

ON VANISHING CLASS SIZES IN FINITE GROUPS

MARIAGRAZIA BIANCHI, JULIAN M.A. BROUGH, RACHEL D. CAMINA,
AND EMANUELE PACIFICI

ABSTRACT. Let G be a finite group. An element g of G is called a *vanishing element* if there exists an irreducible character χ of G such that $\chi(g) = 0$; in this case, we say that the conjugacy class of g is a vanishing conjugacy class. In this paper, we discuss some arithmetical properties concerning the sizes of the vanishing conjugacy classes in a finite group.

1. INTRODUCTION

Many authors have investigated the relationship between the structure of a finite group G and arithmetical data connected to G . The arithmetical data can take various forms: for example, authors have considered the set of conjugacy class sizes, or the set of character degrees. The link between these different sets is also of interest, as demonstrated by the following result by C. Casolo and S. Dolfi. Suppose p and q are distinct primes and pq divides the degree of some irreducible complex character of G ; then pq also divides the size of some conjugacy class of G [5, Theorem A]. As an important step in the proof of this result, the authors consider groups for which p and q both divide a conjugacy class size but pq does not, and they show that such groups are $\{p, q\}$ -solvable [5, Theorem B(i)].

Recently, instead of considering all conjugacy class sizes, authors have been considering a subset of conjugacy class sizes “filtered” by the irreducible characters, namely, the set of *vanishing conjugacy class sizes* (see [3], [4], [6] and also [7] for related properties of vanishing elements). An element $g \in G$ is called a vanishing element if there exists an irreducible character χ of G such that $\chi(g) = 0$, and the conjugacy class of such an element is called a vanishing conjugacy class of G . Motivated by Casolo and Dolfi’s results, we investigate some arithmetical properties of the set of vanishing conjugacy class sizes.

This context is neatly portrayed by the *prime graph* of G for class sizes. Recall that, given a finite nonempty set of positive integers X , the prime graph on X has vertex set defined as the set of all prime numbers that are divisors of some element in X , and edge set consisting of pairs $\{p, q\}$ such that pq divides some element of

2000 *Mathematics Subject Classification.* 20E45.

The first and the fourth author are partially supported by the Italian INdAM-GNSAGA, and by the PRIN 2015TW9LSR_006 “Group Theory and Applications”.

X . When $X = \{x^G : x \in G\}$ is the set of conjugacy class sizes of a finite group G , we denote by $\Gamma(G)$ the prime graph on X , and by $V(G)$ the set of vertices of $\Gamma(G)$. Also, we write $\Gamma_v(G)$ and $V_v(G)$ for the corresponding objects in the case when X is the set of vanishing conjugacy class sizes of G .

In [6] the authors investigate when, for a finite group G , a prime p is not an element of $V_v(G)$; they prove that such a group G is p -nilpotent, with abelian Sylow p -subgroups. Note that $V_v(G)$ can be strictly smaller than $V(G)$, as shown for instance by the symmetric group on three objects. This can actually occur also for nonsolvable groups (see [6, Example 4.1]). However, our first result gives a condition to ensure this does not happen.

Proposition. *Let G be a finite group, and suppose G has a nonabelian minimal normal subgroup. Then $V(G) = V_v(G)$.*

Thus, for the two vertex sets to be the same in a nonsolvable group, it seems important “where” the nonsolvability of the group lies.

Still in the spirit of the work by Casolo and Dolfi, we carry out an investigation concerning the edges of the vanishing graph. In particular, is the “vanishing version” of [5, Theorem B(i)] true? That is, if the edge $\{p, q\}$ is missing in the vanishing graph, but both p and q are vertices, is the group $\{p, q\}$ -solvable?

As our main result shows, the answer is affirmative under the same assumptions as in the above Proposition.

Theorem A. *Let G be a finite group, and suppose G has a nonabelian minimal normal subgroup. If p and q are in $V(G)$, but there is no vanishing conjugacy class of G whose size is divisible by pq , then G is $\{p, q\}$ -solvable.*

As shown by [6, Example 4.1], Theorem A fails in general if G does not have a nonabelian minimal normal subgroup. An important step in the proof of the above result is the following Theorem B, that is the vanishing version of [5, Theorem 9].

Theorem B. *Let G be a finite group with trivial Fitting subgroup. Then every prime divisor of $|G|$ is in $V_v(G)$, and $\Gamma_v(G)$ is a complete graph.*

We point out that another key ingredient in the proof of Theorem A is Corollary 4.4, that turns out to be a useful tool in locating vanishing elements, and may be of interest in its own right.

