An investigation into the properties of multi-valued spectral logic.

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AN INVESTIGATION INTO THE PROPERTIES OF MULTI-VALUED SPECTRAL LOGIC

submitted by V. H. TOKMEN
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SUMMARY

This thesis is concerned with the properties of a particular discrete transform, and its applications to the classification of multi-valued ("m-ary") logic functions and m-ary combinatorial logic analysis and synthesis. The transform used is composed of a complete set of orthogonal functions, namely Chrestenson Functions, and the methods developed are applicable for all m, m = 2, 3, ...

The definition of multi-valued systems and some examples of multi-valued circuits are given in chapter 1. The necessity of a generalised design method which is not based on a particular algebra is considered, and the scope of the thesis is stated.

Chapter 2 introduces the algebraic notation, and continues to show the expansions of fully specified m-ary functions in (i) Lagrange form, (ii) generalised Reed-Muller form, and (iii) as polynomials over the field of real numbers.

Chapter 3 is an application of the mathematical developments covered in the previous chapter. Based on generalised Reed-Muller coefficients, a realisation of m-ary functions using Universal-Logic-Modules is described. The realisation in this case is restricted to m being a power of a prime.

The complex polynomial expansion of m-ary functions is considered in chapter 4. The coefficient set obtained is termed the "spectrum" of the given function. The effects of various operations in the function domain on the spectral values are investigated, and a classification of m-ary functions is described. Applications of spectral properties developed for m-ary combinatorial logic design are shown in examples.

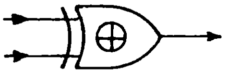
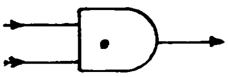
The implementation of any m-ary function involves some form of

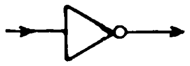
decomposition using physically available logic functions. The spectral properties developed in chapter 4 are further pursued in chapter 5 with an investigation into the relationships between the spectra of the logic functions involved in such a decomposition, and the spectrum of the overall function being realised. With the development of these spectral decomposition relationships, the range of tools for the spectral analysis of m-ary combinatorial logic is completed.

Throughout this thesis emphasis is placed on the generality of techniques developed, such that these techniques may be applicable to whatever higher-valued logic microelectronic circuit realisations may evolve in the future.

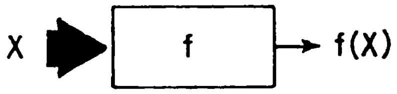
LIST OF SYMBOLS

$[\;]$	matrix.
$]$	column vector.
$\otimes$	Kronecker Product
$\bigotimes_{p=0}^{n-1} \left[A_p\right] = \left[A_{n-1}\right] \otimes \cdots \otimes \left[A_0\right]$	Kronecker Product of n matrices, the order in which the matrices appear on the right hand side of the equation should be noted.
$+$	addition.
x	multiplication (omitted).
$V = \{0,1,....,m-1\}$	set with m elements, integers mod-m.
$X = \{x_{n-1},....,x_0\}$	vector of independent variables over V.
$(I_{n-1},....,I_0)$	m-ary expansion of $0 \le i \le m^n-1$, such that $i = \sum_{p=0}^{n-1} I_p m^p$.
$f(X) : V^n \rightarrow V$	n variable m-ary function.
$\oplus$	GF(m) addition (Ch. 2-3), integers mod-m addition (Ch. 4).
$\cdot$	GF(m) multiplication.
$F]$	column vector whose entries are the local values of f(X) in decimal order.

$[Tr\text{-}m]$	Real transform matrix.
$[Tm\text{-}m]$	Modular transform matrix.
$\left.A\right]$	coefficient vector, such that $\left.A\right] = [Tr\text{-}m]^{\otimes n} \ \left.F\right]$
$\left.A_m\right]$	coefficient vector, such that $\left.A_m\right] = [Tm\text{-}m]^{\otimes n} \ \left.F\right]$
$y = e^{j\frac{2\pi}{m}x}$	character of x.
$c: x \rightarrow y$	an isomorphism between additive group of integers mod-m and multiplicative group of complex numbers.
$[Tc\text{-}m]$	Complex transform matrix.
$\left.S_f\right]$	The spectrum of f(X), such that $\left.S_f\right] = \frac{1}{m^n} [Tc\text{-}m]^{\otimes n} \ \left.cF\right]$
$(i \underline{\star} j)$	$\star$ operation between the corresponding elements of m-ary expansions of i and j.
$(L \underline{\times} i)$	matrix product in mod-m between the m-ary expansion of i considered as a column vector and the matrix $[L]$ of order nxn.
	GF(m) Adder (Ch. 3), integers mod-m Adder (Ch. 4).
	GF(m) Multiplier.



mod-m Simple-negation gate.



gate implementing function f(X).

CONTENTS

CHAPTER 1

Introduction

1. Introduction.

In this chapter the multi-valued logic system is defined and its advantage over a binary system is considered. Some examples of existing multi-valued logic circuits in Complementary Metal-Oxide Semiconductor field effect transistor ("CMOS") and integrated-injection-logic ("I^2L") technologies are given. The necessity of a general design method which is not based on a particular algebra for the design of multi-valued logic systems is discussed and the scope of the following investigation is outlined.

1.1 Definition of multi-valued logic systems

In conventional binary logic systems the information on the interconnecting paths of logic circuits is determined by the existence or non-existence of a signal, usually a signal voltage. The limitation of information at any one time to either one or other of these two states implies that n binary digits (bits) may be used to assign at most 2^n objects, and conversely binary representation of n symbols will require at least $\log_2 n$ bits. Thus a parallel transmission of information in words, i.e. $\log_2 n$ bits, will require $\log_2 n$ lines. It is widely known that a major obstacle which influences the size of integrated circuits is the limit on number of interconnection, both internal and input/output, possible on one chip. Consequently, complexity and the size of systems that may be handled with a single chip are limited by the number of interconnections.

A natural solution to this problem is to increase the information content on any single line by having more than two meaningful signal levels. If, for example, the permissible signal levels are increased to four, the same information will be conveyed on half the lines that would be required by a binary system. Similarly the number of memory units needed to store the same amount of information in 4-level memories will be halved. A prerequisite to the encoding and store of information in multi-levels is to have available logic circuits which process the multi-level information. Such a system "wherein m discrete signal levels shall be found under appropriate conditions on the one output line, the logic circuits responding in some predetermined manner to these m chosen signal levels"[27] is called a multiple-valued (m-valued m-ary) system.

Although the philosophical discussions on multiple-valued logics go as far back as Aristotle[39], the algebraic work on the subject was initiated in this century by Post[31] in the 1920's. Subsequently other researchers developed various multiple-valued algebraic structures for the synthesis and analysis of m-valued logic systems. On the engineering side, the development of m-ary circuits has followed closely the mathematical work, and a large amount of research has been devoted to the implementation of the basic connectives of the different algebras using the currently available technology. One of the first major works in this area is due to Lowenschuss[40]. We will look briefly at some CMOS and I^2L implementations in the following section.

There have been two attempts to build m-ary computers. In both cases m was chosen to be three, because of the advantages offered by a balanced ternary arithmetic[41]. The first of these computers, named SETUN, was designed and built at Moscow State University in 1958. The reports published on the performance of this machine indicate that the software used was complicated and hence not practical, and the hardware proved to be unreliable[28]. The second ternary computer was implemented in 1973 at Suny, Buffalo, U.S.A. The implementation of this computer "was intended primarily to discover if the implementation of a non-binary structure on a binary computer is feasible, and to discover the cost in memory storage and time for such implementation.... . As a feasibility exercise, this effort was successful, and the first version of this implementation proved that both the speed and price are of the order of the speed and price of binary computers"[30]. This development, comparison and assessment was, of course, done before the full impact of binary l.s.i circuits in the digital area was present.

1.2 Examples of m-ary circuits

It is a formidable task to include all the multi-valued circuits reported in the past in this thesis. Instead we will give below some of the most frequently used set of function implementations in CMOS-resistor and I^2L technologies. Even then the reader is reminded that these implementations are not unique and different designs for the implementation of the same function within the same technology may exist.

Unary operators: These are functions operating on one variable. In a m-ary system, for each of m possible values at one input, the output may take one of m possible values thus the making total number of one-variable operators m^m. However, there are only a few of them which appear freqently in many algebraic structures. These are literals, delta-functions, simple-negation and cyclic-negation. Note that in two-valued (binary) system the only functionally useful unary operator is the Invertor (or Not) gate.

Literals $x^{i,i}$ may be considered as threshold detectors. For a circuit implementing literal $x^{i,i}$ the output takes the value (m-1) if input is i, otherwise it remains zero. For example, in a ternary system:

x	$x^{0,0}$	$x^{1,1}$	$x^{2,2}$
0	2	0	0
1	0	2	0
2	0	0	2

Delta functions $x^{i,j}$ are similar to literals, this time the output being (m-1) when $i \leq x \leq j$, otherwise it is zero. Again for example, in a ternary system:

x	$x^{0,1}$	$x^{0,2}$	$x^{1,2}$
0	2	2	0
1	2	2	2
2	0	2	2

The simple negation and cyclic negation are given by the equations:

$\bar{x} = (m-1) - x$ (simple)

$x^{\rightarrow} = (x+1)$ mod-m (cyclic)

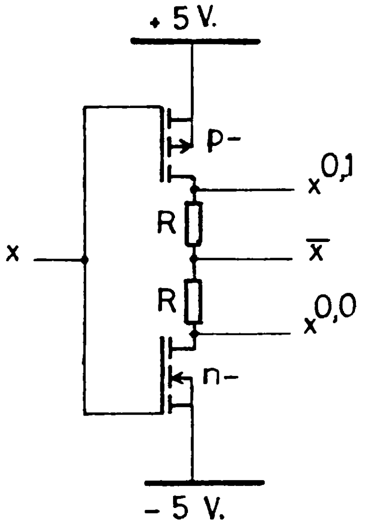
Again, for example, in a ternary system:

x	$\bar{x}$	$x^{\rightarrow}$
0	2	1
1	1	2
2	0	0

As a circuit example, Fig. 1.1 shows the CMOS-resistor technology implementation of the ternary literal $x^{0,0}$, the delta function $x^{0,1}$, and the simple negation $\bar{x}$. Note that all these three functions are realised by the same circuitry. The operation of this circuit may be explained as follows:

We assign logic values 0,1,2 to voltages -5V,0V and +5V respectively. When input is logic 0 the p-type transistor conducts and all three output lines take the logic value 2. If the input is logic 1 (0V) then both transistors conduct, and depending upon which rail the output point we are considering is closest to we obtain at the output logic value 2,1 or 0. In the case when the input is logic 2 (+5V) the n-type transistor conducts and all three outputs take the logic value 0.

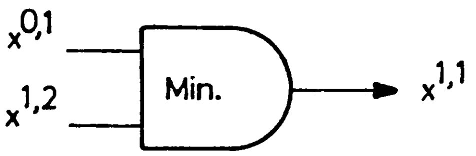
Note that taking the simple negation of output $x^{0,1}$ we obtain $x^{2,2}$, and similarly simple negation of $x^{0,0}$ gives $x^{1,2}$. The literal $x^{1,1}$ may then be obtained by taking binary conjunction of $x^{0,1}$ and $x^{1,2}$.



x	$\bar{x}$	$x^{0,1}$	$x^{0,0}$
0	2	2	2
1	1	2	0
2	0	0	0

Fig. 1.1 CMOS-resistor implementation of some unary functions

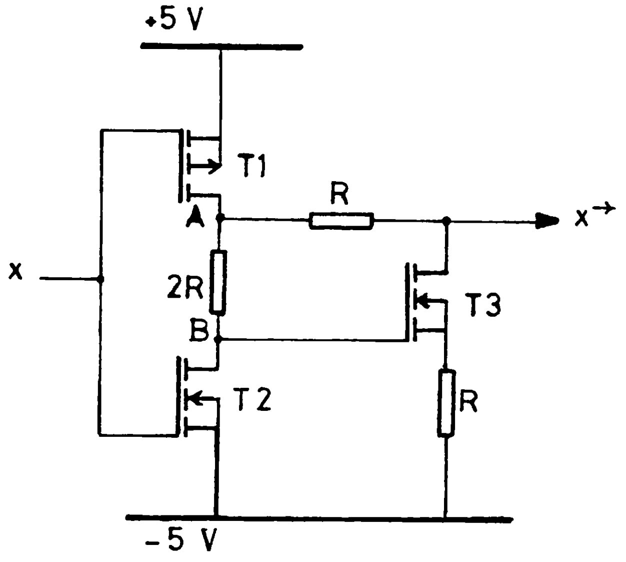
Thus the circuit diagram for $x^{1,1}$ becomes:



The details of the "Min". circuit above will be shown later.

The realisation of cyclic-negation is shown in Fig. 1.2[33]. When input is logic 0 (-5V), T1 conducts and both A and B are at the positive rail potential. Hence T3 is turned on, giving at the output logic 1 (OV). If x is logic 1 (OV) then both T1 and T2 conduct; thus point B has a negative potential turning T3 off, and the output takes the logic value 2(+5V). When x is logic 2(+5V) only T2 conducts and thus the output is at the negative rail potential -5V (logic 0).

We can show current-mode bipolar technology integrated injection logic (I^2L) as an alternative to unipolar implementation.[6] This time we have units of currents as opposed to voltages to represent the logic levels. The I^2L realisation uses three fundamental circuit operations, namely the replication of signals using current mirrors, linear summation and



x	$x^{\rightarrow}$
0	1
1	2
2	0

Fig. 1.2 CMOS-resistor three-valued cycling gate

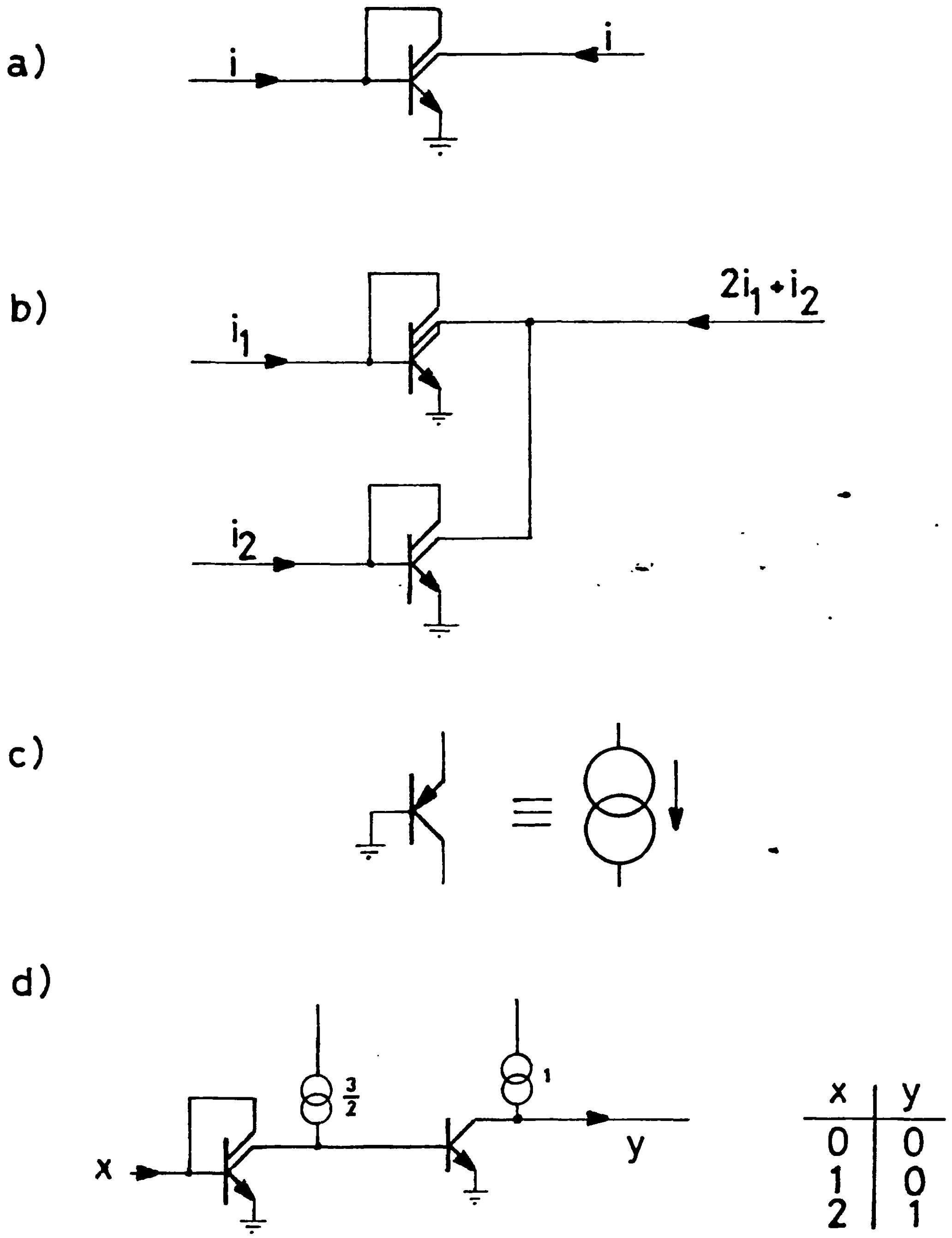


Fig. 1.3 a) I^2L current mirror,

b) Linear summation example,

c) Current source circuit,

d) Threshold detector example.

the detection of threshold values[14]. These circuits are shown in Fig. 1.3. Note that use of currents to represent the logic values limits the fan out of these circuits to one. This problem is overcome by replicating the currents using current mirrors. As the name "current mirror" suggests, the polarity of current i is reversed in the replicated copies.

Fig. 1.4 shows implementation of literals, simple negation and cyclic negation operations for the three-valued case. Similar circuits may be used to implement unary functions in $m = 4$ quaternary logic systems.[6] However, necessary adjustments have to be made to current source values. The operations of these circuits may be explained as follows:

The upper branch of the literal circuit implements the delta function $x^{i,2}$ whilst the lower branch implements $x^{0,i}$. These two functions are then "wire-min"-ned to give $x^{i,i}$ (Fig. 1.4a). The implementation of simple negation uses the "mirror-image"ing property of I^2L, and is readily achieved by adding a current source to the mirror-image circuit (Fig. 1.4b).

The operation of cyclic negation circuit is similar to the literal circuit. In this case the upper branch adds one to the incoming current value, whilst the lower half implements a function similar to the delta function $x^{0,1}$. The two branches are then "wire min"-ned to give $x^{\rightarrow}$.

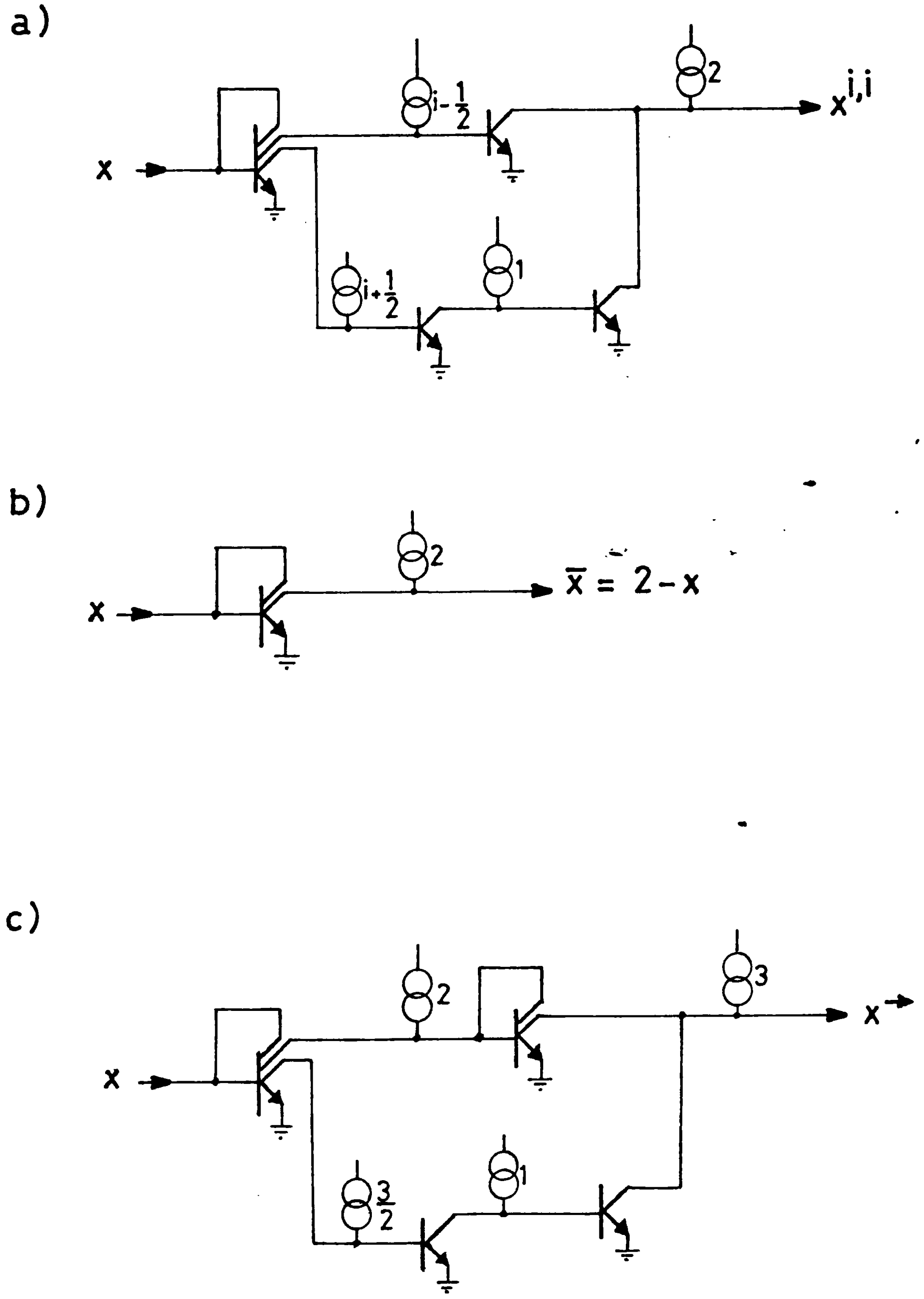


Fig. 1.4 a) Literals $x^{i,i}$,

b) Simple-negation,

c) Cyclic-negation.

Two-input gates: Because the functions they implement are relatively easy to manipulate Max., Min. and mod-m Addition gates have been given the major attention. The CMOS-resistor realisation of ternary $\overline{\text{Max}}$. and $\overline{\text{Min}}$. (simple negation of Max. and Min. respectively) functions are shown in Fig. 1.5.[32] Ternary Max. and Min. are then obtained by taking simple-negation of the outputs. The operation of the $\overline{\text{Max}}$. circuit may be described as follows:

i) When either of the inputs is logic 2, one of the n-type transistors conducts, giving logic 0 at the output.

ii) If both inputs are logic 1 then all transistors conduct, giving logic 1 at the output.

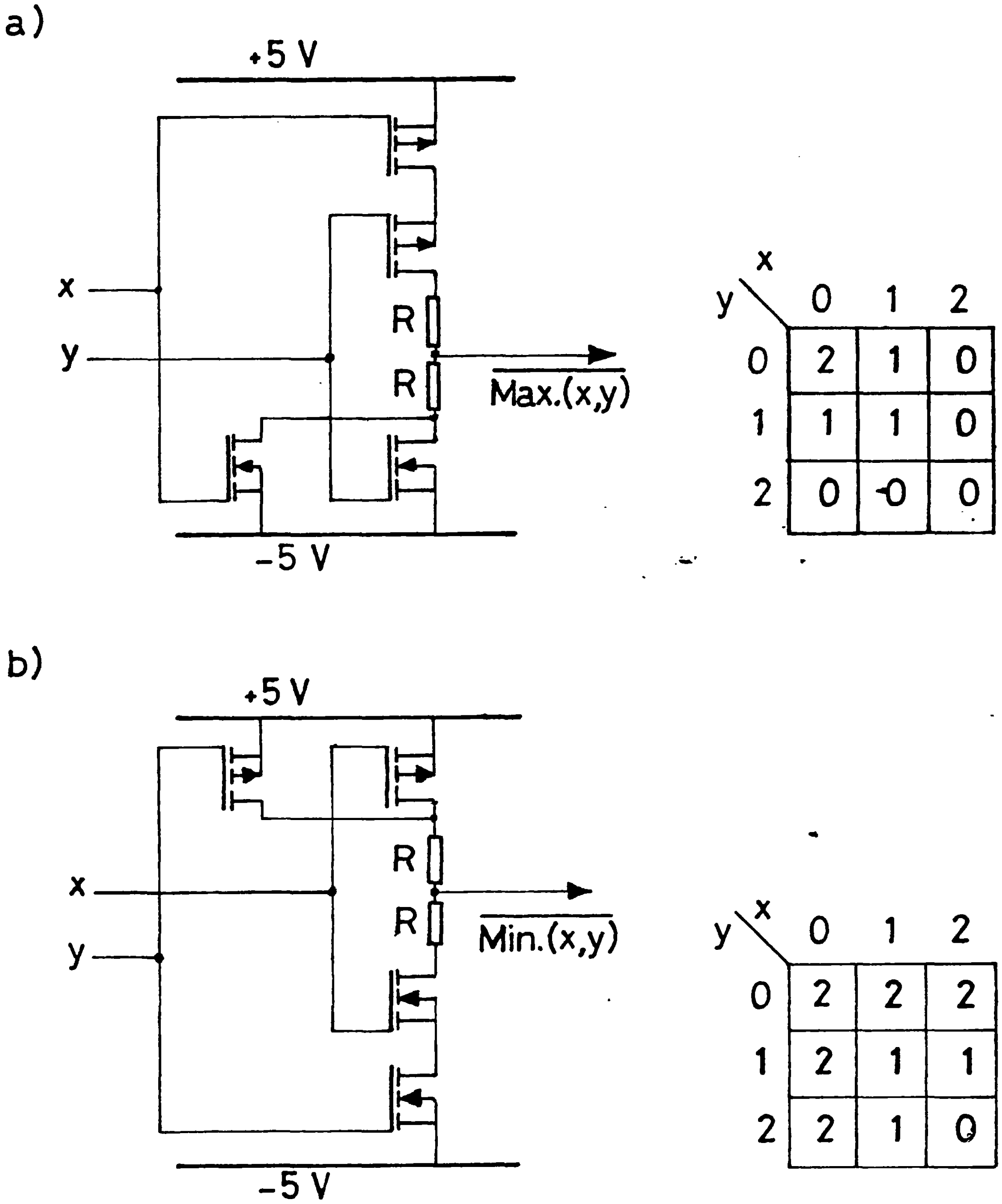
iii) If either of the inputs is logic 1 and the other is logic 0 then both p-type transistors and one of the n-type transistors conduct, again giving logic 1 at the output.

iv) When both inputs are logic 0, only the p-type transistors conduct, giving logic 2 at the output.

Interchanging 2 ↔ 0 and p ↔ n in above arguments, the operation of the $\overline{\text{Min}}$. gate may be described similarly.

A CMOS-resistor implementation of a mod-3 Adder utilises the T-gate which will be described later. Another possible realisation may be based on algebraic expression of mod-3 addition using literals, Max. and Min. operators. However, the expression obtained in this particular case is long[42], and the realisation is uneconomical.

Fig. 1.6 shows a two-input Max. gate and a two-input mod-3 Adder realisation using I^2L. The currents at different parts of the circuit



a)

y\x	0	1	2
0	2	1	0
1	1	1	0
2	0	0	0

b)

y\x	0	1	2
0	2	2	2
1	2	1	1
2	2	1	0

Fig. 1.5 a) $\overline{\text{Max (x,y)}}$ implementation in CMOS,

b) $\overline{\text{Min (x,y)}}$ implementation in CMOS.

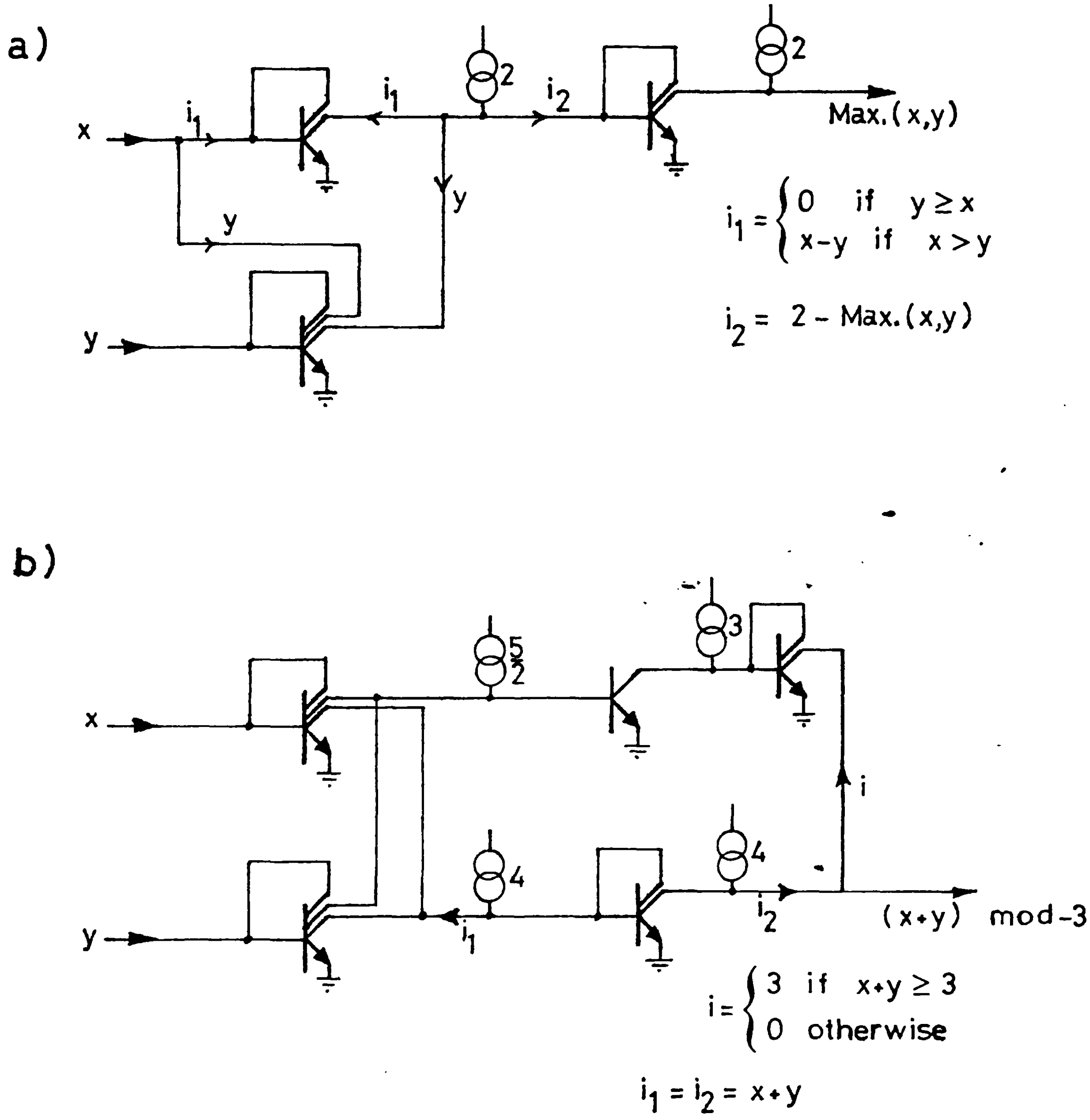


Fig. 1.6 a) Max (x,y) using I^2L,

b) mod-3 Adder.

have been included in the figure so that the operation of these circuits can easily be explained following current values. Note that Min. and Max. functions obey De Morgan's law, which may be stated as:

$$\min (x,y) = \overline{\max (\bar{x},\bar{y})}$$

Hence, the Min. function may be obtained from Max. function following the above relationship.

T-gate: The T-gate is basically a m-ary multiplexer.[34] Its representation in ternary takes the form T(a,b,c;x) and the function takes the same value as one of the data inputs a,b or c depending on the value of the control variable x being 0,1 or 2 respectively. The T-gate may be used to restore signal levels when the signal sources for logic 0,1 and 2 values are connected at a,b and c data inputs.

The CMOS-resistor implementation of the T-gate is shown in Fig. 1.7. The bilateral switches employed in the realisation have one control input, one data input and one output. When the control signal is positive the switch is turned on and the output takes the same values as the data input. If the control signal is negative then the switch is turned off. In the realisation of the T-gate operator the bilateral switches are controlled by signal from literal gates implementing $x^{0,0}$, $x^{1,1}$ and $x^{2,2}$. The realisation of mod-3 Addition using the T-gate operator should now be obvious; in this case the inputs a,b and c are replaced by y, $y^{\rightarrow}$ and $(y^{\rightarrow})^{\rightarrow}$. The implementation of $(y^{\rightarrow})^{\rightarrow}$ is similar to $y^{\rightarrow}$, and may be found in Carmona, et al.[33]

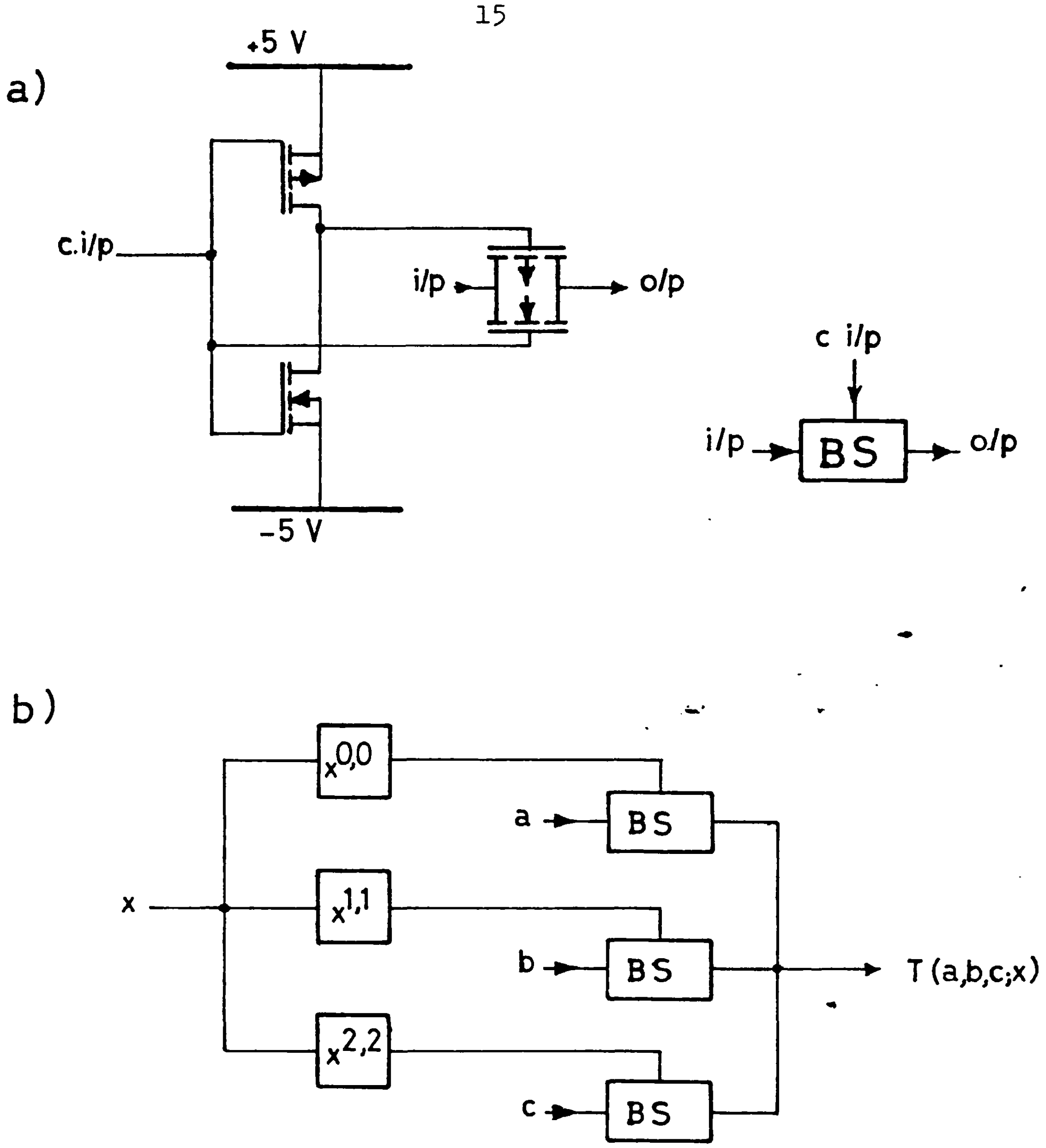


Fig. 1.7 a) CMOS bilateral switch,

b) T-Gate in CMOS.

Fig. 1.8 shows the T-gate in I^2L. Two threshold detectors at the input end of the circuit control the transistors which will ground the unwanted data inputs depending on the value of x. The column of current-followers isolate the data inputs a,b,c from each other, and transmits the selected data to the output.[14]

There are a number of different implementations of the above operators in various technologies such as current-mode-logic CML and Transistor-Transistor Logic T^2L. Most of these circuits may be found reported in the Proceedings of the Multiple-Valued Logic Symposia 1971-80. The CMOS and T^2L voltage mode circuits are in general limited to ternary implementations. T^2L may be expanded for quaternary ($m = 4$), but for higher radices the CML and I^2L current mode circuits are better suited. The CMOS circuits have slow operational speeds in comparison with the bipolar circuits. The ones we have shown in the above examples are designed to operate at 100 kHz with $R = 12\ k\Omega$. CML on the other hand, offers high speed operations at relatively high cost[35].

Recently charge-coupled-devices ("CCD's") have been suggested for use in multi-valued logic design[36]. CCD memory elements which use four discrete charge levels have been built and tested[37]. These devices have the additional advantage that they are MOS compatible.

In conclusion it may be stated that with the current expertise available in semiconductor technology, there would appear to be no fundamental reasons why efficient multi-valued logic circuits should not be fabricated, were it not for the entrenched status of normal binary circuits and systems.

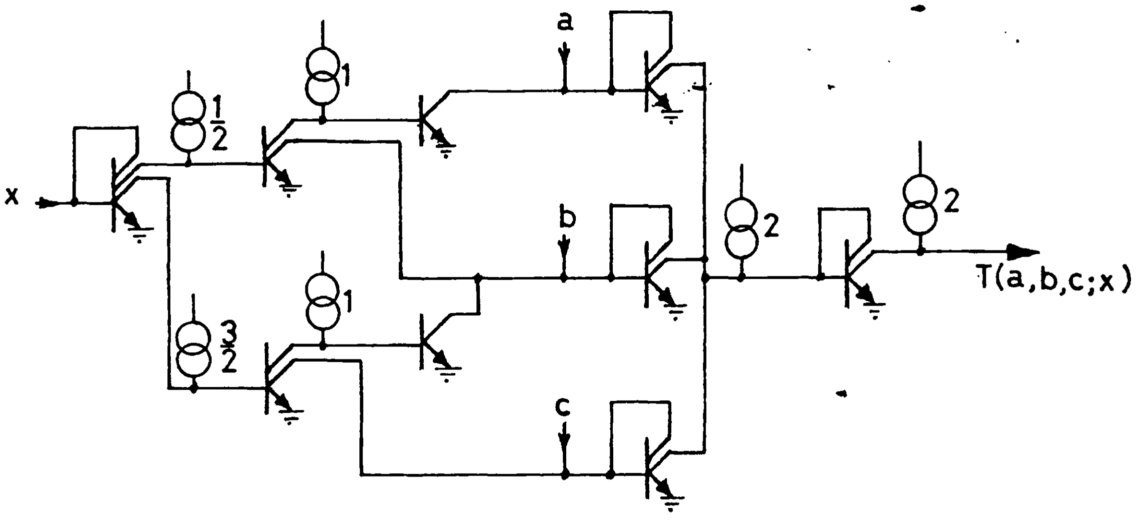


Fig. 1.8 T-Gate in I^2L.

1.3 The reasons and the scope of following investigation

Most work in multi-valued circuit developments has been devoted to implementation[46] of a few functions, namely literals, Max., Min. and base-m Addition, for which we gave the examples of implementations in the last section. These functions are easy to comprehend, although the physical realisations may not be straightforward, and intuitive design using them for small problems is relatively easy. The minimization techniques based upon algebras which contain literals, Max. and Min. functions in their set of basic functions may be found in literature[43,44,45]. The use of a particular technology for the implementation may influence the choice of the basic set of functions. For example, in I^2L technology the realisation of base-4 addition rather than addition operation in a field with four elements is encouraged by the fact that I^2L is naturally capable of adding currents which represent logic values, in base-4 addition fashion.

