CONJUGACY CLASSES OF FINITE GROUPS AND GRAPH REGULARITY

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Dedicated to the memory of David Chillag

Abstract. Given a finite group G, denote by $\Gamma(G)$ the simple undirected graph whose vertices are the distinct sizes of noncentral conjugacy classes of G, and set two vertices of $\Gamma(G)$ to be adjacent if and only if they are not coprime numbers. In this note we prove that, if $\Gamma(G)$ is a k-regular graph with $k \geq 1$, then $\Gamma(G)$ is a complete graph with $k+1$ vertices.

Keywords: finite groups, conjugacy class sizes.

1. Introduction

Given a finite group G, let $\Gamma(G)$ be the simple undirected graph whose vertices are the distinct sizes of *noncentral* conjugacy classes of G, two of them being adjacent if and only if they are not coprime numbers. The interplay between certain properties of this graph and the group structure of G has been widely studied in the past decades, and it is nowadays a classical topic in finite group theory (see, for instance, [4]). The present note is a contribution in this direction.

In [2] it is conjectured that, for every integer $k \geq 1$, the graph $\Gamma(G)$ is k-regular if and only if $\Gamma(G)$ is a complete graph with $k+1$ vertices. That paper settles the case $k \leq 3$, whereas in this note, using a different approach, we provide an affirmative answer to the conjecture in full generality.

Theorem. *Let G be a finite group, and assume that $\Gamma(G)$ is a k-regular graph with $k \geq 1$. Then $\Gamma(G)$ is a complete graph with $k+1$ vertices.*

Another graph related to the conjugacy classes of finite groups, that has been extensively studied in the literature, is the *prime graph* $\Delta(G)$: in this case the vertices are the primes dividing some class size of G, and two distinct vertices p, q are adjacent if and only if there exists a class size of G that is divisible by pq. It is well known that the graphs $\Gamma(G)$ and $\Delta(G)$ share some relevant properties (for instance, they have the same number of connected components, and the diameters of the two graphs differ by at most 1). Nevertheless we remark that, in contrast to the situation described by the main result of this note, the class of finite groups G such that $\Delta(G)$ is a non-complete, connected and regular graph is not empty. Such groups have been classified in Therorem D of [3].

2. The results

Every group considered in the following discussion is tacitly assumed to be a finite group. We start by introducing a definition.

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Definition 1. *Let Γ be a graph and, for a given vertex X of Γ, denote by $\nu(X)$ the set of neighbors of X in Γ (i.e., the set of vertices of Γ that are adjacent to X). Let A, B be vertices of Γ: we say that A and B are* partners *if $\nu(A)\cup\{A\} = \nu(B)\cup\{B\}$. This clearly defines an equivalence relation on the vertex set of Γ.*

Next, given a group G, we focus on the common divisor graph $\Gamma(G)$ built on the set of conjugacy class sizes of G, as defined in the Introduction. For $g \in G$, the G-conjugacy class of g will be denoted by g^G.

Lemma 2. *Let G be a group, and assume that $\Gamma(G)$ is a k-regular graph with $k \geq 1$. Then $\Gamma(G)$ is a connected graph.*

Proof. If $\Gamma(G)$ is not connected, then by [1] (or [6]) it consists of two isolated vertices, and hence it is not k-regular for any $k \geq 1$. ■

Lemma 3. *Let G be a group, and assume that $\Gamma(G)$ is a k-regular graph with $k \geq 1$. If there exists $x \in G \setminus \mathbf{Z}(G)$ such that $|x^G|$ is a prime power, then $\Gamma(G)$ is a complete graph.*

Proof. Assume that $|x^G|$ is a power of the prime p. Then the k neighbors of $|x^G|$ are all divisible by p, and the subgraph of $\Gamma(G)$ induced by them is complete. If the whole $\Gamma(G)$ is not complete, the connectedness of $\Gamma(G)$ (ensured by Lemma 2) implies the existence of $y \in G \setminus \mathbf{Z}(G)$ such that $|y^G|$ is not divisible by p but is adjacent to a neighbor of $|x^G|$; this neighbor will have now valency at least $k+1$, thus violating the regularity of $\Gamma(G)$. ■

