)). Each pupil in Group FM responded consistently at Level 1 (23 pupils in School A and 21 pupils in School B).

The range of responses for pupils (2:4) School B is however:

A(PLT) _ A(ZT) _ C(ET 1) _ C(ET 2) _
B(ET 3) _ C(LNT 1) _ C(LNT 2) _
C(LNT 3).
(See Appendix III).

This pupil made two 'excursions' into Level II. Pupils giving at least one, but not more than five responses belonging to Level II are classified in the Intermediate Group FM/D (14 pupils in School A and 20 pupils in School B - see Tables 27(a) and (b)).

Pupil (3:2) School A responded consistently at Level II:

T(PLT) _ C(ZT) _ D(ET 1) _ D(ET 2) _
C(ET 3) _ D(LNT 1) _ D(LNT 2) _
D(i)(LNT 3).
(See Appendix III).

and is classified in Group D. Each pupil in Group D (13 pupils in School A and 11 pupils in School B - see Tables 27(a) and (b) - responded consistently at Level II.).

Pupils who gave not more than two responses belonging to Level III are classified in the Intermediate Group D/S (see Tables 27 (a) and (b)).

For example, pupil (6:10) School A made an 'excursion' into Level III in the ZT:

T(PLT) _ D(ZT) _ D(ET 1) _
D(ET 2) _ C(ET 3) _ D(LNT 1) _
D(LNT 2) _ D(i)(LNT 3).
(See Appendix III).

and so is classified in Group D/S. 12 pupils in each school belong to this group (Tables 27(a) and (b)).

The final group, Group S, comprises those pupils giving at least three responses belonging to Level III. Pupil (5:5), School B is an example:

F(PLT) _ D(ZT) _ D(ET 1) _
D(ET 2) _ C(ET 3) _ D(LNT 1) _
D(LNT 2) _ D(ii)(LNT 3).

(See Appendix III and Tables 27(a) and (b)).
10 pupils in School A and 8 in School B belong to Group S.

One of the most striking aspects indicated by Tables 27(a) and (b) is that there are pupils in each Year-group1through 5 classified in the two 'lowest' level groups. There is a total of 10 pupils in all in Year five in this group - 41% of all fifth-year pupils in the study. Equally, however there are pupils in each Year-group 1 through 5 classified in the two 'highest' level groups. Although the percentage number of pupils in these latter groups increases from 4% to 41% from Year 1 to Year 5, the indication is that many pupils complete their mathematical studies at the fifty-year level devoid of any clear understanding of the language of algebra.

GROUP	YEAR						
	1	2	3	4	5	6	Totals
FM	5,7,8, 9,10, 11,12	7,8,10, 11,12	7,11, 12	6,7,8, 10,11, 12	10,12		
Totals	7	5	3	6	2	0	23
FM/D	4,6	4,6	4,5,6, 8,9,10	4,9	8,9		
Totals	2	2	6	2	2	0	14
D	2,3	2,3,5, 9	2,3	2,5	6,7,11		
Totals	2	4	2	2	3	0	13
D/S	1		1	1,3	3,4,5	7,9, 10,11, 12	
Totals	1	0	1	2	3	5	12
S		1			1,2	1,2,3, 5,6,8	
Totals	0	1	0	0	2	7	10

Table 27 (a); Pupil classification - School A.

GROUP	YEAR						
	1	2	3	4	5	6	Totals
FM	5,6, 7,8, 9,10, 11,12	5,9, 11, 12	8,11, 12,	5,7, 10, 11,12	6		
Totals	8	4	3	5	1	0	21
FM/D	2,3	3,4, 6,7, 8,10	5,6, 7,9, 10	6,8,	7,9, 10,11, 12		
Totals	2	6	5	2	5	0	20
D	1,4,	2	2,3, 4	1,3, 9	8	8	
Totals	2	1	3	3	1	1	11
D/S		1	1	4	2,3 4,5	1,4 11,12	
Totals	0	1	1	1	4	4	12
S				2	1	2,3, 5,6, 7,10	
Totals	0	0	0	1	1	6	8

Table 27(b); Pupil classification - School B.