Consider the circuits given in Fig. 1.9. The first of these circuits was originally developed by Edwards[38] as a binary EX-OR gate, but with an appropriate selection of signal levels to represent the logic values 0,1,....,m-1 this circuit may be used to implement a 2-variable m-valued Exclusive-OR function m-OR defined as:

$$\text{m-OR} = \begin{cases} \max(x,y) & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

The second circuit in Fig. 1.9 shows I^2L implementation of the 2-variable m-ary function $\overline{\text{Plus}}$. This function is given as:

$$\overline{\text{Plus}}\ (x,y) = \begin{cases} 0 & \text{if } (x + y) \geq m-1 \\ (m-1) - (x + y) & \text{otherwise} \end{cases}$$

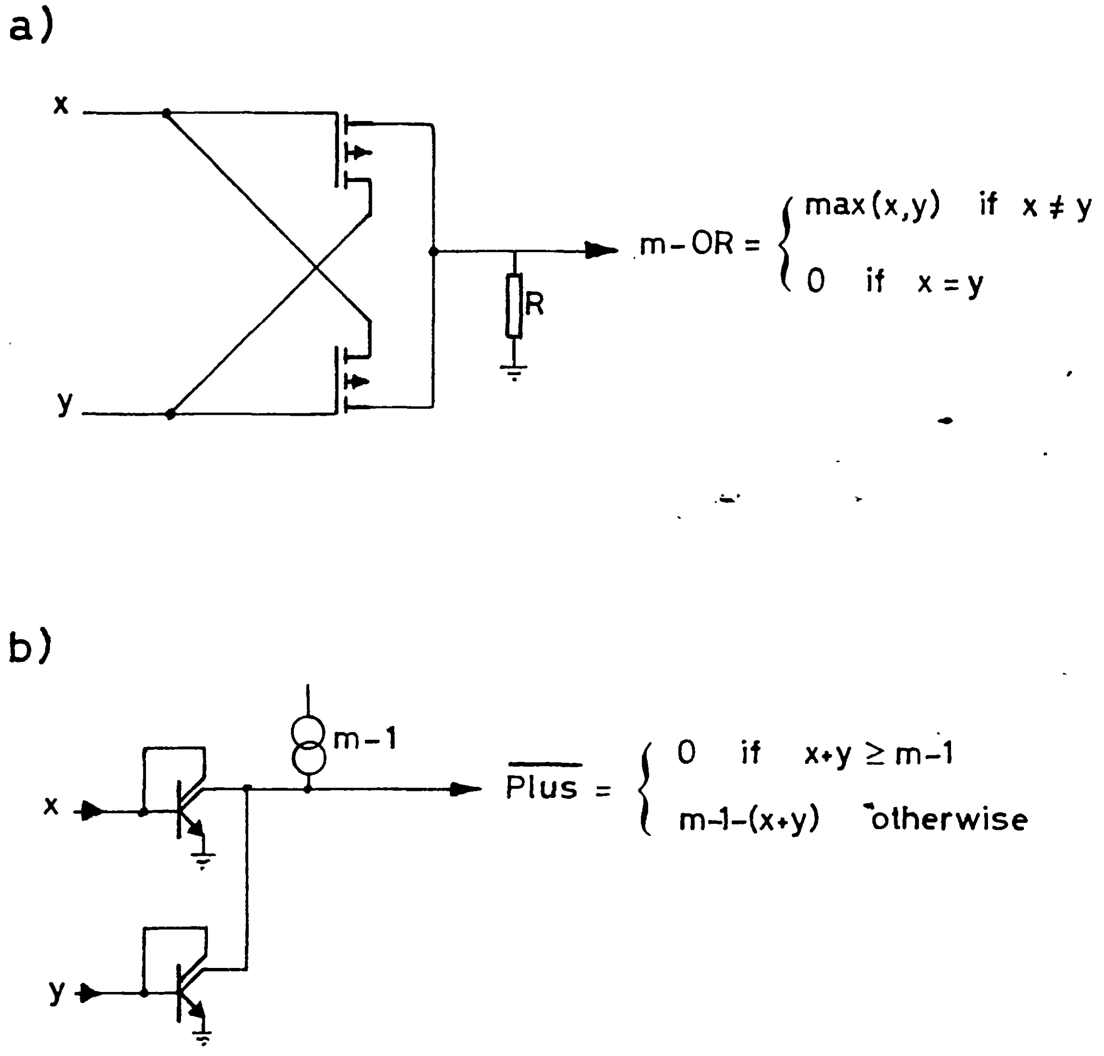


Fig. 1.9 a) M-OR using MOS-FET,

b) $\overline{\text{Plus}}$ gate in I^2L.

The development of these two circuits is based upon the natural capabilities of corresponding technologies, and the functions implemented by them are not included in the basic set of functions of algebras so far developed for synthesis methods. Evidently many other like circuits which have the property of low cost, using small on-chip silicon area, may be suggested. From an engineering point of view the employment of a large variety of electronically simple basic functions may be a big advantage, and thus a design algorithm which makes full use of the advantages offered by these functions is highly desirable. Such a design algorithm must be either independent of or easily adaptable to a change in the basic set of functions since functions with simple realisations will vary depending on the technology and technological advancements.

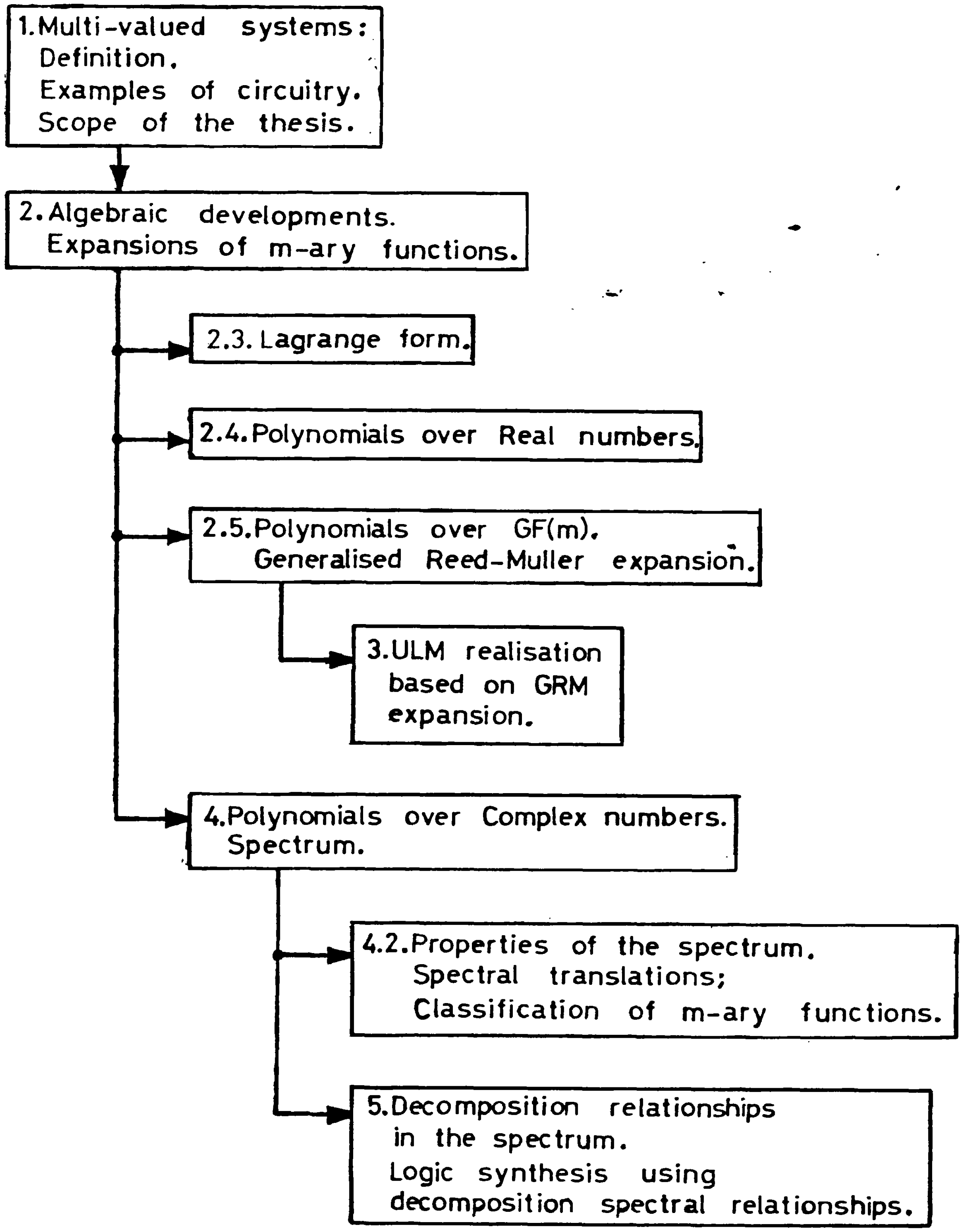
A possible approach to solve this problem, which constitutes the principal material for the following investigation, may be based upon classification of all functions. All m-ary functions are split into classes, such that functions of a class may be obtained from a representative class function by simple and inexpensive operations. For example, a simple modification to a gate with n inputs is the permutation of its input connections. This corresponds to the rearrangement of variables in the mathematical expression which represents the gate, and the functions generated as such may be labelled as a permutation class.

In this thesis an orthogonal series of discrete functions, similar to Fourier Transform, will be used to obtain the spectrum of a function, and the modifications to the spectral coefficients under certain operations in the function domain will be investigated. Under the transformations used the members of a class of functions are identified by the values of their spectral coefficients.

A realisation of any m-ary function is generated by a composition of functions from a basic set. Algebraic relationships between the spectra of functions involved in a composition will be developed. Examples will be shown where the algebraic developments considered may be used to detect functions with certain properties. As a side work, a Universal-Logic-Module realisation of combinatorial m-ary logic functions based upon expansions in Galois Field will be considered.

The work of this thesis may therefore be summarised by the following flow chart:

CONTENTS SUMMARY



CHAPTER 2

Algebraic Developments

2. Algebraic Developments

Mathematical representations of a digital system are essential tools for synthesis and analysis purposes. Shannon expansions, Reed-Muller expansions, spectra etc. in binary (two-valued) logic and Lagrange forms, generalised Reed-Muller expansions, spectra etc. in multi-valued (m-ary) logic are examples of mathematical representations of related systems[1,2,3,4,5]. In practice it is usually the case that the electronic circuits ("gates") which implement the basic connectives that make up such expressions are available. Thus once an expression (eg. a Shannon expansion in binary) that represents a function is evaluated, then the physical implementation of the function readily follows using the gates which implement the required basic connectives (eg. AND, OR, NOT gates for Shannon expansion case).

In this chapter we shall first define the Kronecker Product (Sect. 2.1) and then proceed to look into various expansions of multi-valued functions. The expansions we will discuss are:

i) Lagrange canonical form (Sect. 2.3),
ii) Polynomials over the field of real numbers (Sect. 2.4),
iii) Polynomials over the finite field $GF(q_r)$ (Sect. 2.5).

The expressions for these expansions will in general be of the form:

$$f(X) = \left[\text{some basis functions}\right]\left[\text{Transform}\right]\,F\Big]$$

where f(X) is a m-valued discrete function whose local values are given as a column vector $F\Big]$.

The vectors obtained by multiplication of the transform matrix with the function value vector $F]$ will be called the coefficient vectors for real and modular polynomial expansions in the case when m is a prime.

2.1 Kronecker Product of matrices

Before formally defining the Kronecker Product, a particular notation which will be used throughout this thesis for the representation of matrices will be introduced.

The usual ordering of a matrix $[A]$ of order m x n is:

$$[A] = \begin{bmatrix} a_{1,1} a_{1,2} \cdot\cdot\cdot a_{1,n} \\ \cdot \\ \cdot \\ a_{m,1} \qquad a_{m,n} \end{bmatrix}_{m \times n} \qquad \ldots\ldots\ldots(2.1)$$

However, in our notation, the subscripts i,j of an element $a_{i,j}$ in the above matrix will be altered to subscripts (i-1),(j-1) respectively. Hence for the above matrix the elements will remain in their original places but their identifiers, i.e. their subscripts, will be altered as described, giving us:

$$[A] = \begin{bmatrix} a_{0,0}\ a_{0,1} \ldots a_{0,(n-1)} \\ \cdot \\ \cdot \\ a_{(m-1),0} \qquad a_{(m-1),(n-1)} \end{bmatrix}_{m \times n} \ldots\ldots\ldots(2.2)$$

Let $[A]$ and $[B]$ be two matrices of orders $m_1 \times n_1$ and $m_0 \times n_0$ respectively. The Kronecker Product, $[A] \otimes [B]$ is a matrix $[C]$ of order $m_2 \times n_2$ such that:

$$m_2 = m_1 m_0$$

$$n_2 = n_1 n_0$$

and the elements $c_{i,j}$ of $[C]$ are given by:

$$c_{i,j} = a_{I_1,J_1} \, b_{I_0,J_0} \quad ,$$

where

$$i = I_1 m_0 + I_0 \quad ,$$

$$j = J_1 n_0 + J_0 \quad .$$

For example, let $[A] = \begin{bmatrix} 1 & X \end{bmatrix}_{1 \times 2}$ and $[B] = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}_{2 \times 2}$

Then the Kronecker product $[A] \otimes [B]$ will be:

$$[C] = \begin{bmatrix} 1 & X \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$= \left[1 \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \; X \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \right]$$

$$= \begin{bmatrix} 1 & 0 & X & 0 \\ -1 & 1 & -X & X \end{bmatrix}_{2 \times 4}$$

With the following theorem we state some basic but important properties of Kronecker product without giving the proofs. The reader is referred to the references cited for the detailed proofs.

Theorem 2.1[4]

a) for any three matrices $[A_2]$, $[A_1]$, $[A_0]$, we have:

$$[A_2] \otimes ([A_1] \otimes [A_0]) = ([A_2] \otimes [A_1]) \otimes [A_0] \qquad \ldots\ldots.(2.3)$$

b) for any four matrices $[A_3]$, $[A_2]$, $[A_1]$, $[A_0]$, where matrices $[A_2]$ and $[A_1]$ are of the same order, we have:

$$[A_3] \otimes ([A_2] + [A_1]) = [A_3] \otimes [A_2] + [A_3] \otimes [A_1]$$

and

$$([A_2] + [A_1]) \otimes [A_0] = [A_2] \otimes [A_0] + [A_1] \otimes [A_0] \qquad \ldots\ldots.(2.4)$$

c) for any p matrices $[A_{p-1}]$, ..., $[A_0]$ and any p matrices $[B_{p-1}]$, ..., $[B_0]$ where, if $[A_i]$ is of order k x 1 then $[B_i]$ is of order 1 x m for all i = 0.1,....,(p-1) we have:

$$([A_{p-1}] \otimes \cdots \otimes [A_0]) ([B_{p-1}] \otimes \cdots \otimes [B_0]) =$$
$$([A_{p-1}] \; [B_{p-1}]) \otimes \cdots \otimes ([A_0] \; [B_0]) \qquad \ldots\ldots.(2.5)$$

In the general case of Kronecker product of k matrices the expression will be of the form:

$$[B] = [A_{(k-1)}] \otimes [A_{(k-2)}] \otimes \cdots \otimes [A_0] \qquad \ldots\ldots.(2.6)$$

$$= \bigotimes_{p=0}^{(k-1)} [A_p]$$

Now, let the matrices $\left[A_i\right]$ be of order M_i x N_i. It can easily be seen from above definitions of the Kronecker product that an element $b_{i,j}$ of the matrix $[B]$ in terms of the elements ${}_p a_{I_p,J_p}$; $p = 0,1,$ ...., (k-1), of the matrices $\left[A_p\right]$ is given by:

$$b_{i,j} = \prod_{p=0}^{(k-1)} {}_p a_{I_p,J_p} \qquad \text{.......(2.7)}$$

where

$$i = I_{(k-1)} \prod_{q=0}^{(k-2)} M_q + I_{(k-2)} \prod_{q=0}^{(k-3)} M_q + \ldots + I_1 M_0 + I_0$$

.......(2.8)

and

$$j = J_{(k-1)} \prod_{q=0}^{(k-2)} N_q + J_{(k-2)} \prod_{q=0}^{(k-3)} N_q + \ldots + J_1 N_0 + J_0$$

.......(2.9)

Note that

$$(M_p - 1) \geq I_p \geq 0 \quad ,$$

and similarly

$(N_p - 1) \geq J_p \geq 0$, for all $p = 0,1,\ldots,(k-1)$. Given i and j, the two k-tuples $(I_{(k-1)},\ldots, I_0)$ and $(J_{(k-1)},\ldots, J_0)$ that satisfy equations (2.8) and (2.9) can be found by series of divisions. This can best be illustrated by the following example.

Assume M_5, M_4,...., M_0 have the values 5, 2, 8, 6, 3, 4 respectively. It is required to find the 6-tuple that satisfies equation (2.8) for the decimal number 3867. We divide this number first by M_0, (= 4); then divide the quotient by M_1, (= 3); and so on, giving us:

$3867 \div 4 = 966$, remainder 3
$966 \div 3 = 322$, remainder 0
$322 \div 6 = 53$, remainder 4
$53 \div 8 = 6$, remainder 5
$6 \div 2 = 3$, remainder 0
$3 \div 5 = 0$, remainder 3

The k-tuple is the remainders written in the order bottom-to-top, i.e. (3, 0, 5, 4, 0, 3).

Indeed, we will note that

$$3867 = 3(2 \times 8 \times 6 \times 3 \times 4) + 0(8 \times 6 \times 3 \times 4) + 5(6 \times 3 \times 4) + 4(3 \times 4) + 0(4) + 3 ,$$
$$= 3 \times 1152 + 0 \times 576 + 5 \times 72 + 4 \times 12 + 0 \times 4 + 3 ,$$
$$= 3456 + 360 + 48 + 3$$

Now, let us consider a special case when all matrices $[A_p]$; p = 0,1,...., (k-1) are identical and of order m x n, that is:

$$[B] = \bigotimes_{p=0}^{(k-1)} [A]$$
$$= [A]^{\otimes k} \qquad ....(2.10)$$

Then an element $b_{i,j}$ of the matrix $[B]$ in terms of the elements a_{I_p,J_p} of the matrix $[A]$ will be given by:

$$b_{i,j} = \prod_{p=0}^{(k-1)} a_{I_p,J_p} \qquad ....(2.11)$$

where

$$i = \sum_{p=0}^{(k-1)} I_p m^p , \qquad ....(2.12)$$

and

$$j = \sum_{p=0}^{(k-1)} J_p n^p . \qquad ....(2.13)$$

Therefore the k-tuple $(I_{(k-1)}, ..., I_0)$ is the integer mod-m expansion of i, and similarly the k-tuple $(J_{(k-1)}, ..., J_0)$ is the integer mod-n expansion of j.

<u>Example</u>: Let $[A]$ be the Hadamard matrix[1]

$$[A] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad ....(2.14)$$

Each element $a_{i,j}$ of this matrix is given by $a_{i,j} = (-1)^{ij}$.
The kth order Hadamard matrix $[H_k]$ is the k-Kronecker product of $[A]$, namely:

$$\left[H_k\right] = \left[A\right]^{\otimes k}$$

Let $h_{i,j}$ be an element of $\left[H_k\right]$, then

$$h_{i,j} = \prod_{p=0}^{(k-1)} a_{I_p, J_p} ,$$

$$= \prod_{p=0}^{(k-1)} (-1)^{I_p J_p} .$$

Hence

$$h_{i,j} = (-1)^{\sum_{p=0}^{(k-1)} I_p J_p} , \qquad ....(2.15)$$

where

$$i = I_{(k-1)} 2^{(k-1)} + I_{(k-2)} 2^{(k-2)} + ... +I_0,$$

$$j = J_{(k-1)} 2^{(k-1)} + J_{(k-2)} 2^{(k-2)} + ... +J_0.$$

On the next page is the detailed third order Hadamard matrix. The column and row numbers are written in mod-2 above and on the left of the matrix respectively.

$$[H_3] = \begin{array}{c} \\ 000 \\ 001 \\ 010 \\ 011 \\ 100 \\ 101 \\ 110 \\ 111 \end{array} \begin{array}{c} \begin{array}{cccccccc} 000 & 001 & 010 & 011 & 100 & 101 & 110 & 111 \end{array} \\ \left[\begin{array}{cccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & \boxed{-1} & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{array}\right] \end{array} .$$

Note that by above definitions an element $h_{i,j}$ of this matrix is evaluated by first summing the element-by-element multiplication of binary (mod-2) expansions of i and j (call this s) then taking the sth power of -1. To check this take for example $h_{2,3}$ (encircled)

$$2 = (0,1,0) \bmod 2$$

$$3 = (0,1,1) \bmod 2$$

and therefore $s = 00 + 11 + 01 = 1$

$$h_{2,3} = (-1)^1 = -1 .$$

2.2 Tabular representation of multi-valued functions

Let $V = \{0,1,\ldots,(m-1)\}$ be a set with m-elements (integers modulo-m) and x_i, $i = 0,1,\ldots,(n-1)$, be independent variables over this set. A fully specified m-valued (m-ary) n-variable function $f(x_{n-1},\ldots,x_0)$ is defined to be the mapping:

$$f: V^n \longrightarrow V ,$$

where V^n is the n-cartesian product of V. We shall also use $f(X)$ to represent $f(x_{n-1},\ldots,x_0)$.

A n-variable m-ary function can be represented using a map, termed a Karnaugh map, in which every entry of the map represents a unique point in n-dimensional space V^n. For example with a two variable ternary function $f(x_1,x_0)$, mod-3 addition is defined by the following map:

x_0 \ x_1	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

An alternative representation for a function is a tabulation giving its truth-table. This is a listing of all n-tuples of the set V^n with corresponding values of the function at those points. For example the truth table for above functions is:

x_1	x_0	$f(x_1,x_0)$
0	0	0
0	1	1
0	2	2
1	0	1
1	1	2
1	2	0
2	0	2
2	1	0
2	2	1

In general we represent the n-variable m-ary function f(X) as a column vector F] such that the entries f_{i0} are the values the function takes at the point $f(I_{n-1},\dots, I_0)$, where $i = \sum_{k=0}^{n-1} m^k I_k$. This ordering of the function values as a column vector will be called decimal ordering. We shall say that the n-tuple $(I_{n-1},\dots,I_0)$ is the m-ary expansion of i and, in turn, i is the coding of $(I_{n-1},\dots, I_0)$. Obviously, with this definition, the m-ary expansion of i exists if and only if i (a natural number) is in the closed interval $0,m^n-1$. In the above example the function arguments values are written in decimal order of 0,1,2,....,8.

It can easily be observed that a n-variable m-ary function is defined at m^n points. At each point the function can take one of m-possible values and hence there are $m^{(m^n)}$ possible n-variable m-ary functions. The exponential growth of the number of functions with both m and n can be seen in table 2.1.

m \ n	1	2	3	4
2 binary	4	16	256	65536
3 ternary	27	19683	7.63×10^{12}	4.43×10^{38}
4 quaternary	256	4.29×10^{9}	3.40×10^{38}	$\sim e^{355}$
5 quinary	3125	2.98×10^{17}	2.35×10^{87}	$\sim e^{1006}$

Table 2.1 Number of n-variable m-ary functions.

2.3 A canonical expansion of m-ary functions

We define one variable m-ary functions $x^{i,i}$, called literals, as:[4,6,7]

$$x^{i,i} = \begin{cases} 1 & \text{if } x = i \,, \\ 0 & \text{otherwise} \end{cases} \qquad \ldots\ldots\ldots(2.16)$$

The following tables show the binary and quaternary literals:

x	$x^{0,0}$	$x^{1,1}$
0	1	0
1	0	1

a) binary literals

x	$x^{0,0}$	$x^{1,1}$	$x^{2,2}$	$x^{3,3}$
0	1	0	0	0
1	0	1	0	0
2	0	0	1	0
3	0	0	0	1

b) quaternary literals

Theorem 2.2: Any n-variable m-ary function f(X) can be expanded in the form:

$$f(X) = \sum_{i=0}^{m^n-1} x_{n-1}^{I_{n-1},I_{n-1}} \ldots x_0^{I_0,I_0} \quad f(i) \,, \qquad \ldots\ldots\ldots(2.17)$$

where $(I_{n-1}, I_{n-2},\ldots,I_0)$ is the m-ary expansion of i. This expansion is unique.

Proof: In expression of (2.17) every minterm $x_{n-1}^{I_{n-1},I_{n-1}} \ldots\ldots x_0^{I_0,I_0}$ takes the value 1 if and only if all $x_p = I_p$, $p = 0,\ldots, (n-1)$, by the definition of literals. Therefore the right hand side of the equation is equal to f(i) at each and every point i. The expression is unique, since the coefficient f(i) of each minterm $x_{n-1}^{I_{n-1},I_{n-1}} \ldots\ldots x_0^{I_0,I_0}$ is uniquely defined by the local values of the function.

This expression is called the Lagrange canonical form of the function f(X).

Example: The canonical expansion for the function of Section 2.2, page 33 is:

$$f(x_1, x_0) = \{0\ x_1^{0,0}\ x_0^{0,0} + 1\ x_1^{0,0}\ x_0^{1,1} + 2\ x_1^{0,0}\ x_0^{2,2}$$

$$+\ 1\ x_1^{1,1}\ x_0^{0,0} + 2\ x_1^{1,1}\ x_0^{1,1} + 0\ x_1^{1,1}\ x_0^{2,2}$$

$$+\ 2\ x_1^{2,2}\ x_0^{0,0} + 0\ x_1^{2,2}\ x_0^{1,1} + 1\ x_1^{2,2}\ x_0^{2,2}\}\ .$$

Omitting the terms with zero coefficient we obtain the final canonical expansion:

$$f(X) = \{x_1^{0,0}\ x_0^{1,1} + 2\ x_1^{0,0}\ x_0^{2,2} + \ x_1^{1,1}\ x_0^{0,0} + 2\ x_1^{1,1}\ x_0^{1,1}$$

$$+\ 2\ x_1^{2,2}\ x_0^{0,0} + \ x_1^{2,2}\ x_0^{2,2}\}$$

The expression of (2.17) can also be written as:

$$f(X) = \sum_{i=0}^{m^n-1} x_{n-1}^{I_{n-1},I_{n-1}} \cdots x_0^{I_0,I_0}\ f(i)\ ,$$

$$= \left(\bigotimes_{p=0}^{n-1} \left[x_p^{0,0} \ .\ .\ .\ x_p^{m-1,m-1}\right] \right) F]$$

$$= \left(\left[x_{n-1}^{0,0}\ .\ .\ .\ x_{n-1}^{m-1,m-1}\right] \otimes \ldots\ldots \otimes \left[x_0^{0,0}\ .\ .\ x_0^{m-1,m-1}\right] \right) F] \qquad \ldots\ldots.(2.18)$$

Indeed, this follows from the definitions of Kronecker product (see Section 2.1) and the decimal ordering. For example, the above function can be represented also as:

$$f(X) = \left(\bigotimes_{p=0}^{1} \left[x_p^{0,0}\ x_p^{1,1}\ x_p^{2,2} \right] \right) \left[0\ 1\ 2\ 1\ 2\ 0\ 2\ 0\ 1 \right]^t$$

where $]^t$ stands for the transpose of the row vector.

It is the consequence of theorem 2.1 that the functions $x_{n-1}^{I_{n-1},I_{n-1}} \ldots\ldots x_0^{I_0,I_0}$, $i = 0,1,\ldots.m^n-1$, form a basis for the n-variable m-ary functions.

2.4 The real polynomial expansions of m-ary functions

Equation 2.16 defines the one-variable functions, namely literals, which are used to generate a basis for the n-variable m-ary functions. Here we will show that it is possible to associate a polynomial defined at m-points over the field of real numbers with every one-variable literal. Replacing the literals with appropriate polynomials we obtain the real polynomial expansion of one-variable functions. The generalisation of this polynomial form to n-variable case follows directly from the distributive property of Kronecker product (Theorem 2.1).

Let the polynomial f(x) be:

$$f(x) = a_0 + a_1x + a_2x^2 \ldots. + a_{(m-1)}x^{(m-1)} \qquad \ldots\ldots\ldots(2.19)$$

where the variable x is restricted to take values in the set $V = \{0,1,\ldots.,(m-1)\}$ and a_i, $i = 0,\ldots.,(m-1)$, is element of real numbers.

Our aim is to find the set of coefficients, such that at m-points, $x = 0,1,\ldots,(m-1)$, the polynomial has the same values as the one-variable m-ary function $f(x)$. Now, the equation (2.19) may be written as:

$$f(x) = \begin{bmatrix} 1 & x & \ldots & x^{(m-1)} \end{bmatrix} \begin{bmatrix} a_0 & \ldots & a_{(m-1)} \end{bmatrix}^t \qquad \ldots\ldots\ldots(2.20)$$

Replacing the m-values which the variable x can take the following set of m-simultaneous equations are obtained:

$$f(0) = \begin{bmatrix} 1 & 0 & \ldots\ldots\ldots 0 \end{bmatrix} \left. A \right],$$

$$f(1) = \begin{bmatrix} 1 & 1 & \ldots\ldots\ldots 1 \end{bmatrix} \left. A \right],$$

$$\vdots$$

$$f(m-1) = \begin{bmatrix} 1 & (m-1) & \ldots\ldots\ldots & (m-1)^{(m-1)} \end{bmatrix} \left. A \right] \qquad \ldots\ldots\ldots(2.21)$$

where $\left. A \right]$ is the coefficient vector.

More conveniently this set of simultaneous equations can be re-expressed as:

$$\left. F \right] = \begin{bmatrix} Sr\text{-}m \end{bmatrix} \left. A \right] \qquad \ldots\ldots\ldots(2.21)$$

where the elements $s_{i,j}$ of the matrix $\begin{bmatrix} Sr\text{-}m \end{bmatrix}$ are:

$$s_{i,j} = i^j \qquad \ldots\ldots\ldots(2.21\ a)$$

For example for the quaternary case $(m = 4)$ the matrix $\begin{bmatrix} Sr\text{-}4 \end{bmatrix}$ will be:

$$\left[\text{Sr-4}\right] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{bmatrix}$$

Let $\left[\text{Tr-m}\right]$ be the inverse of $\left[\text{Sr-m}\right]$,i.e.:

$$\left[\text{Tr-m}\right] = \left[\text{Sr-m}\right]^{-1} \qquad \text{.......(2.22)}$$

Therefore from equation (2.21):

$$\left.A\right] = \left[\text{Tr-m}\right] \left.F\right] \qquad \text{.......(2.23)}$$

Table 2.2 shows the matrices $\left[\text{Tr-m}\right]$ for m = 2,3,4 cases.

$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ -1.5 & 2 & -0.5 \\ 0.5 & -1 & 0.5 \end{bmatrix} \qquad \frac{1}{6}\begin{bmatrix} 6 & 0 & 0 & 0 \\ -11 & 18 & -9 & 2 \\ 6 & -15 & 12 & -3 \\ -1 & 3 & -3 & 1 \end{bmatrix}$$

a) $\left[\text{Tr-2}\right]$ b) $\left[\text{Tr-3}\right]$ c) $\left[\text{Tr-4}\right]$

Table 2.2: Matrices a) $\left[\text{Tr-2}\right]$ b) $\left[\text{Tr-3}\right]$ c) $\left[\text{Tr-4}\right]$

Replacing equation (2.23) in equation (2.20) we obtain:

$$f(x) = \left[1 \ x \ \ldots\ldots \ x^{(m-1)}\right] \left[\text{Tr-m}\right] \left.F\right] \qquad \text{.......(2.24)}$$

A comparison of this expansion with the expansion using literals gives us the following relationship:

$$\left[x^{0,0}\ x^{1,1}\ \ldots.\ x^{(m-1),(m-1)}\right] = \left[1\ x\ \ldots.\ x^{(m-1)}\right]\left[\text{Tr-m}\right]$$

$$\ldots\ldots.(2.25)$$

If now the literals in equation (2.18) are replaced by their polynomial expansions given by above relationship the following real polynomial expansion for n-variable m-ary functions is obtained:

$$f(X) = \left(\bigotimes_{p=0}^{(n-1)} \left[1\ x_p\ \ldots.\ x_p^{(m-1)}\right]\left[\text{Tr-m}\right] \right) F] \qquad \ldots\ldots.(2.26)$$

Using the properties of Kronecker product (Theorem 2.1) this equation may be reexpressed as:

$$f(X) = \left(\bigotimes_{p=0}^{(n-1)} \left[1\ x_p\ \ldots.\ x_p^{(m-1)}\right] \right) \left[\text{Tr-m}\right]^{\otimes n}\ F] \qquad \ldots\ldots.(2.27)$$

where

$$\left[\text{Tr-m}\right]^{\otimes n} = \underbrace{\left[\text{Tr-m}\right]\otimes\left[\text{Tr-m}\right]}_{\text{n-times}}\otimes\ \ldots.\ \otimes\left[\text{Tr-m}\right]$$

Finally the above development may be summarized with the following theorem:

Theorem 2.3: Every n-variable m-ary function has a unique polynomial expansion of the form:

$$f(X) = \left(\bigotimes_{p=0}^{(n-1)} \left[1\ x_p\ \ldots.\ x_p^{(m-1)}\right] \right) A] \qquad \ldots\ldots.(2.28)$$

where the coefficient vector $A]$ is given by

$$A] = \left[\text{Tr-m}\right]^{\otimes n}\ F] \qquad \ldots\ldots.(2.29)$$

Example: Consider the three-variable binary $(m = 2)$ function given by the following map:

x_0 \ x_2x_1	0 0	0 1	1 1	1 0
0	0	0	1	0
1	1	1	1	0

The coefficient vector $A\rbrack$ for this particular function is evaluated by:

$$A\rbrack = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}^{\otimes 3} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 \\ -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

and hence:

$$A\,] = \; 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad -1 \quad 1 \quad 0\,]^t$$

Therefore the polynomial expansion for this particular function is given by the following expression:

$$f(x) = x_0 - x_2x_0 + x_2x_1 \qquad .$$

2.5 Polynomial expansions of m-ary functions over finite fields:
In Section 2.4 the polynomial expansions of m-ary functions over the field of real numbers were considered. The final expressions have in general rational coefficients varying over wide range of numbers. Here we will consider the polynomial expansion over finite fields denoted GF(q_r), Galois field with q_r elements; hence the coefficients in the final expression will be the elements of GF(q_r). The addition and multiplication operations will be as defined over this field whereas earlier they were the operations over real numbers.

There is a fundamental theorem concerning the order of finite fields, which states:

Theorem 2.4[4,8] The order q_r of any finite field GF(q_r) is either a prime p_r or a power of a prime p_r^k, where k is a positive integer.

Therefore the following developments in this section assumes m is a power of a prime p_r^k, and the addition and multiplication over the set V are the field operations, which are addition modulo-m and multiplication modulo-m if m is a prime p_r.

As an example, the following are the multiplication and addition tables over fields with i) 2,ii) 3,iii) 4 elements.

i) GF(2)

$\oplus$	0	1
0	0	1
1	1	0

.	0	1
0	0	0
1	0	1

ii) GF(3)

$\oplus$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

.	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

iii) GF(4)

$\oplus$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

.	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	3	1
3	0	3	1	2

Let us now consider the following polynomial over the field GF(m):

$$f(x) = a_0 \oplus a_1 . x \oplus \ldots . \oplus a_{(m-1)} . x^{(m-1)} \qquad \ldots\ldots.(2.30)$$

where $\oplus$ and . are field operations addition and multiplication and a_i, i = 0,1,....,(m-1) are the elements of GF(m). Above equation may be reexpressed as:

$$f(x) = \Big[1 \; x \; \ldots . \; x^{(m-1)}\Big] . A_m\Big] \qquad \ldots\ldots.(2.31)$$

Replacing the m-values the variable x can take in the equation of (2.31) we obtain following simultaneous equations:

$$f(0) = \left[1\ 0\ \ldots\ldots\ 0\right] . \left.A_m\right]$$

$$f(1) = \left[1\ 1\ \ldots\ldots\ 1\right] . \left.A_m\right]$$

$$\vdots$$

$$f(m-1) = \left[1\ (m-1)\ldots(m-1)^{(m-1)}\right] . \left.A_m\right]$$

More conveniently this set of simultaneous equations can be re-expressed as:

$$\left.F\right] = \left[Sm\text{-}m\right] . \left.A_m\right], \qquad \ldots\ldots.(2.32)$$

where the elements $s_{i,j}$ of $\left[Sm\text{-}m\right]$ are:

$$s_{i,j} = i^j . \qquad \ldots\ldots.(2.32\ a)$$

Notice the similarity between the developments of this transform and the transform in Section 2.4. In this case, however, the multiplication operation is the finite field GF(m) operation and hence the elements of $\left[Sm\text{-}m\right]$ are the elements of the set $\{0,1,\ldots,(m-1)\}$. For example, below are the matrices $\left[Sm\text{-}m\right]$ for i) m = 2 ii) m = 3 iii) m = 4 cases:

$$\text{i)} \quad \left[Sm\text{-}2\right] = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\text{ii)} \quad \left[Sm\text{-}3\right] = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

$$\text{iii)} \quad \left[Sm\text{-}4\right] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 1 \\ 1 & 3 & 2 & 1 \end{bmatrix}$$

Let $\left[\mathrm{Tm\text{-}m}\right]$ be the inverse of $\left[\mathrm{Sm\text{-}m}\right]$. Several authors[2,4,9,10] have found the inverse of the matrix $\left[\mathrm{Sm\text{-}m}\right]$ following different approaches. It can be shown to be:

$$\left[\mathrm{Tm\text{-}m}\right] = \begin{bmatrix} 1 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & -1 & -(2^{-1}) & & & & & -((m-1)^{-1}) \\ 0 & -1 & -(2^{-2}) & & & & & \\ \vdots & \vdots & \vdots & & & & & \vdots \\ 0 & -1 & \cdot & & & & & -((m-1)^{-(m-2)}) \\ -1 & -1 & \cdot & \cdot & \cdot & \cdot & & -1 \end{bmatrix} . \qquad \ldots\ldots\ldots(2.33)$$

In this matrix $-\alpha$, $\alpha = 0,\ldots,(m-1)$, is the additive inverse whereas α^{-1}, $\alpha = 1,\ldots,(m-1)$, is the multiplicative inverse of $\alpha \in GF(m)$.

For example, below are the matrices $\left[\mathrm{Tm\text{-}m}\right]$ for i) m = 2 ii) m = 3 iii) m = 4 cases:

$$\text{i)} \quad \left[\mathrm{Tm\text{-}2}\right] = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\text{ii)} \quad \left[\mathrm{Tm\text{-}3}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 2 & 2 \end{bmatrix}$$

$$\text{iii)} \quad \left[\mathrm{Tm\text{-}4}\right] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Now, multiplying both sides of the equation (2.32) with the matrix $\left[\text{Tm-m}\right]$ we obtain the following:

$$\left.\text{Am}\right] = \left[\text{Tm-m}\right] . \left.\text{F}\right] \qquad ....(2.34)$$

Above equation is used to evaluate the coefficients $a_i; i = 0, ..., (n-1)$ necessary for the polynomial to represent a function f(x) in terms of the local values of the function f(x). Replacing this equation in (2.31) we obtain:

$$f(x) = \left[1 \; x \; x^2 .... x^{(m-1)}\right] . \left[\text{Tm-m}\right] . \left.\text{F}\right] \qquad ....(2.35)$$

A comparison of this equation with the canonical expansion of the function (eq 2.18) shows that

$$\left[x^{0,0} \; x^{1,1} \; ...... \; x^{(m-1),(m-1)}\right] = \left[1 \; x \; x^2 \; .... \; x^{(m-1)}\right] . \left[\text{Tm-m}\right] \qquad ....(2.36)$$

Note that we can do this comparison since when all multiplications in equation (2.18) are replaced by GF(m) multiplication operation the canonical expansion of (2.18) remains true. This is because of the functions, namely literals, used in that expansion. Every minterm of the form $x_0^{I_0,I_0} ..... x_{n-1}^{I_{n-1},I_{n-1}}$ takes either the value 0 or 1 which are respectively either the zero element or unity element in both fields, i.e. i) field of real numbers ii) finite field of GF(m). Finally, replacing equation (2.36) in equation (2.18) for every vector $\left[x_p^{0,0} \; ..... \; x_p^{(m-1),(m-1)}\right]$ we obtain:

$$f(X) = \left(\bigotimes_{p=0}^{(n-1)} \left[1 \; x_p \; .... \; x_p^{(m-1)}\right] . \left[\text{Tm-m}\right] \right) . \left.\text{F}\right] \qquad ....(2.37)$$

And using the properties of the Kronecker product this may be re-expressed as:

$$f(X) = \left(\bigotimes_{p\,=\,0}^{(n-1)} \left[1 \; x_p \; .... \; x_p^{(m-1)} \right] \right) . \left[\text{Tm-m} \right]^{\otimes\, n} . \left. F \right] \qquad ......(2.38)$$

The result of the multiplication of nth order $\left[\text{Tm-m}\right]$ with the column vector $\left. F \right]$ will be denoted $\left. A_m \right]$ and called the coefficient vector.

Example: Let the two-variable ternary function whose GF(3) polynomial expansion is required be defined by the following map:

$x_0 \backslash x_1$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

The coefficient vector for the polynomial which represents this function is evaluated as follows:

$$\left. A_m \right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 2 & 2 \end{bmatrix}^{\otimes\, 2} . \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \\ 2 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\ 2 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \\ 2 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} .$$

And hence the polynomial to represent above function is evaluated to be:

$$f(x_1,x_0) = \begin{bmatrix} 1 & x_0 & x_0^2 & x_1 & x_1.x_0 & x_1.x_0^2 & x_1^2 & x_1^2.x_0 & x_1^2.x_0^2 \end{bmatrix} . \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

i.e.

$$f(x_1,x_0) = x_0 \oplus x_1$$

It has been mentioned earlier in this section that the developments to derive the transform matrices $[\text{Sr-m}]$ $([\text{Tr-m}])$ and $[\text{Sm-m}]$ $([\text{Tm-m}])$ are very similar. We will now show that the transform matrices $[\text{Sr-m}]$ $([\text{Tr-m}])$ and $[\text{Sm-m}]$ $([\text{Tm-m}])$ are in fact equivalent in mod-m if m is a prime.

Let a and b be two integers, and m be a prime. Then:

$$(a + b) \text{ mod-m} = (a) \text{ mod-m} \oplus (b) \text{ mod-m} \qquad \text{......(2.39)}$$

and

$$(a\ b) \text{ mod-m} = (a) \text{ mod-m} . (b) \text{ mod-m} \qquad \text{......(2.40)}$$

Note that $\oplus$ and . are the symbols to denote GF(m) field operations addition and multiplication. If the order of the field GF(m) is a power k (≥ 2) of a prime then these operations are not the same as modulo addition and multiplication, and hence the above properties do not hold. To illustrate this consider the following example over the set V with 4 elements. Now, if we assume that $\oplus$ denotes the GF(4) addition operation then:

$$3 \oplus 1 = 2 \qquad \text{GF(4)},$$

but

$$3 \oplus 1 = 0 \qquad \text{in mod-4},$$

Also 2.2 = 3 if . is GF(4) multiplication operation but 2.2 = 0 in mod-4.