As an immediate consequence of Theorem A, we get the following p -solvability criterion. Recall that a vertex of a graph is called *complete* if it is adjacent to all the other vertices.

Corollary. *Let G be a finite group and p a prime. Suppose G has a nonabelian minimal normal subgroup. If p is not a complete vertex of $\Gamma_v(G)$, then G is p -solvable.*

Throughout this paper, every group is assumed to be a finite group.

2. PRELIMINARIES

In this section we gather together previously known results that will be of use. We denote the set of all vanishing elements of the group G by $\text{Van}(G)$, whereas, as customary, $\pi(G)$ denotes the set of prime divisors of $|G|$.

Lemma 2.1. *Let N be a normal subgroup of G . Then $\Gamma_v(G/N)$ is a subgraph of $\Gamma_v(G)$.*

Proof. We combine two results. Firstly note that $|(xN)^{G/N}|$ divides $|x^G|$. Secondly, by [7, Lemma 2.1], if $xN \in \text{Van}(G/N)$ then $xN \subseteq \text{Van}(G)$. ■

Lemma 2.2. [7, Proposition 2.5] *Let M and N be normal subgroups of a group G with $M \cap N = 1$. Suppose there exists $m \in M \cap \text{Van}(G)$. Then $mn \in \text{Van}(G)$ for all $n \in N$.*

Proposition 2.3. [5, Lemma 8] *Let G be a permutation group on a finite set Ω and p, q distinct primes. Then there exist two nonempty subsets Γ_1, Γ_2 of Ω with $\Gamma_1 \cap \Gamma_2 = \emptyset$ and $\{p, q\} \cap \pi(G) \subseteq \pi(|G : G_{\Gamma_1} \cap G_{\Gamma_2}|)$.*

(Here, for $i \in \{1, 2\}$, G_{Γ_i} denotes the setwise stabilizer of Γ_i in G .)

Recall, an irreducible character χ of G is said to have q -defect zero for some prime q , if q does not divide $|G|/\chi(1)$. If χ is such a character and g is an element of G with order divisible by q then $\chi(g) = 0$, i.e. g is a vanishing element [15, Theorem 8.17]. Thus the following is useful.

Theorem 2.4. [12, Corollary 2] *Let G be a nonabelian simple group and q a prime divisor of $|G|$. Then G has an irreducible character of q -defect zero unless one of the following holds.*

- (a) *The prime q is 2 and S is isomorphic to either $M_{12}, M_{22}, M_{24}, J_2, \text{HS}, \text{Suz}, \text{Ru}, \text{Co}_1, \text{Co}_3, \text{BM}$ or $\text{Alt}(n)$ for various values of $n \geq 7$*
- (b) *The prime q is 3 and S is isomorphic to either Suz, Co_3 or $\text{Alt}(n)$ for various values of $n \geq 7$.*

The above result will often be used in conjunction with the following.

Lemma 2.5. [7, Lemma 2.7] *Let G be a group, N a normal subgroup of G and q a prime divisor of $|N|$. If N has an irreducible character of q -defect zero, then every element of N of order divisible by q is a vanishing element of G .*

As an immediate consequence of the two previous statements, we get the following result.

Proposition 2.6. [7, Corollary 2.9] *Let M be a nonabelian minimal normal subgroup of a group G and suppose p is a prime divisor of $|M|$. If $p \geq 5$ then every element of M with order divisible by p is a vanishing element of G .*

Finally, we will freely use without references some basic facts of Character Theory such as Clifford Correspondence, properties of coprime actions, and elementary properties of conjugacy class sizes; for instance, recall that a prime p does not divide the size of any conjugacy class of a group G if and only if G has a central Sylow p -subgroup.

3. THEOREM B

We start with a simple lemma.

Lemma 3.1. *Let G be a group, M a normal subgroup of G with trivial centre and $C = \mathbf{C}_G(M)$. Then for any $g \in C$ and $h \in M$ the conjugacy class size $|(gh)^G|$ is divisible by both $|g^C|$ and $|h^M|$. Furthermore if h or $g \in \text{Van}(G)$ then $gh \in \text{Van}(G)$.*

Proof. As M has trivial centre $MC \cong M \times C$. Thus $|(gh)^{MC}| = |g^C||h^M|$. As MC is normal in G it follows that $|(gh)^{MC}|$ divides $|(gh)^G|$. Finally, if h or $g \in \text{Van}(G)$ apply Lemma 2.2. ■

The following lemma helps us to identify vanishing elements.