APPENDIX III

Pupil	PLT	ET1	ET2	ET3	LNT 1	LNT 2	LNT 3	ZT
2:1	F	D	D	C	D	D	Dii	D
4:3	F	D	C	B	D	D	Dii	C
5:1	F	D	D	D	D	D	Dii	D
5:2	F	D	D	D	D	D	Dii	D
6:1	F	D	D	D	D	D	Dii	D
6:2	F	D	D	D	D	D	Dii	D
6:4	F	D	D	D	D	D	Dii	D
6:5	F	D	D	C	D	D	Dii	D
6:6	F	D	D	B	D	D	Dii	D
6:8	F	D	D	C	D	D	Dii	D
6:9	F	C	D	C	D	D	Dii	C

Table 28(a): Response-type of each 'F' group pupil to the remaining Subtasks - School A

Pupil	PLT	ET1	ET2	ET3	LNT 1	LNT 2	LNT 3	ZT
2:1	F	D	D	C	D	D	Dii	B
3:1	F	D	D	C	D	D	Di	C
4:2	F	D	D	D	D	D	Dii	D
5:5	F	D	D	C	D	D	Dii	D
6:2	F	D	A	C	D	D	Dii	D
6:3	F	D	D	D	D	D	Dii	D
6:5	F	D	D	D	D	D	Dii	D
6:6	F	D	D	C	D	D	Dii	D
6:7	F	D	D	D	D	D	Dii	D

Table 28(a), cont.: Response-type of each 'F' group pupil to the remaining Subtasks - School B

Pupil	PLT	ET1	ET2	ET3	LNT 1	LNT 2	LNT 3	ZT
1:1	T	D	C	C	D	D	Dii	B
1:2	T	C	C	C	D	D	Di	A
2:3	T	D	C	C	D	D	Di	C
2:9	T	D	C	A	D	D	Di	A
3:1	T	D	D	C	D	D	Dii	C
3:2	T	D	D	C	D	D	Di	C
3:3	T	D	C	C	D	D	Di	B
3:8	T	C	C	A	D	D	C	A
4:1	T	D	D	D	D	D	Dii	C
4:2	T	B	C	C	D	D	Di	A
4:5	T	D	D	B	D	D	Di	A
5:3	T	D	D	A	D	D	Di	D
5:4	T	D	D	C	D	D	Dii	A
6:3	T	D	D	D	D	D	Dii	D
6:7	T	D	D	D	D	D	Dii	C
6:10	T	D	D	C	D	D	Di	D
6:12	T	D	D	D	D	D	Di	D

Table 28(b): Response-type of each 'T' group pupil to the remaining Subtasks - School A

Pupil	PLT	ET1	ET2	ET3	LNT 1	LNT 2	LNT 3	ZT
2:3	T	B	D	B	D	D	Di	A
3:2	T	D	D	C	D	D	Di	B
4:1	T	D	D	C	D	D	Di	A
4:3	T	D	D	B	D	D	Di	C
4:4	T	C	C	D	D	D	Dii	C
4:9	T	D	C	A	D	D	Di	B
5:1	T	D	D	D	D	D	Dii	D
5:3	T	C	C	D	D	D	Di	D
5:4	T	D	D	C	D	D	Dii	C
5:8	T	D	B	B	D	D	Di	C
6:1	T	D	B	B	D	D	Dii	D
6:4	T	D	D	C	D	D	Dii	D
6:10	T	C	D	D	D	D	Dii	D
6:11	T	D	D	D	D	D	Di	C

Table 28(b), cont.: Response type of each 'T' group pupil to the remaining Subtasks - School B