When m is a prime it can be shown from equations (2.21a) and (2.32a) and using above properties that:

$$\begin{aligned}(s_{i,j})_{\text{mod-m}} &= (i^j)_{\text{mod-m}} \\ &= i \underbrace{. i . .... .}_{\text{j-times}} i \\ &= s'_{i,j} \qquad \text{.......(2.41)}\end{aligned}$$

where $s_{i,j}$ and $s'_{i,j}$ are the elements of $\left[\text{Sr-m}\right]$ and $\left[\text{Sm-m}\right]$ respectively

Therefore;

$$\left(\left[\text{Sr-m}\right]\right)_{\text{mod-m}} = \left[\text{Sm-m}\right] \quad \text{(prime m)} \qquad \text{.......(2.42)}$$

and similarly

$$\left(\left[\text{Tr-m}\right]\right)_{\text{mod-m}} = \left[\text{Tm-m}\right] \quad \text{(prime m)} \qquad \text{.......(2.43)}$$

We can use the mod-m equivalence of the two transform matrices $\left[\text{Tr-m}\right]$ and $\left[\text{Tm-m}\right]$ to show the equivalence of the coefficient vectors. Let $\left.A\right]$ and $\left.A_m\right]$ be the coefficient vectors for real polynomial expansion and modular polynomial expansion respectively. Then:

$$\begin{aligned}
\left(\left.A\right]\right)_{\text{mod-m}} &= \left(\left[\text{Tr-m}\right] \left.F\right]\right)_{\text{mod-m}} \\
&= \left(\left[\text{Tr-m}\right]\right)_{\text{mod-m}} \cdot \left.F\right] \\
&= \left[\text{Tm-m}\right] \cdot \left.F\right] \\
&= \left.A_m\right] \qquad \text{.......(2.44)}
\end{aligned}$$

Example: Consider the two-variable ternary function ($\overline{\text{Plus}}$) given by the following map:

x_0 \ x_1	0	1	2
0	2	1	0
1	1	0	0
2	0	0	0

The coefficient vector $A]$ for the real polynomial to represent this function is evaluated using equation (2.29):

$$A] = \begin{bmatrix} 1 & 0 & 0 \\ -1.5 & 2 & -0.5 \\ 0.5 & -1 & 0.5 \end{bmatrix}^{\otimes 2} \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \\ -1 \\ -1,5 \\ 1 \\ 0 \\ 1 \\ -0.5 \end{bmatrix} .$$

and hence the real polynomial to represent this function will be of the form:

$$f_r(x_1,x_0) = 2 - x_0 - x_1 - 1.5\, x_0x_1 + x_0^2x_1 + x_0x_1^2 - 0.5\, x_0^2x_1^2$$

The coefficient vector $A_m]$ for the modular polynomial expansion of this function is found from the vector $A]$ as follows:

$$A_m] = \left(A]\right)_{\text{mod-m}}$$

$$= \left(\begin{bmatrix} 2 & -1 & 0 & -1 & -1.5 & 1 & 0 & 1 & -0.5 \end{bmatrix}^t\right)_{\text{mod-m}}$$

$$= \begin{bmatrix} 2 & 2 & 0 & 2 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}^t$$

since $(-1)_{\text{mod-m}} = 2$,

$$(-1,5)_{\text{mod-m}} = (-3 \;\; 2^{-1})_{\text{mod-m}}$$

$$= (-3)_{\text{mod-m}} \cdot (2^{-1})_{\text{mod-m}}$$

$= 0 . 2$

$= 0$, etc.

Hence, the modular expansion of the function will be of the form:

$$f_m(x_1,x_0) = 2 + 2.x_0 + 2.x_1 + x_0^2.x_1 + x_0.x_1^2 + x_0^2.x_1^2$$

The evaluation of the coefficient vector $A_m]$ for this function using equation (2.38) confirms this result. Indeed, using (2.38) the coefficient vector $A_m]$ is found to be:

$$A_m] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 2 & 2 \end{bmatrix}^{\otimes 2} . \left.\begin{matrix} 2 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix}\right] = \left.\begin{matrix} 2 \\ 2 \\ 0 \\ 2 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{matrix}\right] .$$

The algebraic developments covered in this chapter will be used as required in the subsequent chapters of this thesis.

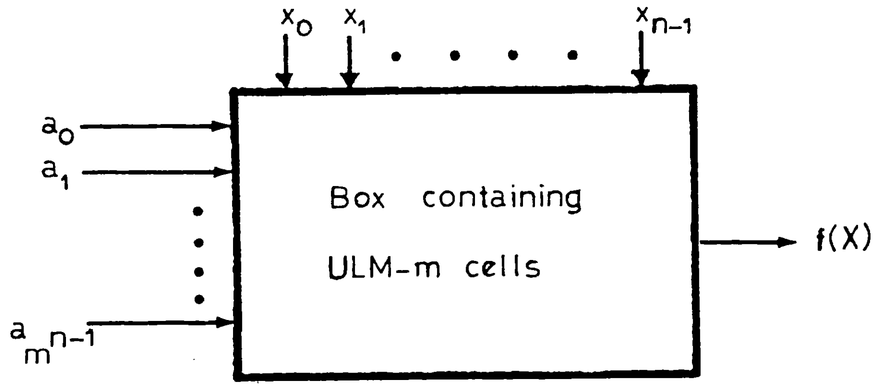
CHAPTER 3

A Universal Logic Module Consideration

3. A Universal Logic Module Consideration:

The continuing developments in integrated circuit technology are expected to achieve chips containing one million binary gates by the year 2,000[11]. With such a large number of gates available on a single chip design methods based on savings at gate level become unnecessary and uneconomical. Instead a modular design procedure which uses simple cells as basic components, allows the construction of complex special-purpose chips, at the same time keeping the design costs at low levels. ULM (Universal Logic Module) implementations are well suited for this purpose. A ULM is a cell with M inputs which is capable of implementing all n-variable ($n \leq M$) functions. The simple and regular circuit topology to which ULM realisations lead is very important, since with such a layout the network interconnections between the cells are kept short and hence long distance or irregular communication is minimized[12].

The m-ary universal logic module ULM-m proposed here is an extension of the binary ULM proposed by Murugeson, [13] and is based on the modular expansion of m-ary functions studied in the last chapter (Sect. 2.5). The general block diagram realisation of a function f(X) would be as follows:



where a_i's, $i = 0,1,\ldots,(m^n-1)$ are set at values given by Generalised Reed-Muller expansion coefficients of the function f(X) and x_p's, $p = 0,1,\ldots,(n-1)$ are the independent input variables.

In the following sections a detailed study of the above block diagram will be made and a method to minimize the number of ULM-m cells in the realisation of m-ary functions will be described.

3.1 Universal Logic Modules for the realisation of m-ary functions:

It was shown in Section 2.5 that in the case when m is a power of a prime q_r a polynomial expansion of the form

$$f(X) = \left(\bigotimes_{p=0}^{n-1} \left[1 \; x_p \; \ldots \; x_p^{(m-1)} \right] \right) . A\Big] \qquad \ldots\ldots.(3.1)$$

is possible for the representation of any n-variable m-ary function. It was also shown that the coefficient vector $A\Big]$ for a particular function f(X) can be evaluated by an appropriate transformation $\left[Tm\text{-}m \right]$ from the vector $F\Big]$ which is constructed by decimal ordering of the function local values, as follows:

$$A\Big] = \left[Tm\text{-}m \right]^{\otimes n} . \; F\Big] \qquad \ldots\ldots\ldots(3.2)$$

where the addition and multiplication operations are the operations over the Galois Field $GF(q_r)$. This polynomial expression of a m-ary function is also referred to as Generalised Reed-Muller form[2,4] and the coefficients are called Reed-Muller coefficients.

The encouraging developments[14] in I^2L technology in recent years have made the physical implementations of modular operations (addition and multiplication) more feasible. With the basic connectives of addition and multiplication available, a module whose circuit diagram is given in Fig. 3.1 may be built to implement all one-variable (n = 1) m-ary functions with the appropriate settings at the a_i, i = 0,....,(m-1) inputs. Indeed, the behaviour of this circuit can be represented mathematically by a polynomial of the form

$$f(x) = a_0 \oplus a_1.x \oplus a_2.x^2 \oplus .... \oplus a_{(m-1)}.x^{(m-1)} \qquad ....(3.3)$$

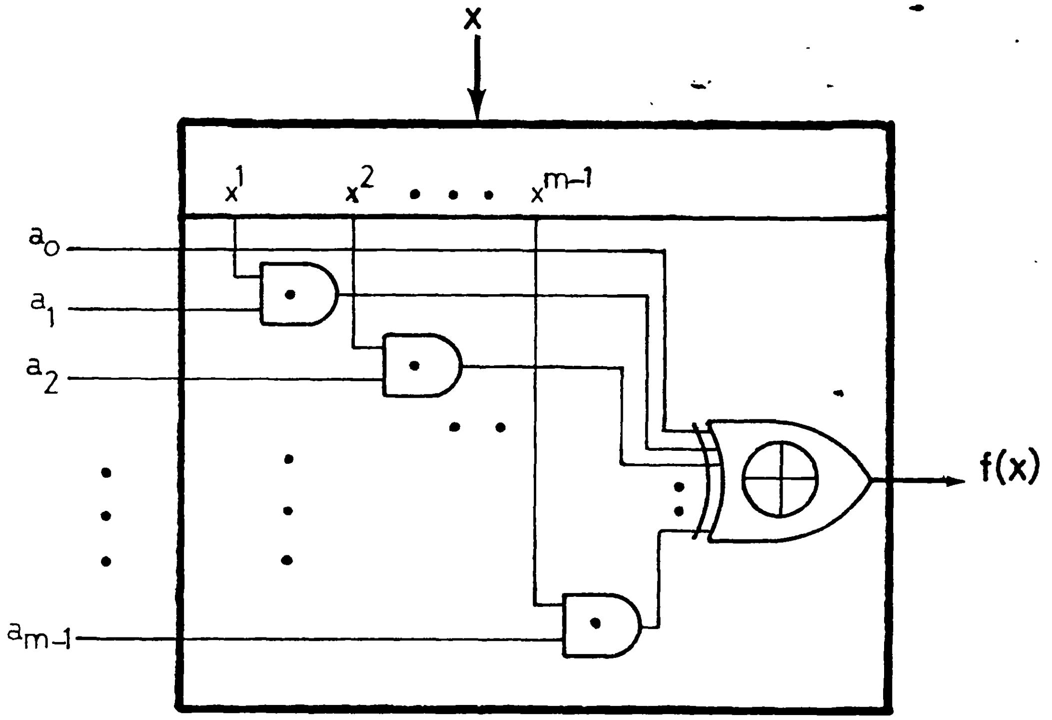


Fig. 3.1. Fundamental ULM-m module for m-ary logic functions, n = 1.

Such a module which has M-inputs (in this specific case M = m + 1), and is capable of implementing all n-variable m-ary functions, is called a universal logic module (ULM-m).

The triangular topology as shown in Fig. 3.2 for the realisation of n-variable, $n > 1$, m-ary functions using ULM-m's follows immediately from the corresponding polynomial expression form. Indeed, the polynomial expression for n-variable m-ary function representation takes the form:

$$f(X) = f_0(x_{n-2},\ldots,x_0) \oplus x_{n-1}.f_1(x_{n-2},\ldots,x_0) \oplus \ldots.$$

$$\ldots. \oplus x_{n-1}^{(m-1)}.f_{(m-1)}(x_{n-2},\ldots,x_0) \qquad \ldots\ldots.(3.4)$$

where $f_k(x_{n-2},\ldots,x_0)$, $k = 0,\ldots,(m-1)$, are all $(n-1)$ variable functions. Once again the appropriate set of coefficients for the realisation of a given function f(X) is computed by the equation (2.38).

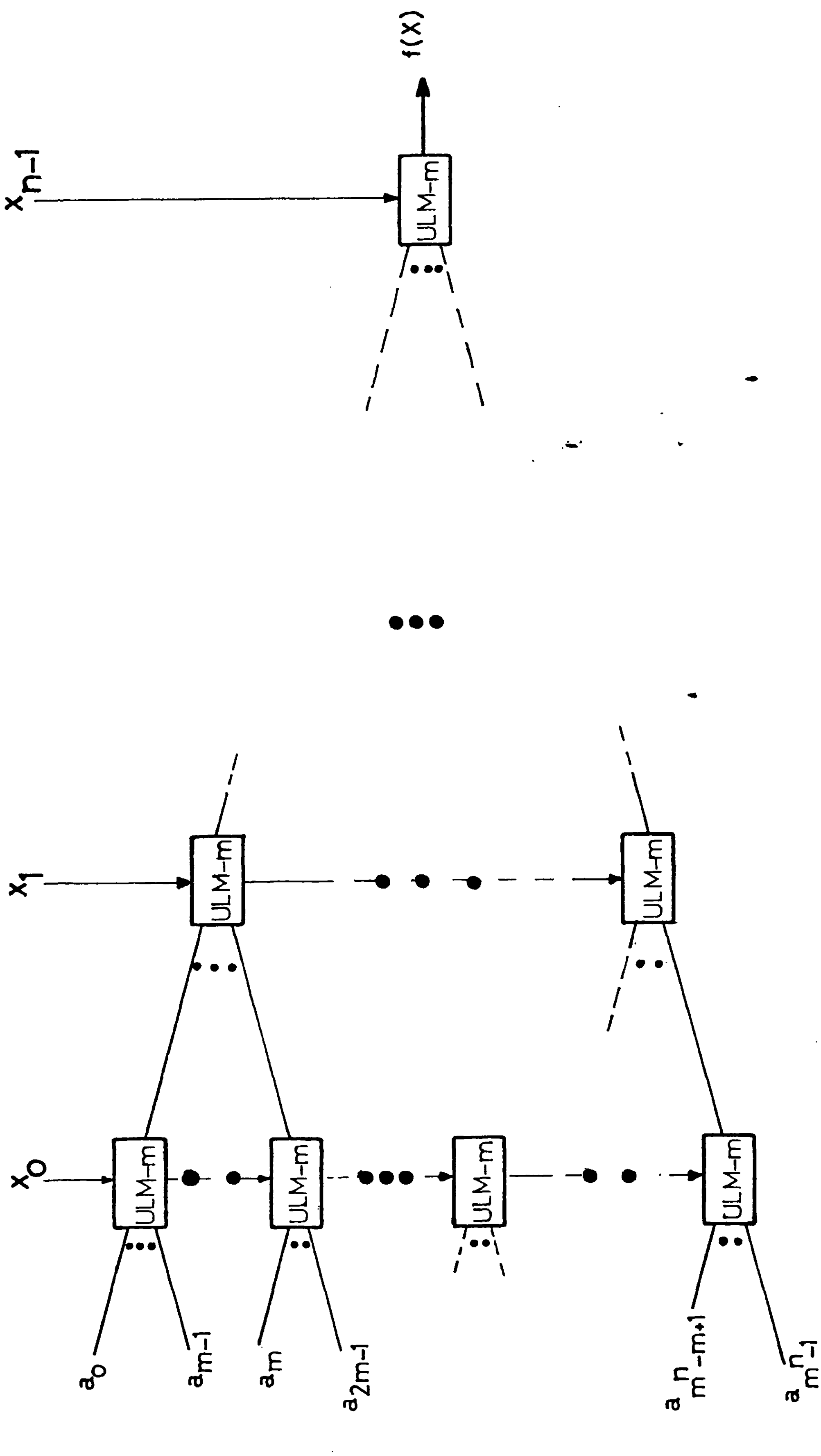
Example: Consider the two-variable ternary function ($n = 2$, $m = 3$) given by the map below:

x_0 \ x_1	0	1	2
0	1	0	2
1	2	1	1
2	0	2	0

The Reed-Muller coefficients for this function are computed to be:

$$A\Big] = \Big[\mathrm{Tm}\text{-}3\Big]^{\otimes 2}. \Big[1 \quad 2 \quad 0 \quad 0 \quad 1 \quad 2 \quad 2 \quad 1 \quad 0\Big]^t$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 2 & 2 \end{bmatrix}^{\otimes 2} .\Big[1 \quad 2 \quad 0 \quad 0 \quad 1 \quad 2 \quad 2 \quad 1 \quad 0\Big]^t$$



<u>Fig. 3.2:</u> Triangular topology for the realisation of n variable m-ary functions using ULM-m .

$$= \begin{bmatrix}1 & 1 & 0 & 2 & 1 & 0 & 0 & 2 & 0\end{bmatrix}^t$$

The circuit diagram of the proposed ULM for the ternary case and the realisation of above function using ULM-3's are shown in Fig. 3.3(a) and (b) respectively.

3.2 The effect of interchange of variables on ULM-m realisations

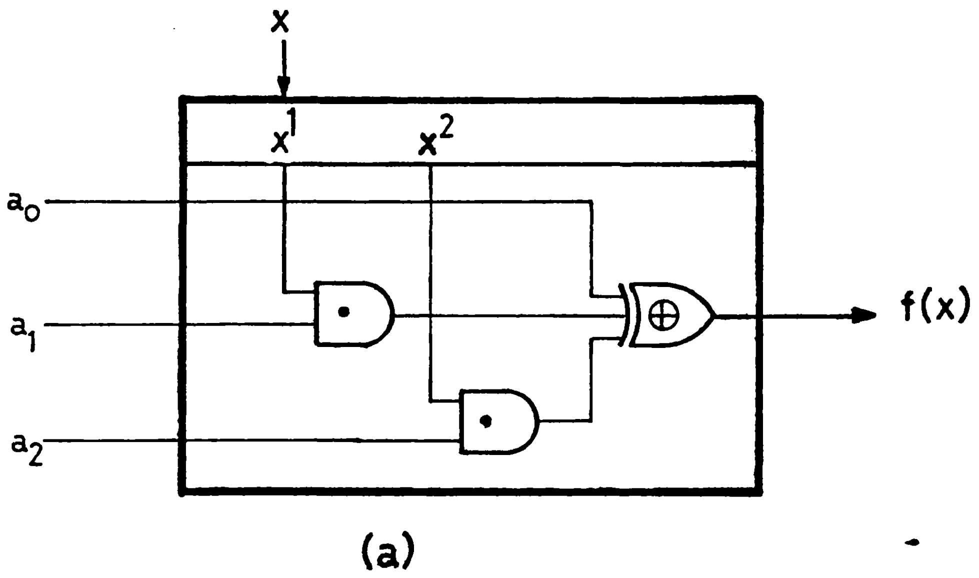
Once the Reed-Muller coefficients necessary to realise a fully specified function f(X) are evaluated by use of the appropriate transformation $\left[Tm\text{-}m\right]$ the circuit configuration to implement the function f(X) using ULM-m's is straightforward. It should be noted that there is one ULM-m at the final level, m ULM-m's at the (n - 1)th level, (m x m) ULM-m's at the (n - 2)th level... (and so on) of the final realisation, see Fig. 3.2. Therefore in general the number of ULM-m's necessary to implement a n-variable m-ary function can never exceed:

$$1 + m + m^2 + \dots + m^{(n-1)} = \frac{m^n - 1}{m - 1} \quad .\qquad \dots\dots\dots(3.4)$$

The exponential growth of number of ULM-m's necessary for the implementation of any given function with the number of variables n upon which the function depends, is undesirable from the practical point of view. However, the total number of modules necessary for the final realisation may be reduced if one or more of the following conditions are satisfied.

i) If all the coefficients of a particular ULM-m module are zero-valued then clearly it is unnecessary to include this module in the final realisation,

ii) Equally, if all but the a_0 coefficient (see Fig. 3.1) are zero, then a_0 is fed through unchanged and the ULM-m is again unnecessary,



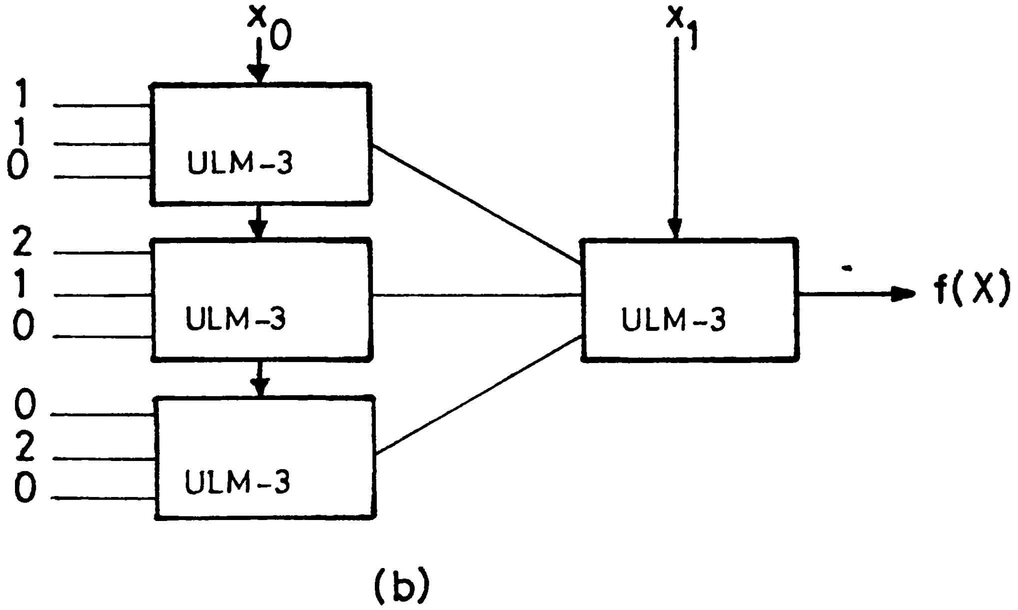


Fig. 3.3: a) The circuit diagram for ULM-3,

b) The realisation of the two variable ternary function

$F] = [1 \quad 2 \quad 0 \quad 0 \quad 1 \quad 2 \quad 2 \quad 1 \quad 0]^t$ using ULM-3 s.

iii) Finally, ~~the~~ two or more ULM-m's with identical a_i inputs at any level of the realisation can be combined into a single ULM-m.

A basic operation we can perform to utilise above three conditions which leads to a possible minimisation of modules without effecting the fundamental ULM-m realisation topology of Fig. 3.2, is the interchange of inputs in the final realisation. Assume that the x_p and x_q inputs to the final ULM-m realisation of the function f(X) are interchanged, see Fig. 3.4. This in turn requires the rearrangement of the order of coefficients such that the function realised by ULM-m remains unaltered. Indeed this requirement can readily be seen from the polynomial expansion of the function f(X):

$$f(X) = \left(\bigotimes_{p=0}^{n-1} \left[1 \ x_p \ . \ . \ x_p^{(m-1)} \right] \right) . \ A \rbrack \qquad ....(3.5)$$

Here an entry a_i of the column vector $A\rbrack$ is the coefficient of the term $x_{n-1}^{I_{n-1}} . \ . \ x_p^{I_p} . \ x_q^{I_q} \ . \ . \ x_0^{I_0}$, such that

$$i = \sum_{p=0}^{(n-1)} I_p m^p \ . \qquad ....(3.6)$$

With the interchange of variables x_p and x_q, a new function $f'(X)$ is obtained such that:

$$f'(x_{n-1},....,x_p,....,x_q,....,x_0) = f(x_{n-1},....,x_q,....,x_p,....,x_0) \qquad ....(3.7)$$

Hence with the original ordering of the coefficients a_i will now appear with the monomial $x_{n-1}^{I_{n-1}} .... \ x_q^{I_p} .... \ x_p^{I_q} .... \ x_0^{I_0}$. Therefore for the

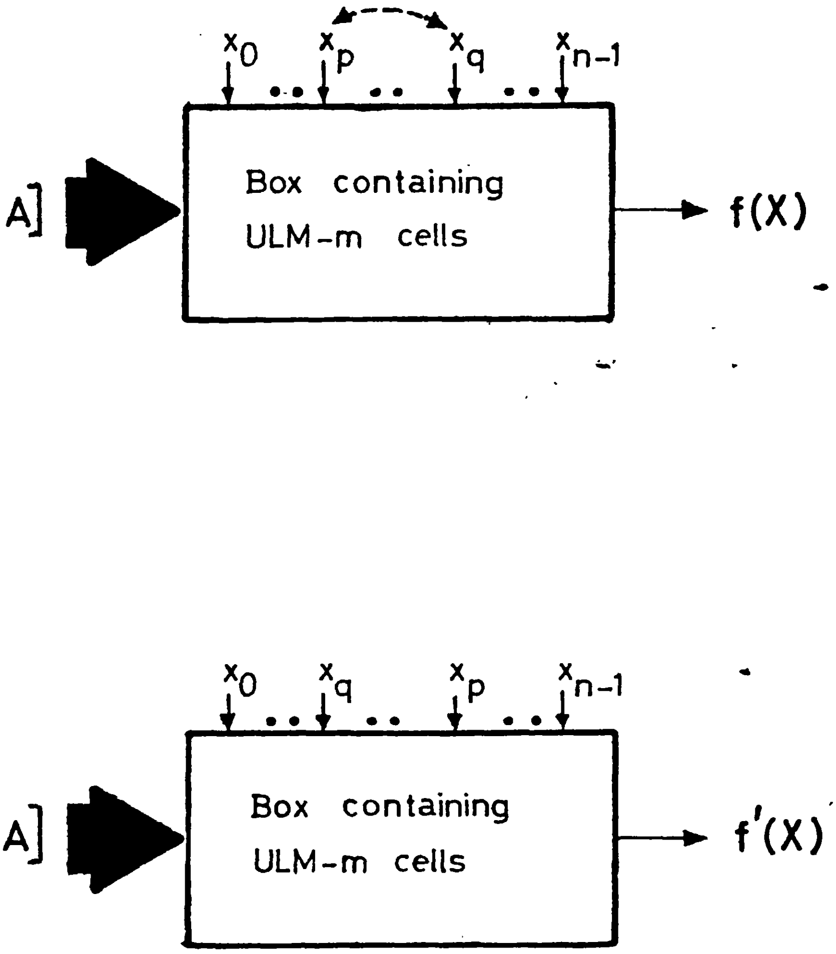


Fig 3.4: Interchange of input connections to ULM-m realisation

ULM-m block to realise the original function f(X), the "coefficient" inputs to the circuit should be rearranged such that when the input values are written as a column vector $A']$ with entries a_i', the entries a_i' correspond to entries a_i of the original coefficient vector $A]$ as follows:

$$a_i' = a_i$$

where, $i' = I_{n-1}\ m^{(n-1)} + \ldots. + I_p\ m^p + \ldots. + I_q\ m^q + \ldots. + I_0\ m^0$

and $\quad i = I_{n-1}\ m^{(n-1)} + \ldots. + I_q m^p + \ldots. + I_p\ m^q + \ldots. + I_0\ m^0$.

Because the inputs to each ULM-m are altered as described without effecting the original function being implemented, the three conditions that lead to the minimisation may be sought and utilised.

To illustrate this with an example, consider the two-variable ternary function $f(x_1,x_0)$ given in the example of Section 3.1: We repeat the function value vector $F]$ and the coefficient vector $A]$ for convenience:

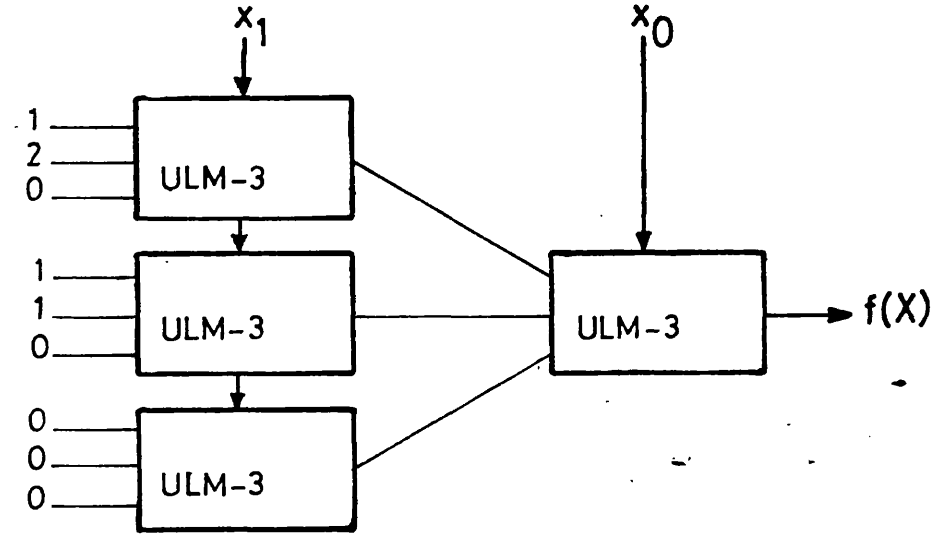
$$F] = \begin{bmatrix}1 & 2 & 0 & 0 & 1 & 2 & 2 & 1 & 0\end{bmatrix}^t$$

$$A] = \begin{bmatrix}1 & 1 & 0 & 2 & 1 & 0 & 0 & 2 & 0\end{bmatrix}^t$$

Clearly the realisation shown in Fig. 3.3 is minimal for the particular chosen pattern of x_0 and x_1 input connections, since none of the three conditions apply for this particular realisation and the number of ULM-m's therefore required is 4. However, with the interchange of $x_1 \leftrightarrow x_0$ the following re-ordered coefficient vector $A']$ is obtained:

$$A'] = \begin{bmatrix}1 & 2 & 0 & 1 & 1 & 2 & 0 & 0 & 0\end{bmatrix}^t$$

This now gives the ULM-3 realisation as follows:



Here the lowest module in the first-level of realisation is redundant because all its coefficients are zero-valued, and thus the realisation now requires a total of three rather than four ULM cells.

It should be noted that in the n-variable case the number of combinations of possible input connections is 1 x 2 x .... x(n-1) x n = n!. This is a function that grows faster than exponential and for each combination of input connections the check for the possible minimisation is obviously a very difficult task. To the author's knowledge there is no efficient algorithm that solves this problem, once the number of variables becomes greater than can be goemetrically plotted in map form.

In conclusion we should note that the physical realisations of GF(m) operations are not backed by available hardware if m is a power of a prime, as in the case m = 4. However, base-4 adder and multiplier circuits in reference 14 may be modified to implement addition and multiplication in base-3 which are in this case the same as GF(3) operations.

CHAPTER 4

Spectral Considerations

4. Spectral Considerations

In the previous chapters the real and modular polynomial expansions of m-ary functions were considered and a universal logic module realisation of m-ary functions discussed. It was shown that ULM-m realisations are restricted to m-being a power of a prime.

Here we will consider the polynomial expansions of m-ary functions over the field of complex numbers. In this case the coefficient set obtained will be called the spectrum of the function. Spectral transformation of binary functions and binary logic synthesis using spectral data has been reported by a number of authors.[1,16,18,19,20] The extension of the work from binary to m-ary logic area has been considered by Karpovsky[16] and Moraga[5]. Here we pursue further this general area. A summary of the spectral properties proved in this chapter is given in Appendix A.

4.1 The polynomial expansions of m-ary functions over the field of complex numbers:

We define an isomorphism between the additive group of integers mod-m and the multiplicative group of complex numbers as follows:

$$c : x \longrightarrow a^x$$

where $a = e^{j\frac{2\pi}{m}}$ and

$$x \in V = \{0,1,\dots\dots,(m-1)\}$$

We shall denote by y the image a^x of x obtained by the isomorphism c. The codomain of this mapping is called the character group of V.[5]

Here we derive the transformation required to express a m-ary function by a complex polynomial, following a development similar to that of Section 2.4 and Section 2.5. In the one variable case such a polynomial will be of the form:

$$\begin{aligned} f(x) &= s'_0 + s'_1\, y + \dots\dots + s'_{m-1}\, y^{m-1} \\ &= \left[1\ y\ y^2\ \dots\dots\dots\ y^{m-1}\right] S'] \end{aligned} \qquad \dots\dots\dots(4.1)$$

where $S'] = \left[s'_0\ s'_1\ \dots\dots\dots\ s'_{m-1}\right]^t$,

and $y = a^x$, character of x as defined above.

Now substituting the m-values which the variable x can take we obtain the following simultaneous equations:

$$\begin{aligned} f(0) &= \left[1\quad 1\quad \dots\dots\quad 1\right] S'] \\ f(1) &= \left[1\quad a\quad a^2 \qquad a^{m-1}\right] S'] \\ &\ \ \vdots \\ f(m-1) &= \left[1\quad a^{m-1}\quad \dots\dots\quad a^{(m-1)^2}\right] S'] \end{aligned} \qquad \dots\dots\dots(4.2)$$

In matrix form this set of simultaneous equations may be expressed as:

$$F] = \left[\overline{Tc-m}\right] S'] \qquad \dots\dots\dots(4.3)$$

where the elements $\bar{t}_{i,j}$ of the matrix $\left[\overline{Tc-m}\right]$ are:

$$\bar{t}_{i,j} = a^{ij} \qquad \dots\dots\dots(4.4)$$

A careful examination of the matrix $\left[\overline{Tc-m}\right]$ will show that:

a) it is symmetric, since;

$$\bar{t}_{i,j} = a^{ij}$$

$$= a^{ji}$$

$$= \bar{t}_{j,i}$$

b) the sum of elements of each row except the first row is equal to zero. For the k'th row this sum will be:

$$\sum_{i=0}^{m-1} a^{ki} = 1 + a^{k} + a^{2k} + \ldots + a^{(m-1)k}$$

$$= \frac{1 - a^{mk}}{1 - a}$$

but $a^{mk} = e^{(j\frac{2\pi}{m})\, mk}$

= 1, making the numerator of above expression zero, and therefore

$$\sum_{i=0}^{(m-1)} \bar{t}_{k,i} = 0$$

c) the element by element multiplication of each row with complex conjugate of any other row gives another row of the matrix $[\overline{Tc-m}]$.

For proof, consider the pth and qth rows of $[\overline{Tc-m}]$:

pth row = $[1 \quad a^{p} \quad \ldots\ldots \quad a^{(m-1)p}]$,

qth row = $[1 \quad a^{q} \quad \ldots\ldots \quad a^{(m-1)q}]$,

the complex conjugate of the qth row gives:

(qth row)* = $[1 \quad a^{-q}. \quad \ldots \quad a^{-(m-1)q}]$, and element by element multiplication of the two vectors will give:

$$\left[\,1 \quad a^{p-q} \quad . \; . \; . \; . \; . \; . \; . \quad a^{(m-1)(p-q)}\right].$$

This obviously is another row of $[\overline{Tc-m}]$, and since p is not equal to q this row is not the first row which contains all "ones".

d) the inner product of two rows is zero. This is because the element by element multiplication of one row with the complex conjugate of any other row gives another row other than the first row of $[\overline{Tc-m}]$ and the sum of the elements of any row other than the first is zero i.e. the rows of $[\overline{Tc-m}]$ are mutually orthogonal. Note that inner product of any row with itself gives m.

The inverse of $[\overline{Tc-m}]$ can now be easily established from above observations a,b,c and d.

<u>Theorem 4.1</u>: The inverse of $[\overline{Tc-m}]$ is given by:

$$[\overline{Tc-m}]^{-1} = \frac{1}{m}\,[Tc-m]$$

where $[Tc-m]$ is the complex conjugate of $[\overline{Tc-m}]$

Indeed, since $[\overline{Tc-m}]$ has mutually orthogonal rows the inverse of $[\overline{Tc-m}]$ is given by:

$$[\overline{Tc-m}]^{-1} = \frac{1}{m}\,[Tc-m]^{t} \;,$$

but because $[\overline{Tc-m}]$ is symmetric, so also is $[Tc-m]$, giving;

$$[\overline{Tc-m}]^{-1} = \frac{1}{m}\,[Tc-m] \qquad .....(4.5)$$

The scaling factor $\frac{1}{m}$ is due to the fact that the inner product of any row with itself gives m.

Below are some examples of the matrix $\left[Tc-m\right]$;

i) m = 2 (binary)

$$\left[Tc-2\right] = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

ii) m = 3 (ternary)

$$\left[Tc-3\right] = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix}$$

where $a = e^{j\frac{2\pi}{3}}$, $= 1 \underline{/120^{\circ}}$

iii) m = 4 (quaternary)

$$\left[Tc-4\right] = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$

where $j = e^{j\frac{2\pi}{4}}$, $= \sqrt{-1}$

$= 1 \underline{/90^{\circ}}$

If we multiply both sides of equation (4.3) with $\left[Tc-m\right]$ we obtain:

$$S'\Big] = \frac{1}{m} \quad \left[Tc-m\right] F\Big] \qquad \text{........(4.6)}$$

Replacing (4.6) in equation (4.1) we have:

$$f(x) = \frac{1}{m}\left[1 \; y \; . \; . \; . \; . \; y^{m-1}\right]\left[Tc-m\right] \; F\Big] \qquad \text{.......(4.7)}$$

Comparing this equation with equation (2.18) we find that:

$$\left[x^{0,0} \; x^{1,1} \; . \; . \; . \; . \; x^{(m-1),(m-1)} \right] = \frac{1}{m} \left[1 \; y \; . \; . \; . \; . \; y^{m-1} \right] \left[Tc\text{-}m \right] \quad ....(4.8)$$

Finally, replacing equation (4.8) for every vector $\left[x_p^{0,0} \; . \; . \; x_p^{(m-1),(m-1)} \right]$ in equation (2.18) and using the distributive properties of Kronecker product we obtain:

$$f(X) = \left(\bigotimes_{p=0}^{(n-1)} \frac{1}{m} \left[1 \; y_p \; y_p^2 \; . \; . \; . \; . \; y_p^{m-1} \right] \left[Tc\text{-}m \right] \right) F] \quad ......(4.9)$$

$$= \frac{1}{m^n} \left(\bigotimes_{p=0}^{(n-1)} \left[1 \; y_p \; y_p^2 \; . \; . \; . \; . \; y_p^{m-1} \right] \right) \left[Tc\text{-}m \right]^{\otimes n} \; F] \quad ......(4.10)$$

Let cF] denote the vector obtained from vector F] by replacing all its elements with their images obtained by the mapping c. In the following we will call the vector S] obtained by the product of [Tc-m] and cF] and scaling factor $\frac{1}{m^n}$ the <u>spectrum</u>, i.e.

$$S] = \frac{1}{m^n} \left(\left[Tc\text{-}m \right]^{\otimes n} \right) cF] \quad ....(4.11)$$

Equation (4.10) will then be altered as:

$$cf(X) = \left(\bigotimes_{p=0}^{(n-1)} \left[1 \; y_p \; y_p^2 \; . \; . \; . \; . \; y_p^{m-1} \right] \right) S] \quad ....(4.12)$$

where $S] = \left[s_0 \quad s_1 \; . \; . \; . \; . \; s_{m^n-1} \right]^t$.

Multiplying both sides of (4.11) with the inverse transform matrix, the matrix relationship between the spectrum and the local values of the function is obtained. This relationship would be of the form:

$$cF] = \left(\left[Tc\text{-}m \right]^{\otimes n} \right)^{*} S]$$

where * stands for the complex conjugate.

Example 1. The truth table of a two variable binary function EX-OR is given below:

x_1	x_0	$f(x_1,x_0)$	$cf = (-1)^{f(x_1,x_0)}$
0	0	0	1
0	1	1	-1
1	0	1	-1
1	1	0	1

The spectrum of this function is given by:

$$S] = \frac{1}{4} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{\otimes 2} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} .$$

Example 2. Consider the two-variable ternary function $\overline{\text{Plus}}$, given by the following truth table.

x_1 x_0	0	1	2
0	2	1	0
1	1	0	0
2	0	0	0

If now the function values 0,1 and 2 are mapped onto 1, a and a^2 as defined by the mapping c, the truth table will be:

x_1 x_0	0	1	2
0	a^2	a	1
1	a	1	1
2	1	1	1

The spectrum for this function is computed by the transform:

$$S] = \frac{1}{3^2} \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix}^{\otimes 2} \left. \begin{matrix} a^2 \\ a \\ 1 \\ a \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{matrix} \right]$$

giving:

$$S]^t = [\ a+5 \quad a^2+2a \quad 4a^2+2a \quad a^2+2a \quad a+2 \quad 2a^2+1 \quad 4a^2+2a \quad 2a^2+1 \quad 5a^2+1\ ]$$

where $a = e^{j\frac{2\pi}{3}}$.

4.2 Some properties of spectrum

We have shown in Section 4.1 that an element $t_{i,j}$ of the m-ary transform matrix $\left[\text{Tc-m}\right]$ is given by:

$$t_{i,j} = (a^{ij})^*$$

where $a = e^{j\frac{2\pi}{m}}$ and * stands for the complex conjugate.

The nth order m-ary transform matrix is the nth Kronecker power of $\left[\text{Tc-m}\right]$ and by definition of nt Kronecker product (Sect. 2.1) an element $t_{i,j}$ of the nth order transform matrix is given by:

$$t_{i,j} = \prod_{p=0}^{(n-1)} t_{I_p,J_p} \qquad \text{........(4.13)}$$

where $(I_{n-1},\ldots\ldots, I_0)$ and $(J_{n-1},\ldots\ldots, J_0)$ are the m-ary (integers mod-m) expansions of i and j.

Replacing the values of t_{I_p,J_p} in equation (4.13) we have:

$$t_{i,j} = \prod_{p=0}^{(n-1)} (a^{I_pJ_p})^*$$

$$= \left(a^{Ch(i,j)}\right)^* \qquad \text{.........(4.14)}$$

where $Ch(i,j) = \sum_{p=0}^{(n-1)} I_pJ_p$.

The step functions $t_i(j) = t_{i,j}$ defined in the close interval $[0,m^n)$ as above form a complete orthogonal basis and are a different ordering of Chrestenson functions[5,16,17]. Chrestenson functions (call them $h_i(j)$ in order to distinguish them from $t_i(j)$) are defined in references 5 and 16 as follows:

$$h_i(j) = \left(a^{\sum_{p=0}^{n-1} I_{n-1-p} J_p} \right)^* \qquad \text{.......(4.15)}$$

Let m-ary expansions of i and i' be $(I_{n-1}, \ldots., I_0)$ and $(I_0, \ldots., I_{n-1})$ respectively. It is clear from above definitions of $h_i(j)$ and $t_i(j)$ that:

$$h_{i'}(j) = t_i(j)$$

and $t_{i'}(j) = h_i(j)$.

A n x n matrix $[Jd]$ may be constructed such that the elements $Jd_{p,q}$ are given by:

$$Jd_{p,q} = \begin{cases} 1 & \text{if } p = n-q \\ 0 & \text{otherwise.} \end{cases} \qquad \text{........(4.16)}$$

The relationships between the two vectors $[I_{n-1} \ldots . I_0]$ and $[I_0 \ldots . I_{n-1}]$ is then shown to be:

$$[I_{n-1} \ldots . I_0] = [I_0 \ldots . I_{n-1}] [Jd] \qquad \text{........(4.17)}$$

This shows that the spectrum obtained using the functions $h_i(j)$ will contain the same spectral coefficients in a different order as the spectrum obtained using the functions $t_i(j)$. The recursive construction of the nth order transform matrix $[Tc\text{-}m]^{\otimes n}$ defines a fast algorithm to compute the spectrum with the minimum number $mm^n(m-1)$ of operations[16].

Let us now consider that we have a set of m-ary logic gates available. The set 0_k of functions that correspond to these gates is called a functionally complete set if all m-ary functions can be generated by a combination of following operations[15];

i) interchange of variables

$$f'(x_{n-1},..,x_p,.,x_q,..,x_0) = f(x_{n-1},..,x_q,.,x_p,..,x_0)$$

where $f(X) \varepsilon 0_k$

ii) making two or more variables identical

$$f'(x_{n-2},x_{n-3},..,x_0) = f(x_{n-2},x_{n-2},x_{n-3},..,x_0) \text{ where } f(X) \varepsilon 0_k$$

iii) setting variable(s) to a constant(s)

$$f'(x_{n-2},....,x_0) = f(h,x_{n-2},....,x_0) \text{ where } f(X) \varepsilon 0_k \text{ and } h \varepsilon V$$

iv) composition of functions

$$f'(x_{n-1},....,x_0) = f(g(x_{n-1},..,x_p),x_{p-1},..,x_0) \text{ where } f(x_p,....,x_0)$$

and $g(x_{n-1},....,x_p) \varepsilon 0_k$.

The corresponding operations for the gates are illustrated in Fig. 4.1. It is clear that any m-ary function is physically realisable utilising gates which are physical implementations of the functions from the set 0_k if and only if the set 0_k is functionally complete. The functional completeness properties of m-ary algebras are summarised in reference 15. Reference 15 also contains a comprehensive bibliography on this subject.