Lemma 3.2. *Suppose G has a unique minimal normal subgroup M which is non-abelian. Then for all $p \in \pi(G)$ there exists $g \in \text{Van}(G)$ such that p divides $|g^G|$. Thus $\pi(G) = \mathbf{V}_v(G)$. Furthermore, g can be chosen to lie in M .*

Proof. As M is nonabelian, we have $M = S_1 \times \cdots \times S_n$ where every S_i is isomorphic to a nonabelian simple group S . We will denote by N the kernel of the action of G by conjugation on $\{S_1, \dots, S_n\}$.

Let p be a prime divisor of $|G|$; our aim is to find an element $g \in M \cap \text{Van}(G)$ such that p divides $|g^G|$. We will treat separately the cases $p \mid |M|$ and $p \nmid |M|$.

Let us start from the latter case. In this situation, p divides $|G/M|$, and we first suppose that p actually divides $|G/N|$. Using Proposition 2.3, we can choose two nonempty subsets Γ_1, Γ_2 of $\Omega = \{S_1, \dots, S_n\}$, such that $\Gamma_1 \cap \Gamma_2 = \emptyset$ and $p \in \pi(|G : G_{\Gamma_1} \cap G_{\Gamma_2}|)$. Certainly we can find nontrivial elements u and v in S of different orders and such that the order of v is divisible by some prime r greater than 3. For $S_\alpha \in \Gamma_1$ and $S_\beta \in \Gamma_2$ let u_α and v_β correspond, respectively, to u and v (via the isomorphisms $S_i \simeq S$), and set g to be the element in M given by

$$g = \Pi u_\alpha \Pi v_\beta.$$

As r divides the order of v , it also divides the order of g and hence g is vanishing in G by Proposition 2.6. Let x be an element in $\mathbf{C}_G(g)$, and consider $S_\alpha \in \Gamma_1$. Let x act on M by conjugation; since M is nonabelian, the factors of g are permuted. As the orders of u and v are different, it follows that $u_\alpha^x = u_{\alpha'}$ for some α' with $S_{\alpha'} \in \Gamma_1$. Moreover it follows that $S_\alpha^x = S_{\alpha'} \in \Gamma_1$ and so $x \in G_{\Gamma_1}$. Similarly we have $x \in G_{\Gamma_2}$, whence $\mathbf{C}_G(g) \leq G_{\Gamma_1} \cap G_{\Gamma_2}$. As a consequence, $|G : G_{\Gamma_1} \cap G_{\Gamma_2}|$ divides $|g^G|$ and, in particular, p divides $|g^G|$.

Suppose now that p does not divide $|G/N|$, so p divides $|N|$. Take i in $\{1, \dots, n\}$, and adopt the bar convention for the factor group $N/\mathbf{C}_N(S_i)$; then \overline{N} is an almost simple group with socle $\overline{S}_i \simeq S$. Moreover, since $\bigcap \mathbf{C}_N(S_i) = \mathbf{C}_N(M) = 1$, the subgroup N can be embedded in the direct product of the factor groups $N/\mathbf{C}_N(S_i)$ and, as these factor groups all have the same order (because G transitively permutes the S_i), it is easy to see that p divides $|\overline{N}|$. However, p does not divide $|M|$, and thus it does not divide $|S_i|$; as a consequence, S is a simple group of Lie type (see for instance [11]). Now, by [5, Lemma 6(a)], there exists an element $\overline{g} \in \overline{S}_i$ such that p divides $|\overline{g}^{\overline{N}}|$, which is in turn a divisor of $|g^G|$ (here g can be chosen in S_i , hence in M). As g is a vanishing element of G by Theorem 2.4 and Lemma 2.5, we are done in this case as well.

Finally, let us assume $p \mid |M|$. If M has an irreducible character of q -defect zero for every prime divisor q of $|M|$, then every nontrivial element of M lies in $\text{Van}(G)$ by Lemma 2.5. Now, just take an element $g \in S_1$ such that $p \mid |g^{S_1}| = |g^M|$, and we are done. On the other hand, if there exists a prime $q \in \pi(M)$ such that M does not have an irreducible character of q -defect zero (so the same holds for S_1), then we apply Lemma 2.2 in [6]: there exists an element g of S_1 whose conjugacy class in M has size divisible by all primes in $\pi(M)$, and there exists an irreducible character θ of S_1 such that $\theta(g) = 0$ and θ extends irreducibly to $\text{Aut}(S_1)$. Moreover, Lemma 5 of [1] yields that $\theta \times \theta \times \dots \times \theta \in \text{Irr}(M)$ extends irreducibly to G , and thus g is vanishing in G . ■

We are now ready to prove the proposition mentioned in the Introduction, that we state again.