Pupil	PLT	ET1	ET2	ET3	LNT 1	LNT 2	LNT 3	ZT
1:3	A	D	D	B	D	D	Di	B
1:4	A	C	C	B	C	C	C	A
1:5	A	B	B	B	C	C	C	A
1:6	A	C	B	B	B	B	C	A
1:7	A	A	A	A	A	B	B	A
1:8	A	B	B	A	C	C	C	A
1:9	A	B	A	A	A	B	A	A
1:10	A	B	B	B	B	B	C	A
1:11	A	B	A	A	A	A	A	A
1:12	A	A	A	A	A	A	A	A
2:2	A	D	C	D	D	D	Di	A
2:4	A	C	C	B	C	C	C	A
2:5	A	C	C	C	D	D	C	B
2:6	A	C	B	A	D	D	Di	A
2:7	A	B	B	B	C	C	C	A
2:8	A	B	B	B	C	C	B	A
2:10	A	B	A	B	C	C	B	A
2:11	A	B	A	A	A	A	A	A
2:12	A	A	A	A	B	B	A	A
3:4	A	B	B	B	D	D	Di	A
3:5	A	C	C	B	D	C	C	A
3:6	A	B	B	C	D	D	C	A
3:7	A	B	B	A	B	B	C	A
3:9	A	A	A	B	D	D	B	A
3:10	A	D	B	A	A	A	A	A
3:11	A	B	A	A	B	B	B	A
3:12	A	A	B	A	A	A	A	A
4:4	A	B	B	A	D	D	D	C
4:6	A	B	B	B	C	C	C	A
4:7	A	A	A	A	B	B	A	A
4:8	A	B	A	A	C	C	C	A
4:9	A	D	D	B	B	C	C	A
4:10	A	B	B	B	A	A	A	A
4:11	A	A	A	C	C	C	C	A
4:12	A	A	B	A	A	B	A	A
5:5	A	D	C	C	D	D	Dii	A
5:6	A	D	D	C	D	D	Di	A
5:7	A	D	D	C	D	D	C	C
5:8	A	D	B	B	B	A	A	A
5:9	A	B	B	B	D	D	Di	C
5:10	A	B	B	A	A	A	A	B
5:11	A	D	C	C	D	D	Di	C
5:12	A	B	A	A	A	A	A	A
6:11	A	D	B	C	D	D	Di	D

Table 28(c): Response-type of each 'A' group pupil to the remaining Subtasks - School A

Pupil	PLT	ET1	ET2	ET3	LNT 1	LNT 2	LNT 3	ZT
1:1	A	D	D	C	D	D	Di	B
1:2	A	B	B	A	C	C	A	C
1:3	A	C	B	A	D	D	Di	A
1:4	A	D	C	C	D	D	Di	B
1:5	A	B	B	B	C	C	C	A
1:6	A	C	B	B	C	C	C	A
1:7	A	B	B	B	C	C	C	A
1:8	A	B	A	A	B	B	C	A
1:9	A	B	B	A	A	B	B	A
1:10	A	B	A	A	A	A	A	A
1:11	A	A	B	A	A	A	A	A
1:12	A	B	A	A	A	A	A	A
2:2	A	C	C	C	D	D	Di	B
2:4	A	B	B	B	D	D	Di	A
2:5	A	A	A	A	A	A	A	A
2:6	A	C	B	B	D	D	Di	B
2:7	A	D	B	C	D	D	Di	A
2:8	A	D	C	C	B	D	C	A
2:9	A	A	A	B	C	C	B	A
2:10	A	B	B	B	D	D	C	A
2:11	A	B	A	A	B	B	A	A
2:12	A	B	A	A	A	A	A	A
3:3	A	D	D	B	D	D	Di	B
3:4	A	D	D	C	D	D	Di	A
3:5	A	C	C	A	D	C	Di	A
3:6	A	B	B	B	D	C	C	A
3:7	A	B	A	A	D	D	C	B
3:8	A	B	B	A	A	A	A	A
3:9	A	D	B	A	D	D	Di	A
3:10	A	C	C	A	D	C	C	A
3:11	A	B	A	A	B	B	C	A
3:12	A	B	B	A	A	A	A	A
4:5	A	B	B	A	B	B	B	A
4:6	A	D	D	A	D	D	Di	A
4:7	A	B	B	B	B	A	A	A
4:8	A	D	D	A	C	C	C	A
4:10	A	B	A	A	A	C	A	A
4:11	A	B	B	B	B	C	C	A
4:12	A	B	B	A	C	C	C	A
5:2	A	D	D	D	D	D	Dii	B
5:6	A	A	A	A	A	A	A	A
5:7	A	B	B	B	C	D	C	A
5:9	A	B	A	A	D	D	Di	B
5:10	A	D	D	A	A	A	A	A
5:11	A	D	B	C	B	C	C	B
5:12	A	B	D	C	A	A	A	A
6:8	A	B	B	C	D	D	Di	C
6:9	A	D	D	A	D	D	Dii	D
6:12	A	D	D	C	D	D	C	D

Table 28(c)(cont.): Response-type of each 'A' group pupil to the remaining Subtasks - School B

R E F E R E N C E S

ANTHONY, W.S.: 1977 — Activity in the learning of Piagetian operational thinking. Br. J. Educ. Psychol. 47, 18-24.