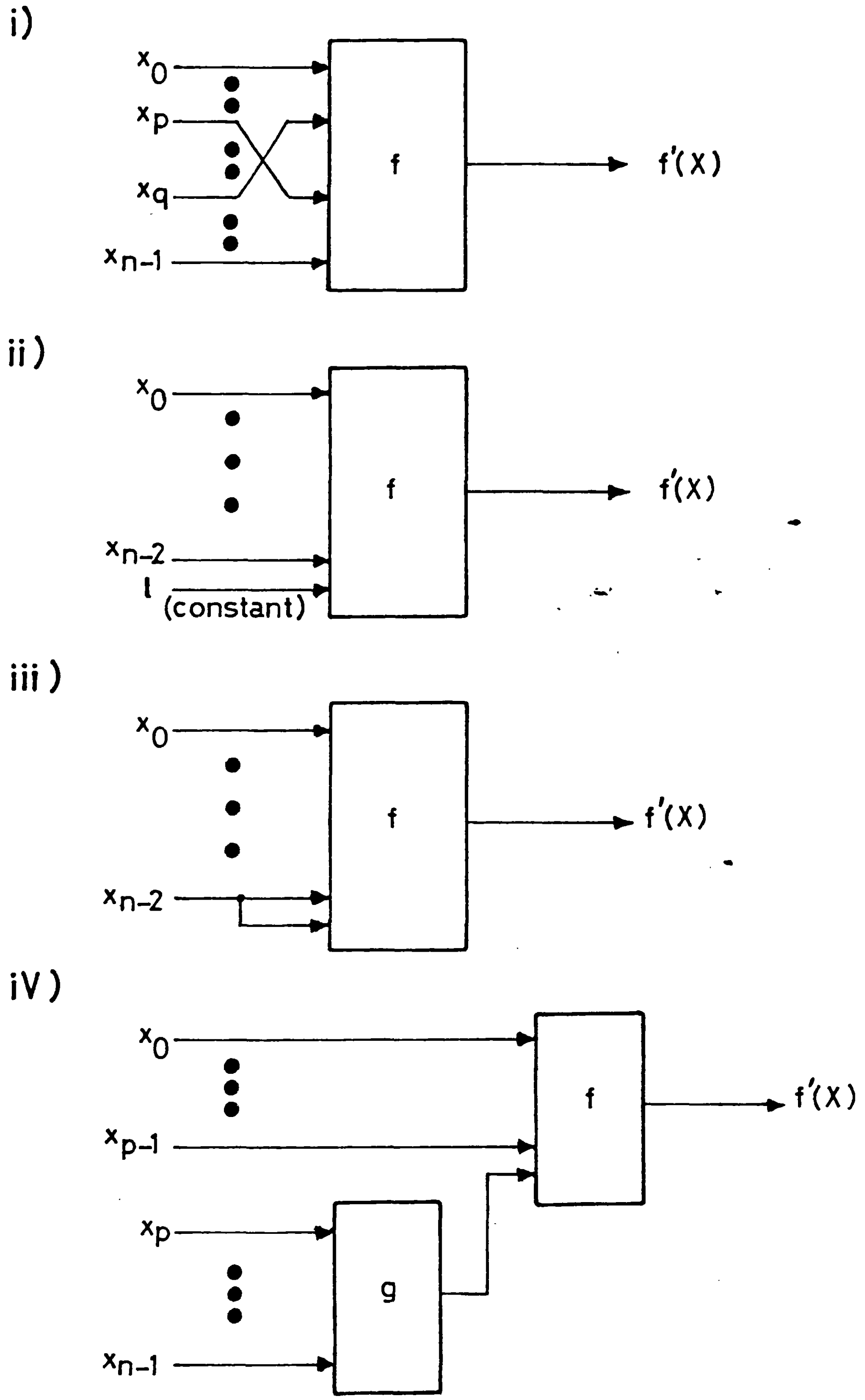


Fig. 4.1 The operations considered i) Interchange of inputs,

ii) "Held-at" one gate input,

iii) Commoning two inputs,

iv) Cascade composition of gates.

Some examples of a functionally complete set of functions used in this thesis are:

a) The set containing unary functions as defined in Section 2.3 together with the two-variable functions multiplication and addition,

b) The set containing Galois Field operations addition and multiplication in the case when m is a power of a prime.

Now, let us continue with developments to investigate the modifications to spectral coefficients with the operations i) through iv). Unless otherwise noted we shall denote the spectrum of the function f(X) by $S_f]$ and the ith element of the vector $S_f]$ by $s_f(i)$. $S_{f'}]$ and $s_{f'}(i)$ are defined likewise.

Theorem 4.2 (Interchange of Variables).

Let $f'(x_{n-1},..,x_p,..,x_q,..,x_0) = f(x_{n-1},...,x_q,..,x_p,..,x_0)$

then $s_{f'}(i) = s_f(i')$

where $i = I_{n-1}m^{n-1} + ... + I_pm^q + .. + I_qm^p + ... + I_0$

and $i' = I_{n-1}m^{n-1} + ... + I_qm^q + .. + I_pm^p + ... + I_0$

.......(4.18)

This theorem is a special case of linear transformation of arguments theorem 4.7 and proof of the latter will be found subsequently.

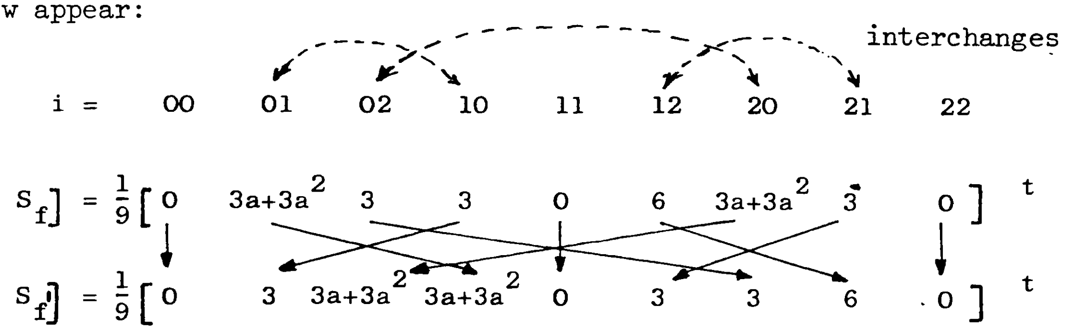
Example. Consider the two-variable ternary function $f(x_1,x_0)$ given by the following map.

$x_0 \backslash x_1$	0	1	2
0	0	1	2
1	2	0	2
2	1	1	0

The spectrum of this function is evaluated to be:

$$S_f] = \left[0 \quad 3a + 3a^2 \quad 3 \quad 3 \quad 0 \quad 6 \quad 3a + 3a^2 \quad 3 \quad 0\right]^t$$

If now x_1 and x_0 variables are interchanged a function $f'(x_1,x_0) = f(x_0,x_1)$ is obtained. The spectrum $S_{f'}]$ for this function is found by reordering the above vector elements. The vectors $S_{f'}]$ and $S_f]$ are shown below, arrows indicating where the original spectral values $s_f(i)$ now appear:



$$S_f] = \tfrac{1}{9}\left[0 \quad 3a+3a^2 \quad 3 \quad 3 \quad 0 \quad 6 \quad 3a+3a^2 \quad 3 \quad 0\right]^t$$

$$S_{f'}] = \tfrac{1}{9}\left[0 \quad 3 \quad 3a+3a^2 \quad 3a+3a^2 \quad 0 \quad 3 \quad 3 \quad 6 \quad 0\right]^t$$

Theorem 4.3 (Making two variables identical)

Let $f'(x_{n-2},\ldots,x_0) = f(x_{n-2},x_{n-2},\ldots,x_0)$

then $s_{f'}(i) = s_f(i_0) + s_f(i_1) + \ldots. + s_f(i_{m-1})$ ....(4.19)

where $i = I_{n-2}m^{n-2} + I_{n-3}m^{n-3} + \ldots + I_0$

and $i_k = k\ m^{n-2} + (I_{n-2} \oplus (m-k))\ m^{n-2} + I_{n-3}m^{n-3} + \ldots + I_0$ for all $k = 0,1, \ldots , m-1$.

Proof. Consider the complex polynomial expansion of the original function f(X):

$$cf(X) = \left(\bigotimes_{p=0}^{(n-1)} \begin{bmatrix} 1 & y_p & \dots & y_p^{m-1} \end{bmatrix} \right) S_f] \qquad \dots\dots.(4.20)$$

Let $i_k = k\, m^{n-1} + (I_{n-2} \oplus (m-k))\, m^{n-2} + \dots + I_0$ for all $k = 0,1,\dots,$ $m-1$.

If now the nth and (n-1)th variables in equation (4.20) are made identical then the spectral coefficients $s_f(i_k)$, $k = 0,1,\dots, m-1$, will appear with the same monomial $y_{n-2}^{I_{n-2}} \dots y_0^{I_0}$. This is so, since:

$$y_{n-2}^{k}\; y_{n-2}^{(I_{n-2} \oplus (m-k))}\; y_{n-3}^{I_{n-3}} \dots y_0^{I_0} = y_{n-2}^{I_{n-2}} \dots y_0^{I_0}$$

Therefore $s_f(i_k)$ $k = 0,\dots, m-1$ may be collected in one bracket to give:

$$s_{f'}(i) = s_f(i_0) + s_f(i\ ) + \dots + s_f(i_{m-1})$$

where $i = I_{n-2}m^{n-2} + \dots + I_0$.

The complex polynomial expansion for the function obtained by making nth and (n-1)th variables of f(X) may now be expressed as:

$$cf(x_{n-2}, \dots , x_0) = \left(\bigotimes_{p=0}^{(n-2)} \begin{bmatrix} 1 & y_p & \dots & y_p^{m-1} \end{bmatrix} \right) S_{f'}] \qquad \dots\dots.(4.21)$$

Example. Consider the function $f(x_1,x_0)$ of the previous example on page 80 . If now the x_1 variable is commonned with x_0 we obtain the function $f(x_0,x_0)$ with spectral values:

$$s_f'(0) = s_f(00) + s_f(12) + s_f(21) ,$$

$$= \frac{1}{9} (0 + 6 + 3) ,$$

$$= 1 .$$

$$s_f'(1) = s_f(01) + s_f(10) + s_f(22) ,$$

$$= \frac{1}{9} (3a + 3a^2 + 3 + 0) ,$$

$$= 0 .$$

$$s_f'(2) = s_f(02) + s_f(20) + s_f(11) ,$$

$$= \frac{1}{9} (3 + 3a + 3a^2 + 0) ,$$

$$= 0 .$$

Note that we have now obtained a spectrum which contains at most $m^{(n-1)}$ non-zero spectral coefficients. Indeed the function obtained by making two variables of the original function identical has (n-1) arguments and thus the number of spectral coefficients required to define it is reduced to one mth of the number of coefficients in the original spectrum. This is not a surprising result since the spectral coefficients are the set of numbers for the complex polynomial expansion of the function, and if the function is not dependent on a variable, say x_k, then all the coefficients appearing with monomials involving x_k must be zero valued.

Theorem 4.4 (Setting a variable to a constant)

Let $f_h(x_{n-2},\dots,x_0) = f(h,x_{n-2},\dots,x_0)$ where h = 0,1,...., (m-1).

Then,

$$\left.\begin{matrix} \left.S_{f_0}\right] \\ \left.S_{f_1}\right] \\ \cdot \\ \cdot \\ \left.S_{f_{m-1}}\right] \end{matrix}\right] = \left(\left[\overline{\mathrm{Tc{-}m}}\right] \otimes \left[\mathrm{Id}\right]^{\otimes\, n-1}_{m\mathrm{x}m} \right) \left.S_f\right] \qquad \ldots\ldots\ldots(4.22)$$

Proof. Consider the matrix relationship between the function f(X) and its spectrum $\left.S_f\right]$ (equation (4.11) rpt.):

$$\left.S_f\right] = \frac{1}{m^n} \left[\mathrm{Tc{-}m}\right]^{\otimes\, n} \left.\mathrm{cF}\right] , \qquad \ldots\ldots\ldots\ldots(4.23)$$

multiplying both sides of the equation with the inverse transform we obtain:

$$\left.\mathrm{cF}\right] = \left[\overline{\mathrm{Tc{-}m}}\right]^{\otimes\, n} \left.S_f\right] \qquad \ldots\ldots\ldots(4.24)$$

We know that by definition the vector $\left.\mathrm{cF}\right]$ contains the local values (mapped on unit circle in complex plane with mapping c) of the function f(X), decimally ordered. Therefore first $m^{(n-1)}$ entries of $\left.\mathrm{cF}\right]$ are the local values of the function $f(0,x_{n-2},\ldots,x_0)$ and the next $m^{(n-1)}$ entries are the local values of the function $f(1,x_{n-2},\ldots,x_0)$, and so on, and hence the corresponding spectra $\left.S_{f_h}\right]$ for functions $f_h(x_{n-2},\ldots,x_0)$, $h = 0,1,\ldots,(m-1)$, may be evaluated as follows:

$$\left[\begin{array}{c} S_{f_0}] \\ S_{f_1}] \\ \cdot \\ \cdot \\ \cdot \\ S_{f_{m-1}}] \end{array}\right] = \frac{1}{m^{n-1}} \left[\begin{array}{cccc} [Tc\text{-}m]^{\otimes\, n-1} & & & 0 \\ & [Tc\text{-}m]^{\otimes\, n-1} & & \\ & & \ddots & \\ 0 & & & [Tc\text{-}m]^{\otimes\, n-1} \end{array}\right] \left[cF \right] .$$

More conveniently the right hand side of this equation may be expressed as:

$$= \frac{1}{m^{n-1}} \left(\left[Id \right] \otimes \left[Tc\text{-}m \right]^{\otimes\, n-1} \right) cF] \qquad \ldots\ldots\ldots(4.25)$$

where $[Id]$ is the mxm identity matrix.

Replacing equation (4.23) in above for the vector $cF]$ we obtain:

$$= \frac{1}{m^{n-1}} \left(\left[Id \right] \otimes \left[Tc\text{-}m \right]^{\otimes\, n-1} \right) \left[\overline{Tc\text{-}m} \right]^{\otimes\, n} \; S_f] \quad ,$$

and using the distributive properties of Kronecker product (Theorem 2.1) we finally obtain:

$$\left.\begin{matrix} S_{f_0}\rbrack \\ S_{f_1}\rbrack \\ \cdot \\ \cdot \\ \cdot \\ S_{f_{m-1}}\rbrack \end{matrix}\right] = \left(\left[\overline{Tc-m}\right] \otimes \left[\,Id\right]^{\otimes\, n-1} \right) S_f\rbrack \qquad \ldots\ldots\ldots(4.26)$$

Example: Consider again the function given in the example on page 80. Replacing the appropriate values in equation (4.26) we obtain the spectra $S_{f_h}\rbrack$ for the functions $f_h(x_0) = f(x_1,x_0)$ as follows:

$$\left.\begin{matrix} S_{f_0}\rbrack \\ S_{f_1}\rbrack \\ S_{f_2}\rbrack \end{matrix}\right] = \left(\begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) S_f\rbrack$$

$$= \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & a & 0 & 0 & a^2 & 0 & 0 \\ 0 & 1 & 0 & 0 & a & 0 & 0 & a^2 & 0 \\ 0 & 0 & 1 & 0 & 0 & a & 0 & 0 & a^2 \\ 1 & 0 & 0 & a^2 & 0 & 0 & a & 0 & 0 \\ 0 & 1 & 0 & 0 & a^2 & 0 & 0 & a & 0 \\ 0 & 0 & 1 & 0 & 0 & a^2 & 0 & 0 & a \end{bmatrix} \left.\begin{matrix} 0 \\ 3a + 3a^2 \\ 3 \\ 3 \\ 0 \\ 6 \\ 3a + 3a^2 \\ 3 \\ 0 \end{matrix}\right] \quad \frac{1}{9} ,$$

giving:

$$S_{f_0}\Big] = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} , S_{f_1}\Big] = \begin{bmatrix} 2a + 1 \\ a + 2a^2 \\ 1 + 2a \end{bmatrix} \frac{1}{3} , S_{f_2}\Big] = \begin{bmatrix} 1 + 2a^2 \\ 2a + a^2 \\ 1 + 2a^2 \end{bmatrix} \frac{1}{3} .$$

Repeated application of above theorem gives the spectra of the decomposition about a subset of variables. In the case when a function $f(x_{n-1},\ldots,x_0)$ has a disjoint decomposition we have the following lemma:

Lemma 4.4 (Cascade composition of functions).

Let $f'(X)$ be a cascade composition of the form

$$f'(x_{n-1},\ldots,x_0) = f(g(x_{n-1},\ldots,x_k),x_{k-1},\ldots,x_0)$$

and let $S_{f_{g(i)}}\Big]$, $i = 0$ to $(m^{n-k}-1)$, be the spectra of the functions $f(g(i),x_{k-1},\ldots,x_0)$. Then:

$$S_{f'}\Big] = \left(\Big[\mathrm{Tc}\text{-}m\Big]^{\otimes\, n-k+1} \otimes \Big[\mathrm{Id}\Big]^{\otimes\, k-1}_{m\times m}\right) \begin{bmatrix} S_{f_{g(0)}}\Big] \\ S_{f_{g(1)}}\Big] \\ \cdot \\ \cdot \\ \cdot \\ S_{f_{g(m^{n-k}-1)}}\Big] \end{bmatrix}$$

.......(4.27)

We shall in chapter 5 give a more precise relationship between the spectra $S_{f'}]$ and $S_f]$ and $S_g]$ when we study the decomposition of functions in terms of their related spectra in greater detail. But now we continue proving further properties of the spectrum studied in references 5 and 16. Our first definition will be used in certain theorems following.

Let ★ be a defined operation, e.g. mod-m addition, multiplication, etc., and i and j be two numbers. The expression $(i \underline{\bigstar} j)$ means ★ operation between the corresponding elements of the m-ary expansions of i and j. For example:

$$(i \underline{\oplus} j) = (I_{n-1} \oplus J_{n-1}, I_{n-2} \oplus J_{n-2}, \dots, I_0 \oplus J_0).$$

If $\left[L\right]$ is a square matrix then $L \underline{x}\, i$ $(i \underline{x}\, L)$ is matrix product between m-ary expansion of i considered as a column (row) vector and the matrix $\left[L\right]$. For example, let $\left[Jd\right]$ be the Jordan matrix defined on page 76, and $(I_{n-1}, \dots, I_0)$ be the m-ary expansion of i as usual, then:

$$(Jd \underline{x}\, i) = \begin{bmatrix} & & & 1 \\ 0 & & 1 & \\ & \cdot & & \\ \cdot & & 0 & \\ 1 & & & \end{bmatrix} \begin{bmatrix} I_{n-1} \\ I_{n-2} \\ \cdot \\ \cdot \\ I_0 \end{bmatrix}$$

$$= (I_0, I_1, \dots, I_{n-1})$$

Theorem 4.5[16]. Let $f(0,0,\dots,0) = 0$ and

$$f'(X) = \begin{cases} k & \text{if } (x_{n-1}, \dots, x_0) = (0,0,\dots,0) \\ f(X) & \text{otherwise} \end{cases}$$

where $k \in V$.

Then $s_{f'}(i) = s_f\,(i) + m^{-n}(ck-1)$ ....(4.28)

Proof. The spectrum of $f'(X)$ is given by:

$$S_{f'}\Big] = \frac{1}{m^n} \Big[Tc-m\Big]^{\otimes n} \quad cF'\Big]$$

but $cf'(0,0,\ldots,0) = ck$

$$= ck - 1 + cf(0,0,\ldots,0),$$

since

$$cf(0,0,\ldots,0) = e^{j\frac{2\pi}{m} f(0,0,\ldots,0)}$$

$$= 1.$$

Thus the right hand side of above equation may be written as:

$$= \frac{1}{m^n} \Big[Tc-m\Big]^{\otimes n} \left(cF\Big] + \begin{matrix} (ck-1) \\ 0 \\ . \\ . \\ 0 \end{matrix}\Bigg] \right),$$

and hence:

$$S_{f'}\Big] = S_f\Big] + \begin{matrix} (ck-1) \\ (ck-1) \\ . \\ . \\ (ck-1) \end{matrix}\Bigg] \frac{1}{m^n},$$

i.e. $s_{f'}(i) = s_f(i) + m^{-n}(ck-1)$

Theorem 4.6[16] (Argument translation)

Let $f'(X) = f(X \underline{\oplus} w)$

then $s_{f'}(i) = t^{*}_{i,w}\, s_f(i)$

where $t_{i,w}$ is the (i,w)th element of the nth order transform matrix.

Proof. To calculate the spectrum of the function $f'(X)$ the equation (4.11) may be reexpressed as:

$$s_{f'}(i) = \frac{1}{m^n} \sum_{j=0}^{m^n-1} t_{i,j} \quad cf'(j) \; , \qquad \text{..........(4.29)}$$

replacing $f'(j)$ with $f(j \underline{\oplus} w)$ we obtain:

$$s_{f'}(i) = \frac{1}{m^n} \sum_{j=0}^{m^n-1} t_{i,j} \quad cf(j \underline{\oplus} w) \qquad \text{..........(4.30)}$$

It was shown in the beginning of this section that an element $t_{i,j}$ of the transform matrix is given by:

$$t_{i,j} = \left(a^{Ch(i,j)} \right)^{*}$$

$$\text{where } Ch(i,j) = \sum_{p=0}^{n-1} I_p J_p \quad .$$

Therefore:

$$t_{i,(j \underline{\ominus} w)} = \left(a^{Ch(i,j \underline{\ominus} w)} \right)^{*},$$

$$\text{but } Ch(i,j \underline{\ominus} w) = \sum_{p=0}^{n-1} I_p (J_p - W_p)$$

$$= Ch(i,j) - Ch(i,w)$$

where $(W_{n-1},\dots,W_0)$ is the m-ary expansion of w.

And hence:

$$t_{i,(j\ \underline{\theta}\ w)} = \left(a^{Ch(i,j)} - Ch(i,w) \right)^{*}$$

$$= \ t_{i,j}\ t_{i,w}^{*} \qquad \dots\dots\dots(4.31)$$

Now equation (4.30) may be reexpressed as follows:

$$s_{f'}(i) = \frac{1}{m^n} \sum_{j=0}^{m^n-1} t_{i,(j\ \underline{\theta}\ w)}\ cf(j) \qquad \dots\dots\dots(4.32)$$

Replacing equation (4.31) in equation (4.32) we obtain:

$$s_{f'}(i) = \frac{1}{m^n}\ t_{i,w}^{*} \sum_{j=0}^{m^n-1} t_{i,j}\ \ cf(j)$$

$$= t_{i,w}^{*}\ s_f(i) \ .$$

Theorem 4.7[5,16] (Linear transformation of argument)

Let $[L]$ be a nxn non-singular matrix containing elements which are non-zero dividers in V, and let:

$$f'(X) = f(X\ \underline{x}\ L),$$

then $s_{f'}(i) = s_f(L^{-1}\ \underline{x}\ i)$ ....(4.33)

The above restriction on the elements of [L] being non-zero dividers in V guarantees that the only effect considered is that of the permutation of the values of f(X).

Proof. First we show that $Ch(i,j \underline{x} L) = Ch(L \underline{x} i,j)$ as follows:

$$Ch(i,j \underline{x} L) = \sum_{p=0}^{n-1} I_p \sum_{q=0}^{n-1} J_q \, l_{q,p}$$

$$= \sum_{q=0}^{n-1} J_q \sum_{p=0}^{n-1} I_p \, l_{q,p}$$

$$= Ch(L \underline{x} i,j)$$

where $l_{p,q}$ are the elements of $\left[L \right]$.

Thus:

$$t_{i,(j \underline{x} L)} = \left({}_{a}Ch(i,j \underline{x} L) \right)^{*}$$

$$= \left({}_{a}Ch(L \underline{x} i,j) \right)^{*}$$

$$= t_{(L \underline{x} i),j} \qquad \text{........(4.34)}$$

Now, the spectrum $S_{f'}]$ of the function $f'(X)$ is given by:

$$s_{f'}(i) = \frac{1}{m^n} \sum_{j=0}^{m^n-1} t_{i,j} \; cf'(j)$$

$$= \frac{1}{m^n} \sum_{j=0}^{m^n-1} t_{i,j} \; cf(j \underline{x} L)$$

$$= \frac{1}{m^n} \sum_{j=0}^{m^n-1} t_{i,(j \underline{x} L^{-1})} \; cf(j) \qquad \text{........(4.35)}$$

Replacing equation (4.34) in equation (4.35) we obtain:

$$s_{f'}(i) = \frac{1}{m^n} \sum_{j=0}^{m^n-1} t_{(L^{-1} \underline{x}\ i),j} \quad cf(j)$$

$$= s_f(L^{-1} \underline{x}\ i)$$

Note that theorem 4.2 is a special case of theorem 4.7 where $[L]$ is a permutation matrix $[L_p]$. Indeed in this case:

$$[L_p]^{-1} = [L_p]^t$$

giving:

$$s_{f'}(i) = s_f(L_p^t \underline{x}\ i)$$

But the integer which corresponds to $(L_p^t \underline{x}\ i)$ and its transpose $(i \underline{x}\ L_p)$ is the same and hence, in the case when $[L_p]$ is a permutation matrix we have

$$s_{f'}(i) = s_f(i \underline{x}\ L_p)$$

<u>Theorem 4.8</u>[5,18] (Disjoint spectral translation)

Let $f'(X) = f(X) \oplus x_k$,

then $s_{f'}(i) = s_f(i \underline{\theta}\ m^k)$ ....(4.36)

<u>Proof</u>. The value of x_k at the point j will be equal to J_k and hence the spectrum $S_{f'}]$ is given by:

$$s_{f'}(i) = \frac{1}{m^n} \sum_{j=0}^{m^n-1} t_{i,j} \quad c(f(j) + J_k) \qquad ....(4.37)$$

The mapping c is an isomorphism between the additive group of integers mod-m and the multiplicative group of complex numbers on the unit circle (Sect. 4.1). Therefore equation (4.37) may be rewritten as:

$$s_{f'}(i) = \frac{1}{m^n} \sum_{j=0}^{m^n-1} t_{i,j} \quad cf(j) \quad cJ_k$$

$$= \frac{1}{m^n} \sum_{j=0}^{m^n-1} t_{i,j} \quad cf(j) \; a^{J_k} \qquad \text{..........(4.38)}$$

Note that $t_{i,j} \; a^{J_k} = \left(a^{Ch(i,j)}\right)^* \; a^{J_k}$

$$= \left(a^{Ch(i,j)} - J_k\right)^* ,$$

but $Ch(i,j) - J_k = \sum_{p=0}^{n-1} \left(I_p J_p - J_k\right)$

$$= I_{n-1} \, J_{n-1} + \ldots + (I_k - 1) \; J_k + \ldots + I_0 J_0$$

$$= Ch(i \; \underline{\theta} \; m^k, j) \; ,$$

and hence $t_{i,j} \; a^{J_k} = t_{(i \; \underline{\theta} \; m^k), j}$ ....(4.39)

Replacing (4.39) in (4.38) we obtain:

$$s_{f'}(i) = \frac{1}{m^n} \sum_{j=0}^{m^n-1} t_{(i \; \underline{\theta} \; m^k), j} \quad cf(j)$$

$$= s_f(i \; \underline{\theta} \; m^k)$$

Corollary. If $f'(X) = f(X) \oplus g(X)$

where $g(X)$ is a linear function of the form:

$$g(X) = W_{n-1}\, x^{n-1} \oplus W_{n-2}\, x^{n-2} \oplus \ldots \oplus W_0 x^0$$

and w is an integer whose m-ary expansion is $(W_{n-1}, \ldots, W_0)$, then the spectrum $S_{f'}]$ is given by:

$$s_{f'}(i) = s_f\,(i \underline{\theta}\, w)$$

This result is obtained by successive application of theorem 4.8.

Theorem 4.9 (Cyclic negation of the function f(X)

Let $f'(X) = f(X) \oplus k$ where $k \in V$

then $s_{f'}(i) = s_f(i)\, a^k$ ....(4.40)

Proof. Proof follows directly from the definition of isomorphism c:

$$s_{f'}(i) = \frac{1}{m^n} \sum_{j=0}^{m^n-1} t_{i,j}\;\; c(f(j) \oplus k)$$

$$= \frac{1}{m^n} \sum_{j=0}^{m^n-1} t_{i,j}\;\; cf(j)\; ck$$

$$= s_f(i)\; a^k$$

We define the inverse $\bar{x}$ of a variable x over the set V as follows:

$$\bar{x} = (m-1) - x$$

For example in quaternary (m = 4) the truth table for the inverse operation would be as follows:

x	$\overline{x}$
0	3
1	2
2	1
3	0

Theorem 4.10 Let $f'(X) = \overline{f(X)}$,

then $s_{f'}(i) = c(-1)\ s^{*}_{f}(D \underline{x}\ i)$ ....(4.41)

where $[D] = (m-1)\ [Id]_{nxn}$

Proof.

$$s_{f'}(i) = \frac{1}{m^n} \sum_{j=0}^{m^n-1} t_{i,j}\ c((m-1) - f(j))$$

$$= \frac{1}{m^n} \sum_{j=0}^{m^n-1} t_{i,j}\ cm\ c(-1)\ c(-f(j)) \qquad \text{...........(4.42)}$$

But $cm = a^m$

$= 1$,

and $c(-f(j)) = a^{-f(j)}$

$= (a^{f(j)})^*$

$= (cf(j))^*$.

Therefore equation (4.42) may be reexpressed as:

$$s_{f'}(i) = c(-1)\frac{1}{m^n}\sum_{j=0}^{m^n-1} t_{i,j}\ (cf(j))^*$$

$$= c(-1)\frac{1}{m^n}\sum_{j=0}^{m^n-1} t_{i,j}^*\ \left(cf(j)\right)^*$$

..........(4.43)

Note that $t_{i,j}^* = a^{Ch(i,j)}$

$$= \left(a^{(m-1)Ch(i,j)}\right)^*$$

$$= t_{(D\ \underline{x}\ i),j}$$ ....(4.44)

Replacing equation (4.44) in (4.43) we finally obtain:

$$s_{f'}(i) = c(-1)\ s_f^*(D\ \underline{x}\ i)$$

<u>Theorem 4.11</u> Let $f'(X) = f(x_{n-1},\ldots,\bar{x}_k,\ldots,x_0)$,

then $s_{f'}(i) = a^{I_k}\ s_f(L_k\ \underline{x}\ i)$ ....(4.45)

where $\left[L_k\right]$ is an nxn matrix whose elements $l_{p,q}$ are given by:

$$l_{p,q} = \begin{cases} (m-1) & \text{if } p = q = (n-1) - k \\ 1 & \text{if } p = q \neq (n-1) - k \\ 0 & \text{otherwise} \end{cases}$$ ....(4.46)

<u>Proof</u>. The inverse $\bar{x}$ of x can be expressed as;

$$\bar{x} = (m-1) - x$$

$$= (m-1) \oplus (m-1).x$$

Therefore $(x_{n-1},\dots,\bar{x}_k,\dots,x_0)$ may be obtained from $(x_{n-1},\dots,x_k,\dots,x_0) = (X)$ as follows:

$$(x_{n-1},\dots,\bar{x}_k,\dots,x_0) = \Big((X \underline{x} L_k) \underline{\oplus} (m-1)m^k\Big) \qquad \dots\dots\dots\dots(4.47)$$

Note that, since $(m-1).(m-1) = 1$, we see from above definition of $[L_k]$ that the inverse of $[L_k]$ is $[L_k]$ itself.

Now, let $f''(X) = f(X \underline{x} L_k)$,

hence $f'(X) = f''(X \underline{\oplus} (m-1)m^k)$.

Applying theorem 4.6 we obtain:

$$s_{f'}(i) = t^*_{i,(m-1)m^k} \; s_{f''}(i)$$

But, by theorem 4.7 $s_{f''}(i) = s_f(L_k \underline{x} i)$. Therefore:

$$s_{f'}(i) = t^*_{(L_k \underline{x} i),(m-1)m^k} \; s_f(L_k \underline{x} i) \qquad \dots\dots\dots\dots(4.48)$$

Note that $Ch(L_k \underline{x} i,(m-1)m^k) = 0\, I_{n-1} + \dots + (m-1)^2\, I_k + \dots + 0\, I_0$

$$= (m-1)^2\, I_k \,,$$

and hence $t^*_{(L_k \underline{x} i),(m-1)m^k} = a^{(m-1)^2 I_k}$

$$= a^{I_k}.$$

Replacing this in equation (4.48) the result of theorem 4.11 is obtained.

<u>Theorem 4.12</u> Let $f'(X) = f(\bar{X})$

where $(\bar{X}) = (\overline{x_{n-1}},\dots,\bar{x}_0)$

then $s_{f'}(i) = t^*_{i,(m^n-1)} \; s_f(D \underline{x} i)$ $\qquad \dots\dots\dots\dots(4.49)$

where $[D] = (m-1)\,[Id]_{nxn}$

Proof. The proof of this theorem is similar to proof of theorem 4.11. In this case $(\overline{X})$ is given by:

$$(\overline{X}) = \Big((X \underline{x} D) \underline{\oplus} (m^n-1)\Big) \qquad \ldots\ldots\ldots(4.50)$$

With the successive applications of theorem 4.6 and 4.7 we obtain:

$$s_{f'}(i) = t^{*}_{i,(m^n-1)} \quad s_f(D \underline{x} i)$$

The results of theorem 4.10 and 4.12 may now be combined to give following lemma.

Lemma 4.12 Let $f'(X) = \overline{f(\overline{X})}$

then $s_{f'}(i) = c(-1)\ t_{(D \underline{x} i),(m^n-1)} \quad s^*_f(i)$ ....(4.51)

Corollary. A function is self-dual, i.e. $f(X) = \overline{f(\overline{X})}$ if and only if

$$s_f(i) = c(-1)\ t_{(D \underline{x} i),(m^n-1)} \quad s^*_f(i)$$

$$= \left(\left(1+ \sum_{p=0}^{n-1} I_p\right)_a\right)^{*} \quad s^*_f(i)$$

Theorem 4.13 (Convolution theorem)

Let $f(X) = f_1(X) \oplus f_2(X)$

then $s_f(i) = \sum_{\tau=0}^{m^n-1} s_{f_1}(\tau) \quad s_{f_2}(i \underline{\theta} \tau)$ ....(4.52a)

$$= \sum_{\tau=0}^{m^n-1} s_{f_1}(i \underline{\theta} \tau) \quad s_{f_2}(\tau) \qquad \text{..........(4.52b)}$$

Proof. The spectrum of f(X) is given by:

$$s_f(i) = \frac{1}{m^n} \sum_{j=0}^{m^n-1} t_{i,j} \quad c(f_1(j) \oplus f_2(j))$$

$$= \frac{1}{m^n} \sum_{j=0}^{m^n-1} t_{i,j} \quad cf_1(j) \quad cf_2(j) \qquad \text{..........(4..53)}$$

The inverse transform given in equation 4.12 may be expressed as:

$$cf(j) = \sum_{\tau=0}^{m^n-1} t^*_{j,\tau} \; s_f(\tau) \qquad \text{..........(4.54)}$$

Replacing (4.54) in (4.53) for $f_2(j)$ and rearranging the summation we obtain:

$$s_f(i) = \frac{1}{m^n} \sum_{\tau=0}^{m^n-1} s_{f_2}(\tau) \sum_{j=0}^{m^n-1} t_{i,j} \quad t^*_{j,\tau} \quad cf_1(j) \qquad \text{.....(4.55)}$$

But the product $t_{i,j} \; t^*_{j,\tau}$ gives:

$$t_{i,j} \quad t^*_{j,\tau} = a^{Ch(i,j)} \left(a^{Ch(j,\tau)} \right)^*$$

$$= a^{Ch(i \underline{\theta} \tau, j)} \; .$$

$$= t_{(i \underline{\theta} \tau),j}$$

Hence equation (4.55) may be reexpressed as:

$$s_f(i) = \frac{1}{m^n} \sum_{\tau=0}^{m^n-1} s_{f_2}(\tau) \sum_{j=0}^{m^n-1} t_{(i \underline{\theta} \tau),j} \; cf(j)$$

$$= \sum_{\tau=0}^{m^n-1} s_{f_2}(\tau) \quad s_{f_1}(i \underline{\theta} \tau)$$

Equation (4.52a) is obtained by interchanging $f_1(j)$ and $f_2(j)$ wherever they occur in above development.

4.3 Classification of m-ary functions and a design example

The spectral properties discussed in theorems 4.6 through 4.12 have an important common feature. The functions $f'(X)$ obtained by applying the operations stated in these theorems from a function $f(X)$, have spectra which are permutations and/or complex scaling of the spectrum $S_{f}]$ of the original function $f(X)$. In general these operations are called spectral translations. In the classification of logic functions an equivalence relationship is defined such that two functions are the members of the same class if one can be obtained from the other by spectral translations. The equivalence classes generated as such are represented by a member of the class. The tables which list these classes have been constructed by Edwards[25] for binary functions up to 5 variables and by Moraga[5] for 2-variable ternary functions. To the author's knowledge there is no theoretical method to enumerate the equivalence classes as a function of the number of variables and the radix m. Both Edwards and Moraga used exhaustive search mechanisms to obtain the list of classes. The very small number of classes, 48 for all the 4.3×10^9 variable binary functions using theorems 4.6 to 4.9 and 12 for all the 19683 2-variable ternary functions using theorems 4.6 to 4.12, is important to note.

Below is an example of application of above theorems for ternary logic analysis.

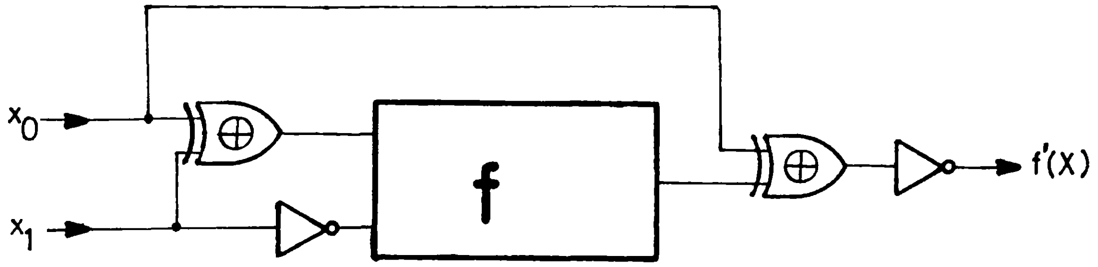
Example. Let us consider the following two-variable ternary function $f(x_1,x_0)$ given by the map:

x_0 \ x_1	0	1	2
0	0	1	0
1	1	1	2
2	2	1	1

Assume that gate f is a physical realisation of this function. The spectrum $S_f]$ is computed to be:

$$S_f] = \frac{1}{9}\left[3a \quad 3 \quad 3 \quad 3 \quad 3 \quad 3a \quad 3a^2 \quad 3 \quad 3a^2\right]^t.$$

Suppose now the following realisation of a function $f'(X)$ is given by the following figure:



We find the spectrum $S_{f'}]$ of the function f'(X) which is algebraically given by:

$$f'(x_1,x_0) = \overline{f(\overline{x_1},\ x_1 \oplus x_0) \oplus x_0}$$

from the spectrum $S_f]$ with successive applications of relevant theorems as follows:

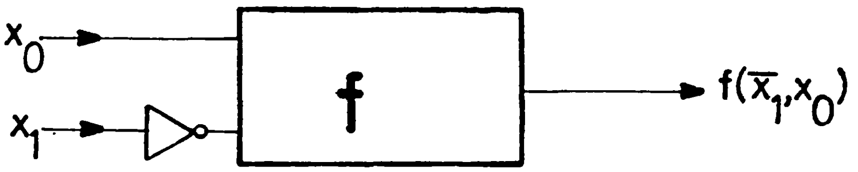
i) First the spectrum $S_1]$ of the function $f(\overline{x_1},x_0)$ is found by using theorem 4.11. In this case the matrix $\left[L_k\right]$ is given by:

$$\left[L_k\right] = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

and therefore the spectrum $S_1]$ is obtained as follows:

i	$9\ s_f(i)$	$L_k \underline{x}\ i$	a^{I_1}	$9\ s_1(i) = 9\ a^{I_1}\ s_f(L_k \underline{x}\ i)$
00	3a	00	1	3a
01	3	01	1	3
02	3	02	1	3
10	3	20	a	3
11	3	21	a	3a
12	3a	22	a	3
20	$3a^2$	10	a^2	$3a^2$
21	3	11	a^2	$3a^2$
22	$3a^2$	12	a^2	3

The circuit realisation of $f(\overline{x}_1,x_0)$ will be as shown below:



ii) Now the spectrum $S_2]$ of the function $f(\overline{x}_1, x_1 \oplus x_0)$ is evaluated from the spectrum $S_1]$ with the application of theorem 4.7. In this case the matrix $[L]$ is given by:

$$[L] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

so that:

$$[x_1 \quad x_0] \, [L] = [x_1 \quad x_1 \oplus x_0] \; .$$

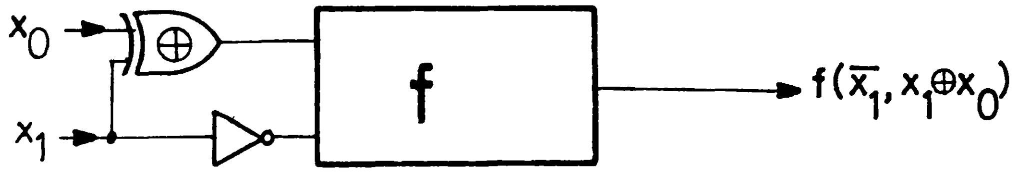
The inverse $[L]^{-1}$ of $[L]$ is found to be:

$$[L]^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad \text{(mod-3)},$$

and hence the spectrum of the function $f(\overline{x}_1, x_1 \oplus x_0)$ is found as:

i	$L^{-1} \underline{x}$ i	9 $s_1(i)$	9 $s_2(i) = 9\, s_1(L^{-1} \underline{x}\, i)$
00	00	3a	3a
01	21	3	$3a^2$
02	12	3	3
10	10	3	3
11	02	3a	3
12	22	3	3
20	20	$3a^2$	$3a^2$
21	11	$3a^2$	3a
22	02	3	3

The circuit realisation of $f(\overline{x}_1, x_1 \oplus x_0)$ is obtained from the previous circuit simply by combining two inputs through a mod-3 adder to x_0 input as shown on the next page:



iii) Next the spectrum $\left.S_3\right]$ of the function $f(\bar{x}_1, x_1 \oplus x_0) \oplus x_0$ is found using theorem 4.8 as follows:

i	$i \underline{\theta} 1$	$9\ s_2(i)$	$9\ s_3(i) = 9\ s_2(i \underline{\theta} 1)$
00	02	3a	3
01	00	$3a^2$	3a
02	01	3	$3a^2$
10	12	3	3
11	10	3	3
12	11	3	3
20	22	$3a^2$	3
21	20	3a	$3a^2$
22	21	3	3a

iv) And finally we obtain the spectrum $\left.S_{f'}\right]$ of the function $f'(x_1, x_0) = \overline{f(\bar{x}_1, x_1 \oplus x_0)} \oplus x_0$ using theorem 4.10. In this case the matrix $[D]$ is:

$$[D] = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix},$$

Giving:

i	$D \underline{x} i$	$9\ s_3(i)$	$9\ s_{f'}(i) = 9\ c(-1)\ s_3^*(D \underline{x} i)$
00	00	3	$3a^2$
01	02	$3a$	3
02	01	$3a^2$	$3a$
10	20	3	$3a^2$
11	22	3	$3a$
12	21	3	3
20	10	3	$3a^2$
21	12	$3a^2$	$3a^2$
22	11	$3a$	$3a^2$

In the example above, the functions $f'(x_1,x_0)$ and $f(x_1,x_0)$ are in the same class by definition of spectral classification, and the spectrum of $f'(x_1,x_0)$ has been obtained from the spectrum of $f(x_1,x_0)$ by application of relevant theorems following the circuit diagram for the realisation of $f'(x_1,x_0)$. Thus example shown illustrates m-ary combinatorial logic circuit <u>analysis</u> using spectral methods. The spectral <u>synthesis</u> of m-ary combinatorial logic may be described on the same example as follows:

Suppose that $f'(x_1,x_0)$ is to be realised and $f(x_1,x_0)$ is the spectral class representative according to some complexity criterion. It is found by comparing the spectrum of $f'(x_1,x_0)$ with the spectra of different class representatives that $f'(x_1,x_0)$ belongs to the same class as $f(x_1,x_0)$. $f'(x_1,x_0)$ may now be obtained from $f(x_1,x_0)$ with spectral translations. Note that this involves a lot of computation since the class which some spectra belong to is difficult to recognise if the spectral translation requires argument translation or complementation, see theorem 4.6 and 4.11 respectively. In concluding, the spectral logic design methods assume that simple and reliable realisations for modulo adders exist. The computatations involved in spectral translations are not suited for hand-work and use of m-ary computers for this purpose would be most advantageous.