Proposition. *Let G be a group, and suppose G has a nonabelian minimal normal subgroup. Then $V(G) = V_v(G)$.*

Proof. Let M be a nonabelian minimal normal subgroup of G , and set $C = \mathbf{C}_G(M)$. Then $\overline{G} = G/C$ has a unique minimal normal subgroup, which is isomorphic to M . Suppose $p \in V(G)$. If $p \in \pi(\overline{G})$ then $p \in V_v(\overline{G})$ by Lemma 3.2, and hence $p \in V_v(G)$ by Lemma 2.1. Thus we can assume that p does not divide $|\overline{G}|$. If C does not have a central Sylow p -subgroup, then there exists $g \in C$ with p dividing $|g^C|$. As M is nonabelian, there exists $h \in M \cap \text{Van}(G)$ by Proposition 2.6. Now apply Lemma 3.1 to get that gh is vanishing in G with conjugacy class size divisible by p , thus $p \in V_v(G)$ as required. Finally, suppose that C has a central Sylow p -subgroup P . Then P , which is a Sylow p -subgroup of G as well, is (abelian and) normal in G . If $p \notin V_v(G)$ then, by [6, Theorem A], G has a normal p -complement, thus P is central in G . But this would contradict $p \in V(G)$, and the proof is complete. ■

Thus, when $\mathbf{F}(G) = 1$ we have established that $V_v(G) = \pi(G)$. We now turn our attention to edges in the vanishing graph and, after the following proposition, we will prove Theorem B.

Proposition 3.3. *Let G be an almost simple group with socle S , and let p, q be distinct primes in $\pi(G)$. Then p and q are adjacent vertices of $\Gamma_v(G)$. Moreover, there exists an element $g \in S$ such that pq divides $|g^G|$ and g is vanishing in G .*

Proof. If $p, q \in \pi(S)$, then p and q are adjacent vertices of $\Gamma(S)$ by [5, Theorem 9]. So there exists $g \in S$ with pq dividing $|g^S|$ and thus $|g^G|$. If S has an irreducible character of q -defect zero for all primes q , then g is vanishing in G by Lemma 2.5 and we are done. Thus, we can assume S does not have an irreducible character of q -defect zero for some prime q , and the same argument as in the last paragraph of Lemma 3.2 yields the conclusion.

If p or q does not divide $|S|$, then S is a simple group of Lie type. We note that in the proof of [5, Proposition 7] the authors produce, in each case, an element of S with conjugacy class size divisible by p and q in G . Moreover, since S is of Lie type, this element will be vanishing in G by Lemma 2.5. ■

Theorem B. *Let G be a group with trivial Fitting subgroup. Then every prime divisor of $|G|$ is in $V_v(G)$, and $\Gamma_v(G)$ is a complete graph.*

Proof. Let G be a counterexample of minimal order to our statement; thus, $\mathbf{F}(G) = 1$ and there exist two distinct prime divisors p and q of $|G|$ such that $\{p, q\}$ is not an edge of $\Gamma_v(G)$. Let $M = S_1 \times \cdots \times S_n$ be a minimal normal subgroup of G (where the S_i are all isomorphic to a nonabelian simple group S) and let N be the kernel of the action of G on $\{S_1, \dots, S_n\}$. Also, denote $\mathbf{C}_G(M)$ by C and let $\overline{G} = G/C$. We proceed through various steps.

(i) C is trivial.

Observe that $\overline{G} = G/C$ has a unique minimal normal subgroup $\overline{M} \cong M$, and therefore $\mathbf{F}(\overline{G}) = 1$. For a proof by contradiction, let us assume $C \neq 1$.

If $\{p, q\} \subseteq \pi(\overline{G})$ then, by the minimality of G , we get that $\{p, q\}$ is an edge of $\Gamma_v(\overline{G})$; thus Lemma 2.1 yields that $\{p, q\}$ is an edge of $\Gamma_v(G)$ as well, against our assumptions.

Suppose now $p, q \in \pi(C)$. As $\mathbf{F}(C) \leq \mathbf{F}(G) = 1$, there exists $g \in C$ with pq dividing $|g^C|$ by [5, Theorem 9]. Choose $h \in M$ with order divisible by some prime greater than 3; then h is vanishing in G by Proposition 2.6. An application of Lemma 3.1 yields that gh is vanishing in G with conjugacy class size divisible by pq , again a contradiction.