Lemma 4. *Let G be a group, and assume that $\Gamma(G)$ is a regular graph. If x and y are noncentral elements of G such that $\mathbf{C}_G(x) \leq \mathbf{C}_G(y)$, then $|x^G|$ and $|y^G|$ are partners in $\Gamma(G)$. In particular, the following conclusions hold.*

(a) *If x and y are noncentral elements of G having coprime orders, and such that $xy = yx$, then $|x^G|$ and $|y^G|$ are partners in $\Gamma(G)$.*
(b) *If x is an element of G and k is an integer such that x^k is noncentral, then $|x^G|$ and $|(x^k)^G|$ are partners in $\Gamma(G)$.*

Proof. As $\mathbf{C}_G(x) \leq \mathbf{C}_G(y)$, we have that $|y^G|$ is a divisor (different from 1) of $|x^G|$. Clearly we have

$$\nu(|y^G|) \cup \{|y^G|\} \subseteq \nu(|x^G|) \cup \{|x^G|\};$$

but the regularity of $\Gamma(G)$ forces those two sets to have the same cardinality. Hence equality holds, and $|x^G|$ and $|y^G|$ are partners. Now, since $\mathbf{C}_G(x) \leq \mathbf{C}_G(x^k)$, Conclusion (b) follows. Moreover, in the setting of Conclusion (a), we have $\mathbf{C}_G(xy) = \mathbf{C}_G(x) \cap \mathbf{C}_G(y)$; thus, taking into account the transitivity of being partners, Conclusion (a) follows as well. ■

Recall that, for every group G, the diameter of the graph $\Gamma(G)$ is at most 3 (see [5] or [6]). The following proposition shows that no graph of this kind can be regular of diameter 3.

Lemma 5. *Let G be a group, and assume that $\Gamma(G)$ is a k-regular graph with $k \geq 1$. Then the diameter of $\Gamma(G)$ is at most 2.*

Proof. The groups G such that the diameter of $\Gamma(G)$ is 3 are classified in [6]: they are direct products $F \times H$ where $(|F|, |H|) = 1$, the graph $\Gamma(F)$ consists of two isolated vertices, and $\Gamma(H)$ is not the empty graph. Now, let $|x^F|$ be a vertex of $\Gamma(F)$, $|y^H|$ a vertex of $\Gamma(H)$, and consider the vertices $|y^G|$, $|(xy)^G|$ of $\Gamma(G)$. Since $\mathbf{C}_G(xy) \leq \mathbf{C}_G(y)$ and $\Gamma(G)$ is regular, it follows by Lemma 4 that $|y^G|$ and $|(xy)^G|$ are partners. This is a contradiction, because $|x^G|$ is adjacent to $|(xy)^G|$ but not to $|y^G|$. ■

Before proving the main result, it will be convenient to introduce some more terminology. If $g \in G$ is such that, for every $h \in G$, the condition $\mathbf{C}_G(h) \leq \mathbf{C}_G(g)$ implies $\mathbf{C}_G(h) = \mathbf{C}_G(g)$, then we say that $\mathbf{C}_G(g)$ is a *minimal centralizer* in G. Also, we say that $g \in G$ is *strongly noncentral* if the order of $g\mathbf{Z}(G)$ (as an element of $G/\mathbf{Z}(G)$) is not a prime power. Finally, given a prime p, we denote by g_p and $g_{p'}$ respectively the *p-part* and the *p'-part* of g, i.e., g_p is a p-element, $g_{p'}$ a p'-element, and $g_p g_{p'} = g = g_{p'} g_p$. Recall that both g_p and $g_{p'}$ are powers of g. Note also that a prime p divides $\mathrm{o}(g\mathbf{Z}(G))$ if and only if $g_p \notin \mathbf{Z}(G)$.

We are now in a position to prove the main theorem of this note, that was stated in the Introduction.