This chapter has therefore taken the published binary work in spectral logic into the general higher-valued case, revising the properties of multi-valued spectral logic and investigating further the modifications to spectral coefficients under the operations i) setting variable(s) to a constant(s) and, ii) making two or more variables identical. We shall in the following pages consider the decomposition relationships in the spectrum using the properties developed in this chapter.

CHAPTER 5

Spectral Decomposition Theorems

5. Spectral Decomposition Theorems

The realisation of any m-ary function takes place by some form of decomposition into functions which are readily available in physically-implemented form. Mathematically the overall function is represented as a function of functions. The general block diagram of any realisation is shown later in Fig. 5.3.

In this chapter we look at some of the existing work and general results concerning the decomposition of discrete functions, and investigate and prove further relationships between the spectra of functions involved in the decomposition topology.

5.1 Decomposition

Consider the block diagram realisation of a function f(X) shown in Fig. 5.1. The function f(X) implemented as such is a composition of functions $h(X_h)$ and $g_l(X_l)$, l=0,.., k-1; mathematical representation of this will be of the form:

$$f(X) = h(g_{k-1}(X_{k-1}), g_{k-2}(X_{k-2}), \dots , g_0(X_0)) ,$$

where each X_l, l=0,...., k-1 is a subset of $X = \{x_{n-1},\dots,x_0\}$.
In many cases some inputs at the first level of the implementation are preceded by other gates or systems, creating a more complex overall complete picture.

In order to differentiate between various decompositions we have the following definitions.

A composition is said to be a) simple or b) complex if it is a composite function of functions in the form:

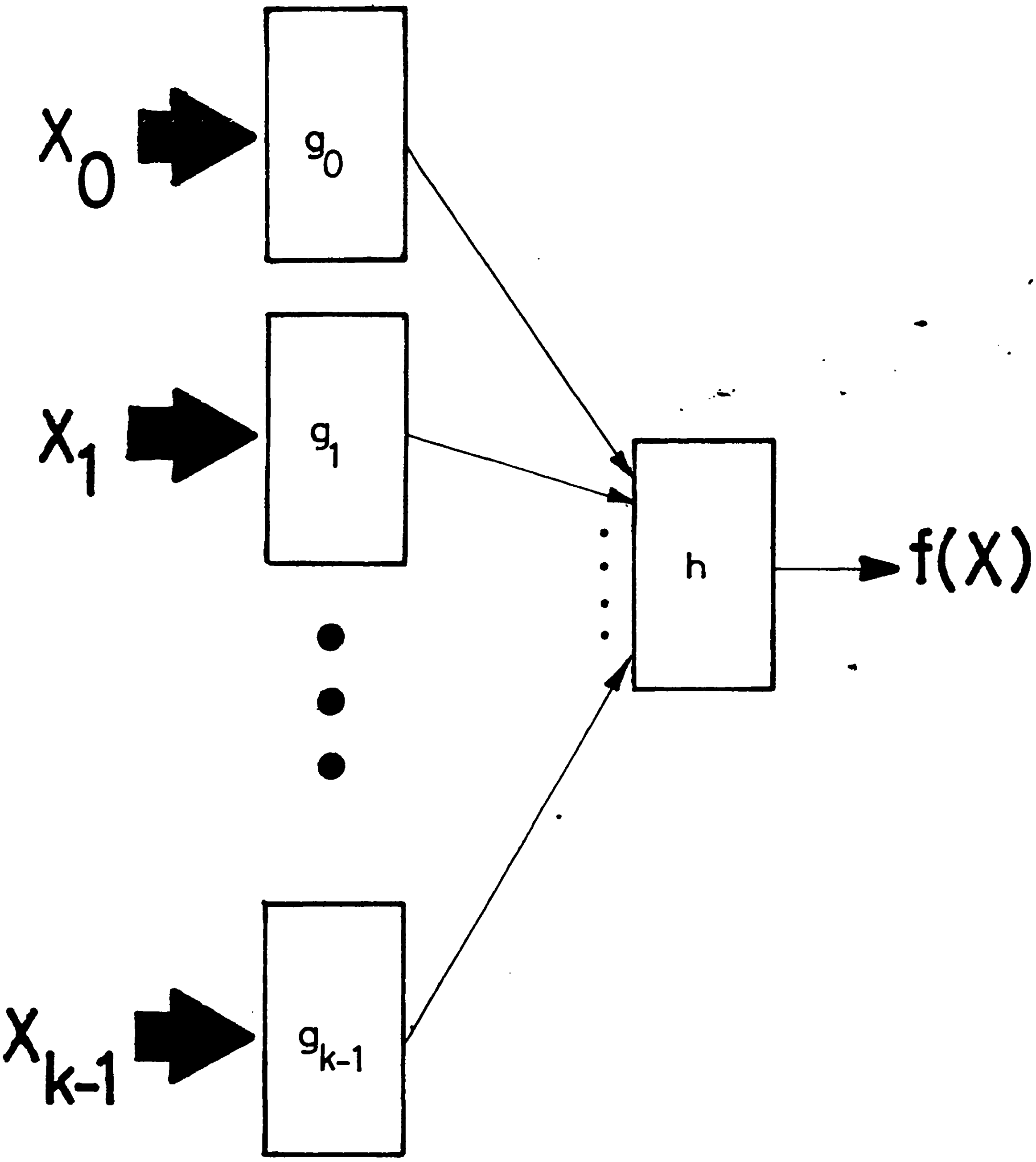


Fig. 5.1 Block diagram realisation of function f(X) as composite function of functions.

a) $f(X) = h(X_1, g(X_0))$, or ....(5.1)

b) $f(X) = h(g_{k-1}(X_{k-1}), \ldots, g_0(X_0))$, ...(5.2)

respectively. A complex decomposition may be either multiple, in which case it will be of the form (5.2), or iterative in which case the variables in the multiple decomposition are also replaced by other functions. Furthermore, a composition is disjunctive if in above for every pair p,q the sets $\{X_p\}$ and $\{X_q\}$ are disjoint, otherwise it is non-disjunctive.

Historically the problem of finding decompositions of a given function was first studied by Ashenhurst[21] for the case of binary logic functions. A fast algorithm for detection of non-decomposability was developed by Shen, et al[22]. Karp[23] studied the problem in a more general frame and stated necessary and sufficient conditions for decomposability for the case of logic functions. Some important points from these works are summarised below.

Let f(X) be a n-variable m-ary function and $X_1 = \{x_{n-1}, \ldots, x_q\}$ $(q < n-1)$ and $X_0 = \{x_{q-1}, \ldots, x_0\}$ be a partition of $X = \{x_{n-1}, \ldots, x_0\}$. We define an equivalence relation $r(X_0, f)$ on the set V^q as follows:

$(i, i') \in r(X_0, f)$ if and only if

$$f(x_{n-1}, \ldots, x_q, I_{q-1}, \ldots, I_0) = f(x_{n-1}, \ldots, x_q, I'_{q-1}, \ldots I'_0) \qquad \text{..........(5.3)}$$

for all values of $\{x_{n-1}, \ldots, x_q\}$, where the q-tuples $(I_{q-1}, \ldots, I_0)$ and $(I'_{q-1}, \ldots, I'_0)$ are the m-ary encodings (expansions) of i and i' respectively.

The number of equivalence classes generated by $r(X_0,f)$ is denoted $k(X_0,f)$, and is the number of different functions generated by fixing first q-variables of the function $f(X)$. The following theorem follows from the definition of $k(X_0,f)$.

Theorem 5.1[23,4] A n-variable m-ary function $f(X)$ has a disjoint decomposition of the form:

$$f(X) = h(X_1,g(X_0))$$

if and only if $k(X_0,f) \leq m$.

We can see this more clearly by constructing a partition matrix. This is the same as Karnaugh map representation of a function $f(X)$, but the columns are now identified by the variables of the set X_0 and the rows are labelled with the variables of the set X_1. The column multiplicity of such a map is the number of different columns it contains. Clearly the column multiplicity is the same as $k(X_0,f)$ defined above and if the column multiplicity is less than or equal to m then the column variables may be replaced by a function $g(X_0)$.

For example, consider the following 3-variable binary function.

x_2 \ x_1 x_0	0 0	0 1	1 1	1 0
0	0	1	0	1
1	1	0	1	0

The column multiplicity of this map is 2; therefore we can replace the column variables with the function,

$$g(x_1,x_0) = x_1 \oplus x_0 \quad ,$$

giving us a reduced map:

x_2 \ $g(x_1,x_0)$	0	1
0	0	1
1	1	0

A fast algorithm which compares the rows rather than the columns of the partition matrix was developed by Shen, et al[22] for the m = 2 binary case. The algorithm is based on the following theorem:

Theorem 5.2 Let $\{X_1,X_0\}$ be a partition of X, then the n-variable binary function f(X) has a disjoint decomposition iff for all $x_p,x_q \in X_0$ and $x_k \in X_1$ of the partition matrix one of the following holds:

a) one of the rows is all 0's

b) one of the rows is all 1's

c) two rows are identical

d) one row is inverse of the other.

A proof of this theorem may be found in reference 22. Note that in the example given above condition d is satisfied.

Now consider a section of the partition matrix, which is obtained by setting the remaining (n-3) variables to some constants.

This will be of the form:

x_k \ $x_p\ x_q$	0 0	0 1	1 1	1 0
0	m_0	m_1	m_3	m_2
1	m_4	m_5	m_7	m_6

Every square (minterm) in this map can be either logic value 0 or 1, thus making the number of possible functions $2^8 = 256$. Among these functions 88 will satisfy the conditions of theorem 5.2[4]. Therefore [22] "for a randomly given function the probability that a pair of rows will satisfy the conditions of theorem 5.2 is (11/32) and hence the probability that the conditions a) to d) are satisfied for all possible values of remaining (n-3) variables is $(11/32)^{2^{(n-3)}}$. On the average only a small number of pairs of rows (less than 32/21) will have to be checked to show that it does not satisfy the conditions of theorem 5.2. Thus one would expect that the time required to check all possible combinations of three variables $\frac{n!}{(n-3)!3!} \sim n^3$ would grow approximately as n^3.

The running times for this algorithm in the case when f(X) has a disjoint decomposition grow very rapidly, since the number of points that the conditions of theorem 5.2 satisfied becomes exponentially large, and each of them has to be checked completely".

A generalisation of theorem 5.2 to the m-valued case would be very lengthy, mainly because the number of one variable functions m^m grows very rapidly as m-increases. However, the probability that a randomly given function has a disjoint decomposition may be calculated as follows:

If Q_u^v is the number of ways of partitioning a set of u elements into v non-empty subsets, then

$$Q_u^v = \frac{1}{V!} \sum_{k=0}^{v} (-1)^{v-k} \; C_k^v \, . \, k^u.$$

These are called Stirling numbers of the second kind, and this formula may be found in handbooks of mathematical functions[26]. A function which admits disjoint decomposition will have a partition matrix with column multiplicity of less than or equal to m. For the three variable case the partition matrix will have m-rows and m^2-columns, each column defining any one of the m^m one-variable functions. The number N of the functions whose partition matrices will have column multiplicity 1,2,....,m is therefore given by:

$$= \sum_{v=1}^{m} C^{m^m}_{v} \; v! \; Q^{v}_{m^2}$$

Indeed, we have m^2-columns and v-symbols (v = 1,....,m), which may be distributed in these columns v! $Q^{v}_{m^2}$ ways. We multiply this with $C^{m^m}_{v}$ since there are $C^{m^m}_{v}$ combinations of one-variable functions. The table below shows N-as a function of m; the fraction ration $N/m^{(m^3)}$ is the probability that a randomly given three-variable m-ary function will admit disjoint decomposition of the form $h(x_2,g(x_1,x_0))$.

m	N	$\frac{N}{m^{m^3}}$
2	88	$3,44 \times 10^{-1}$
3	$5,33 \times 10^{7}$	$6,79 \times 10^{-7}$
4	$7,21 \times 10^{17}$	$2,12 \times 10^{-21}$
5	$7,24 \times 10^{32}$	$3,08 \times 10^{-55}$

It can be seen from this table that although the number of disjoint decomposable functions is large, they constitute a very small proportion of all three-variable functions.

5.2 Evaluation of the spectra of multi-level functions

We now consider the case when a function $f(X)$ has a two-level realisation as shown in Fig. 5.2. We incorporate the spectra $S_f]$, $S_h]$ and $S_{g_p}]$, $p = 0,\ldots,k-1$ with the functions $f(X), h(x_{k-1},\ldots,x_0)$ and $g_p(X), p=0,\ldots,k-1$ respectively.

Let us denote the k-tuple $(g_{k-1}(i),\ldots,g_0(i))$ by an integer z_i such that:

$$z_i = \sum_{p=0}^{k-1} g_p(i)\, m^p \qquad \ldots\ldots\ldots\ldots(5.4)$$

At a point i the functions $g_p(X), p = 0,\ldots,k-1$ take the values $g_p(i)$, and the value of the function $f(X)$ at the point i is determined by the value of the function $h(x_{k-1},\ldots,x_0)$ at point z_i. Therefore the function values $f(i)$ may be computed from the spectrum $S_h]$ as follows:

$$\begin{bmatrix} z_i\text{th row of} \\ \text{kth order inverse} \\ \text{transform matrix } \left[\overline{Tc-m}\right]^{\otimes k} \end{bmatrix} S_h] = cF] \qquad \ldots\ldots\ldots\ldots(5.5)$$

The z_ith row elements $t_{z_i,j}$ of the inverse transform matrix are given by (see Section 4.2)

$$t_{z_i,j} = a^{\left(\sum_{p=0}^{k-1} g_p(i)\, J_p\right)}, \qquad \ldots\ldots\ldots\ldots(5.6)$$

since $(g_{k-1}(i),\ldots,g_0(i))$ is the m-ary expansion of z_i by equation (5.4).

Note that $t_{z_i,j}$ given in equation (5.6) is merely the image of some linear combination of functions $g_p(X)$ under the mapping c. Therefore

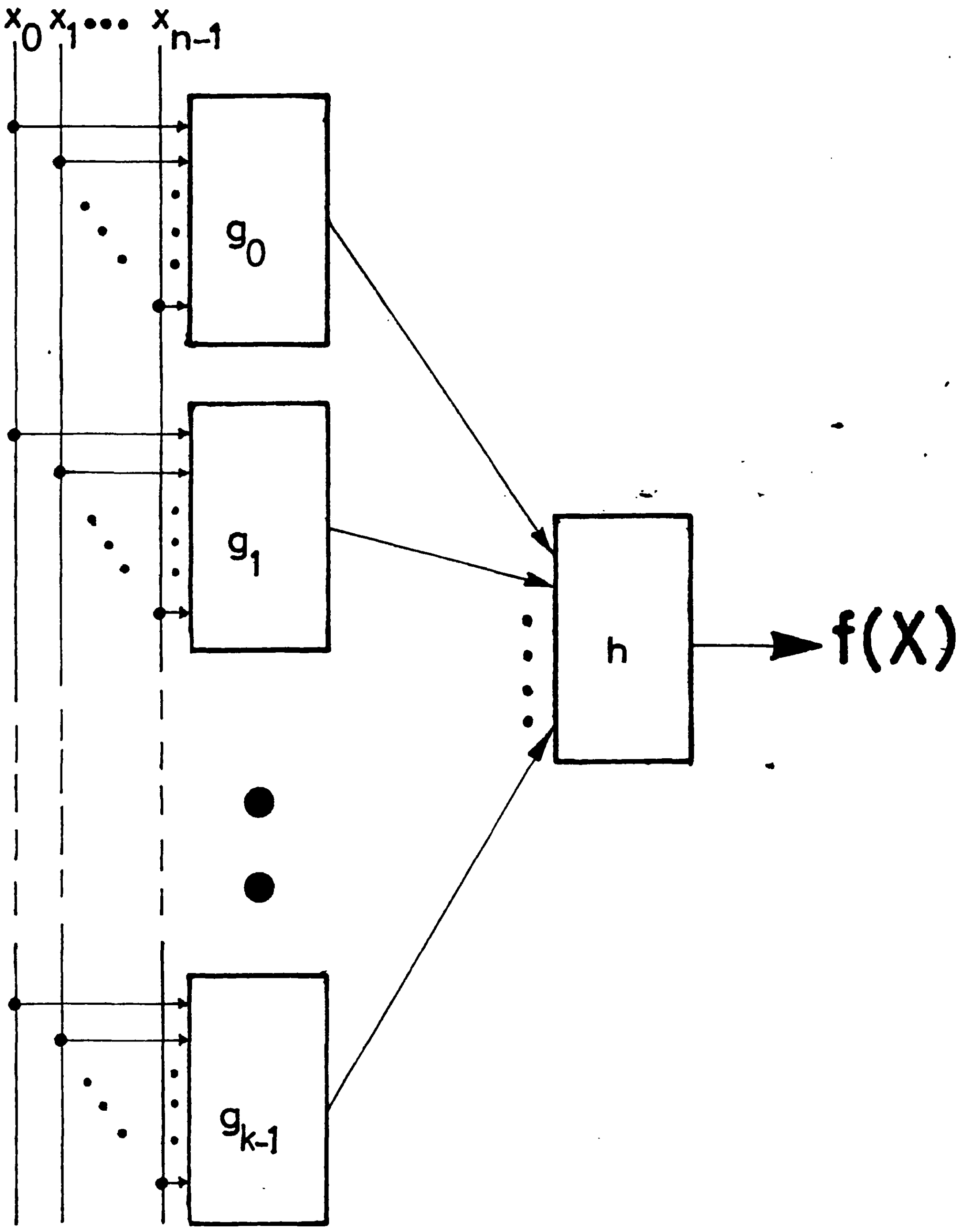


Fig. 5.2: Two level realisation of f(X) in the form:

$$f(X) = h(g_{k-1}(X), \ldots, g_1(X), g_0(X))$$

the columns of this matrix are merely the local values of some linear combination of functions $g_p(X)$, and hence equation (5.5) may now be reexpressed as:

$$\left[\left.\begin{matrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \end{matrix}\right] \quad cG_0\Big] \quad \cdots \quad c\left(\sum_{p=0}^{k-1} G_pJ_p\right)\Bigg] \quad \cdots \right] \quad S_h\Big] = cF\Big] ,$$

$$\ldots\ldots\ldots\ldots(5.7)$$

where $G_p\Big]$,$p = 0,\ldots,k-1$ are the vectors which contain local values of $g_p(X)$ in decimal order.

Multiplying both sides of equation (5.7) with the nth order transform matrix the following theorem is obtained:

<u>Theorem 5.3</u> A function $f(X)$ is a composite function of functions of the form:

$$f(X) = h(g_{k-1}(X),\ldots,g_0(X)), \qquad \ldots\ldots\ldots\ldots(5.8)$$

iff the corresponding spectra satisfy the relationship:

$$\left[\left.\begin{matrix} 1 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \end{matrix}\right] \quad S_{g_0}\Big] \quad \cdots \quad S_{\sum_{p=0}^{k-1} g_pJ_p}\Bigg] \quad \cdots \right] \quad S_h\Big] = S_f\Big]$$

$$\ldots\ldots\ldots\ldots(5.9)$$

Above development has shown that equation (5.9) is satisfied when f(X) has a decomposition of the form (5.8). If there is a relationship of the form (5.9), then f(X) has a decomposition of the form (5.8) comes from the fact that spectrum-function pair is unique.

Example. Let $f_1(X)$ and $f_2(X)$ be two fully specified n-variable binary functions with spectra $S_{f_1}]$ and $S_{f_2}]$ respectively. It is required to find the spectrum $S_f]$ of the function f(X) which is the Boolean disjunction ("Or") of the two functions $f_1(X)$ and $f_2(X)$. In this case the spectrum $S_g]$ of a two input Or function is given by:

$$S_g] = \frac{1}{4} \quad [-2 \quad 2 \quad 2 \quad 2]^t \quad ,$$

putting these values in equation (5.9) we obtain:

$$S_f] = \frac{1}{4} \left[\begin{array}{c|c|c|c|} 1 & S_{f_1} & S_{f_2} & S_{f_1 \oplus f_2} \\ 0 & & & \\ 0 & & & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \end{array} \right] \left. \begin{array}{c} -2 \\ 2 \\ 2 \\ 2 \end{array} \right] \quad ,$$

and hence:

$$S_f] = \frac{1}{2} \left(\left. \begin{array}{c} -1 \\ 0 \\ 0 \\ \cdot \\ \cdot \end{array} \right] + \; S_{f_1}] + \; S_{f_2}] + \; S_{f_1 \oplus f_2}] \right)$$

.........(5.10)

This relationship for the Boolean sum was found by Eris[24] following a dissimilar approach.

Continuing with the example, if now the two functions $f_1(X)$ and $f_2(X)$ are, say, the following two-valued Boolean functions:

$$f_1(X) = \bar{x}_0 x_1 \bar{x}_2 \vee x_0 x_2$$

and $$f_2(X) = x_0 \bar{x}_1 \vee \bar{x}_1 x_2 \quad ,$$

then the corresponding spectra are computed to give:

$$S_{f_1}\Big] = \frac{1}{8}\begin{bmatrix} 2 & 2 & 2 & 2 & 2 & -6 & 2 & 2 \end{bmatrix}^t$$

$$S_{f_2}\Big] = \frac{1}{8}\begin{bmatrix} 2 & 2 & -6 & 2 & 2 & 2 & 2 & 2 \end{bmatrix}^t$$

We may use the convolution theorem (theorem 4.13) to evaluate the resulting spectrum of $f_1(X) \oplus f_2(X)$:

$$s_{f_1 \oplus f_2}(i) = \sum_{\tau=0}^{7} s_{f_1}(\tau) \; s_{f_2}(i \underline{\theta} \tau) \; ,$$

giving:

$$S_{f_1 \oplus f_2}\Big] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^t.$$

Hence, we finally obtain the spectrum $S_f\Big]$ of the function $f(X) = f_1(X) \vee f_2(X)$ by putting above values in equation (5.10):

$$S_f\Big] = \frac{1}{2}\left(\begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ -6 \\ 2 \\ 2 \end{bmatrix} \frac{1}{8} + \begin{bmatrix} 2 \\ 2 \\ -6 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} \frac{1}{8} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

whence:

$$S_f\Big] = \frac{1}{8}\begin{bmatrix} -2 & 2 & -2 & 2 & 2 & -2 & 2 & 6 \end{bmatrix}^t$$

A natural extension of this theorem gives the relationships between the spectra of functions which are realised in more than two-levels. For example, with the three-level composition shown in Fig. 5.3 the mathematical representation would be of the form:

$$f(X) = h\Big(g_{l-1}(e_{k-1}(X),\dots,e_0(X)),\dots,g_0(e_{k-1}(X),\dots,e_0(X)\Big) \qquad \dots\dots\dots(5.11)$$

The relationship between the corresponding spectra is therefore obtained by repeated application of above theorem 5.3, giving:

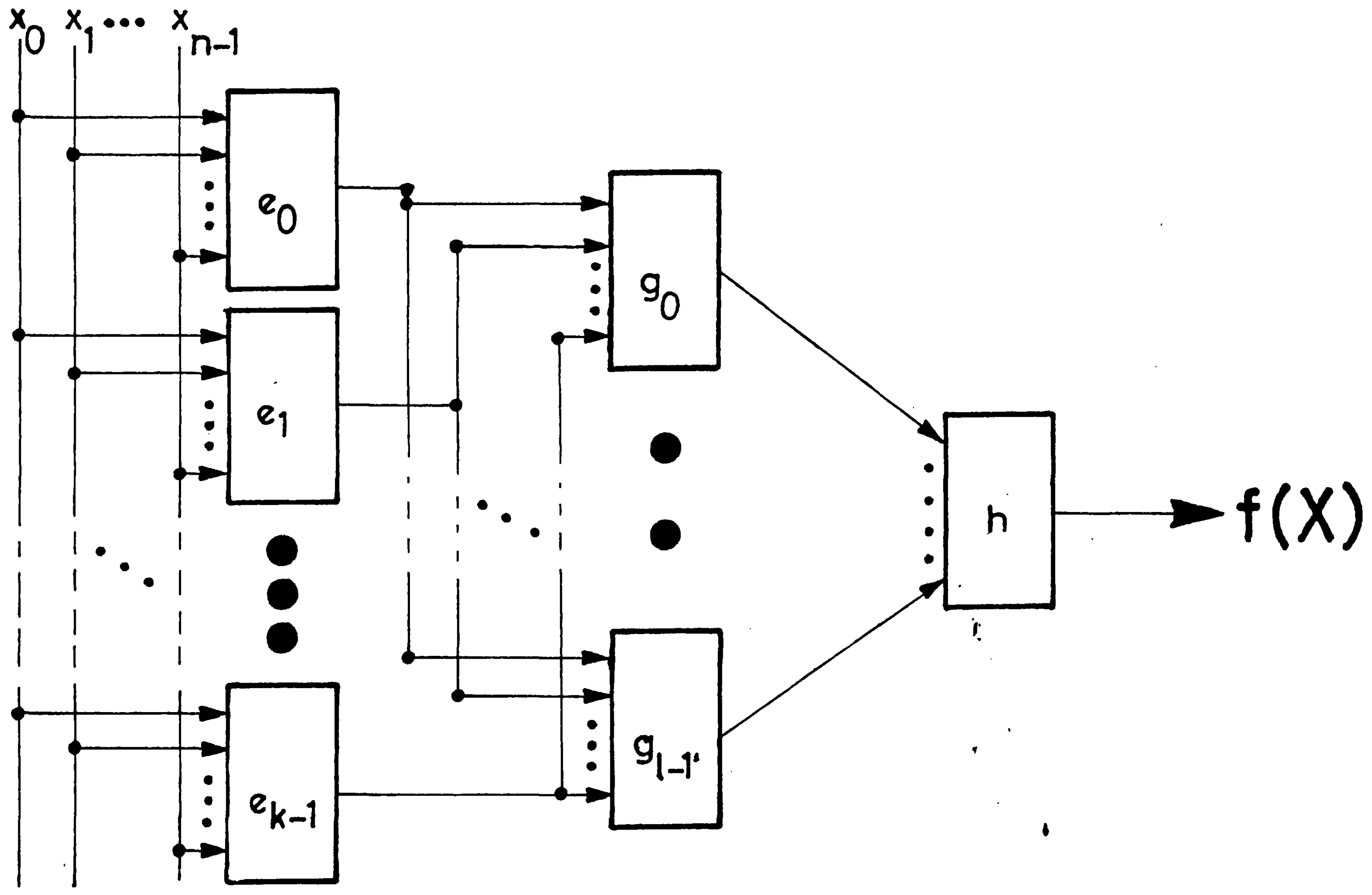


Fig. 5.3 Multi level realisation of f(X).

$$\left[\left.\begin{matrix}1\\0\\0\\0\\\cdot\\\cdot\end{matrix}\right] \cdots\; s_{\sum e_p I_p} \Bigg] \cdots \right] \left[\left.\begin{matrix}1\\0\\0\\0\\\cdot\\\cdot\end{matrix}\right] \cdots\; s_{\sum g_p J_p} \Bigg] \cdots \right] s_h] = s_f] \qquad \text{..........(5.12)}$$

where $(I_{k-1},....,I_0)$ and $(J_{1-1},....,J_0)$ are the integer mod-m expansions of column numbers i and j respectively.

5.3 Disjoint decomposability of m-ary functions by spectral means

We now consider a simple disjoint decomposition of a function f(X). The mathematical representation of this will be of the form:

$$f(X) = h(X_1, g(X_0)) \qquad \text{..........(5.13)}$$

where $X_1 = (x_{n-1},....,x_k)$, and $X_2 = (x_{k-1},....,x_0)$ is a partition of $X = (x_{n-1},....,x_0)$.

The block diagram realisation that makes use of this decomposition is shown in Fig. 5.4.

Theorem 5.4 A m-valued n-variable function f(X) has a simple disjoint decomposition of the form (5.13) iff the spectra $s_f]$, $s_g]$, $s_h]$ of the respective functions satisfy the relationship:

$$\left([Id]^{\otimes\, n-k} \otimes [s_G]\right) s_h] = s_f] \qquad \text{..........(5.14)}$$

where $[Id]$ is m x m identity matrix,

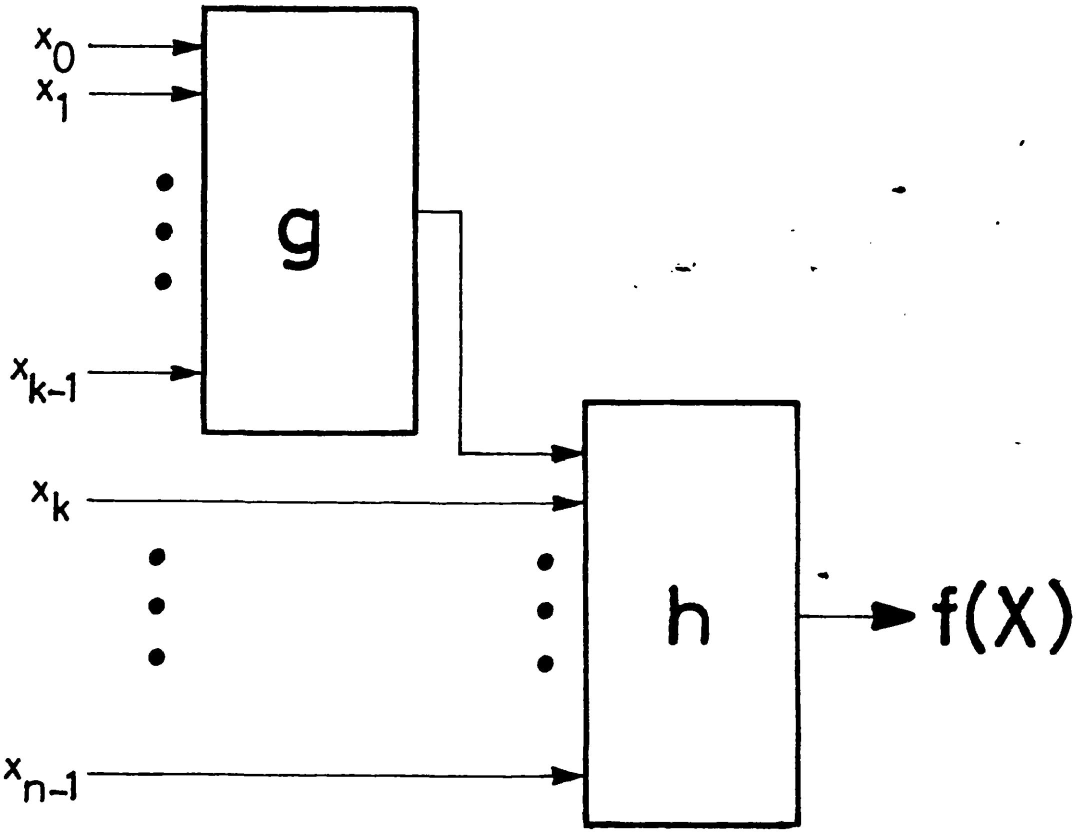


Fig. 5.4 Simple disjoint decomposition of f(X).

and $$\left[S_G\right] = \left[\begin{bmatrix}1\\0\\\cdot\\\cdot\\\cdot\end{bmatrix} \quad S_{1.g}\Big] \quad \cdots \quad S_{(m-1).g}\Big]\right] \qquad \ldots\ldots\ldots(5.15)$$

Proof. Let us consider that the last $(n-k)$ variables of $f(X)$ are assigned to some constant logic values so that the functions $f_z(X_0)$ are obtained such that:

$$f(Z_{n-1},\ldots,Z_k,x_{k-1},\ldots,x_0) = f_z(X_0).$$

where $z = \sum_{p=0}^{n-k-1} m^p \; Z_{p+k}$, and $Z_q \; \varepsilon \; V \quad q = k,\ldots,n-1$

By theorem 4.4 the spectra $S_{f_z}\Big]$ and the spectrum $S_f\Big]$ are related as follows:

$$\left(\left[\overline{Tc\text{-}m}\right]^{\otimes\, n-k} \otimes \left[Id\right]^{\otimes\, k}\right) S_f\Big] = \left[\begin{matrix}S_{f_0}\Big]\\\cdot\\\cdot\\S_{f_z}\Big]\\\cdot\\\cdot\end{matrix}\right] \qquad \ldots\ldots\ldots\ldots(5.16)$$

Similarly the spectra $S_{h_z}\Big]$ of the functions $h_z(x_0) = h(Z_{n-k},\ldots,Z_1,x_0)$ in terms of the spectrum $S_h\Big]$ are given by:

$$\left(\left[\overline{Tc-m}\right]^{\otimes\, n-k} \otimes [Id]\right) \left. S_h \right] = \left. \begin{matrix} \cdot \\ \cdot \\ S_{h_z} \\ \cdot \\ \cdot \\ \cdot \end{matrix} \right]$$

$$\ldots\ldots\ldots\ldots(5.17)$$

We also have the relationship between the spectra $\left. S_{f_z} \right]$, $\left. S_{h_z} \right]$ and $\left. S_g \right]$ given by theorem 5.3 above:

$$[S_G] \quad \left. S_{h_z} \right] = \left. S_{f_z} \right] \qquad \ldots\ldots\ldots(5.18)$$

Equation (5.18) is valid for all $z = 0, m^{n-k}-1$ and hence it may be extended to give:

$$\left([Id]^{\otimes\, n-k} \otimes [S_G]\right) \left. \begin{matrix} \cdot \\ \cdot \\ \cdot \\ S_{h_z} \\ \cdot \\ \cdot \\ \cdot \end{matrix} \right] = \left. \begin{matrix} \cdot \\ \cdot \\ \cdot \\ S_{f_z} \\ \cdot \\ \cdot \\ \cdot \end{matrix} \right]$$

$$\ldots\ldots\ldots(5.19)$$

Replacing (5.16) and (5.17) in (5.19) and multiplying both sides with $\frac{1}{m^{n-k}} \left([Tc-m]^{\otimes\, n-k} \otimes [Id]^{\otimes\, k}\right)$ we obtain:

$$\frac{1}{m^{n-k}} \left([Tc-m]^{\otimes\, n-k} \otimes [Id]^{\otimes\, k}\right)\left([Id]^{\otimes\, n-k} \otimes [S_G]\right)$$

$$\left(\left[\overline{Tc-m}\right]^{\otimes\, n-k} \otimes [Id]\right) \left. S_h \right] = \left. S_f \right] ,$$

using distributive properties of Kronecker product we obtain:

$$\frac{1}{m^{n-k}}\left(\left[Tc\text{-}m\right]^{\otimes n-k}\otimes\left[S_G\right]\right)\left(\left[\overline{Tc\text{-}m}\right]^{\otimes n-k}\otimes\left[Id\right]\right)S_h\right] = S_f\right]$$

and finally:

$$\left(\left[Id\right]^{\otimes n-k}\otimes\left[S_G\right]\right)\ S_h\right] = S_f\right]$$

The sufficiency part of the theorem, namely if there is a relationship of the form (5.14) between the corresponding spectra then a simple disjoint decomposition of the form (5.13) exists, stems from the fact that the spectrum-function pair is unique.

Equation (5.14) gives us a system of simultaneous equations whose solutions will completely specify the unknown function for the realisation in the form shown in Fig. 5.4. The unknown function may be $g(X_0)$ or $h(X_1,x_0)$ and there may be a unique solution, no solution or many solutions to the system of simultaneous equations. These various cases are illustrated with the following examples.

Example 1. A binary example with unique solution:

Consider a simple 4-variable (n = 4) binary (m = 2) function f(X) given by the following map:

$x_2\,x_3$ \ $x_0\,x_1$	0 0	0 1	1 1	1 0
0 0	0	0	1	0
0 1	0	0	1	0
1 1	1	1	1	1
1 0	1	1	0	1

Using conventional techniques the following realisation for this function may be obtained:

$$f(X) = x_2.x_3 \vee \bar{x}_1.x_2 \vee \bar{x}_0.x_2 \vee x_0.x_1.\bar{x}_2 \qquad \text{..........(5.20)}$$

computation of the spectrum $S_f]$ gives:

$$S_f] = \frac{1}{16} \left[-2 \quad 2 \quad 2 \quad -2 \quad 10 \quad 6 \quad 6 \quad -6 \quad 2 \quad -2 \quad -2 \quad 2 \quad -2 \quad 2 \quad 2 \quad -2 \right]^t$$

Now let us choose $g(x_1,x_0)$ as a two input And function whose spectrum is:

$$S_g] = \frac{1}{4} \left[2 \quad 2 \quad 2 \quad -2 \right]^t$$

Hence for the binary case under consideration:

$$[S_G] = \begin{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} & S_g] \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 4 & 2 \\ 0 & 2 \\ 0 & 2 \\ 0 & -2 \end{bmatrix}$$

since only $S_{1.g}] = S_g] = S_{(m-1).g}]$ exists in this simple example.

Putting the latter values in equation (5.14) the following simultaneous equations are obtained:

$$
\begin{aligned}
2\,s_0 + s_1 &= -2\\
s_1 &= 2\\
s_1 &= 2\\
-s_1 &= -2\\
2\,s_2 + s_3 &= 10\\
s_3 &= 6\\
s_3 &= 6\\
-s_3 &= -6\\
2\,s_4 + s_5 &= 2\\
s_5 &= -2\\
s_5 &= -2\\
-s_5 &= 2\\
2\,s_6 + s_7 &= -2\\
s_7 &= 2\\
s_7 &= 2\\
-s_7 &= -2
\end{aligned}
$$

where $S_h] = \frac{1}{8}\left[s_0 \;\; s_1 \;\; s_2 \;\; s_3 \;\; s_4 \;\; s_5 \;\; s_6 \;\; s_7 \right]^t$

The solution for above set of simultaneous equations is clearly:

$$S_h] = \frac{1}{8}\left[-2 \;\; 2 \;\; 2 \;\; 6 \;\; 2 \;\; -2 \;\; -2 \;\; 2 \right]^t$$

and hence the given function f(X) is reduced to the following decomposition where $g(x_1,x_0)$ is the And function $x_1.x_0$:

$g = x_1.x_0$

$x_2\ x_3$	0 0	0 1	1 1	1 0
0	0	0	1	1
1	1	1	1	0

Therefore

$$f(X) = h(x_3,x_1,g(x_1,x_0))$$

$$= g.\overline{x}_2 \ V \ g.x_3 \ V \ \overline{g}.x_2 \qquad \text{.........(5.21)}$$

A gate count of realisations using (5.20) and (5.21) shows that the decomposition realisation uses one 2-input And gate and one NOT gate less than the conventional design, assuming 2-input And and 2-input Or gates are used only.

Example II. A binary example with no solution.

Let us now obtain a function $f'(X)$ from $f(X)$ of the above example by interchanging the variables x_0 and x_2. i.e:

$$f'(x_3,x_2,x_1,x_0) = f(x_3,x_0,x_1,x_2)$$

The spectrum $S_{f'}]$ of $f'(X)$ can be found from the spectrum $S_f]$ of $f(X)$ with the application of theorem 4.2 in chapter 4, giving:

$$S_{f'}] = \frac{1}{8}\left[-2 \quad 2 \quad 10 \quad 6 \quad 2 \quad -2 \quad 6 \quad -6 \quad 2 \quad -2 \quad -2 \quad 2 \quad -2 \quad 2 \quad 2 \quad -2\right]^t$$

Again, putting these values in the equation of (5.14) we obtain the following simultaneous equations:

$$2\ s_0 + s_1 = -2$$

$$s_1 = 2$$

$$s_1 = 10$$

$$-s_1 = 6$$

etc.

Obviously there is no s_1 (and s_0) to satisfy above equations and we can terminate our calculations at this point concluding that $f'(X)$ does not have a disjoint decomposition of the form:

$$f'(X) = h(x_3,x_2,g(x_1,x_0))$$

if $g(x_1,x_0)$ is the Boolean conjunction $x_1 . x_0$.

Example III. A ternary example where the function $g(X_0)$ is to be found.