Thus, we can assume that p divides $|\overline{G}|$ and q does not. As \overline{G} has a unique minimal normal subgroup \overline{M} , by Lemma 3.2, there exists an element $\overline{g} \in \overline{M}$ with p dividing $|\overline{g}^{\overline{G}}|$ and \overline{g} vanishing in \overline{G} . As C does not have a central Sylow q -subgroup (because $\mathbf{F}(C) = 1$), there exists an element $h \in C$ with q dividing $|h^C|$. For g a preimage of \overline{g} in M , by Lemma 3.1, we have that gh is vanishing in G with conjugacy class size divisible by pq . This contradiction proves the claim.

Note that, for every $i \in \{1, \dots, n\}$, the factor group $N/\mathbf{C}_N(S_i)$ is an almost simple group with socle isomorphic to S . Moreover, as $C = 1$, N can be embedded in the direct product of the factor groups $N/\mathbf{C}_N(S_i)$, and since these all have the same order, we get $\pi(N) = \pi(N/\mathbf{C}_N(S_i))$.

(ii) N is a proper subgroup of G .

In fact, if $N = G$, then M is a simple group and G is an almost simple group with socle M . The desired conclusion now follows from Proposition 3.3.

(iii) We have $\{p, q\} \not\subseteq \pi(G/N)$.

In fact, assuming the contrary, choose nonempty $\Gamma_1, \Gamma_2 \subseteq \Omega = \{S_1, \dots, S_n\}$ with $\Gamma_1 \cap \Gamma_2 = \emptyset$ and $\{p, q\} \subseteq \pi(|G : G_{\Gamma_1} \cap G_{\Gamma_2}|)$ (see Proposition 2.3). Also, let u and v be nontrivial elements of different orders in $S \simeq S_i$, such that the order of v is divisible by some prime r greater than 3. Now, for $S_\alpha \in \Gamma_1$ and $S_\beta \in \Gamma_2$ let u_α and v_β correspond, respectively, to u and v (via the isomorphisms $S_i \simeq S$). As in the proof of Lemma 3.2, we see that $g = \Pi u_\alpha \Pi v_\beta$ is vanishing in G and its conjugacy class size is divisible by pq , contradicting the hypotheses.

(iv) We have $\{p, q\} \not\subseteq \pi(N)$.

In view of the last paragraph of Claim (i), if p and q both divide $|N|$, then they both divide the order of $\overline{N} = N/\mathbf{C}_N(S_1)$ as well. By Proposition 3.3, there exists $\overline{g} \in \overline{S}_1$ such that $pq \mid |\overline{g}^{\overline{N}}|$. Let $g \in S_1$ be a preimage of \overline{g} , and set $h \in S_2$ to be an element with order divisible by a prime greater than 3. Then gh is vanishing in G by Proposition 2.6; moreover, as $\overline{gh} = \overline{g}$, we get $pq \mid |\overline{gh}^{\overline{N}}|$. But $|\overline{gh}^{\overline{N}}|$ divides $|(gh)^N|$, which in turn divides $|(gh)^G|$, against the hypotheses.

(v) *Final contradiction.*

Our conclusion so far is that we can assume $p \in \pi(G/N)$ and $q \in \pi(N)$. Then q divides $|N/\mathbf{C}_N(S_1)|$ and, as in the previous claim, there exists an element $u \in S_1$ such that q divides $|u^N|$ (note that u is a vanishing element of G unless possibly when it is a $\{2, 3\}$ -element). Now choose nonempty subsets Γ_1 and Γ_2 of $\Omega = \{S_1, \dots, S_n\}$ such that p divides $|G : G_{\Gamma_1} \cap G_{\Gamma_2}|$. Take $v \in S$ with $o(v) \neq o(u)$ and such that, in the case when u is a $\{2, 3\}$ -element, the order of v is divisible by a prime greater than 3. As in the proof of Lemma 3.2, define $g = \Pi u_\alpha \Pi v_\beta$ where u_α and v_β correspond to u and v respectively. As already seen, g is vanishing in G and $|g^G|$ is divisible by p . Moreover, as we can assume $S_1 \in \Gamma_1$, the image of g in $N/\mathbf{C}_N(S_1)$ is u , and therefore q divides $|g^N|$, which in turn divides $|g^G|$. Thus g is a vanishing element of G with conjugacy class size divisible by pq , and this contradiction completes the proof of the theorem. \blacksquare

4. THEOREM A

In this section we prove the main result of this paper. Our key tool will be Lemma 4.3 and its consequence Corollary 4.4. We start with a lemma concerning vanishing elements of alternating groups.