Proof of the Theorem. Let G be a group with centre Z and, aiming at a contradiction, assume that $\Gamma(G)$ is a non-complete k-regular graph with $k \geq 1$. We start by proving two claims.

Claim 1. *For every minimal centralizer $\mathbf{C}_G(g)$ of G, there exists a strongly noncentral element $x \in G$ such that $\mathbf{C}_G(x) = \mathbf{C}_G(g)$.*

Denoting by p a prime divisor of the order of $gZ \in G/Z$, observe first that $\mathbf{C}_G(g)/Z$ cannot be a p-group. In fact, if $|\mathbf{C}_G(g)/Z| = p^n$, then $|g^G|$ is a multiple of all the primes dividing some conjugacy class size of G, except perhaps p. As $\Gamma(G)$ is non-complete and regular, there exists $y \in G \setminus Z$ such that $|y^G|$ is coprime to $|g^G|$; in other words, $|y^G|$ is a p-power, and Lemma 3 yields a contradiction.

Now, let $q \neq p$ be a prime divisor of $|\mathbf{C}_G(g)/Z|$, and let $yZ \in \mathbf{C}_G(g)/Z$ be an element of order q. We can clearly assume $g_{p'} \in Z$, thus we have $\mathbf{C}_G(g_p) = \mathbf{C}_G(g)$. Moreover, g_p and y_q commute, whence, setting $x = g_p y_q$, we get

$$\mathbf{C}_G(x) = \mathbf{C}_G(g_p y_q) = \mathbf{C}_G(g_p) \cap \mathbf{C}_G(y_q) = \mathbf{C}_G(g) \cap \mathbf{C}_G(y_q).$$

The minimality of $\mathbf{C}_G(g)$ forces $\mathbf{C}_G(x) = \mathbf{C}_G(g)$, and x is strongly noncentral in G, as desired.

Claim 2. *There exist two elements x, y such that $|x^G|$ and $|y^G|$ are coprime, and both $\mathbf{C}_G(x)$, $\mathbf{C}_G(y)$ are minimal centralizers.*

Choose $x \in G$ so that $\mathbf{C}_G(x)$ is minimal. As $\Gamma(G)$ is non-complete and regular, we can find $w \in G \setminus Z$ such that $|x^G|$ and $|w^G|$ are coprime. Now, let $y \in G$ be such that $\mathbf{C}_G(y)$ is minimal and contained in $\mathbf{C}_G(w)$. Lemma 4 yields that $|y^G|$ and $|w^G|$ are partners and, since $|x^G|$ and $|w^G|$ are coprime, it follows that also $|x^G|$ and $|y^G|$ are coprime, as required.

In view of the previous claims, we can now conclude the proof. Let us choose two elements x and y as in Claim 2, i.e., $|x^G|$ and $|y^G|$ are coprime, and both $\mathbf{C}_G(x)$, $\mathbf{C}_G(y)$ are minimal centralizers. In particular, by Claim 1, we can assume that both x and y are strongly noncentral in G.

Since, by Lemma 5, the diameter of $\Gamma(G)$ is 2, we can find $w \in G$ such that $|w^G|$ is a common neighbor for $|x^G|$ and $|y^G|$. Also, let p be a prime divisor of $\mathrm{o}(wZ) \in G/Z$. As $|x^G|$ and $|y^G|$ are coprime, we may assume that $\mathbf{C}_G(x)$ contains a Sylow p-subgroup of G; thus, up to replacing x by a conjugate of it, we have $w_p \in \mathbf{C}_G(x)$. Now, since x is strongly noncentral, there exists a prime $q \neq p$ which divides the order of xZ; Lemma 4 implies that $|x^G|$ and $|(x_q)^G|$ are partners, but $|(x_q)^G|$ is also a partner of $|(w_p)^G|$, and the latter is a partner of $|w^G|$. As a consequence, $|x^G|$ and $|w^G|$ are partners: this is the final contradiction, as $|y^G|$ is adjacent to $|w^G|$ but not to $|x^G|$. ∎

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