Consider a three variable ternary function $f(x_2,x_1,x_0)$ given by the following table:

x_0 \ x_2 x_1	0 0	0 1	0 2	1 0	1 1	1 2	2 0	2 1	2 2
0	2	1	0	2	1	1	2	2	2
1	1	0	0	1	1	1	2	2	2
2	0	0	0	1	1	1	2	2	2

It is desired to find if a disjoint decomposition exists where $h(x_2, g)$ is a two variable max. function x_0 V x_1. The spectra $S_f]$ and $S_h]$ are evaluated using the ternary transform and the spectral values are replaced in equation (5.14) giving the following set of simultaneous equations:

$$
\begin{aligned}
9(4a^2 + 2a) + s_0(a + 5) + s'_0(a^2 + 2a) &= 15a^2 + 12a \\
s_1(a + 5) + s'_1(a^2 + 2a) &= 6a^2 + 3a \\
s_2(a + 5) + s'_2(a^2 + 2a) &= 15a^2 + 3a \\
s_3(a + 5) + s'_3(a^2 + 2a) &= 6a^2 + 3a \\
s_4(a + 5) + s'_4(a^2 + 2a) &= 3a^2 + 6 \\
s_5(a + 5) + s'_5(a^2 + 2a) &= 12a^2 + 6 \\
s_6(a + 5) + s'_6(a^2 + 2a) &= 15a^2 + 3a \\
s_7(a + 5) + s'_7(a^2 + 2a) &= 12a^2 + 6 \\
s_8(a + 5) + s'_8(a^2 + 2a) &= 21a^2 + 6
\end{aligned}
$$

$$
\begin{aligned}
9(a + 5) + s_0(2a^2+4) + s'_0(a + 2) &= 6a + 66 \\
s_1(2a^2+4) + s'_1(a + 2) &= 6a^2 + 12a \\
s_2(2a^2+4) + s'_2(a + 2) &= 15a^2 + 12a \\
s_3(2a^2+4) + s'_3(a + 2) &= 6a^2 + 12a \\
s_4(2a^2+4) + s'_4(a + 2) &= 6a + 3 \\
s_5(2a^2+4) + s'_5(a + 2) &= 6a^2 + 3a \\
s_6(2a^2+4) + s'_6(a + 2) &= 15a^2 + 12a \\
s_7(2a^2+4) + s'_7(a + 2) &= 6a^2 + 3a \\
s_8(2a^2+4) + s'_8(a + 2) &= 15a^2 + 3a
\end{aligned}
$$

$$
\begin{aligned}
9(a^2 + 2a) + s_0(a + 2) + s'_0(2a^2 + 1) &= 6a^2 + 12a \\
s_1(a + 2) + s'_1(2a^2 + 1) &= 6a + 3 \\
s_2(a + 2) + s'_2(2a^2 + 1) &= 6a^2 + 3a \\
s_3(a + 2) + s'_3(2a^2 + 1) &= 6a + 3 \\
s_4(a + 2) + s'_4(2a^2 + 1) &= 6a + 12 \\
s_5(a + 2) + s'_5(2a^2 + 1) &= 3a^2 + 6 \\
s_6(a + 2) + s'_6(2a^2 + 1) &= 6a^2 + 3a \\
s_7(a + 2) + s'_7(2a^2 + 1) &= 3a^2 + 6 \\
s_8(a + 2) + s'_8(2a^2 + 1) &= 12a^2 + 6
\end{aligned}
$$

where $S_g \Big] = \frac{1}{9} \left[s_0 \; s_1 \; . \; . \; . \; . \; s_8 \right]^t$

and $S_{2.g}\Big] = \frac{1}{9} \left[s'_0 \; s'_1 \; . \; . \; . \; . \; s'_8 \right]^t$

Note that we only need to compute the values of $s_p, p = 0,8$ but it is necessary to confirm that these values satisfy all equations. A unique solution to above set of equations is possible, and will be found to be:

$$s_0 = a + 5$$
$$s_1 = a^2 + 2a$$
$$s_2 = 4a^2 + 2a$$
$$s_3 = a^2 + 2a$$
$$s_4 = a^2 + 2$$
$$s_5 = 2a^2 + 1$$
$$s_6 = 4a^2 + 2a$$
$$s_7 = 2a^2 + 1$$
$$s_8 = 5a^2 + 1$$

The inverse transformation on this set of spectral values will show, if not already recognised, that it is the spectrum of the $\overline{\text{Plus}}$ operator whose truth table is given below:

$x_0 \backslash x_1$	0	1	2
0	2	1	0
1	1	0	0
2	0	0	0

and hence the final decomposition synthesis for the function is:

$f(X) = \{(x_0 \; \psi \; x_1) \; V \; x_2\}$, where Ψ denotes $\overline{\text{Plus}}$.

Example IV. A ternary example with many decompositions.

The set of simultaneous equations obtained by replacing spectral values in (5.14) may in some cases be consistent but with infinite solutions. The case arises under certain circumstances which will be discussed after the following example.

Consider a 3-variable ternary function $f(x_2,x_1,x_0)$ given by the following map:

x_2 \ x_1 x_0	0 0	0 1	0 2	1 0	1 1	1 2	2 0	2 1	2 2
0	2	2	0	0	0	2	2	0	2
1	2	2	1	1	1	2	2	1	2
2	0	0	1	1	1	0	0	1	0

It is desired to know whether this function has a simple disjoint decomposition of the form below in which $g(x_1,x_0)$ is defined as:

x_1 \ x_0	0	1	2
0	1	1	2
1	2	2	1
2	1	2	1

Let $\left. s_h \right] = \frac{1}{9}\left[s_0 \; s_1 \; . \; . \; . \; . \; s_8 \right]^t$ be the spectrum of a two variable ternary function $h(x_1,x_0)$. The set of simultaneous equations following, is obtained by incorporating the relevant spectra in equation (5.14) for the decomposition of the form:

$$f(x_2,x_1,x_0) = h(x_2,g(x_1,x_0)).$$

$$\begin{aligned}
9s_0 + (5a + 4a^2)\ s_1 + (4a + 5a^2)\ s_2 &= 3 + 6a^2 \\
(2a + a^2)\ s_1 + (2 + a^2)\ s_2 &= 12 + 6a^2 \\
(2 + a)\ s_1 + (a + 2a^2)\ s_2 &= 6a + 12a^2 \\
(2a + a^2)\ s_1 + (2 + a^2)\ s_2 &= 12 + 6a^2 \\
(2 + a)\ s_1 + (a + 2a^2)\ s_2 &= 6a + 12a^2 \\
(5a + 4a^2)\ s_1 + (5 + a^2)\ s_2 &= 30 + 6a^2 \\
(2 + a)\ s_1 + (a + 2a^2)\ s_2 &= 6a + 12a^2 \\
(5 + a)\ s_1 + (4a + 5a^2)\ s_2 &= 24a + 10a^2 \\
(2a + a^2)\ s_1 + (2 + a^2)\ s_2 &= 12 + 6a^2
\end{aligned}$$

$$\begin{aligned}
9s_3 + (5a + 4a^2)\ s_4 + (4a + 5a^2)\ s_5 &= 6a + 3a^2 \\
(2a + a^2)\ s_4 + (2 + a^2)\ s_5 &= 6a + 12a^2 \\
(2 + a)\ s_4 + (a + 2a^2)\ s_5 &= 6 + 12a \\
(2a + a^2)\ s_4 + (2 + a^2)\ s_5 &= 6a + 12a^2 \\
(2 + a)\ s_4 + (a + 2a^2)\ s_5 &= 6 + 12a \\
(5a + 4a^2)\ s_4 + (5 + a^2)\ s_5 &= 6a + 30a^2 \\
(2 + a)\ s_4 + (a + 2a^2)\ s_5 &= 6 + 12a \\
(5 + a)\ s_4 + (4a + 5a^2)\ s_5 &= 24 + 30a \\
(2a + a^2)\ s_4 + (2 + a^2)\ s_5 &= 6a + 12a^2
\end{aligned}$$

$$\begin{aligned}
9s_6 + (5 + 4a^2)\ s_7 + (4a + 5a^2)\ s_8 &= 39 + 42a^2 \\
(2a + a^2)\ s_7 + (2 + a^2)\ s_8 &= 3 + 6a^2 \\
(2 + a)\ s_7 + (a + 2a^2)\ s_8 &= 6a + 3a^2 \\
(2a + a^2)\ s_7 + (2 + a^2)\ s_8 &= 3 + 6a^2 \\
(2 + a)\ s_7 + (a + 2a^2)\ s_8 &= 6a + 3a^2 \\
(5a + 4a^2)\ s_7 + (5 + a^2)\ s_8 &= 12 + 15a^2 \\
(2 + a)\ s_7 + (a + 2a^2)\ s_8 &= 6a + 3a^2 \\
(5 + a)\ s_7 + (4a + 5a^2)\ s_8 &= 15a + 3a^2 \\
(2a + a^2)\ s_7 + (2 + a^2)\ s_8 &= 3 + 6a^2
\end{aligned}$$

Eliminating the linearly independent rows, the following reduced set of simultaneous equations are obtained:

$$\begin{aligned}
9s_0 + (5a + 4a^2)\ s_1 + (4a + 5a^2)\ s_2 &= 3 + 6a^2 \\
(2a + a^2)\ s_1 + (2 + a^2)\ s_2 &= 12 + 6a^2 \\
9s_3 + (5a + 4a^2)\ s_4 + (4a + 5a^2)\ s_5 &= 6a + 3a^2 \\
(2a + a^2)\ s_4 + (2 + a^2)\ s_5 &= 6a + 12a^2 \\
9s_6 + (5a + 4a^2)\ s_7 + (4a + 5a^2)\ s_8 &= 39 + 42a^2 \\
(2a + a^2)\ s_7 + (2 + a^2)\ s_8 &= 3 + 6a^2
\end{aligned}$$

Obviously there are infinite number of solutions to this set of equations. One solution may be obtained by putting $s_1 = s_4 = 0$, giving:

$$S_h\Big] = \frac{1}{3}\Big[\ 3\quad 0\quad 6\quad 3a^2\quad 0\quad 6a^2\quad 6 + 6a^2\quad 0\quad 3 + 3a^2\ \Big]^t \ .$$

Taking the inverse transform on these values we obtain:

$$cH\Big] = \Big[\ 2 + 2a^2\quad a^2\quad 1\quad 1\quad a^2\quad a\quad 2 + 2a\quad 1\quad a\ \Big]^t \ .$$

Note that at two points $ch(0,0) = 2 + 2a^2$ and $ch(2,0) = 2 + 2a$ there are undefined values under the mapping c. (Under this mapping, the values should be of the form a^k where $k \in V = \{0,1,\ldots,m-1\}$. $(2 + 2a)$ is not an element of the codomain of mapping c). This is because of the random values we have chosen for three of the spectral values to solve the simultaneous equations above. In fact, the values at h(0,0) and h(2,0) do not affect the final realisation of the function f(X) and therefore may be assumed to be "don't cares". Any function which agrees with the local values of the function $h(x_1,x_0)$ at the points where $h(x_1,x_0)$ is specified, will be suitable to replace $h(x_1,x_0)$. For the particular case in this example a $h(x_1,x_0)$ will be given as:

$$H] = [1 \; 2 \; 0 \; 0 \; 2 \; 1 \; 2 \; 0 \; 1]^t$$

The infinite solution case occurs when the range of the mapping defined by the function $g(X_0)$ is a proper subset of V. If the function $g(X_0)$ never takes the value $k \in V$, then in a composition of the form:

$$f(X) = h(X_1,g(X_0)) \; ,$$

the values of the function $h(X_1,x_0)$ at points $h(X_1,k)$ will be irrelevant to the specification of f(X). The infinite solutions occur because the function $h(X_1,x_0)$ can take any value at points $h(X_1,k)$. The range of the mapping defined by $g(x_1,x_0)$ in the above example is {1,2}, and hence the function values $h(x_1,0)$ do not effect the final realisation.

Note that in the case of binary systems a proper subset of {0,1} is either {0} or {1}, and if the range of $g(X_0)$ contains only one element g then $g(X_0)$ is independent of all the variables $\{X_0\}$ and can be replaced by the constant g. Above developments assume that the function f(X) has no redundant variables and hence the infinite solution case in the disjoint decomposition synthesis of binary systems cannot occur.

5.4 Discussion

The simultaneous equations of (5.14) contain m^k or m^{n-k+1} unknowns, depending on whether the function $g(X_0)$ or $h(X_1,x_0)$ is to be determined. However, a careful examination of (5.14) shows that the equations are separated into groups each containing m or $(m-1)$ unknowns. Indeed the examples in Section 5.3 illustrate that, for example in $m = 2$ binary case, each group of equations contain two unknowns, one of which is readily obtained without any computation. Advance knowledge of either $g(X_0)$ or $h(X_1,x_0)$ will ease the construction of a heuristic algorithm to search for decomposibility. For example, a binary function $f(X)$ has a decomposition of the form $h(x_{n-1},g(X_0))$ where $h(x_1,x_0)$ is a two input And function if $s(0) + s(2^n-1) = 1$. This may be shown by incorporating the relevant spectra in (5.14) as follows:

$$\frac{1}{4}\left(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\otimes\left[\begin{array}{c|c}1 & \\ 0 & S_{g}\rbrack \\ 0 & \\ \cdot & \\ \cdot & \end{array}\right]\right)\begin{bmatrix}2\\ 2\\ 2\\ -2\end{bmatrix} = S_{f}\rbrack$$

Let $s(0) \varepsilon S_{g}\rbrack$ and $s(0),s(2^n-1) \varepsilon S_{f}\rbrack$; then the above set of equations give us:

$$1 + s(0) = 2\, s(0)$$

$$1 - s(0) = 2\, s(2^n-1)$$

i.e. $s(0) + s(2^n-1) = 1$

The sufficiency of this condition stems from the fact that by theorem 4.4 adding corresponding entries of the bottom half of $S_{f}\rbrack$ to the top half gives the spectrum of $f(0,x_{n-2},\ldots,x_0)$, and if a component of this spectrum is 1 then the remainder will be zero. Note that this is possible if and only if in the spectrum $S_{f}\rbrack$ the following holds:

$s(i) = -s(2^n-1 + i)$ for all $i = 1,\ldots,2^{n-1}-2$

The spectral analysis methods of m-ary combinatorial logic systems have now been completed with the spectral properties developed in this chapter in addition to those discussed in chapter 4.

Any decomposition of f(X) by successive evaluation of functions involved in the decomposition defines a synthesis algorithm. Obviously the time required to perform a complete search for minimum-cost solutions will be prohibitive due to the exponentially growing number of functions that may be involved in the decomposition. A more conventional method may use look-up tables which list all possible decompositions of classes so that, once the class of a function is identified the synthesis will readily follow using the decomposition charts.

A summary of the significance of the spectral techniques developed in this chapter and the preceding chapter 4 will be given in the following pages.

CHAPTER 6

General Conclusions

6. General Conclusions:

In Chapter 1 multi-valued systems were defined, and examples of several existing ternary (3-valued) combinatorial circuits were given. An overall assessment of the published material on combinatorial circuits covered the following:

(a) The development of combinatorial circuits followed closely the mathematical developments in this area. Many designs are implementations of a few primitive sets of functions, which include the Delta literals, Max., Min. and Addition operations, and a multiplexer, namely the T-Gate operator. Evidence shows that efficient multi-valued logic circuits can be fabricated with the current expertise available in semiconductor technology.

(b) There are m^{m^n} m valued n variable functions. Further examples in Chapter 1 showed that there exists inexpensive circuit implementations for some of these functions which are not included in the primitive sets of gates noted above. It is concluded that the development of a general design method which allows maximum use of the advantages offered by a large set of primitive gates will encourage the circuit designers to invent, at reasonable cost and using small on-chip silicon area, novel m-ary circuits.

In Chapter 2 various expansions of multi-valued functions were considered. The local values of the function, expressed as a vector, were transformed into a set of coefficients using an appropriate transform, such that the function and the corresponding set of coefficients were uniquely related. The expansion of a given m-ary function over the field of real numbers was considered to be impractical, since the coefficients obtained vary over wide range of rational numbers. The expansions over a finite field GF(m) with m elements overcame this problem, since each coefficient may now take only one of m values. However, GF(m) expansion is possible if and only if m is a power of a prime.

A practical application of the mathematical developments in this Chapter was demonstrated in Chapter 3, and a Universal-Logic-Module implementation based on GF(m) expansion was considered. The set of primitive gates in this case consists of the GF(m) Multiplier and the GF(m) Adder. For the ternary case the Mod-3 Adder and Multiplier circuits reported elsewhere are suitable for this purpose. However, for m = 4, GF(4) Adder and Multiplier circuits have to be developed for a Universal-Logic-Module realisation of quaternary logic functions. It was shown by means of an example that the interchange of input variable connections to a ULM realisation may result in a reduction on number of modules necessary for a given function implementation. A fundamental problem of finding the minimum-cost connection pattern of the input variables remains unsolved.

Chapter 4 introduced the function spectrum; the set of spectral values are the coefficients of the complex polynomial expansion of m-ary functions such that the variables and the function now take complex values. Chapter 5 continued with developments in this spectral logic area, investigating particularly the spectral relationships in a composite function. A summary of the significant results of these latter two chapters may be outlined as follows:

(i) Spectral Classification: Many operations in the function domain correspond to permutations and/or complex scaling of the spectral coefficients in the spectral domain. The latter are generally known as spectral translations. The m-ary functions whose spectra are equivalent under spectral translations constitute a class. Existing information indicates that the number of equivalence classes generated under spectral translations is small, but further research is required to find a method to generate n-variable classes with the information on (n-1) and less variable classes. Tables listing all 5-variable classes in binary[18] and all 2-variable classes in ternary[5] were constructed using exhaustive search mechanisms which required a fair amount of computation.

(ii) Spectral Synthesis: The spectral synthesis method based on spectral classification assumes that spectral translations have inexpensive and practical implementations. This assumption seems to be reasonable since realisations for Mod-m Adders and Inverters have already been reported in I^2L and CCD technologies. Each class may be represented by a member for which a simple realisation exists. A synthesis algorithm first identifies the class to which a function belongs, and implementation readily follows using appropriate spectral translation operations.

Further research in this area may be the construction of charts, which will show the decomposability relationships between the classes.

(iii) Spectral Analysis: The main contribution of the investigation in this thesis to the spectral logic has been in the spectral analysis area. The composition relationships between the spectra given in

Chapter 5 and the other properties pursued in Chapter 4 are considered to be invaluable mathematical tools for m-ary combinatorial logic analysis. The spectral analysis methods have the characteristic that the circuits to be analysed may contain any gate which implements a m-ary function, and is not confined to any given algebra.

Finally, it is concluded that the spectral information enjoys many further properties, which find applications in classification, synthesis through classification and decomposition, and analysis of m-ary combinatorial functions. Increasing commercial interests in higher-valued logic than binary are likely to be felt in the future, and therefore the research area of the work reported herewith is likely to be of increasing future significance.

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APPENDIX A

Summary of
The Spectral Properties

SUMMARY OF SPECTRAL PROPERTIES

Definitions

$V = \{0,1,\ldots,m-1\}$	set with m elements, integers mod-m
$X = (x_{n-1},\ldots,x_1,x_0)$	vector of independent variables over V
$f(X) = V^n \to V$	n-variable m-ary function
$F]$	column vector whose entries are the local values of f(X) in decimal order
$i = (I_{n-1},\ldots,I_1,I_0)$	m-ary expansion of i such that $i = \sum_{p=0}^{n-1} m^p I_p$ and $I_p \in V$ for all $p = 0,1,\ldots,n-1$
Similarly $j = (J_{n-1},\ldots,J_0)$	
and $w = (W_{n-1},\ldots,W_0)$	
$X \underset{-}{\oplus} w = ((x_{n-1} \oplus W_{n-1}),\ldots,(x_0 \oplus W_0))$	
$L \underset{-}{x} i = [L] x [I_{n-1} \ldots\ldots I_0]^t$	
$a = e^{J \frac{2\pi}{m}}$	mth primitive root of unity on complex plane
$c:k \to a^k$	character of $k \in V$
$Ch(i,j) = \sum_{p=0}^{n-1} I_p J_p$	
$t_i(j) = (a^{Ch(i,j)})*$ where * stands for complex conjugate	set of stepping functions defined in the interval $[0,m^n)$

$[Tc\text{-}m]_{m \times m}$ m x m transform matrix such that elements $t_{i,j}$ are given by:

$$t_{i,j} = (a^{ij})*$$

$[Tc\text{-}m]^{\otimes n}_{m \times m} = \underbrace{[Tc\text{-}m] \otimes \cdots \otimes [Tc\text{-}m]}_{n\text{-times}}$ nth Kronecker power of $[Tc\text{-}m]$

nth order transform matrix

$s_f(i) = \frac{1}{m^n} \sum_{j=0}^{m^n-1} t_i(j) \quad cf(j)$ spectrum of f(X)

Or in vector form:

$$S_f] = \frac{1}{m^n} [Tc\text{-}m]^{\otimes n} \quad cF]$$

$S_{f'}]$, $S_g]$, etc. are defined similarly

<u>Theorems</u>

1) Let $f(0) = 0$ and

$$f'(x) = \begin{cases} k & \text{if } X = (0) \text{ where } k \in V \\ f(X) & \text{otherwise} \end{cases}$$

Then $s_{f'}(i) = s_f(i) + m^{-n}(ck - 1)$

2) Argument translation:

$f'(X) = f(X \underline{\oplus} w)$

Then $s_{f'}(i) = t^*_{i,w}\ s_f(i)$

3) Linear transformation of argument:

$f'(X) = f(X \underline{x} L)$

Then $s_{f'}(i) = s_f(L^{-1} \underline{x}\ i)$

where [L] is a n x n non-singular matrix containing elements which are non-zero dividers in V.

4) Disjoint spectral translation

$f'(X) \; = \; f(X) \oplus x_k$ where $k = 0,1,\ldots,n-1$

Then $s_{f'}(i) \; = \; s_f(i \underline{\ominus} m^k)$

5) Cyclic negation of a function

$f'(X) \; = \; f(X) \oplus k$ where $k \in V$

Then $s_{f'}(i) \; = \; s_f(i) \, a^k$

6) Inverse (simple negation) of a function

$f'(X) \; = \; \overline{f(X)}$

Then $s_{f'}(i) \; = \; c(-1) \, s_f^* \, (D \underline{x} \, i)$

where $[D] \; = \; \text{diag}\,(m-1)$

7) $f'(X) \; = \; f(x_{n-1},\ldots,\overline{x}_k,\ldots,x_0)$

Then $s_{f'}(i) \; = \; a^{I_k} \, s_f(L_k \underline{x} \, i)$

where $\ell_{p,q} \in L_k$ is given by:

$$\ell_{p,q} = \begin{cases} (m-1) & \text{if } p = q = (n-1) - k \\ 1 & \text{if } p = q \\ 0 & \text{if otherwise} \end{cases}$$

8) Making two variables identical

$f'(x_{n-2},x_{n-3},\ldots,x_0) \; = \; f(x_{n-2},x_{n-2},x_{n-3},\ldots,x_0)$

Then $s_{f'}(i) \; = \; s_f(i_0) + s_f(i_1) + \ldots\ldots + s_f(i_{m-1})$

where $i_k \; = \; km^{n-1} + (I_{n-2} \oplus (m-k))m^{n-2} + I_{n-3}m^{n-3} + \ldots\ldots + I_0 m^o$

and $i \; = \; I_{n-2}m^{n-2} + I_{n-3}m^{n-3} + \ldots\ldots + I_0 m^o$, for all $k = 0,1,\ldots,m-1$

9) Setting a variable to a constant

$f'_k(x_{n-2},\ldots,x_0) \; = \; f(k, x_{n-2},\ldots,x_0)$ where $k \in V$

Then

$$\left.\begin{matrix} S_{f'_0}\big] \\ S_{f'_1}\big] \\ \vdots \\ S_{f'_{m-1}}\big] \end{matrix}\right] = \left([Tc-m]^* \otimes [Id]^{\otimes n-1} \right) S_f\big]$$

10) Convolution theorem

$$f'(X) = f_1(X) \oplus f_2(X)$$

Then $s_{f'}(i) = \sum_{w=0}^{m^n-1} s_{f_1}(w)\, s_{f_2}(i \ominus w)$

$$= \sum_{w=0}^{m^n-1} s_{f_1}(i \ominus w)\, s_{f_2}(w)$$

11) Decomposition

$$f'(X) = k(g_{k-1}(X), g_{k-2}(X), \ldots, g_0(X))$$

Then

$$\left[\begin{array}{cccc} \left.\begin{matrix}1\\0\\0\\ \vdots\end{matrix}\right] & \left.S_{g_0}\right] & \cdots \left.\underset{\uparrow \text{jth column}}{S_{\sum_{p=0}^{k-1} g_p J_p}}\right] & \cdots \end{array}\right] S_h\big] = S_f\big]$$

12) Disjoint decomposition

$f'(X) = k(g(X_1), X_0)$ where $\{X_1, X_0\}$ is a partition of $\{X\}$

Then

$$([Id]^{\otimes n-k} \otimes [S_G])\ S_h\big] = S_f\big]$$

where

$$\left[S_G \right] = \left[\left. \begin{matrix}1\\0\\0\\ \cdot\\ \cdot\\ \cdot\end{matrix}\right] \quad \left. S_{1.g} \right] \quad \left. S_{2.g} \right] \quad \cdots\cdot \; \left. S_{(m-1).g} \right] \right]$$

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APPENDIX B

Copies of
Published Material

REPORT NO 1 "A functionally complete ternary system„

(Reprinted from:
I.E.E. Electronic Letters,
Vol.14, 1978, No.3, pp.69-71).

A FUNCTIONALLY-COMPLETE TERNARY SYSTEM

Indexing terms: Integrated logic circuits, Ternary logic

A single two-input ternary operator and a single one-input ternary operator are proposed, which together with the logic values 1 and 2 form a functionally-complete set of ternary operators. A particular feature of the proposed operators is their ease of realisation in semiconductor integrated-circuit form, the two-input operator being circuit-wise similar to recently-developed binary exclusive-OR realisations.

Introduction: The theoretical advantages of three-valued (ternary) logic over two-valued (binary) logic are well known: these advantages promise to become increasingly desirable due to pin limitations on complex digital integrated-circuit packages. Circuit realisations of ternary functions have been investigated by several authorities,[1-10] this area representing the greatest challenge to the practical adoption of three-valued systems. The difficulties of realisation of certain previously proposed circuits and slow operating speeds characterise many of these proposals. Here we introduce two ternary operators which together with logic 1 and 2 enable any ternary function $f(X)$ to be realised, the particular features of these operators being their ease of realisation in monolithic i.c. form.

Proposed operators: The two proposed ternary operators are as follows:

(a) the two-input TOR function defined by

$$\begin{aligned} f(X_i, X_j) &= X_i \tau X_j, \\ &= \max\{X_i, X_j\} \text{ if } X_i \neq X_j \\ &= 0 \text{ if } X_i = X_j \end{aligned}$$

As will be shown later, this TOR ('Ternary exclusive-OR') function may be realised by an exclusive-OR type circuit configuration.

(b) the single-variable (unary) operator defined by

$$\begin{aligned} {}^0(X_i)^0 &= 2 \text{ if } X_i = 0 \\ &= 0 \text{ if } X_i \neq 0 \end{aligned}$$

This operator may also be termed a 'literal' or a 'threshold' operator by certain authorities

These two functions therefore have truth values as below:

	X_i		
X_j	0	1	2
0	0	1	2
1	1	0	2
2	2	2	0

TOR operator $X_i \tau X_j$

X_i	$^0(X_i)^0$
0	2
1	0
2	0

Unary operator

The function-completeness of the above operators for all possible 27 single-variable ternary functions may be checked by simple evaluation. The full listing is given in Table 1.

Functional completeness for the 2-variable case may be demonstrated by the following.

Consider the function of two variables

$$f(X_i, X_j) = ((X_i \tau X_j) \tau X_j) \tau X_i$$

Let this output be Z. Then $Z = 1$ when $X_i = 1, X_j = 2$, and is 0 on all other input minterms. A further operation

$$^0(Z)^0 \tau 2$$

will convert this $Z = 1$ output minterm to 2, leaving the 0-valued minterms unchanged.

Now by permutation of the truth-values of the input variables by appropriate single-variable operations on X_i and/or X_j (see Table 1), this single output minterm of value 1 or 2 may be realised in any chosen minterm position in the truth-table for $f(X_i, X_j)$. The full truth table for any required function $f(X_i, X_j)$ can therefore be realised by the minterm expansion:

$$f(X_i, X_j) = f_1(X_i, X_j) \tau f_2(X_i, X_j) \tau \ldots \tau f_n(X_i, X_j),$$

where

$$f_1(X_i, X_j), \quad f_2(X_i, X_j) \ldots$$

are each functions with a single output minterm of value 1 or 2, otherwise 0, and $n \leqslant 9$. Hence functional completeness for two-variable case is shown. Note that this minterm expansion is not necessarily minimal; certain minimisation procedures are available with the TOR operator to give reduced-length expressions.

With functional-completeness present for both single-variable and two-variable cases, functional-completeness for functions of any number of input variables is assured.

Circuit realisations: The particular merit of the above operators is their ease of realisation in bipolar or m.o.s., technology. In particular the TOR operator is based upon previously-discussed exclusive-OR circuits,[9,11,12] and takes the forms shown in Fig. 1. The single-variable function $^0(X_i)^0$ is a simple inverter circuit, also as shown in Fig. 1.

The propagation time of these circuits will be seen to be comparable with their times when used in binary situations, unlike a number of previously-disclosed ternary circuits, which due to increased circuit complexity exhibit correspondingly increased propagation times. Cascading of the TOR circuits to realise functions such as $(X_i \tau X_j) \tau X_k$ is available, subject to similar limits as in the binary case.[11,12]

Table 1 GENERATION OF ALL SINGLE-VARIABLE FUNCTIONS OF $f(X_i)$ USING TOR AND UNARY OPERATORS

X_i	0	1	2	
f_0	0	0	0	(trivial case)
f_1	0	0	1	$((X_i \tau 1) \tau X_i) \tau 1$
f_2	0	0	2	$(X_i \tau 2) \tau 2$
f_3	0	1	0	$((X_i \tau 2) \tau 2) \tau X_i$
f_4	0	1	1	${}^0(X_i)^0 \tau ({}^0(X_i)^0 \tau 1)$
f_5	0	1	2	X_i
f_6	0	2	0	${}^0(X_i)^0 \tau (X_i \tau 2)$
f_7	0	2	1	$({}^0(X_i)^0 \tau 1) \tau (X_i \tau 2)$
f_8	0	2	2	${}^0(X_i)^0 \tau 2$
f_9	1	0	0	$(X_i \tau 1) \tau ((X_i \tau 2) \tau 2)$
f_{10}	1	0	1	$(((X_i \tau 2) \tau 2) \tau X_i) \tau 1$
f_{11}	1	0	2	$X_i \tau 1$
f_{12}	1	1	0	$(X_i \tau 1) \tau X_i$
f_{13}	1	1	1	(trivial case)
f_{14}	1	1	2	$((X_i \tau 2) \tau 2) \tau 1$
f_{15}	1	2	0	$({}^0(X_i)^0 \tau 2) \tau (X_i \tau 1)$
f_{16}	1	2	1	$({}^0(X_i)^0 \tau (X_i \tau 2)) \tau 1$
f_{17}	1	2	2	$({}^0(X_i)^0 \tau 2) \tau 1$
f_{18}	2	0	0	${}^0(X_i)^0$
f_{19}	2	0	1	$({}^0(X_i)^0 \tau 1) \tau ((X_i \tau 1) \tau X_i)$
f_{20}	2	0	2	${}^0(X_i)^0 \tau (X_i \tau 1)$
f_{21}	2	1	0	$({}^0(X_i)^0 \tau X_i) \tau (X_i \tau 1)$
f_{22}	2	1	1	${}^0(X_i)^0 \tau 1$
f_{23}	2	1	2	${}^0(X_i)^0 \tau X_i$
f_{24}	2	2	0	$X_i \tau 2$
f_{25}	2	2	1	$(X_i \tau 2) \tau 1$
f_{26}	2	2	2	(trivial case)

Further considerations: While functional completeness ensures that any ternary function $f(X)$ can be realised by the use of the above two operators only, many authorities have previously considered the advantages of an augmented set of operators above the minimum-necessary set. In particular, the immediate availability of a set of unary functions is advantageous, often considerably reducing the algebraic and overall circuit complexity of a given function. Among the commonly encountered are the set of six unary operators:

X_i	$X_i\downarrow$	$X_i\uparrow$	X_i'	${}^0(X_i)^0$	${}^1(X_i)^1$	${}^2(X_i)^2$
0	2	1	2	2	0	0
1	0	2	1	0	2	0
2	1	0	0	0	0	2

The availability of a choice of unary operations eases the realisation of ternary tristable circuit elements. The specification of the 'best' tristable circuit, however, is a subject for continuing development, but a simple unclocked type of circuit which provides steady ternary output signals may be proposed as in Fig. 2. In order that the two halves of Fig. 2a shall be circuitwise identical requires outputs Q_1 and Q_2 to be the diagonal-inversion of each other, that is $Q_2 = Q_1'$, with logic 1 being the quiescent input level on both inputs S_1 and S_2. Raising input S_1 to logic 2 forces output Q_1 to 2; lowering S_1 to logic 0 forces Q_1 to 1. Similarly, raising S_2 to logic 2 forces Q_2 to 2, and lowering S_2 to 0 forces Q_2 to 1. The required logic for each half of Fig. 2a is mapped in Fig. 2b, which is realisable as follows:

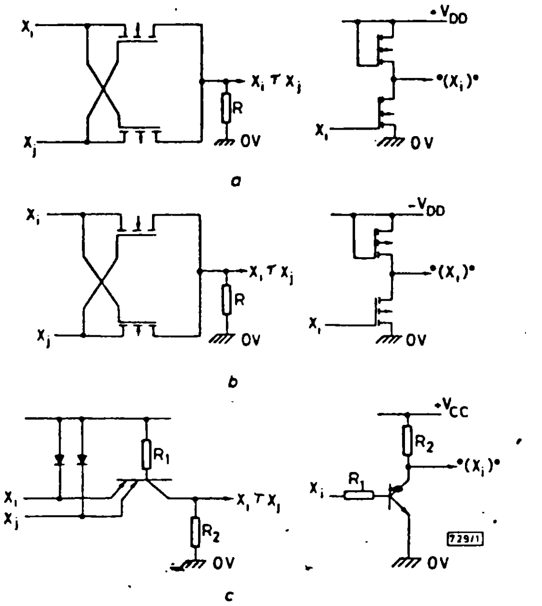
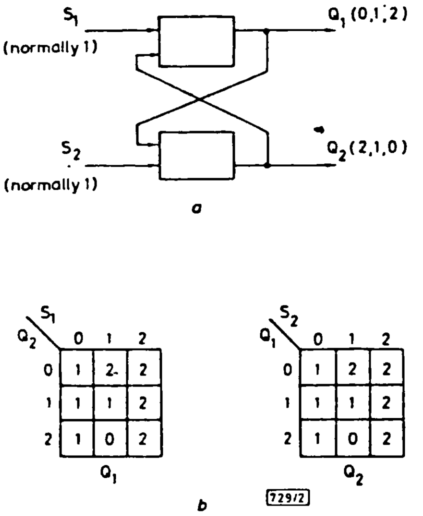


Fig. 1 *TOR and unary operator realisations in enhancement-m.o.s. and bipolar form*

a m.o.s., positive logic, say 0 = 0V, 1 = +5V, 2 = +10V

b m.o.s., negative logic, say 0 = 0V, 1 = −5V, 2 = −10V

c bipolar, positive logic (negative logic circuits using opposite polarity devices equally possible)



Q_2 \ S_1	0	1	2
0	1	2	2
1	1	1	2
2	1	0	2

Q_1

Q_1 \ S_2	0	1	2
0	1	2	2
1	1	1	2
2	1	0	2

Q_2

Fig. 2 *Simple unclocked symmetrical 'set-reset' type tristable circuit*

a schematic

b truth table for each output Q_1 and Q_2

$$Q_1 = \{(S_1 \tau 1) \tau (((S_1 \tau Q_2\uparrow) \tau Q_2\uparrow) \tau S_1) \tau \\ ({}^0\{((S_1 \tau Q_2\downarrow) \tau Q_2\downarrow) \tau S_1\}^0 \tau 2)\}$$

Q_2 identical except inputs S_2 and Q_1 instead of S_1 and Q_2.

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2nd December 1977

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REPORT NO 2 "Some properties of the spectra of ternary logic functions„

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SOME PROPERTIES OF THE SPECTRA OF TERNARY LOGIC FUNCTIONS

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Abstract

The transformation of conventional digital data into an alternative mathematical domain, the spectral domain, has previously been considered for both binary and higher-valued logic functions. The coefficient values in the spectral domain maintain the same information content as the original digital domain data, and may be used for logic analysis and synthesis purposes. Here we investigate how four logical operations on ternary input variables in the 3-valued digital domain modify the resultant coefficient values in the ternary spectral domain. The objective of this development is its subsequent application to the problem of synthesis of ternary (and higher-valued) logic functions.

1. Introduction

Spectral transformation of binary functions and binary logic synthesis using spectral data has been reported by a number of authors[1,2,3,4]. The extension of these spectral techniques to higher-valued combinational logic synthesis is also being pursued[5,6,7]. These techniques involve the transform of the function domain data into spectral domain data, using one of many possible transform matrices constructed from orthogonal functions. Mathematically this may be expressed as:

$$[T]\;Q] = S]$$

where $[T]$ = the chosen orthogonal transform,

$Q]$ = the function domain data arranged as a column vector (truth-table),

$S]$ = the resultant spectral domain data for the given function.

The spectral coefficients of $S]$ may each be considered as a measure of the dependence of the function on one or other variable of the function, or mod.m addition of combinations of these variables[3,4,7]. If, for example, all the spectral coefficients involving the $i^{\underline{th}}$ variable are zero-valued, then the $i^{\underline{th}}$ variable is redundant.

At this stage of development, it is difficult to define which subsets of spectral coefficients are more important than others when using the coefficients for synthesis purposes, as the type(s) of logic gate available must be taken into consideration. However, the effect on the spectral coefficients of certain logical operations in the function domain, for example linear transformation of the variables, has been reported at a previous MVL Symposium[6]. In this paper we will continue to investigate these relationships, as a necessary prerequisite to the use of spectral data for logic synthesis purposes.

In particular, we will consider the effect of the following universal algebraic operations in the ternary function domain on the resultant ternary spectral domain:

(a) the interchange (permutation) of variables of the function, see Figure 1(b),

(b) the "commoning" (making two or more identical) of variables of the function, see Figure 1(c),

(c) the holding of a variable at one of the logic levels ("held-at" conditions), see Figure 1(d),

and

(d) cascade composition of functions, see Figure 1(e).

These four operations are illustrated in Figure 1. Note that they jointly cover the operations possible to synthesise any given function from a (potentially unlimited) supply of physically realizable logic gates. Operations (a) to (c) concern functions of a lower degree than n which are realizable by a n-input gate.

2. Definitions

2.1) V = {0,1,2} is a set with three elements.

2.2) $\mathcal{Z} = (Z_{n-1}Z_{n-2}\ldots Z_0)$ is a n-tuple, where $Z_i \; \varepsilon \; V$, i = 0, ..., n-1, with

$$z = \sum_{i=0}^{n-1} 3^i Z_i \; .$$

2.3) $\mathcal{X} = (X_n X_{n-1}\ldots X_1)$ is the input vector, where $X_i \; \varepsilon \; V$, i = 1, ..., n.

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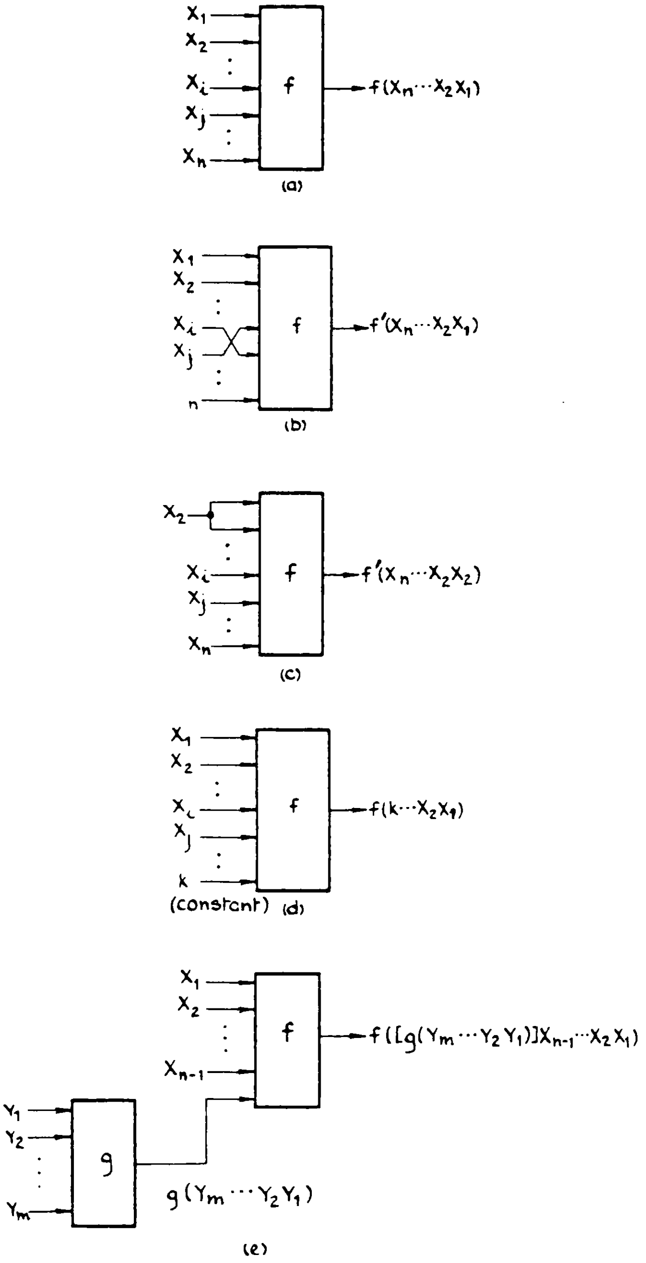

Figure 1 The logical operations considered:

(a) original gate, realising a n-variable function f(X),

(b) interchange of two X_i input variables,

(c) commoning of two X_i inputs,

(d) "held-at" at one gate input,

(e) cascade composition.

2.4) $f: V^n \to V$ is a fully-specified n-variable function, the truthtable of f being a column vector $F]$ whose entries $f(Z)$ are the local values of the function $f(X)$, ordered decimally.

2.5) $Q]$ is a column vector, whose entries are q(z) ordered decimally, where $q(z) = u^{f(z)}$ and $u = e^{j\frac{2\pi}{3}}$.

2.6) Chrestenson functions[6] are a set of orthogonal stepping functions defined in the interval $[0,3^n)$ and $(0 \leq w \leq 3^n-1)$: $t_w(z) = u^{Ch(w,z)}$,

$$\text{where } Ch(w,z) = \sum_{s=0}^{n-1} W_{n-1-s} Z_s \; , \; ,$$

$$w = \sum_{i=0}^{n-1} W_i 3^i ,$$

$$z = \sum_{i=0}^{n-1} Z_i 3^i$$

2.7) The spectral coefficients of a n-variable ternary function using the Chrestenson functions as the transform are individually given by:

$$S_w = \sum_{z=0}^{3^n-1} \overline{t_w(z)}.q(z),$$

where $\overline{t_w(z)}$ is the complex conjugate of $t_w(z)$. Re-expressed in vector form, this is merely $S = T Q$, which is one transformation of the complete transform $S] = [T]Q]$ previously noted, and where the elements of $[T]$ are $t_{wz} = \overline{t_w(z)}$.