Lemma 4.1. *For $n \geq 7$ and $t \geq 2$, let x be a permutation in $\text{Alt}(n)$ whose type (including fixed points) is (t, \dots, t) or $(t, \dots, t, 1)$. Then there exists an irreducible character θ of $\text{Alt}(n)$ which has an extension to $\text{Sym}(n)$, and such that $\theta(x) = 0$.*

Proof. Assume that x is of type (t, \dots, t) , whence $n = kt$ for a suitable integer k . If $k \geq 3$, consider the partition $\mu = (n - t - 1, t, 1)$ of n (which is not self-associate), and let $\chi_\mu \in \text{Irr}(\text{Sym}(n))$ be the corresponding character. Then the Murnaghan-Nakayama formula (see for instance [18, Theorem 4.10.2]) yields $\chi_\mu(x) = 0$, and θ can be chosen to be the (irreducible) restriction of χ_μ to $\text{Alt}(n)$. In the case when $k \leq 2$, we can argue as above using the partition $(n - 3, 2, 1)$.

Finally, if x is of type $(t, \dots, t, 1)$, then we use the partition $(n - 1, 1)$. ■

Let V be an n -dimensional vector space over a field \mathbb{E} , and let D be the set consisting of the elements $(\alpha_1, \dots, \alpha_n) \in V$ such that $\sum \alpha_i = 0$. The $(n - 1)$ -dimensional vector space D has the natural structure of an $\mathbb{E}[\text{Alt}(n)]$ -module, where $\text{Alt}(n)$ acts by permuting coordinates, and we will be interested in the case when the characteristic q of \mathbb{E} is coprime with $|\text{Alt}(n)|$, i.e., $q > n$; in this case, D turns out to be an irreducible $\mathbb{E}[\text{Alt}(n)]$ -module, which is called the *deleted permutation module* over \mathbb{E} (see [9]).

We will deal with the module D regarded as an (irreducible) $\mathbb{E}^\times \times \text{Alt}(n)$ module over \mathbb{E} , where \mathbb{E}^\times denotes the multiplicative group of \mathbb{E} , acting on D by scalar multiplication.

Lemma 4.2. *Let D be the deleted permutation module for $\text{Alt}(n)$ over the field \mathbb{E} , where the characteristic of \mathbb{E} is larger than n . Let $d = (\alpha_1, \alpha_2, \dots, \alpha_{n-1}, \beta) \in D$ be such that the α_i are pairwise distinct elements of \mathbb{E}^\times . Regarding D as a module for $G = \mathbb{E}^\times \times \text{Alt}(n)$, assume that the element $(\lambda, x) \in G$ centralizes d . Then, if t is the order of x , the type of x (including fixed points) is either (t, \dots, t) or $(t, \dots, t, 1)$.*

Proof. Let d be an element of D as in the statement, and assume $(\lambda, x) \in \mathbf{C}_G(d)$. We first observe that, if $\lambda = 1$, then x is also 1. In fact, if x maps the symbol $j \in \{1, \dots, n - 1\}$ to $jx \neq j$ then, as $(1, x) \in \mathbf{C}_G(d)$, the corresponding entries of d must coincide; but, the α_i being pairwise distinct, we get $jx = n$ and $\alpha_j = \beta$. As a consequence, the only nontrivial cycle in x is possibly (j, n) , but this is a contradiction since x is an even permutation.

Next, let $j \in \{1, \dots, n - 1\}$ be a symbol lying in an $\langle x \rangle$ -orbit of length s , so s is a divisor of $t = o(x)$. Since (λ^s, x^s) centralizes d , the j th entries of d and of $d^{(\lambda^s, x^s)}$ must coincide; therefore we get $\lambda^s \alpha_j = \alpha_j$, whence $\lambda^s = 1$. This yields that $(1, x^s) = (\lambda^s, x^s)$ centralizes d , therefore $x^s = 1$ by the paragraph above, and $s = t$.