BLASI, A. and HOEFFEL, E.C.: 1974 — Adolescent and formal operations. Human Dev. 17, 344 - 363.

BOYER, C.B.: 1968 — History of Mathematics New York: Wiley.

BROWN, G. and DESFORGES, C.: 1977 — Piagetian psychology and education: time for revision. Br. J. Educ. Psychol., 47, 7-17.

BROWN, M. and KUCHEMANN, D.: 1976/7 — Is it an add, miss? Mathematics in School, 5,5 and 6,1.

BRUNER, J.S. and POSTMAN, L.: 1949 — On the perception of incongruity: a paradigm. J. Pers. 18, 206-223.

BRUNER, J.S., POSTMAN, L and RODRIGUES, J.: 1951 — Expectations and the perception of colour. <u>Amer. J. Psychol.</u>, 64, 216-227

BRUNER, J.S., et. al.: 1966 — <u>Studies in cognitive growth</u>. New York: Wiley.

BRUNER, J.S.: 1974 — <u>Relevance of education</u>. Middlesex, Eng.: Penguin Education.

BYERS, V. and HERSCOVICS, N.: 1978 — Understanding school mathematics. <u>Mathematics Teaching</u>. 81, 24-27.

CHANNON, J.B. and McLEISH SMITH, A.: 1948 — <u>A school algebra</u>. London: Longmans.

CHOMSKY N., and HALLE, M. 1968. — <u>Sound patterns of English</u>. New York: Harper and Row.

CHOMSKY N.: 1965 — <u>Aspect of the theory of syntax</u>. Cambridge, Mass.: MIT Press.

CHOMSKY, N.: 1972 — <u>Problems of knowledge and freedom</u>. London: Fontana/Collins.

COLLIS, K.F.: 1969 Concrete operational thinking and formal operational thinking in mathematics. The Australian Mathematics Teacher. 25,3, 77-84.

COLLIS, K.F.: 1971 A study of concrete and formal reasoning in school mathematics. Austral. J. Psychol. 23,3, 289-296.

COLLIS, K.F.: 1973 A study of children's ability to work with elementary mathematical systems. Austral. J. Psychol. 25,2, 121-130.

COLLIS, K.F.: 1975a Cognitive development and mathematics learning. Shell Mathematics Unit Centre for Science Education. Chelsea College, London Univ.

COLLIS, K.F. 1975 The development of formal reasoning. Univ. Newcastle, N.S.W.: Australia.

DANTZIG, T.: 1954 *Number, the language of science.* London: Allen and Unwin.

DIENES, Z.P.: 1961 On abstraction and generalisation. *Harvard Educational Review.*, Summer, 381-399.

DIENES, Z.P.: 1963 *An experimental study of mathematics learning.* London. Hutchinson.

DONALDSON, M.: 1978 *Children's Minds.* London: Croom Helm.

FAWDRY, R.C.: 1931 *Junior Arithmetic.* London: Bell.

FREUDENTHAL, H.: 1973 *Mathematics as an educational task.* Dordrecht: Reidel.

GALVIN, W.P. and BELL, A.W. 1977 Aspects of difficulties in the solution of problems involving the formation of equations. Nottingham: *Shell Centre for Mathematical Education.* University of Nottingham.

GARRETT, H.E.: 1966 — Statistics in Psychology and Education. (6th Edn.). London: Longmans.

GOODSTEIN, R.L.: 1956 — The meaning of counting. Math. Gaz. 40 No. 332.

GOODSTEIN, R.L.: 1965 — Essays in the philosophy of mathematics. Leicester Univ. Press.

GOODSTEIN, R.L.: 1976 — Arithmetic without sets. Math Gaz. 60 No. 413. Presidential Address.

GUILDFORD, J.P. and FRUCHTER, B.: 1973 — Fundamental Statistics in Psychology and Education. Yokyo: McGraw Hill, Kogokusha.

HARPER, Eon.: 1976 — An alternative to formalism in mathematics education. Int. J. Math. Ed. Sci. Technol., 7,4, 455-459.