Note that as $t_{wz} = t_{zw}$ the transform matrix $[T]$ is diagonally symmetric; also the rows of $[T]$ are mutually orthogonal. Therefore the inverse of $[T]$ is simply its conjugate multiplied by a scaling factor $\frac{1}{3^n}$, giving:

$$Q] = \frac{1}{3^n}[\overline{T}]S]$$

Such orthogonal transformations may be shown to be unique, the information content of $S]$ and $Q]$ being identical.

3. The Algebraic Operations and Resultant Spectral Changes

3.1) Permutation of gate inputs, see Figure 1(b) (simplified from Karpovsky[3])

Required:

$$f'(X_n X_{n-1} \ldots X_j X_i \ldots X_1) = f(X_n X_{n-1} \ldots X_i X_j \ldots X_1)$$

Note: This interchange of pairs of variables may be repeated to achieve any desired input permutation.

Then the new spectral coefficients S'_w in terms of the original spectral coefficients S_w are given by:

$$S'_w = S_{w'}, \quad \text{where:}$$

$$w = \{3^{n-1}W_n + \ldots + 3^j W_j + 3^i W_i + \ldots + W_0\},$$

and

$$w' = \{3^{n-1}W_n + \ldots + 3^j W_i + 3^i W_j + \ldots + W_0\}$$

Proof:

From (2.7) we have:

$$S'_w = \sum_{z=0}^{3^n-1} \overline{t_w(z)}.q(z'),$$

where $z = \{3^{n-1}Z_{n-1} + \ldots + 3^j Z_j + 3^i Z_i + \ldots Z_0\}$

and $z' = \{3^{n-1}Z_{n-1} + \ldots + 3^j Z_i + 3^i Z_j + \ldots Z_0\}$

But

$$S'_w = \sum_{z=0}^{3^n-1} \overline{t_w(z)}.q(z') = \sum_{z=0}^{3^n-1} \overline{t_w(z')}.q(z),$$

and from the definition of the Chrestenson functions it follows that:

$$t_{w'}(z) = t_w(z')$$

Therefore:

$$S'_w = \sum_{z=0}^{3^n-1} \overline{t_w(z')}.q(z) = \sum_{z=0}^{3^n-1} \overline{t_{w'}(z)}.q(z) = S_{w'}$$

3.2) Commoning gate inputs, see Figure 1(c)

Required:

$$f'(X_n X_{n-1} \ldots X_2 X_2) = f(X_n X_{n-1} \ldots X_2 X_1)$$

Then the new spectral coefficients S'_w in terms of the original spectral coefficients $_w$ are given by:

$$S'_w = S_{w_0} + S_{w_1} + S_{w_2},$$

where $w_0 = \{W_{n-1}3^{n-1} + W_{n-2}3^{n-2} + \ldots + W_0 3^0\}$

$$w_1 = \{(2 \oplus W_{n-1})3^{n-1} + (1 \oplus W_{n-2})3^{n-2} + \ldots + W_0 3^0\}$$

$$w_2 = \{(1 \oplus W_{n-1})3^{n-1} + (2 \oplus W_{n-2})3^{n-2} + \ldots + W_0 3^0\}$$

and where $\oplus$ = mod.3 addition.

$$S'_w = 0 \text{ if } W_{n-1} \neq 0.$$

Proof:

Consider the function column vector $F]$, where $F] \overset{\Delta}{=} f(X)$. If X_1 is now made equal to X_2, then the column vector $F]$ will be altered to $F']$ such that the original values of $f(X)$ when $X_1 = X_2$ are now repeated in blocks of threes.

Examination of the transform matrix $[T]$ will show that the last $\{3^{n-1} \times 2\}$ rows of the matrix have a structure such that when each row is divided into blocks of threes, these blocks contain 1, u and u^2, which together sum to zero. (Note that these are the rows whose $W_{n-1} \neq 0$.) The multiplication of $Q'] , \overset{\Delta}{=} F']$, with the transform matrix $[T]$ will therefore result in a spectrum with spectral coefficient values $S'_w = 0$ for $W_{n-1} \neq 0$. However, the first 3^{n-1} rows of $[T]$ contain blocks of three of the same value. Hence we require to find three rows of $[T]$ which take the same values for $Z_0 = Z_1$, and take different values for $Z_0 \neq Z_1$, so that when the spectral values associated with these three rows are added, their contribution to the spectral coefficient value S from $Z_0 = Z_1$ is tripled, but their contribution from $Z_0 \neq Z_1$ sums to zero.

These three rows can be found from examination of the exponents of the Chrestenson functions which make up $[T]$, and are:

$$Ch(w_0,z) = Z_0 W_{n-1} + Z_1 W_{n-2} + \ldots + Z_{n-1} W_0,$$

$$Ch(w_1,z) = Z_0(2 + W_{n+1}) + Z_1(1 + W_{n-2}) + \ldots + Z_{n-1} W_0,$$

$$Ch(w_2,z) = Z_0(1 + W_{n+1}) + Z_1(2 + W_{n-2}) + \ldots + Z_{n-1} W_0,$$

that is:

$$Ch(w_0,z) = k, \quad \text{where k is an integer,}$$

$$Ch(w_1,z) = 2Z_0 + Z_1 + k,$$

$$Ch(w_2,z) = Z_0 + 2Z_1 + k.$$

If $Z_0 = Z_1$, then $Ch(w_1,z) = k + 3Z_0$

and $Ch(w_2,z) = k + 3Z_0$

and therefore

$$t_{w_0}(z) = t_{w_1}(z) = t_{w_2}(z).$$

But if $Z_0 \neq Z_1$, then we have the following numerical results:

Z_0	Z_1	$Z_0 + 2Z_1$	$2Z_0 + Z_1$
0	1	2	1
0	2	4	2
1	0	1	2
1	2	5	4
2	0	2	4
2	1	4	5

It can be seen from this table that

$$t_{w_i}(z) = u t_{w_0}(z),$$

and

$$t_{w_j}(z) = u^2 t_{w_0}(z),$$

where $i \neq j$, $i,j \in \{1,2\}$,

and therefore

$$t_{w_0}(z) + t_{w_1}(z) + t_{w_2}(z) = 0.$$

Hence the commoning (making equal) of two function variables reduces the n-variable function to a (n-1) variable function. Therefore, finally, if the spectral coefficient values S'_w whose $W_{n-1} \neq 0$ are omitted, and all remaining values in S_w are divided by three, then the conventional minimum length, minimum coefficient value spectrum for the (n-1) variable function $f'(X)$ is obtained.

3.3) "Held-at" input, see Figure 1(d)

Consider a n-variable function $f(X_n X_{n-1} \ldots X_1)$ with spectrum S_w. If the $n^{\underline{th}}$ input of the logic gate that realizes the function $f(X)$ is now held at a logic level, the new function realized by this gate and corresponding spectrum will be:

$$\frac{1}{3}\begin{bmatrix} [I] & [I] & [I] \\ [I] & u[I] & u^2[I] \\ [I] & u^2[I] & u[I] \end{bmatrix} \Big[S_{\tilde{w}} \Big] = \begin{bmatrix} S_{\tilde{w}_0} \\ S_{\tilde{w}_1} \\ S_{\tilde{w}_2} \end{bmatrix}$$

where $\left.S_{\tilde{w}}\right]$ = the spectrum for the healthy n-variable function, re-ordered as defined below,

$\left.S_{\tilde{w}_0}\right] \Rrightarrow f(0\ X_{n-1} \ldots X_1)$, that is X_n held at 0,

$\left.S_{\tilde{w}_1}\right] \Rrightarrow f(1\ X_{n-1} \ldots X_1)$, that is X_n held at 1,

$\left.S_{\tilde{w}_2}\right] \Rrightarrow f(2\ X_{n-1} \ldots X_1)$, that is X_n held at 2.

Proof:

In order to simplify the mathematics of the proof, a different ordering of the Chrestenson transform matrix $[T]$ will be employed. This re-ordered matrix will be called $[H]$, (Hadamard ordering). We now have:

$$[H_0] = 1$$

$$[H_n] = \begin{bmatrix} [H_{n-1}] & [H_{n-1}] & [H_{n-1}] \\ [H_{n-1}] & u[H_{n-1}] & u^2[H_{n-1}] \\ [H_{n-1}] & u^2[H_{n-1}] & u[H_{n-1}] \end{bmatrix}$$

The elements of this matrix can be shown to be:

$$h_{wz} = u^{p(w,z)}$$

$$\text{where } p(w,z) = \sum_{s=0}^{n-1} W_s Z_s.$$

The $[H]$ transform matrix will produce the same set of spectral coefficients as obtained using the $[T]$ transform matrix, but re-ordered as follows:

$$S_w = S_{\tilde{w}} \quad \text{where } \tilde{w} = \{W_{n-1}3^{n-1} + \ldots + W_0 3^0\}$$

$$= \sum_{i=0}^{n-1} 3^i . W_i \quad ,$$

$$w = \sum_{i=0}^{n-1} 3^i . W_{(n-1)-i}$$

The column vector $\left.Q\right]$ may be written as three adjoining vectors

$$\begin{bmatrix} Q_0 \\ Q_1 \\ Q_2 \end{bmatrix}$$

where $\left.Q_0\right]$, $\left.Q_1\right]$ and $\left.Q_2\right]$ are the column vectors with X_n = 0, 1, and 2, respectively. Then:

$$\left[\begin{array}{ccc} \left[\overline{H_{n-1}}\right] & \left[\,0\,\right] & \left[\,0\,\right] \\ \left[\,0\,\right] & \left[\overline{H_{n-1}}\right] & \left[\,0\,\right] \\ \left[\,0\,\right] & \left[\,0\,\right] & \left[\overline{H_{n-1}}\right] \end{array}\right] \left[\begin{array}{c} \left.Q_0\right] \\ \left.Q_1\right] \\ \left.Q_2\right] \end{array}\right] = \left[\begin{array}{c} \left.S_{\tilde{w}_0}\right] \\ \left.S_{\tilde{w}_1}\right] \\ \left.S_{\tilde{w}_2}\right] \end{array}\right]$$

where $\left[\overline{H_{n-1}}\right]$ is the complex conjugate of $\left[H_{n-1}\right]$

The column vector $\left.Q\right]$ can be replaced by

$$\left.Q\right] = \frac{1}{3^n}\left[H_n\right]\left.S_{\tilde{w}}\right]$$

and hence it follows that

$$\frac{1}{3}\left[\begin{array}{ccc} [I] & [I] & [I] \\ [I] & u[I] & u^2[I] \\ [I] & u^2[I] & u[I] \end{array}\right] \left.S_{\tilde{w}}\right] = \left[\begin{array}{c} \left.S_{\tilde{w}_0}\right] \\ \left.S_{\tilde{w}_1}\right] \\ \left.S_{\tilde{w}_2}\right] \end{array}\right]$$

From the above relationship between the usual Chrestenson ordering of the spectral coefficients and this revised Hadamard ordering, if the Chrestenson ordered set of spectral coefficients are used to calculate the stuck-at coefficients, we finally have:

$$\frac{1}{3}\left[\begin{array}{c|c|c} 1\,1\,1\,0\,0\,0\,. & 0\,0\,0\,.\,.\,.\,. & 0\,0\,0\,.\,.\,.\,. \\ 0\,0\,0\,.\,.\,.\,. & 1\,1\,1\,0\,0\,0\,. & 0\,0\,0\,.\,.\,.\,. \\ 0\,0\,0\,.\,.\,.\,. & 0\,0\,0\,.\,.\,.\,. & 1\,1\,1\,0\,0\,0\,. \\ 0\,0\,0\,1\,1\,1\,. & 0\,0\,0\,.\,.\,.\,. & 0\,0\,0\,.\,.\,.\,. \\ . & . & . \\ . & . & . \\ 0\,0\,0\,.\,.\,.\,. & 0\,0\,0\,.\,.\,.\,. & 0\,0\,0\,.\,1\,1\,1 \\ \hline 1\,u\,u^2\,0\,0\,0\,. & 0\,0\,0\,.\,.\,.\,. & 0\,0\,0\,.\,.\,.\,. \\ 0\,0\,0\,.\,.\,.\,. & 1\,u\,u^2\,.\,.\,.\,. & 0\,0\,0\,.\,.\,.\,. \\ . & . & . \\ . & . & . \\ . & . & . \\ . & . & . \\ 0\,0\,0\,.\,.\,.\,. & 0\,0\,0\,.\,.\,.\,. & 0\,0\,0\,.\,1\,u\,u^2 \\ \hline 1\,u^2\,u\,0\,0\,0\,. & 0\,0\,0\,.\,.\,.\,. & 0\,0\,0\,.\,.\,.\,. \\ 0\,0\,0\,0\,0\,0\,. & 1\,u^2\,u\,.\,.\,.\,. & 0\,0\,0\,.\,.\,.\,. \\ . & . & . \\ . & . & . \\ . & . & . \\ . & . & . \\ 0\,0\,0\,.\,.\,.\,. & 0\,0\,0\,.\,.\,.\,. & 0\,0\,0\,.\,1\,u^2\,u \end{array}\right] \left.S_w\right] = \left[\begin{array}{c} \left.S_{w_0}\right] \\ \left.S_{w_1}\right] \\ \left.S_{w_2}\right] \end{array}\right]$$

3.4) <u>Cascade composition of functions, see Figure 1(e)</u>

Given two functions f and g with n and m variables, respectively, with individual spectra in Hadamard ordering of $\left.S_{\tilde{w}_f}\right]$ and $\left.S_{\tilde{w}_g}\right]$, we may generate a new function as shown in Figure 1(e), given by:

$$f([g(Y_m Y_{m-1} \dots Y_1)]\, X_{n-1} X_{n-2} \dots X_1) \quad ,$$

which we may denote by (f*g). Then the spectrum for (f*g) is given by:

$$\underbrace{\left[\begin{array}{c} \\ H \\ \\ \end{array}\right]}_{m+n-1} \left[\begin{array}{c} \vdots \\ S_{\tilde{w}_{f_i}} \\ \vdots \end{array}\right] = \left.S_{\tilde{w}}\right] \quad ,$$

where $\left[H\right]$ = the Hadamard-ordered transform, of dimensions m+n-1,

$\left.S_{\tilde{w}_{f_i}}\right]$ = the spectra of function f, with its $n^{\underline{th}}$ input 0,1 or 2, repeated 3^m times as defined by $i = g(Y_m Y_{m-1} \dots Y_1)$,

$\left.S_{\tilde{w}}\right]$ = the resultant complete spectrum of the n-input function (f*g), in Hadamard ordering.

<u>Proof</u>:

Proof follows closely from that of section (3.3) above, viz:

$$\left[\begin{array}{ccc} \overline{H_{n-1}} & & 0 \\ & \ddots & \\ 0 & & \overline{H_{n-1}} \end{array}\right] \left.Q_{(f*g)}\right] = \left.S_{\tilde{w}_{f_i}}\right]$$

But

$$\left[\overline{H_{m+n-1}}\right] \left.Q_{(f*g)}\right] = \left.S_{\tilde{w}}\right]$$

Therefore

$$\left.S_{\tilde{w}}\right] = \frac{1}{3^{n-1}} \left[H_{m+n-1} \right] \left[\begin{array}{ccc} \left[\overline{H_{n-1}}\right] & & 0 \\ & \ddots & \\ 0 & & \left[\overline{H_{n-1}}\right] \end{array}\right] \left[\begin{array}{c} \vdots \\ \left.S_{\tilde{w}_{f_i}}\right] \\ \vdots \end{array}\right]$$

whence

$$\left.S_{\tilde{w}}\right] = \left[H_m\right]_{m+n-1} \left.S_{\tilde{w}_{f_i}}\right]$$

where $\left[H_m\right]_{m+n-1}$ is the matrix obtained by multiplying each element of $\left[H_m\right]$ matrix by $\left[I\right]_{3^{n-1}\times 3^{n-1}}$.

4. Closing Remarks

Given a logic gate which realizes a n-variable function, the operations (3.1) to (3.3) above produce functions of (n-1) variables realizable by the same gate. Repeated such operations clearly produce functions of (n-2) variables, and so on. With the addition of the simple logical operation of negation an appropriate n-input gate may be used to implement some or all of the functions of < n inputs. In the binary case the existence of n=3 functions which are each capable of realizing all the binary n=2 functions, and also n=5 functions which are capable of realizing all the binary n=3 functions have been reported[9], the specification of these "master" n-input functions being derived from binary spectral considerations. Corresponding developments for the ternary case are being considered[10-13], but a prerequisite to this complete development is an understanding of ternary spectral manipulations.

Operation (3.4) above is clearly one which may enable the cascade synthesis of required logic functions to be undertaken using spectral data, and therefore is complementary to operations (3.1) to (3.4). It may finally be noted that these developments may be extended to the consideration of the spectra of m-valued functions, m > 3.

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REPORT NO 3 " A consideration of Universal - Logic - Modules for ternary synthesis, based upon, Reed-Muller coefficients „

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A CONSIDERATION OF UNIVERSAL-LOGIC-MODULES FOR TERNARY SYNTHESIS, BASED UPON REED-MULLER COEFFICIENTS

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Abstract

Ternary switching functions may be realised by the use of universal-logic-modules ("ULM's"), the specification and use of such modules being based upon the canonic Reed-Muller ternary expansion. Function realisation, however, requires computation of the Reed-Muller coefficients for the particular function being realised. In this paper a straightforward matrix method of solving the coefficients for any given ternary function is disclosed. The method does not require the lengthy solution of 3^n simultaneous equations, but instead involves the multiplication of three matrices of order $3^m \times 3^m$ to determine the 3^n unknown coefficients, where $m = \frac{n}{2}$.

The method may be shown to be extendable to any q-ary system, where q is a power of a prime.

1. Introduction

The uncomplemented Reed-Muller expansion for a two-variable (n = 2) binary function, namely:

$$f(x_1,x_2) = \{a_0 \oplus a_1x_1 \oplus a_2x_2 \oplus a_3x_1x_2\}, \qquad \ldots(1)$$

where $x_1,x_2 \in \{0,1\}$, $\oplus$ = mod.2 addition (= Exclusive-OR), and a_i, i = 0,1, ..., are the binary Reed-Muller coefficients, $a_i \in \{0,1\}$, may be factorised as:

$$f(x_1,x_2) = \{(a_0 \oplus a_1x_1) \oplus x_2.(a_2 \oplus a_3x_1)\} \qquad \ldots(2)$$

This forms the basis of the binary universal logic module proposed by Murugesan[1,2,3], where it is appreciated that each parenthesis bracket in the factored expansion may take one value from the set $\{0,1,x_1,\bar{x}_1\}$. This is illustrated in Figure 1(a). For functions of more than two variables, an assembly of such modules such as shown in Figure 1(b) develops.

For the ternary case, the corresponding n = 2 Reed-Muller expansion takes the usual form:

$$f(x_1,x_2) = \{a_0 \oplus a_1x_1 \oplus a_2x_1^2 \oplus a_3x_2 \oplus a_4x_1x_2 \oplus a_5x_1^2x_2 \oplus a_6x_2^2 \oplus a_7x_1x_2^2 \oplus a_8x_1^2x_2^2\} \qquad \ldots(3)$$

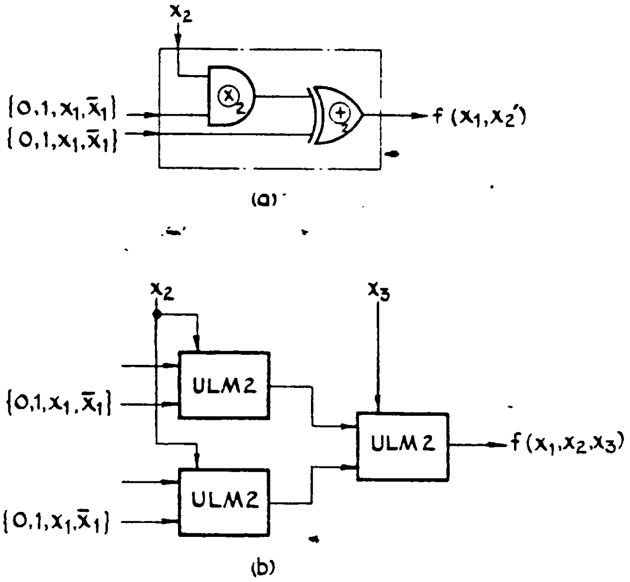

Figure 1 Binary universal-logic-module based upon the binary Reed-Muller canonic expansion.

(a) basic n = 2 ULM of Murugesan.

(b) tree structure of n = 2 ULM's for functions of three variables (Note, x_i inputs may be interchanged, with corresponding modification to the coefficient values).

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where $x_1,x_2 \in \{0,1,2\}$, $\oplus$ = mod.3 addition, and a_i, i = 0,1, ..., are the Reed-Muller coefficients, $a_i \in \{0,1,2\}$. It will be noted that there are 3^n Reed-Muller coefficients to describe any ternary function of n variables. A similar development may be applied to Equation (3) as was applied to the binary Equations (1) and (2), giving the mod. 3 factorisation:

$$f(x_1,x_2) = \{(a_0 \oplus a_1x_1 \oplus a_2x_1^2) \oplus x_2(a_3 \oplus a_4x_1 \oplus a_5x_1^2) \oplus x_2^2(a_6 \oplus a_7x_1 \oplus a_8x_1^2)\} \qquad \ldots(4)$$

Each parenthesis bracket in equation (4) may now be realised by a single-variable universal logic module, as illustrated in Figure 2(a).

The assembly of such modules for the realisation of equation (4) therefore is as shown in Figure 2(b). The triangular topology for more than two ternary variables (n > 2) should be clear, similar to that published for the binary case.

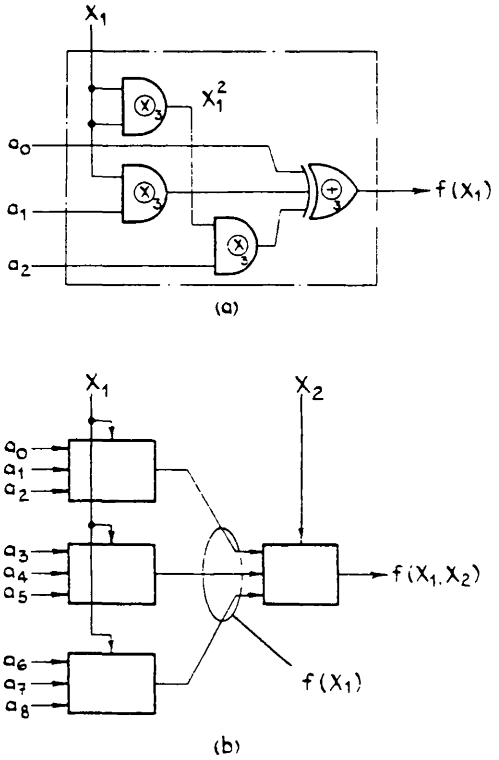

Figure 2 Ternary universal-logic-module based upon ternary Reed-Muller canonic expansion.

(a) basic n = 1 ternary ULM, incorporating mod.3 multiplication and addition.

(b) tree structure of n = 1 ULM's for functions of two variables. (Note, the x_i inputs may be interchanged, with corresponding modification to the coefficient values.)

To apply this form of realisation for any given ternary function f(x) thus requires the calculation of the a_i constants for the particular function. This is the purpose of the algebraic development disclosed in the following sections, finally leading to the establishment of equation (22), which will give the desired coefficients.

The number of universal logic modules ("ULM's") necessary to realise any given ternary function of n variables will not exceed

$$\sum_{i=1}^{n} 3^{i-1} \text{ , } = \tfrac{1}{2}(3^n - 1) \qquad \ldots(5)$$

Should all the a_i coefficient inputs into a particular ULM be zero-valued, then clearly it is unnecessary to include this module in the final hardware realisation. Equally, if a_1 and a_2 but not a_0 (see Figure 2(a)) are zero, then a_0 is fed through unchanged and the ULM is unnecessary. Hence it is desirable from the cost point of view that the number of ULM's with zero-valued a_1 and a_2 coefficients is maximised, which may be achieved by the appropriate permutation of the x_i inputs in the general realisation structure illustrated in Figure 2(b). Further, if two (or more) ULM's at any level of realisation have identical inputs, then such ULM's may be combined into a single ULM with the same input signals. Thus the possible permutations of the x_i inputs may markedly influence the minimum number of ULM's necessary for the realisation of the given function, this feature being considered in the final section.

The physical implementation of the mod.3 multipliers and adder in the ULM of Figure 2(a) in monolithic form would appear to be most promising in I^2L technology[4]. Work in this area is in hand by other authorities.

2. Matrix relationships for the Reed-Muller coefficients

A major problem is the determination of the a_i coefficients for any given function f(x)[2,5,6]. The usually-described method is to solve the set of 3^n simultaneous equations which exist and have to be satisfied when each minterm value of f(x) is listed. This is a lengthy procedure when n is large. Here we will develop an alternative matrix method for solving the a_i values, and then continue further using these values in a ULM realisation.

2.1) Definitions and basic developments

Definition 1

Let V = {0,1,2} be the value set for the ternary algebra, and $x_i \in V$ be the independent ternary input variables, i = 1, 2, ..., n.

Definition 2

If $F:V^n \rightarrow V$ is a n-variable ternary function, then F may be expressed in a modified Reed-Muller form[7], as below:

$$f(x_1,x_2,\ldots,x_n) = \{f_1(x_1,x_2,\ldots x_{n-1}) \oplus x_n.f_2(x_1,x_2,\ldots,x_{n-1}) \oplus x_n^2.f_3(x_1,x_2,\ldots,x_{n-1})\} \qquad \ldots(6)$$

where $\oplus$ = mod.3 addition and . = mod.3 multiplication.

The latter will continue to be omitted from here on.

Similarly, each internal function may be expanded; for example

$$f_1(x_1,x_2,\dots,x_{n-1}) = \{g_1(x_1,x_2,\dots,x_{n-2}) \oplus x_{n-1}g_2(x_1,x_2,\dots x_{n-2}) \oplus x^2_{n-1}g_3(x_1,x_2,\dots,x_{n-2})\} \qquad \dots(7)$$

When this process is repeated n times we have the full expansion containing 3^n terms, namely:

$$\begin{aligned} f(x_1,x_2,\dots x_n) = \{ & a_0 \oplus a_1x_1 \oplus a_2x_1{}^2 \\ & \oplus x_2(a_3 \oplus a_4x_1 \oplus a_5x_1{}^2) \\ & \oplus x_2{}^2(a_6 \oplus a_7x_1 \oplus a_8x_1{}^2) \oplus \dots. \\ & \oplus a_ix_1{}^{p_1}x_2{}^{p_2}x_3{}^{p_3}\dots x_n{}^{p_n} \oplus \dots. \\ & \oplus a_{3^n-1}x_1{}^2x_2{}^2x_3{}^2\dots x_n{}^2 \} \end{aligned} \qquad \dots(8)$$

where $i = p_1 3^0 + p_2 3^1 + \dots + p_n 3^{n-1}$, and a_i and p_i ε {0,1,2}.

Definition 3

(a) $X_k]$ is a row matrix of order 1×3^m, where $m = \frac{n}{2}$; n is assumed to be even integer.*

Then $X_k] = [k_{11}\ k_{12}\ k_{13}\ \dots\ k_{13^m}]$,

where $k_{1h} = x_1{}^{p_1}x_2{}^{p_2}\dots x_m{}^{p_m}$, and $h-1 = p_1 3^0 + p_2 3^1 + \dots p_m 3^{m-1}$

(b) $X_l]$ is a column matrix of order $3^m \times 1$.

l_{j1} ε $X_l]$ where $l_{j1} = x_{m+1}{}^{r_1}x_{m+2}{}^{r_2}\dots x_n{}^{r_m}$, and $j-1 = r_1 3^0 + r_2 3^1 + \dots + r_m 3^{m-1}$.

Definition 4

$[A]$ is the Reed-Muller coefficient matrix of order $3^m \times 3^m$,

$$[A] \triangleq \begin{bmatrix} a_0 & a_{3^m} & \dots & . \\ a_1 & a_{3^m+1} & & \\ \vdots & \vdots & & \\ a_{3^m-1} & . & \dots & a_{3^n-1} \end{bmatrix}$$

where a_i ε {0,1,2} are the same as in Equation (8). Note that all the 3^n a_i coefficients are contained in this matrix.

Hence it follows from Definitions 3 and 4 that Equation (8) is given by:

$$f(x_1,x_2,\dots,x_n) = X_k]\,[A]\,X_l] \qquad \dots(9)$$

Definition 5

$[Y_1]$ and $[Y_2]$ are matrices of the order $3^m \times 3^m$, such that if k_{gh} ε $[Y_1]$ and l_{ij} ε $[Y_2]$ then

$$k_{gh} = x_1{}^{p_1}x_2{}^{p_2}\dots x_m{}^{p_m}$$

where $h-1 = p_1 3^0 + p_2 3^1 + \dots + p_m 3^{m-1}$,

$g-1 = x_1 3^0 + x_2 3^1 + \dots + x_m 3^{m-1}$,

and

$l_{ij} = x_{m+1}{}^{r_1}x_{m+2}{}^{r_2}\dots x_n{}^{r_m}$, where $i-1 = r_1 3^0 + r_2 3^1 + \dots + 3_m 3^{m-1}$

and $j-1 = x_{m+1}3^0 + x_{m+2}3^1 + \dots x_n 3^{m-1}$.

It readily follows that

$$[Y_2] = [Y_1]^t \qquad \dots(10)$$

Let us define a matrix of dimensions b x b as follows, where $\frac{b}{3} = c$.

$$[K]_{b \times b} \triangleq \begin{bmatrix} [I]_{c\times c} & [0]_{c\times c} & [0]_{c\times c} \\ [I]_{c\times c} & [I]_{c\times c} & [I]_{c\times c} \\ [I]_{c\times c} & 2[I]_{c\times c} & [I]_{c\times c} \end{bmatrix}_{b\times b}$$

where $[I]$ is the unity matrix.

Theorem 1

The $3^m \times 3^m$ matrix $[Y_1]$ in terms of the $[K]$ matrices is given by:

---

* Note, should the number of variables n in a given problem not be an even integer, then we may take the next highest integer value, and assume in all the subsequent algebraic developments that this additional variable is eventually redundant, that is it is a dummy variable.

$$[Y_1]_{3^m\times 3^m} = \begin{bmatrix} [K]_{3\times 3} & & \mathbf{0} \\ & [K]_{3\times 3} & \\ \mathbf{0} & \ddots & [K]_{3\times 3} \end{bmatrix}_{3^m\times 3^m} \begin{bmatrix} [K]_{9\times 9} & & \mathbf{0} \\ & [K]_{9\times 9} & \\ \mathbf{0} & \ddots & [K]_{9\times 9} \end{bmatrix}_{3^m\times 3^m} \cdots \begin{bmatrix} [K]_{3^m\times 3^m} \end{bmatrix}_{3^m\times 3^m} \quad \ldots(12)$$

Proof

Each row of $\left[Y_1\right]$ matrix can be considered as an $X_k]$ row vector with vector elements given numerical values. Let k be a real number such that

$k = \sum_{i=1}^{m} x_i 3^{i-1}$, then the first row of $\left[Y_1\right]$ is $X_0]$, the second row of $\left[Y_1\right]$ is $X_1]$, etc.

If $\left[Y_1\right]$ is a 3 x 3 matrix, then $X_k]$ with one variable will be $X_k] = \left[1 \quad x_1 \quad x_1^2\right]$. ...(13)

In particular

$$X_0] = [1 \quad 0 \quad 0],$$

$$X_1] = [1 \quad 1 \quad 1],$$

$$X_2] = [1 \quad 2 \quad 1],$$

giving

$$[Y_1] = \begin{bmatrix} X_0] \\ X_1] \\ X_2] \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix} \quad \ldots(14)$$

With two variables we have:

$$X_k] = [1 \quad x_1 \quad x_1^2 \quad x_2 \quad x_1x_2 \quad x_1^2x_2 \quad x_2^2 \quad x_1x_2^2 \quad x_1^2x_2^2]$$

$$= [1 \quad x_1 \quad x_1^2]_{1\times 3} \begin{bmatrix} 1 & 0 & 0 & x_2 & 0 & 0 & x_2^2 & 0 & 0 \\ 0 & 1 & 0 & 0 & x_2 & 0 & 0 & x_2^2 & 0 \\ 0 & 0 & 1 & 0 & 0 & x_2 & 0 & 0 & x_2^2 \end{bmatrix}_{3\times 9} \quad ..(15)$$

Hence:

$$X_0] = [1 \quad 0 \quad 0] \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$X_1] = [1 \quad 1 \quad 1]\left[[I]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3}\right]$$

$$X_2] = [1 \quad 2 \quad 1]\left[[I]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3}\right]$$

$$X_3] = [1 \quad 0 \quad 0]\left[[I]_{3\times 3} \quad [I]_{3\times 3} \quad [I]_{3\times 3}\right]$$

$$\vdots$$

$$X_8] = [1 \quad 2 \quad 1]\left[[I]_{3\times 3} \quad 2[I]_{3\times 3} \quad [I]_{3\times 3}\right] \quad ---(16)$$

Therefore:

$$[Y_1] = \begin{bmatrix} X_0] \\ X_1] \\ \vdots \\ X_8] \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} [I]_{3\times 3} & [0]_{3\times 3} & [0]_{3\times 3} \\ [I]_{3\times 3} & [I]_{3\times 3} & [I]_{3\times 3} \\ [I]_{3\times 3} & 2[I]_{3\times 3} & [I]_{3\times 3} \end{bmatrix}$$

$$= \begin{bmatrix} [K]_{3\times 3} & [0] & [0] \\ [0] & [K]_{3\times 3} & [0] \\ [0] & [0] & |K|_{3\times 3} \end{bmatrix} \begin{bmatrix} [K]_{9\times 9} \end{bmatrix}$$

Note, all matrix element subscripts are integer, and all matrix multiplications are mod.3.

Definition 6

Let $[F]$ be the function output matrix of order $3^m \times 3^m$, $f_{ij} \in [F]$, such that

$$f_{ij} = f(x_1, x_2, x_3, .., x_n) \qquad ...(17)$$

where

$$i-1 = x_1 3^0 + x_2 3^1 + \dots + x_m 3^{m-1},$$

and

$$j-1 = x_{m+1} 3^0 + x_{m+2} 3^1 + \dots + x_n 3^{m-1}$$

Theorem 2

It follows from the previous definitions that

$$[F] = [Y_1][A][Y_2] \qquad ....(18)$$

Proof

From equations (9) and (17)

$$f_{ij} = X_k][A]X_l] \qquad ...(19)$$

where $\quad k = \sum_{a=1}^{m} x_a 3^{a-1}$

and $\quad l = \sum_{a=m+1}^{n} x_a 3^{a-(m+1)}$

Therefore $i = k$ and $j = l$.

Hence $\quad f_{kl} = X_k][A]X_l]$.

Now $X_k]$'s are the rows of $[Y_1]$ and $X_l]$'s are the columns of $[Y_2]$. Therefore it follows that:

$$[F] = [Y_1][A][Y_2] \qquad ...(18)$$

(repeated)

3. Evaluation of the Reed-Muller coefficients for a given function

Matrix $[K]$ is a non-singular matrix and its inverse therefore is:

$$[K]^{-1} = \begin{bmatrix} [I]_{c\times c} & [0]_{c\times c} & [0]_{c\times c} \\ [0]_{c\times c} & 2[I]_{c\times c} & [I]_{c\times c} \\ 2[I]_{c\times c} & 2[I]_{c\times c} & 2[I]_{c\times c} \end{bmatrix}_{b\times b}$$

where $\frac{b}{3} = c$ as before.

Therefore from equation (18) we have

$$[Y_1]^{-1}\,[F][Y_2]^{-1} = [A] \qquad ...(20)$$

and from equation (10) we have

$$[Y_2]^{-1} = \left([Y_1]^t\right)^{-1} \quad * \qquad ...(21)$$

But for any non-singular matrix $[M]$ we have

$$\left([M]^t\right)^{-1} \equiv \left([M]^{-1}\right)^t ,$$

and hence Equation (20) finally may be written:

$$\begin{aligned} [A] &= [Y_1]^{-1}\,[F]\left([Y_1]^t\right)^{-1} \\ &= [Y_1]^{-1}\,[F]\left([Y_1]^{-1}\right)^{t'} \qquad ...(22) \end{aligned}$$

Finally, expressing $[Y_1]^{-1}$ in terms of $[K]^{-1}$ matrices, we have:

$$[Y_1]^{-1} = \left(\begin{bmatrix} [K]_{3\times3} & & 0 \\ & \ddots & \\ 0 & & [K]_{3\times3} \end{bmatrix} \begin{bmatrix} [K]_{9\times9} & & 0 \\ & \ddots & \\ 0 & & [K]_{9\times9} \end{bmatrix} \cdots \begin{bmatrix} [K]_{3^m\times3^m} \end{bmatrix} \right)^{-1}$$

$$= \begin{bmatrix} [K]^{-1}_{3^m\times3^m} \end{bmatrix} \cdots \begin{bmatrix} [K]^{-1}_{9\times9} & & 0 \\ & \ddots & \\ 0 & & [K]^{-1}_{9\times9} \end{bmatrix} \begin{bmatrix} [K]^{-1}_{3\times3} & & 0 \\ & \ddots & \\ 0 & & [K]^{-1}_{3\times3} \end{bmatrix}$$

...(23)

As an example of the application of these final equations, let us determine the Reed-Muller coefficients for a four-variable ternary function, whose output values expressed in the $3^m \times 3^m$ matrix format $[F]$ are as follows:

---

* Note, the $[Y_1]^{-1}$ and $\left([Y_1]^{-1}\right)^t$ matrices for n = 2, 4 and 6 are given in the final Appendix.

Function

$f(x_1,x_2,x_3,x_4)$:

x_4x_3 / x_2x_1	00	01	02	10	11	12	20	21	22
00	0	1	2	1	2	0	2	0	1
01	1	2	0	2	0	1	0	1	2
02	2	0	1	0	1	2	1	2	0
10	1	2	0	2	0	1	0	1	2
11	2	0	1	0	1	2	1	2	0
12	0	1	2	1	2	0	2	0	1
20	2	0	1	0	1	2	1	2	0
21	0	1	2	1	2	0	2	0	1
22	1	2	0	2	0	1	0	1	2

From Equation (23) we have:

$$[Y_1]^{-1} = \begin{bmatrix} 1&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0 \\ 0&0&0&2&0&0&1&0&0 \\ 0&0&0&0&2&0&0&1&0 \\ 0&0&0&0&0&2&0&0&1 \\ 2&0&0&2&0&0&2&0&0 \\ 0&2&0&0&2&0&0&2&0 \\ 0&0&2&0&0&2&0&0&2 \end{bmatrix} \begin{bmatrix} 1&0&0&0&0&0&0&0&0 \\ 0&2&1&0&0&0&0&0&0 \\ 2&2&2&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&2&1&0&0&0 \\ 0&0&0&2&2&2&0&0&0 \\ 0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&2&1 \\ 0&0&0&0&0&0&2&2&2 \end{bmatrix}$$

$$= \begin{bmatrix} 1&0&0&0&0&0&0&0&0 \\ 0&2&1&0&0&0&0&0&0 \\ 2&2&2&0&0&0&0&0&0 \\ 0&0&0&2&0&0&1&0&0 \\ 0&0&0&0&1&2&0&2&1 \\ 0&0&0&1&1&1&2&2&2 \\ 2&0&0&2&0&0&2&0&0 \\ 0&1&2&0&1&2&0&1&2 \\ 1&1&1&1&1&1&1&1&1 \end{bmatrix}$$

Hence $[Y_2]^{-1} = \left([Y_1]^{-1}\right)^t = \begin{bmatrix} 1&0&2&0&0&0&2&0&1 \\ 0&2&2&0&0&0&0&1&1 \\ 0&1&2&0&0&0&0&2&1 \\ 0&0&0&2&0&1&2&0&1 \\ 0&0&0&0&1&1&0&1&1 \\ 0&0&0&0&2&1&0&2&1 \\ 0&0&0&1&0&2&2&0&1 \\ 0&0&0&0&2&2&0&1&1 \\ 0&0&0&0&1&2&0&2&1 \end{bmatrix}$

The evaluation of Equation (22) therefore gives:

$$[A] = \begin{bmatrix} 0&1&0&1&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \end{bmatrix},$$

and hence the function is merely:

$$f(x_1,x_2,x_3,x_4) = [x_1 \oplus x_2 \oplus x_3 \oplus x_4]$$

4. The effect of interchange of variables on ULM realisations

It has previously been noted that when ULM realisations are required, choice of the x_i inputs at each level of realisation can influence the total number of ULM's necessary in the realisation. Let us now consider this feature.

Let R be a n-tuple $R_nR_{n-1}\ldots R_2R_1$, and r be a real number such that $\sum_{i=1}^{n} R_i 3^{i-1} = r$, where $R_i \in V$.

Let us number the rows and the columns of the $[A]$ matrix with the subscripts of the first elements of each row and column, respectively, in mod.3. Then it can be seen from Equation (9) that the coefficients in the r^{th} row (or column) of $[A]$ appear with the variables $x_i^{R_i}$. The interchange of inputs x_i of such a combinational circuit will cause the row (or column or both) numbers of $[A]$ to be changed.

If then the rows and columns of the $[A]$ matrix are re-arranged so that the row and column numbers are brought back to their original places, the function remains unaltered, since the coefficients will be still appearing with the same variables. This interchange of variables can be done so that the number of "blocks" of zero's are maximised, and hence the number of ULM's to be used to implement the function is reduced.

For example, consider the function $f(x_1,x_2)$ whose truthtable is:

$x_1 \backslash x_2$	0	1	2
0	1	1	0
1	1	0	0
2	2	1	1

From equation (23) we have:

$$[A] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 2 \\ 0 & 1 & 2 \end{bmatrix},$$

$$= \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 0 \\ 2 & 1 & 1 \end{bmatrix}$$

Thus a circuit realisation satisfying the above function is as shown in Figure 3(a), using four ULM blocks in this particular realisation.

However if x_1 and x_2 are interchanged, recalculation of the $[A]$ coefficient matrix will give:

$$[A'] = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

with the corresponding realisation shown in Figure 3(b)

In this simple example no two blocks at the first level of realisation have all-zero or the same set of a_i input coefficients, but the middle ULM is now redundant as only the a_i constant 1 is fed on to the second-level ULM. Thus calculation of all different $[A]$ coefficient matrices and examination of the zero-valued or identical-group-of-three valued coefficients will show the maximum redundant ULM configurations.