We conclude that every cycle of x involving at least one symbol in $\{1, \dots, n - 1\}$ has in fact length t , and the proof is complete. ■

Lemma 4.3. *Let S be a nonabelian simple group that is not of Lie type, and let $M = S_1 \times \dots \times S_k$ be the direct product of k copies of S . Let q be a prime not dividing*

$|S|$, and A a faithful M -module over the field \mathbb{F} with q elements. Assume that there are no regular orbits for the action of M on A . Then S is isomorphic to $\text{Alt}(n)$ for some $n \geq 7$; moreover, there exists $a \in A$ such that, for $x = s_1 \cdots s_k \in \mathbf{C}_M(a)$, each s_i is a permutation of type $(o(s_i), \dots, o(s_i))$ or $(o(s_i), \dots, o(s_i), 1)$.

Proof. Since q is coprime with the order of M , the $\mathbb{F}[M]$ -module A is semisimple. Assuming that there is no regular orbit for the action of M on A , our aim will be to construct an element a of A yielding the desired conclusions; to this end, we will choose suitable vectors from each simple constituent of A .

Let V be such a constituent. Up to renumbering, we can assume that the kernel of the action of M on V is either trivial (and in this case we set $h = k$) or $S_{h+1} \times \cdots \times S_k$ for a suitable $h \in \{1, \dots, k-1\}$, so $N = S_1 \times \cdots \times S_h$ acts faithfully on V . Let \mathbb{E} be a finite field extension of \mathbb{F} that is a splitting field for S , and consider the faithful $\mathbb{E}[N]$ -module $V^{\mathbb{E}} = V \otimes_{\mathbb{F}} \mathbb{E}$; if W is an irreducible constituent of $V^{\mathbb{E}}$ and χ is the corresponding character, we denote by \mathbb{K} the field extension of \mathbb{F} obtained by adjoining the set of values $\{\chi(n) \mid n \in N\}$ to \mathbb{F} (so, \mathbb{K} is a subfield of \mathbb{E}). By Theorem 1.16 in [14, VII], that we will freely use with no further reference throughout this proof, we get

$$V^{\mathbb{E}} \simeq \bigoplus_{\xi \in \text{Gal}(\mathbb{K}|\mathbb{F})} W^{\xi}.$$

Observe that, for $w \in W$, the vector

$$v = \sum_{\xi \in \text{Gal}(\mathbb{K}|\mathbb{F})} w^{\xi}$$

lies in V , and we have $\mathbf{C}_N(v) \leq \mathbf{C}_N(w)$ (see [8, Lemma 3.1]). As W is a simple $S_1 \times \cdots \times S_h$ -module over a splitting field for each of the S_i , Theorem 3.7.1 in [10] yields that the module W decomposes as a tensor product $W_1 \otimes \cdots \otimes W_h$, where each W_i is a simple $\mathbb{E}[S_i]$ -module. In this situation, given an element of $W \setminus \{0\}$ of the form $w = w_1 \otimes \cdots \otimes w_h$, it can be checked that $x = s_1 \cdots s_h \in N$ centralizes w if and only if $w_i^{s_i} = \lambda_i^{-1} w_i$ for all $i \in \{1, \dots, h\}$, where the λ_i are in \mathbb{E} and $\prod \lambda_i = 1$; in other words, if x centralizes w then, regarding W_i as an $\mathbb{E}^{\times} \times S_i$ -module (\mathbb{E}^{\times} acting by scalar multiplication) the element (λ_i, s_i) centralizes w_i for every $i \in \{1, \dots, h\}$.

Now, assume that $\mathbb{E}^{\times} \times S_i$ does not have any regular orbit on W_i , and let \mathbb{K}_i be the field extension of \mathbb{F} obtained by adjoining the values of the character of W_i (as a module for S_i). Denoting by Z an irreducible constituent of W_i regarded as a $\mathbb{K}_i[S_i]$ -module, we get $W_i \simeq Z^{\mathbb{E}}$, and so the field of values of the $\mathbb{K}_i[S_i]$ -module Z is \mathbb{K}_i as well. We claim that the group $\mathbb{K}_i^{\times} \times S_i$ does not have any regular orbit on Z : assuming, for a proof by contradiction, that z lies in such an orbit, it is easy to check that the vector $z \otimes 1 \in W_i$ lies in a regular orbit for the action of $\mathbb{E}^{\times} \times S_i$, which is not possible. Now, if Z_0 denotes the $\mathbb{K}_i[S_i]$ -module Z viewed as an $\mathbb{F}[S_i]$ -module, Z_0 turns out to be irreducible, and $\mathbb{K}_i^{\times} \times S_i$ does not have any regular orbit on