HARPER, Eon.: 1978 — Sketch for a theory of confusion in mathematics learning. Int. J. Math. Ed. Sci. Technol., 9,2, 231-237.

HASTORF, A.H.: 1950 The influence of suggestion on the relationship between stimulus size and perceived distance. *J. Psychol.*, 29, 195-217.

HOLT, M and MARJORAM, D.T.E.: 1966 *Mathematics through experience* (Book 1). London: Blond Educational.

HOOPER, A.: 1956 *Makers of Mathematics.* London: Faber and Faber.

HUNTER, J. and CUNDY, M.: 1978 *The mathematics curriculum a critical review - 'Number'.* Schools Council. Blackie.

INHELDER, B. and PIAGET, J.: 1958 *The growth of logical thinking from childhood to adolescence.* New York: Basic Books.

KASNER, E.D. and NEWMAN, J.: 1965 *Mathematics and the imagination.* London: Bell.

KLEIN, J: 1968 *Greek Mathematical Thought and the Origin of Algebra.* Cambridge, Mass: MIT Press.

KLINE, M.: 1962 — *Mathematics: a cultural approach*. Addison Wesley.

KLINE, M.: 1972 — *Mathematical Thought from Ancient to Modern Times*. London: Oxford Un. Press.

KOESTLER, A.: 1969 — *The act of creation*. London: Hutchinson.

KOESTLER, A.: 1973 — *The sleepwalkers*. London: Penguin Books.

KUCHEMANN, D. 1978 — Children's understanding of numerical variables. *Mathematics in School*. 7,4, 23-26.

KUHN, T.S.: 1970 — *The structure of scientific revolutions*. Chicago: Univ. of Chicago Press.

KRUTETSKII, V.A.1 1976 — *The Psychology of Mathematical Abilities in Schoolchildren*. Univ. of Chicago Press.

LAKATOS, I.: 1976 — *Proofs and refutations*. Cambridge: Cambridge Univ. Press.

LEE, D.D.: 1938 — Conceptual implications of an indian language. <u>Phil. Sci.</u>, 5, 89-102.

LEWIS, D.G.: 1967 — <u>Statistical Methods in Education</u>. London: Univ. London Press.

LUNZER, E.A.: 1968 — Formal reasoning. In Lunzer, E.A. and Morris, J.F., (Ed.). <u>Development in human learning</u>. New York: American Elsevier.

LUNZER, E.A.: 1973 — Formal reasoning: a re-appraisal. <u>Shell Mathematics Unit Centre for Science Education</u>. Chelsea College, London Univ.

MATHMATICAL ASSOCIATION: 1964 — <u>A second report on the teaching of arithmetic in schools</u>. London: Bell.

NIEMARK, E.D.: 1975 — Longitudinal development of formal operational thought. <u>Genet. Psychol. Monogr.</u>, 91, 171-225.

NUNN, T.P.: 1919 — <u>The teaching of algebra</u>. London: Longmans.

PIAGET, J.: 1952 — *The Child's Conception of Number*. London: Routledge and Kegan Paul.

PIAGET, J.: 1953 — *Principles of genetic epistemology*. London: Routledge and Kegan Paul.

PIAGET, J.: 1972 — *Psychology and epistemology: towards a theory of knowledge*. Middlesex, Eng: Penguin Univ. Books.

PIVCEVIC, E.: 1970 — *Husserl and Phenomenology*. London: Hutchinson.

POINCARÉ, H.: 1908 — *Science et Méthode*. Paris: Flammarion. Authorised English translation by G.B. Halstead: *The foundations of science*, Lancaster, Pennsylvania: The Science Press, 546-854.

POLANYI, M.: 1958 — *Personal knowledge*. London: Routledge and K. Paul.

POLANYI, M.: 1966 — *The tacit dimension*. London: Routledge and K. Paul.

POOLE, R.: 1977 — Empiricism in crisis. Tract. No.21. Gryphon Press.

POLYA, G.: 1962 — The Teaching of Mathematics and the biogenetic law., in I.J. Good (Ed.): The scientist speculates. London: Heinemann, 352-356.

POLYA, G.: 1965 — Mathematical Discovery on Understanding Learning and Teaching Problem Solving. (Vol. I and II) New York: Wiley.