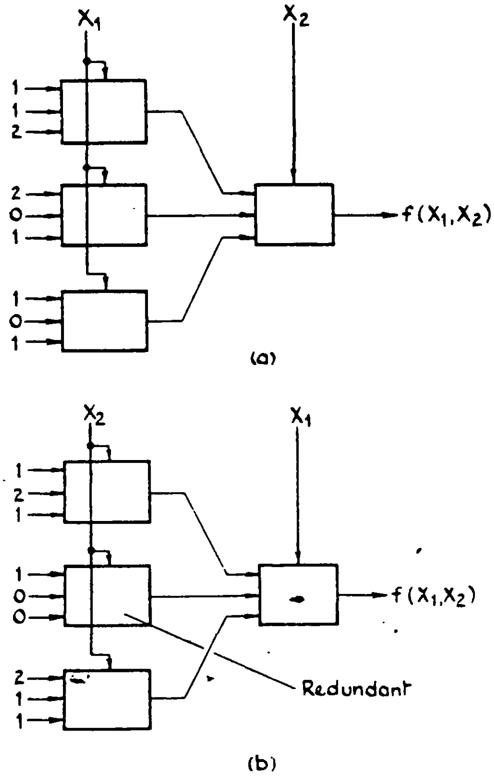

Figure 3 Ternary ULM realisation of $f(x_1,x_2) = 1,1,1,1,0,0,2,1,1$

(a) with x_1 in the first level of realisation

(b) with x_2 in the first level of realisation, the first-level middle ULM now being redundant.

5. Further comments

Functions in q-valued logic, where q is a positive integer power of a prime p, can be represented as polynomials over the Galois field GF(q). For example, the polynomials for a single-variable q-valued function will take the form

$$f(x) = a_0 + a_1.x^1 + \ldots + a_{q-1}.x^{q-1}$$

where $a_i \in V = \{0,1,\ldots,q-1\}$ and +,. are addition and multiplication operations, respectively, over the field GF(q). The computation of the a_i coefficients in terms of the local values of the function f(x) has been described by several authorities[8-11]. In particular, Green et. al. have given matrix solutions for the determination of coefficients, which involves the computation of a $(q^n \times q^n)$ inverse matrix for the n-variable case. The development described in this paper has computational advantages over Green et. al's method in that:

(a) the inverse matrix to be computed is of the order of $(q^{n/2} \times q^{n/2})$, hence reducing the memory space required to store the data,

and

(b) the number of additions for the computation of the coefficients is smaller.

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APPENDIX

The $\left[Y_1\right]^{-1}$ and $\left(\left[Y_1\right]^{-1}\right)^t$ matrices used in the final equation

$$\left[A\right] = \left[Y_1\right]^{-1}\left[F\right]\left(\left[Y_1\right]^{-1}\right)^t \qquad \ldots(22) \text{ repeated}$$

are given below for n = 2, n = 4, and n = 6.

(a) n = 2

$$\left[Y_1\right]^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 2 & 2 \end{bmatrix}$$

$$\left(\left[Y_1\right]^{-1}\right)^t = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

(b) n = 4

$$\left[Y_1\right]^{-1} = \begin{bmatrix} 1&0&0&0&0&0&0&0&0 \\ 0&2&1&0&0&0&0&0&0 \\ 2&2&2&0&0&0&0&0&0 \\ 0&0&0&2&0&0&1&0&0 \\ 0&0&0&0&1&2&0&2&1 \\ 0&0&0&1&1&1&2&2&2 \\ 2&0&0&2&0&0&2&0&0 \\ 0&1&2&0&1&2&0&1&2 \\ 1&1&1&1&1&1&1&1&1 \end{bmatrix}$$

$$\left(\left[Y_1\right]^{-1}\right)^t = \begin{bmatrix} 1&0&2&0&0&0&2&0&1 \\ 0&2&2&0&0&0&0&1&1 \\ 0&1&2&0&0&0&0&2&1 \\ 0&0&0&2&0&1&2&0&1 \\ 0&0&0&0&1&1&0&1&1 \\ 0&0&0&0&2&1&0&2&1 \\ 0&0&0&1&0&2&2&0&1 \\ 0&0&0&0&2&2&0&1&1 \\ 0&0&0&0&1&2&0&2&1 \end{bmatrix}$$

(c) n = 6

$$[Y_1]^{-1} = \left[\begin{array}{ccccccccccccccccccccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 2 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 2 & 1 & 0 & 2 & 1 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 1 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 1 & 2 \\
0 & 0 & 0 & 2 & 2 & 2 & 1 & 1 & 1 & 0 & 0 & 0 & 2 & 2 & 2 & 1 & 1 & 1 & 0 & 0 & 0 & 2 & 2 & 2 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 2 & 1 & 0 & 2 & 1 & 0 & 2 & 1 & 0 & 2 & 1 & 0 & 2 & 1 & 0 & 2 & 1 & 0 & 2 & 1 & 0 & 2 & 1 & 0 & 2 & 1 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2
\end{array}\right]$$

$$\left([Y_1]^{-1}\right)^t = \left[\begin{array}{ccccccccccccccccccccccccccc}
1 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 2 \\
0 & 2 & 2 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 2 & 2 \\
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0 & 0 & 0 & 0 & 2 & 2 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 2 & 2 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 1 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 1 & 1 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 2 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 1 & 2 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 2 & 2 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 2 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 2 & 1 & 0 & 2 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 2 & 0 & 1 & 0 & 0 & 0 & 2 & 0 & 1 & 1 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 2 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 1 & 2
\end{array}\right]$$

□

REPORT NO 4 " The evaluation of the spectrum of multi-level logic networks "

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THE EVALUATION OF THE SPECTRUM OF MULTI-LEVEL LOGIC NETWORKS

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Abstract—The relationship between the spectral coefficients of the functions in a two-level decomposition and the overall function spectrum are considered, and an algebraic relationship established to give the latter from the former. These algebraic relationships hold for binary or higher-valued logic systems. The extension to three- or more-level realisations is mathematically straightforward.

1. INTRODUCTION

The synthesis of logical functions, that is functions which are the mappings from the cartesian products of a set V with n elements onto V, using spectral techniques has been the subject of research by a number of authorities in recent years[1–5]. Such techniques involve the transformation of information given in truthtable form into a spectral domain using orthogonal transforms, and manipulating the resultant spectral domain data so as to achieve an efficient design realisation using the range of available logic gates. Although most research has concentrated upon two-valued (binary) synthesis, an attraction of spectral synthesis methods is that the properties of orthogonal transforms and the design principles can be shown to be applicable for any m-valued system, where $m \geq 2$ is an integer. A number of properties and the theorems concerned have been published[2, 6, 7].

This paper investigates the general relationship between the spectra of $(k+1)$ m-valued functions that are used in a two-level realisation of a function $G(X)$, and the spectrum of $G(X)$. Such a two-level realisation is shown in Fig. 1. The function $G(X)$ is therefore a composition of the individual functions f_i, $i = 0, \dots, k-1$ and g, where g may be any logical function. An extension to more than a two-level realisation follows.

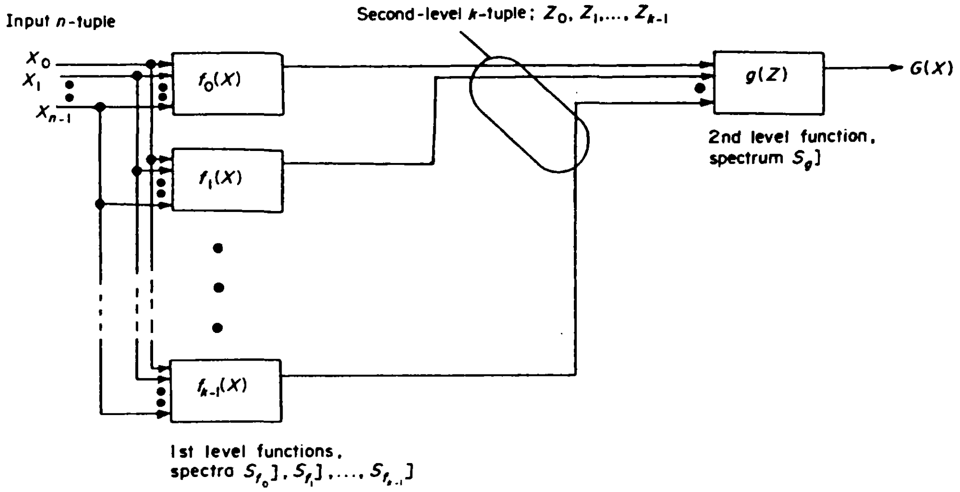

Fig. 1. Two-level realisation of a function $G(X)$, function inputs X_i, $i = 0$ to $n-1$.

2. DEFINITONS

(1) $V = \{0, 1, \dots, m-1\}$ is a set with m elements (integers mod-m).
(2) G, f_i: $V^n \to V$, $i = 0, 1, \dots, k-1$, are a set of k fully-specified n-variable functions.
(3) $X = (X_{n-1}X_{n-2}\dots X_0)$ is a n-tuple, where X_i, $i = 0, \dots, n-1 \in V$.

(4) $x = \sum_{i=0}^{n-1} m^i X_i$. (1)

(5) $g: V^k \to V$ is a fully-specified k-variable function.

(6) The local values of $f_i(X), i = 0, 1, \ldots, k-1$, form a k-tuple which will be denoted by Z, i.e.

$$Z = (f_{k-1}(X) f_{k-2}(X) \ldots f_0(X)) \tag{2}$$

and

$$z = \sum_{i=0}^{k-1} m^i f_i(x) \tag{3}$$

(7) Function G is the composition such that

$$G(X) = g(f_{k-1}(X), f_{k-2}(X), \ldots, f_0(X)). \tag{4}$$

(8) The spectrum $\mathrm{S}_f]$ of a function f is given[2, 6] by the transform

$$[\mathrm{H}]\mathrm{Q}_f] = \mathrm{S}_f], \tag{5}$$

where the entries $q(x)$ of $\mathrm{Q}_f]$ are

$$q(x) = u^{f(x)} \text{ ordered decimally and } u = e^{j(2\pi/m)}, \tag{6}$$

and where the entries h_{wy} of [H] are

$$\bar{h}_{wy} = u^{\xi}, \quad \text{where} \quad \xi = \sum_{i=0}^{n-1} Y_i W_i \tag{7}$$

where $y = \Sigma_{i=0}^{n-1} m^i Y_i$ and $w = \Sigma_{i=0}^{n-1} m^i W_i$, and $\bar{h}_{wy}$ is the complex conjugate of h_{wy}.

This transform matrix [H] is symmetric, and the rows of [H] are mutually orthogonal[2, 6]. Therefore the inverse of [H] is simply its conjugate [$\bar{\mathrm{H}}$] with a scaling factor ($1/m^n$), giving

$$\mathrm{Q}_f] = \frac{1}{m^n} [\bar{\mathrm{H}}]\mathrm{S}_f]. \tag{8}$$

(9) $\mathrm{Q}_{f_i}]$, $\mathrm{S}_{f_i}]$, $\mathrm{Q}_g]$, $\mathrm{S}_g]$, $\mathrm{Q}_G]$ and $\mathrm{S}_G]$ are defined likewise.

(10) $\mathrm{S}_{\sum_{i=0}^{k-1} f_i Y_i}]$, where $Y_i \in V$ is the spectrum of the linear combination of functions $f_i, i = 0, 1, \ldots, k-1$.

(Note, Σ denotes mod-m summation).

3. DEVELOPMENTS

Theorem. The output spectrum $\mathrm{S}_G]$ of a two-level composed function $G(X)$ in terms of the first-level spectra $\mathrm{S}_{f_i}]$, $i = 0, 1, \ldots, k-1$, and the second-level spectrum $\mathrm{S}_g]$, see Fig. 1, is given by

$$\underset{m^n \times m^k}{\frac{1}{m^k}\left[\begin{bmatrix} \vdots \end{bmatrix} \bullet\bullet \mathrm{S}_{\Sigma f_i Y_i}\Big] \bullet\bullet \vdots \right]} \quad \underset{m^k \times 1}{\mathrm{S}_g\Big]} = \underset{m^n \times 1}{\mathrm{S}_G\Big]} \tag{9}$$

where the yth column of the $m^n \times m^k$ matrix is the spectrum column vector $\mathrm{S}_{\otimes f_i Y_i}$ and $y = \Sigma_{i=0}^{k-1} m^i Y_i$.

This relationship therefore may be re-written as

$$\frac{1}{m^k}\left[\begin{bmatrix} m^n\\ 0\\ 0\\ \vdots \end{bmatrix} \bullet\bullet \left.\begin{matrix}\vdots\end{matrix}\right] \bullet\bullet\ \left.\mathrm{S}_{\Sigma(m-1)f_i}\right]\right] \left.\mathrm{S}_g\right] = \left.\mathrm{S}_G\right] \tag{9a}$$

Proof. At a point $X = (X_{n-1}X_{n-2}\dots X_0)$, the functions $f_i(X)$ each take a certain value, which collectively form a k-tuple that determines the value of $g(Z)$ and hence output $G(X)$ at the point X. The value of $g(Z)$ at this point is found by multiplying $\mathrm{S}_g]$ with the zth row of the $m^k \times m^k$ inverse transform matrix. Therefore $\mathrm{Q}_G]$ is given by

$$\frac{1}{m^k}\underset{m^k \times m^k}{\begin{bmatrix} z\text{th row of matrix} \\ [\bar{\mathrm{H}}] \end{bmatrix}} \underset{m^n \times m^k}{\left.\mathrm{S}_g\right]} = \left.\mathrm{Q}_G\right] \tag{10}$$

where $z = \Sigma_{i=0}^{k-1} m^i f_i(X)$, and $[\bar{\mathrm{H}}]$ is the complex conjugate of [H].

We shall call this $m^n \times m^k$ matrix [T]. From eqns (7) and (10) it may be shown that the elements of [T] are

$$t_{xy} = u^{\phi}, \text{ where } \phi = \sum_{i=0}^{k-1} f_i(X) Y_i.$$

Therefore the yth column of the matrix [T] is merely the column vector

$$\left.\mathrm{Q}_{\sum_{i=0}^{k-1} f_i Y_i}\right] \tag{11}$$

The multiplication of both sides of eqn (10) with the transform $[\mathrm{H}]_{m^n \times m^n}$ therefore yields eqn (9).

Example. Required: the spectrum $\mathrm{S}_G]$ of the Boolean product (AND) $f_1 . f_2$ of two fully-specified functions f_1, f_2, with spectra $\mathrm{S}_{f_1}]$ and $\mathrm{S}_{f_2}]$, respectively.

Now in this case, the spectrum $\mathrm{S}_g]$ of a 2-input AND function is 2,2,2,−2]; hence

$$\frac{1}{2^2}\left[\begin{bmatrix} 2^n\\ 0\\ 0\\ \vdots \end{bmatrix} \left.\mathrm{S}_{f_1}\right] \left.\mathrm{S}_{f_2}\right] \left.\mathrm{S}_{f_1 \oplus f_2}\right]\right] \left.\begin{matrix} 2\\ 2\\ 2\\ -2 \end{matrix}\right] = \left.\mathrm{S}_G\right] \tag{12}$$

whence

$$\left.\mathrm{S}_G\right] = \frac{1}{2}\left\{ \begin{bmatrix} 2^n\\ 0\\ 0\\ \vdots \end{bmatrix} + \left.\mathrm{S}_{f_1}\right] + \left.\mathrm{S}_{f_2}\right] - \left.\mathrm{S}_{f_1 \oplus f_2}\right]\right\} \tag{13}$$

It may be noted that this relationship for a Boolean product was found by Eris[8] following a dissimilar approach.

Continuing, if now the two functions f_1 and f_2 are, say, the simple Boolean functions

$$f_1(X) = \bar{X}_1 X_2 \bar{X}_3 \vee X_1 X_3$$

and

$$f_2(X) = X_1 \bar{X}_2 \vee \bar{X}_2 X_3$$

the corresponding spectra $S_{f_1}]$, $S_{f_2}]$ and $S_{f_1 \oplus f_2}]$ may be computed [9], giving

$$S_{f_1}] = 2,2,2,2,2,-6,2,2$$

$$S_{f_2} = 2,2,-6,2,2,2,2,2$$

and

$$S_{f_1} \oplus {}_{f_2}] = 0,0,0,0,0,0,0,8.$$

Hence from eqns (12) and (13) the spectrum of $G = f_1 . f_2$ is given by

$$S_G] = \left.\begin{matrix} \\ \\ \\ \\ \\ \end{matrix}\right] \frac{1}{2} \left\{ \begin{bmatrix} 8 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ -6 \\ 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \\ -6 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} , - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 8 \end{bmatrix} \right\} ,$$

whence

$$S_G] = 6,2,-2,2,2,-2,2,-2.$$

The ease of incorporating any final function of $g(Z)$ with its associated spectrum $S_g]$ will be apparent from this simple Boolean AND example.

4. MULTI-LEVEL >2 REALISATIONS

The above development considered a composed function $G(X)$ with a two-level realisation. It should be clear that the composed spectrum for more than two levels of realisation may be calculated by repeated application of the above development. For example with the three-level composition shown in Fig. 2 where the output may be expressed as

$$G(X) = h\{g_{s-1}(f_{k-1}(X), \ldots, f_0(X)), g_{s-2}(\ldots.), \ldots, g_0(f_{k-1}(X), \ldots, f_0(X))\}$$

then the spectrum $S_G]$ in terms of the preceding spectra is given by

$$\underset{}{\frac{1}{m^{s+k}}} \underset{m^n \times m^k}{\left[\begin{bmatrix} m^k \\ 0 \\ 0 \\ \vdots \end{bmatrix} \quad \bullet\bullet S\Sigma_{f_i\alpha_i} \right]} \underset{m^k \times m^s}{\left[\begin{bmatrix} m^s \\ 0 \\ 0 \\ \vdots \end{bmatrix} \quad \bullet\bullet S\Sigma_{g_i\beta_i} \right]} \underset{m^s \times 1}{\left. S_h \right]} = \underset{m^n \times 1}{\left. S_G \right]} \qquad (14)$$

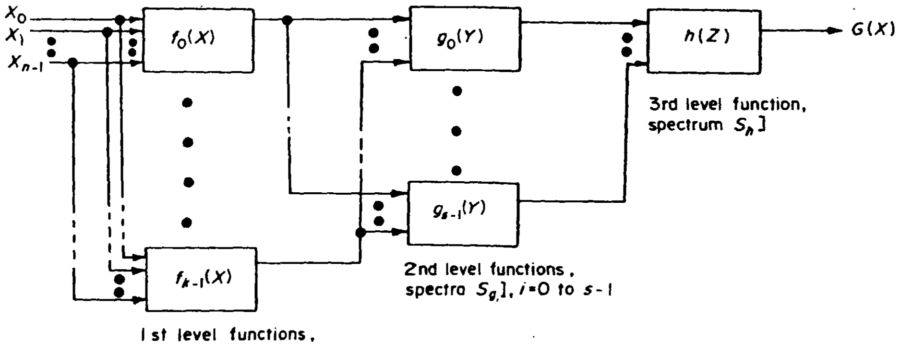

Fig. 2. Three-level realisation of a function $G(X)$.

where $(\alpha_{k-1}, \ldots, \alpha_0)$ and $(\beta_{s-1}, \ldots, \beta_0)$ are the mod-m expansions of the column numbers α and β respectively.

5. CONCLUSIONS

The application of the above equations enables the spectrum of any function $G(X)$ to be determined from the spectra of its decomposition functions, without the need to determine the full truthtable of $G(X)$. Work to apply these mathematical relationships for multivalued synthesis purposes is being pursued.

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REPORT NO 5 " Disjoint decomposability of multi-valued functions by spectral means „

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DISJOINT DECOMPOSABILITY OF MULTI-VALUED FUNCTIONS BY SPECTRAL MEANS

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Abstract

The disjoint decomposability of multi-valued logic functions is investigated in terms of the function spectra. It is shown that if a set of simultaneous equations containing the spectral coefficient values can be satisfied, then a given decomposition of the function is available. Two simple examples of the method are shown.

List of symbols used

n	total number of independent input variables.
m	number of values in the logic system; $m = 2$ for binary, $m = 3$ for ternary etc.
$U = \{0,1,\ldots, m-1\}$	set with m-elements, integer mod-m.
$x_i, i = 0$ to $n-1$	independent input variables taking values on the set U.
$(V_{n-1},V_{n-2},\ldots,V_0)$	n-tuple, an element of U^n, m-ary expansion of v.
v	an integer such that $v = \sum_{i=0}^{n-1} m^i V_i$.
$x_i \vee x_j$	maximum of x_i,x_j
$x_i . x_j$	minimum of x_i,x_j
$x_i \otimes x_j$	multiplication Mod_m of x_i,x_j
$x_i \dotplus x_j$	plus connective, defined as $x_i \dotplus x_j = (m-1) - (x_i+x_j)$ if $(x_i+x_j) \leq (m-1)$, $= 0$ otherwise.
$x^{p,q}$	delta functions, defined as $x^{p,q} = (m-1)$ if $p \leq x \leq q$, $= 0$ otherwise, $p \leq q$, $p,q \in 0,\ldots,(m-1)$.
x^p	as above with $p = q$.
f,g,h	m-ary functions of the x_i inputs.
$S_f], S_g]$, etc.	spectrum of the functions f,g, etc.

1. Introduction

With the introduction of I^2L technology, circuit configurations suitable for multi-valued (m-valued) logic applications have been investigated by several authors[1,2]. Although this technology is naturally capable of readily implementing many functions (e.g. plus, max., mod. addition, etc.) at reasonable cost and using small on-chip silicon area, there is at present no design algorithm to make maximum use of the advantages offered by these functions for combinatorial logic design.

Indeed, consider for example the following three-variable ternary function, expressed in <0,1,2> notation:

	$x_2=0$			$x_2=1$			$x_2=2$		
$x_0 \backslash x_1$	0	1	2	0	1	2	0	1	2
0	2	1	0	2	1	1	2	2	2
1	1	0	0	1	1	1	2	2	2
2	0	0	0	1	1	1	2	2	2

f(X)

Evaluation of the delta functions contained in this example would give the following synthesis:

$$f(x_2,x_1,x_0) = \{x_2^2 \vee x_1^0.x_0^0 \vee 1.x_2^{1,2} \vee 1.x_0^0.x_1^{0,1} \vee 1.x_0^{0,1}.x_1^0\}$$

However it will later be shown that this function is more simply given by:

$$f(x_2,x_1,x_0) = \{(x_0 \dotplus x_1) \vee x_2\}$$

Definition

The simple disjunctive decomposition of a n-variable function f(X), $X = x_{n-1},\ldots,x_1,x_0$, exists if f(X) can be expressed as $f(X) = h\big(x_{n-1},\ldots,x_k,g(x_{k-1},\ldots,x_0)\big)$.

The block diagram of such a realisation is shown in Fig.1.

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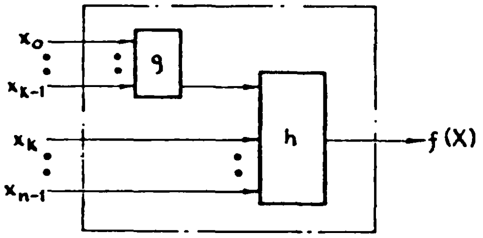

Fig.1 Simple disjunctive decomposition of f(X)

The permutation of the x_i input variables allows the function g to be dependent upon any subset of the variables $x_0,\dots,x_{n-1}$. Our aim is to detect a simple disjunctive decomposition of the given function f(X) if it exists. Here we propose a method to detect such decomposability using spectral rather than break count techniques[9] or other algebraic methods[6].

We will call functions of n-variables (e.g. the Plus, max., mod-addition, etc.) "simple" functions if they are readily available physically. The method proposed will be used to detect disjoint decomposability in particular when g is a simple function.

2. Definitions and notation used

2.1 Let w be an integer between 0 and m^n-1 and $(W_{n-1},\dots,W_0)$ be its m-ary expansion. Then the spectrum s_w of a m-valued n-variable function f(X) is defined as follows:

$$s_w = \sum_{v=0}^{m^n-1} t_w(v)y(v), \qquad \dots(1)$$

where

$$Ch(w,v) = \sum_{\ell=0}^{(n-1)} W_\ell V_\ell ,$$

$$a = e^{-j\frac{2\pi}{m}} ,$$

$$y(v) = a^{f(v)} ,$$

and

$$t_w(v) = \text{Conjg}(a^{Ch(w,v)}) .$$

2.2 Many properties of the spectral transformation and several spectral synthesis techniques have been investigated in recent years[3,4,5], mainly for the binary (m = 2) area. The properties which we here propose to utilise will be stated without proof; for proofs see the cited references. A basic property, however, is that the spectrum-function relationship is unique[3], the function in terms of its spectrum being given by the inverse transformation

$$y(v) = \frac{1}{m^n}\left\{\sum_{v=0}^{m^n-1} \overline{t_w(v)}\, s_w\right\} \qquad \dots(2)$$

2.3 A simpler way to consider and to express these transformations is in terms of matrix relationships. The forward transformation of Eq.(1) may be expressed as:

$$\left.S_f\right] = \left[T_n\right]\left.Y\right] \qquad \dots(3)$$

where the vth entry of the column vectors $\left.S_f\right]$ and $\left.Y\right]$ are s_v and y(v) respectively, and the elements $t_{w,v}$ of the transform matrix $\left[T_n\right]$ are $t_w(v)$.

In the same manner we may re-express Eq.(1) as:

$$\left.Y\right] = \frac{1}{m^n}\left[\overline{T}_n\right]\left.S_f\right] \qquad \dots(4)$$

where $\left[\overline{T}_n\right]$ is the complex conjugate of the matrix $\left[T_n\right]$. For example for the n = 1, m = 3 case the transform matrix $\left[T_n\right]$ will be of the form:

$$\left[T_1\right] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix}$$

where $a = e^{-j\frac{2\pi}{3}}$

It can easily be seen from the definition of spectral transformation that, for all m, the transform matrix has a recursive structure, that is:

$$\left[T_n\right] = \left[T_1\right] * \left[T_{n-1}\right] \qquad \dots(5)$$

where * stands for the Kronecker product. Hence, for example for n = 2, m = 3 the transform matrix $\left[T_2\right]$ is evaluated by:

$$\left[T_2\right] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} * \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & a^2 & a & 1 & a^2 & a & 1 & a^2 & a \\ 1 & a & a^2 & 1 & a & a^2 & 1 & a & a^2 \\ 1 & 1 & 1 & a^2 & a^2 & a^2 & a & a & a \\ 1 & a^2 & a & a^2 & a & 1 & a & 1 & a^2 \\ 1 & a & a^2 & a^2 & 1 & a & a & a^2 & 1 \\ 1 & 1 & 1 & a & a & a & a^2 & a^2 & a^2 \\ 1 & a^2 & a & a & 1 & a^2 & a^2 & a & 1 \\ 1 & a & a^2 & a & a^2 & 1 & a^2 & 1 & a \end{bmatrix}$$

3. Developments

Consider now the block diagram realisation of a function f(X), as shown in Fig.1, where for the moment g is any function, not necessarily a simple function as defined above.

Fixing each of the last (n-k) variables of the function f(X) at one of the m logic levels, we obtain functions f_z, such that:

$$f(Z_{n-1},\ldots,Z_k,x_{k-1},\ldots,x_0) = f_z(x_{k-1},\ldots,x_1,x_0),$$

where $z = \sum_{i=0}^{n-k-1} m^i Z_{i+k}$

and $Z_j \varepsilon U$, $j = k,\ldots,(n-1)$.

If $S_f]$ and $S_z]$, $z = 0,\ldots,m^{(n-k)}-1$, are the spectra of the functions f(X) and f_z respectively, then it can be shown, see Appendix, that:

$$\frac{1}{m^{(n-k)}}\left[\quad \bar{T}_{(n-k)} \quad\right]_n \left.\quad S_f \quad\right] = \left.\begin{matrix} S_{f_0}] \\ S_{f_1}] \\ \cdot \\ \cdot \\ \cdot \\ S_{f_{m^{(n-k)}-1}}] \end{matrix}\right] \qquad (6)$$

where

$$\left[\bar{T}_{(n-k)}\right]_n = \left[\bar{T}_{(n-k)}\right] * \left[I\right]_{m^k \times m^k}$$

Similarly for each z, gate h at the second-level realisation implements a function h_z with spectrum $S_{h_z}]$ related to $S_h]$ as follows:

$$\frac{1}{m^{(n-k)}}\left[\quad \bar{T}_{(n-k)} \quad\right]_{(n-k)+1} \left.\quad S_h \quad\right] = \left.\begin{matrix} S_{h_0}] \\ S_{h_1}] \\ \cdot \\ \cdot \\ \cdot \end{matrix}\right] \qquad (7)$$

Each $S_{h_z}]$ on the other hand is related to $S_{f_z}]$ by [8]:

$$\frac{1}{m}\left[\left.\begin{matrix} m^k \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{matrix}\right] \left. S_{1\otimes g} \right] \left. S_{2\otimes g} \right] \bullet\bullet\bullet \left. S_{(m-1)\otimes g} \right]\right] \left. S_{h_z} \right] = \left. S_{f_z} \right] \qquad (8)$$

By combining Equations (6), (7) and (8) above, we may prove the following theorem:

Theorem

A m-valued, n-variable function f(X), $X = x_{n-1},\ldots x_0$, has a disjunctive decomposition of the form

$$f(x_{n-1},\ldots,x_0) = h\left(x_{n-1},\ldots,x_k,g(x_{k-1},\ldots,x_0)\right)$$

if the spectra $S_f]$, $S_g]$, $S_h]$ of the respective functions satisfy the relationship

$$\frac{1}{m}\begin{bmatrix} [S_G] & & 0 \\ & \ddots & \\ 0 & & [S_G] \end{bmatrix} \left. S_h \right] = \left. S_f \right] \qquad \ldots(9)$$

where

$$[S_G] = \left[\left.\begin{matrix} m^k \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \end{matrix}\right] \left. S_{1\otimes g} \right] \left. S_{2\otimes g} \right] \bullet\bullet\bullet \left. S_{(m-1)\otimes g} \right]\right]$$

Therefore if a particular g is specified for any function f(X), the decomposition of f(X) into g, h is possible if all the simultaneous equations in (8), where the components of $S_h]$ are initially unknown, have unique solutions. Moreover, the solutions completely specify the function h.

4. Example No.1, a Binary Example

Consider a simple 4-variable (n = 4) binary (m=2) function f(X) given by the following map:

x_3x_4 \ x_1x_2	00	01	11	10
00	0	0	1	0
01	0	0	1	0
11	1	1	1	1
10	1	1	0	1

f(X)

Using conventional design techniques we may obtain the following realisation for this function:

$$f(X) = \{x_3.x_4 \ \ V \ \ \bar{x}_2.x_3 \ \ V \ \ \bar{x}_1.x_3 \ \ V \ \ x_1.x_2.\bar{x}_3\} \quad ....(10)$$

Computation of the spectrum of f(X) gives:

$$S_f] = -2,\ 2,\ 2,\ -2,\ 10,\ 6,\ 6,\ -6,\ 2,\ -2,\ -2,\ 2,\ -2,\ 2,\ 2,\ -2]^t$$

Now let us choose g as a simple two-input AND gate, whose spectrum is

$$S_g] = 2,\ 2,\ 2,\ -2]^t$$

Hence for the binary case under consideration:

$$\left[S_G\right] = \begin{bmatrix} 2^2 & \\ 0 & S_g] \\ 0 & \\ 0 & \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 0 & 2 \\ 0 & 2 \\ 0 & -2 \end{bmatrix} ,$$

since only $S_{1\otimes g}] = S_g] = S_{(m-1)\otimes g}]$ exist in this simple example.

Putting the latter values in Eq.(9) we obtain the following simultaneous equations which have to be satisfied, where $S_h] = s_0, s_1, \ldots, s_7]^t$:

$$\begin{aligned} 2s_0 + s_1 &= -2 \\ s_1 &= 2 \\ s_1 &= 2 \\ -s_1 &= -2 \\ 2s_2 + s_3 &= 10 \\ s_3 &= 6 \\ s_3 &= 6 \\ -s_3 &= -6 \\ 2s_4 + s_5 &= 2 \\ s_5 &= -2 \\ s_5 &= -2 \\ -s_5 &= 2 \\ 2s_6 + s_7 &= -2 \\ s_7 &= 2 \\ s_7 &= 2 \\ -s_7 &= -2 \end{aligned}$$

The solution for h in the above simple example is clearly:

$$S_h] = -2,\ 2,\ 2,\ 6,\ 2,\ -2,\ -2,\ 2]^t ,$$

and hence the given function f(X) is reduced to the following decomposition, where g is the AND function $x_1.x_2$:

$g=x_1x_2$ \ $h=x_3,x_4$	00	01	11	10
0	0	0	1	1
1	1	1	1	0

Therefore

$$\begin{aligned} f(X) &= h(x_4, x_3, g(x_2, x_1) \ , \\ &= \{g.\bar{x}_3 \ \ V \ \ g.x_4 \ \ V \ \ \bar{g}.x_3\} \quad ...\ (10) \end{aligned}$$

Note that although Eq.(9) contains $m^{(n-k)+1}$ unknowns for the specified functions g and f, the equations are grouped into distinct sets of simultaneous equations, where each group of equations has m unknowns and m^k equations. For example, in the above problem there are 8 unknowns $(m^{(n-k)+1})$, 4 groups $(m^{m-k)})$ with 2 unknowns (m) and 4 equations (m^k). Therefore, in the general m-valued case the problem is reduced to solving simultaneous equations for m unknowns. The fact that we have m^k equations with m unknowns does not create any additional difficulties for the solution, since assuming g is known in advance (that is it is chosen from one of our available simple functions), then m linearly-independent rows of the matrix $S_G]$ can be selected in advance.

Repeated application of the above decomposition method clearly results in the realisation topologies indicated in Fig.2.

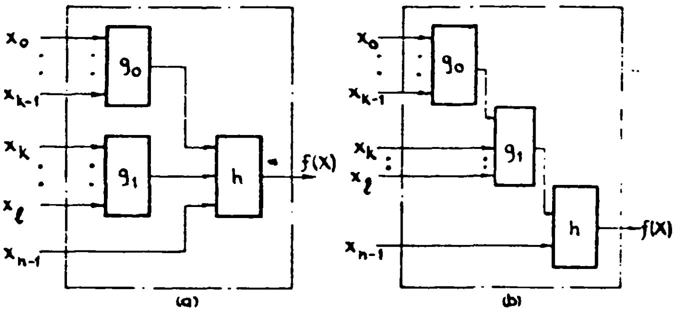

Fig.2 Repeated application of disjoint decomposition

a) $f(X) = h\left(x_{n-1}, \ldots, g_1(x_\ell, \ldots, x_k), g_0(x_{k-1}, \ldots, x_0)\right)$

b) $f(X) = h\left(x_{n-1}, \ldots, g_1\{x_\ell, \ldots, x_k, g_0(x_{k-1}, \ldots, x_0)\}\right)$

5. Example No.2, a Ternary Example

The search for the decomposability of m-ary functions, m > 2, by spectral means involves the solution to a set of equations with complex coefficients[5,7,8]. Consider the ternary (m=3) function f(X) given in the Introduction of this paper.

Let us consider whether this function is decomposable about x_0, x_1 using the Plus function $g = x_0 + x_1$. In this case the matrix $S_G]$ would be:

$$\left[S_G\right] = \left[\begin{bmatrix}3^2\\0\\ \cdot\\ \cdot\\ \cdot\end{bmatrix} \left.S_g\right] \left.S_{2 \otimes g}\right]\right]$$

$$= \begin{bmatrix} 9 & (a + 5) & (a^2 + 5) \\ 0 & (a^2 + 2a) & (2a^2 + 4a) \\ 0 & (4a^2 + 2a) & (2a^2 + a) \\ 0 & (a^2 + 2a) & (2a^2 + 4a) \\ 0 & (a^2 + 2) & (5a + 1) \\ 0 & (2a^2 + 1) & (2a + 1) \\ 0 & (4a^2 + 2a) & (2a^2 + a) \\ 0 & (2a^2 + 1) & (2a + 1) \\ 0 & (5a^2 + 1) & (a^2 + 2) \end{bmatrix}$$

The following simultaneous equations may now be obtained by incorporating the relevant spectra of $\left.S_f\right]$ and $\left.S_g\right]$ in Eq.9 (c.f. the previous binary example). Note that $a = e^{-j\,2\pi/3}$, and the s_w's are the spectral coefficients, also complex, for the residual function h.

$$\begin{aligned}
9\,s_0 + (a + 5)s_1 + (a^2 + 5)s_2 &= 15a^2 + 12a \\
(a^2 + 2a)s_1 + (2a^2 + 4a)s_2 &= 6a^2 + 3a \\
(4a^2 + 2a)s_1 + (2a^2 + a)s_2 &= 15a^2 + 3a \\
(a^2 + 2a)s_1 + (2a^2 + 4a)s_2 &= 6a^2 + 3a \\
(a^2 + 2)s_1 + (5a + 1)s_2 &= 3a^2 + 6 \\
(2a^2 + 1)s_1 + (2a + 1)s_2 &= 12a^2 + 6 \\
(4a^2 + 2a)s_1 + (2a^2 + a)s_2 &= 15a^2 + 3a \\
(2a^2 + 1)s_1 + (2a + 1)s_2 &= 12a^2 + 6 \\
(5a^2 + 1)s_1 + (a^2 + 2)s_2 &= 21a^2 + 6
\end{aligned}$$

$$\begin{aligned}
9\,s_{10} + (a + 5)s_{11} + (a^2 + 5)s_{12} &= 6a + 66 \\
(a^2 + 2a)s_{11} + (2a^2 + 4a)s_{12} &= 6a^2 + 12a \\
(4a^2 + 2a)s_{11} + (2a^2 + a)s_{12} &= 15a^2 + 12a \\
(a^2 + 2a)s_{11} + (2a^2 + 4a)s_{12} &= 6a^2 + 12a \\
(a^2 + 2)s_{11} + (5a + 1)s_{12} &= 6a + 3 \\
(2a^2 + 1)s_{11} + (2a + 1)s_{12} &= 6a^2 + 3a \\
(4a^2 + 2a)s_{11} + (2a^2 + a)s_{12} &= 15a^2 + 12a \\
(2a^2 + 1)s_{11} + (2a + 1)s_{12} &= 6a^2 + 3a \\
(5a^2 + 1)s_{11} + (a^2 + 2)s_{12} &= 15a^2 + 3a
\end{aligned}$$

$$\begin{aligned}
9\,s_{20} + (a + 5)s_{21} + (a^2 + 5)s_{22} &= 6a^2 + 12a \\
(a^2 + 2a)s_{21} + (2a^2 + 4a)s_{22} &= 6a + 3 \\
(4a^2 + 2a)s_{21} + (2a^2 + a)s_{22} &= 6a^2 + 3a \\
(a^2 + 2a)s_{21} + (2a^2 + 4a)s_{22} &= 6a + 3 \\
(a^2 + 2)s_{21} + (5a + 1)s_{22} &= 6a + 12 \\
(2a^2 + 1)s_{21} + (2a + 1)s_{22} &= 3a^2 + 6 \\
(4a^2 + 2a)s_{21} + (2a^2 + a)s_{22} &= 6a^2 + 3a \\
(2a^2 + 1)s_{21} + (2a + 1)s_{22} &= 3a^2 + 6 \\
(5a^2 + 1)s_{21} + (a^2 + 2)s_{22} &= 12a^2 + 6
\end{aligned}$$

A solution to this set of simultaneous equations is possible, and will be found to be:

$$\begin{aligned}
s_0 &= 4a^2 + 2a \\
s_1 &= a + 5 \\
s_2 &= a^2 + 2a \\
s_{10} &= a + 5 \\
s_{11} &= 2a^2 + 4 \\
s_{12} &= a + 2 \\
s_{20} &= a^2 + 2a \\
s_{21} &= a + 2 \\
s_{22} &= 2a^2 + 1
\end{aligned}$$

The inverse transformation on this set of spectral coefficient values will show, if not already recognised, that it is the spectrum of the 2-input Maximum operator. Hence the final decomposition synthesis for the function is:

$$f(X) = \{(x_0 + x_1) \vee x_2\}$$

6. Conclusions

A method is given to detect simple disjoint decomposability of a given m-valued ($m \geqslant 2$) logic function. The method presented here makes use of the spectrum of the given function f(X) and of the functions (gates) with which it is desired to effect the decomposition, and involves the solution to a set of m^k simultaneous equations each with m unknowns for the m-valued case. The decomposition is possible if each and all sets of simultaneous equations have unique solutions. The method is applicable irrespective of the logical complexity or otherwise of the gates which we desire to use in the decomposition.

Further, if it is found that the decomposition using the desired gates is possible, then the detection and proof of this decomposition fully defines the spectrum and hence functional relationships of the next level (remainder) function h. The detection of such simple disjoint decomposability of a given multi-valued function may be considered as the first stage in the design of combinational multi-valued logic.

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Appendix

The nth order transform matrix is obtained by taking nth Kronecker power of the first order transform matrix $\left[T_1\right]$, i.e.

$$\left[T_n\right] = \Big\{ \underbrace{\left[T_1\right] * \cdots\cdots * \left[T_1\right]}_{\longleftarrow\ n \text{ times}\ \longrightarrow} \Big\}$$

$$= \left[T_1\right]^{*n}$$

Consider now the functions f_z, such that;

$$f_z(x_{k-1},\ldots,x_0) = f(Z_{(n-1)},\ldots,Z_k,x_{k-1},\ldots,x_0),$$

$$\text{where} \quad z = \sum_{i=0}^{n-k-1} m^i Z_{i+k}$$

It follows from the ordering of the vector $\left.Y\right]$ that the spectra $\left.S_{f_z}\right]$ corresponding to above functions f_z is found by:

$$\left[\left[I\right]_{m^{(n-k)} \times m^{(n-k)}} * \left[T_k\right]\right] Y\Big] = \left[\begin{matrix} \left.S_{f_0}\right] \\ \left.S_{f_1}\right] \\ \cdot \\ \cdot \\ \left.S_{f_{m^{(n-k)}-1}}\right] \end{matrix}\right]$$

Replacing $\left.Y\right]$ with its spectrum and inverse transform matrix, we obtain on the left-hand side of the above equation:

$$\frac{1}{m^n}\left[\left[I\right]_{m^{(n-k)} \times m^{(n-k)}} * \left[T_k\right]\right]\left[\overline{T}_n\right] S_f\Big]$$

$$= \frac{1}{m^n}\left[\left[I\right]\left[\overline{T}_{(n-k)}\right] * \left[T_k\right]\left[\overline{T}_k\right]\right] S_f\Big]$$

and hence

$$\frac{1}{m^{(n-k)}}\left[\left[T_{(n-k)}\right] * \left[I\right]_{m^k \times m^k}\right] S_f\Big] = \left[\begin{matrix} \left.S_{f_0}\right] \\ \left.S_{f_1}\right] \\ \cdot \\ \cdot \\ \cdot \end{matrix}\right]$$