POPPER, K.R.: 1972 — Objective knowledge. London: Oxford Univ. Press.

QUINE, W.V.O.: 1962 — Mathematical logic. New York: The Science Library, Harper and Row.

RESOURCES FOR LEARNING DEV. UNIT. — Algebra booklets. AL 1,2. Bristol, Avon: Resources for learning development unit.

RUSSELL, B.: 1919 — Introduction to mathematical philosophy. London: Allen and Unwin.

RUSSELL, B.: 1927 — *An outline of philosophy.* Ninth impression (1970). London: Allen and Unwin.

RUSSELL, B.: 1946 — *History of western philosophy.* Tenth impression (1967). London: Allen and Unwin.

RUSSELL, B. and WHITEHEAD, A.N. 1927 — *Principia Mathematica.* London: Cambridge Univ. Press.

SAPIR, E.: 1949 — *Selected writings of Edward Sapir.* D.G. Mendelbaum (Ed.) Berkeley: Univ. of Calif. Press.

SCHOOLS COUNCIL.: 1971 — *Number appreciation: mathematics for the majority project.* London: Chatto and Windus.

SCHOOL MATHEMATICS PROJECT.: 1970 — *Book B.* London: Cambridge Univ. Press.

SEDIVY, J. 1974 — A note on the role of parameters in mathematics teaching. *Educational Studies in Mathematics.* 7, 121-126.

SKEMP, R.R.: 1964 *Understanding mathematics (Book 1)*. London: Univ. of London Press.

SKEMP, R.R.: 1971 *The psychology of learning mathematics*. Middlesex, Eng.: Penguin Books.

SKEMP, R.R. 1976 Relational understanding and instrumental understanding. *Mathematics Teaching*. 77, 20

SMEDLUND, J.: 1970 Circullar relation between understanding and logic. *Scand. J. Psychol*. 11, 217-219.

SMEDLUND, J.: 1977 Piaget's psychology in practice. *Br. J. Educ. Psychol*. 47, 1-6.

SMITH, D.E.: 1958 *History of Mathematics*. (Vol. I and II) London. Dover.

STEINER, M.: 1975 *Mathematical Knowledge*. Ithaca: Cornell Univ. Press.

STRATTON, G.M.: 1897 Vision without inversion of the retinal image. *Psychological Review*, 4, 341-360.

STRUICK, D.J.: 1956 A concise history of mathematics. London: Bell.

WANG, H.: 1974 From mathematics to philsophy. London: Routledge and Kegan Paul.

WASON, P.C. and JOHNSON-LAIRD, P.M.: 1972 Psychology of Reasoning: Structure and Content. London: Batsford.

WHITEHEAD, A.N.: 1911 Introduction to mathematics. London: Oxford Univ. Press.

WHORF, B.L.: 1940 Linguistics as an exact science. Technol. Rev., 61-63.

WHORF, B.L.: 1950 Four articles on metalinguistics. Washington D.C.; Foreign Inst., Dept. of State.

WILDER, R.L.: 1968 Evolution of Mathematical Concepts. New York: Wiley.

BIBLIOGRAPHY

The following articles are relevant to the main theme of the study but are not quoted in the text.

AIKEN, L.R. Jr.: 1972 — Language Factors in Learning Mathematics. Review of Educational Research. 42, 3, 369-384.

ARMITAGE, J.V.: 1972 — The place of geometry in a mathematical education. Math. Gaz. 57, 267.

DAWKINS, R.: 1976 — Memes and the evolution of culture. New Scientist. 28th Oct. 208-210.

HARKIN, J.B. and RISING, G.R.: 1974 — Some psychological and pedogogical aspects of mathematical symbolism. Educational Studies in Mathematics. 5,255-260.

LESGOLD, A.M.: 1974 — Variability in children's comprehension of syntatical structures. J. Educ. Psychol. 66, 3, 333-338.

PAIGE, J.M. and SIMON, H.A.: 1966 — Cognitive processes in solving algebra word problems. In. B. Kleinmuntz (Ed.). Problem solving: research, method and theory. 51-119. New York: Wiley.

VINNER, S.: 1978 — The elimination of some mathematical confusions through the analysis of language. Mathematical Education for Teaching. 3,2.