it: we are in a position to apply a theorem by D. P. M. Goodwin ([9, Theorem 1]; see also [17, Theorem 2.1 and Theorem 2.2]), getting that $S_i \simeq \text{Alt}(n)$ for some $n \geq 7$ and Z_0 is the deleted permutation module for S_i over \mathbb{F} . Since this module is absolutely irreducible, we get $\mathbb{K}_i = \mathbb{F}$, and $W_i \simeq Z_0^{\mathbb{E}}$ is the deleted permutation module for S_i over \mathbb{E} . Whenever we are in this situation (which does occur for some constituent V , as otherwise M would have regular orbits on A against our assumptions), we choose $d_i \in W_i$ as in Lemma 4.2; this can be applied because, q being coprime with S , we certainly have $q > n$. In all other cases, choose d_i lying in a regular orbit for the action of $\mathbb{E}^\times \times S_i$ on W_i . Then set $w = d_1 \otimes \cdots \otimes d_h \in W$, and finally define $v = \sum_{\xi \in \text{Gal}(\mathbb{K}|\mathbb{F})} w^\xi \in V$.

To sum up, let $A = V_1 \oplus \cdots \oplus V_n$ be a decomposition of A into simple $\mathbb{F}[M]$ -constituents. For each V_j , let v_j be a vector as defined in the paragraph above, and let $a = \sum v_j$. In view of Lemma 4.2, it can be checked that such an element a satisfies the conclusions of our statement. \blacksquare

The following consequence of Lemma 4.3 is helpful in locating vanishing elements.

Corollary 4.4. *Let G be a group, and A an abelian minimal normal subgroup of G . Let M/N be a chief factor of G such that $|A|$ is coprime with $|M/N|$, and $N = \mathbf{C}_M(A)$. Then every element of $M \setminus N$ is a vanishing element of G .*

Proof. Set $\overline{G} = G/N$ and, adopting the bar convention, assume that there exists an element a of A lying in a regular orbit for the action of \overline{M} ; then, by coprimality, the same happens for the action of \overline{M} on $\text{Irr}(A)$. In other words, there exists an irreducible character θ of A such that $I_G(\theta) \cap M = N$, and every element x of $M \setminus N$ clearly does not lie in $\bigcup_{g \in G} I_G(\theta^g)$. Now, if ψ is an irreducible character of $I_G(\theta)$ lying over θ , then ψ^G is an irreducible character of G which vanishes on $G \setminus \bigcup_{g \in G} I_G(\theta^g)$, thus in particular $\psi^G(x) = 0$, as required.

Therefore, we may assume that there are no regular orbits for the action of \overline{M} on A . Since, by a well-known consequence of Brodkey's Theorem ([2]), regular orbits do exist if \overline{M} is abelian, we may focus on the case when $\overline{M} = S_1 \times \cdots \times S_k$, where the S_i are pairwise isomorphic nonabelian simple groups.

Observe that every nontrivial element of \overline{M} is a vanishing element of \overline{G} provided S_1 is a group of Lie type (Theorem 2.4 and Lemma 2.5), so we easily get the desired conclusion in this case. Therefore we can assume that S_1 is not of Lie type, and we can apply Lemma 4.3 with respect to the action of \overline{M} on A : we get $S_1 \simeq \text{Alt}(n)$ for some $n \geq 7$, and we can choose an element $a \in A$ satisfying the conclusions of Lemma 4.3. Now, by coprimality, we can consider a character $\theta \in \text{Irr}(A)$ whose inertia subgroup $I_{\overline{M}}(\theta)$ coincides with $\mathbf{C}_{\overline{M}}(a)$.

Let x be an element in $M \setminus N$. If \overline{x} lies in $I_{\overline{M}}(\theta)$, then each factor of \overline{x} (in its decomposition into a product of elements of the S_i) is a permutation with the cyclic structure prescribed by Lemma 4.3, and the same of course holds also if \overline{x}

centralizes a G -conjugate of θ . In other words, either \bar{x} does not centralize any G -conjugate of θ (and in that case x is vanishing in G , by the argument in the first paragraph of this proof), or it is as in the conclusions of Lemma 4.3. In the latter case, an application of Lemma 4.1 (together with Lemma 5 of [1]) yields that \bar{x} is a vanishing element of \bar{G} , and the proof is complete. ■

We are ready to prove Theorem A, that we state again.

Theorem A. *Let G be a group, and suppose G has a nonabelian minimal normal subgroup. If p and q are in $V(G)$, but there is no vanishing conjugacy class of G whose size is divisible by pq , then G is $\{p, q\}$ -solvable.*

Proof. Let G be a counterexample to the statement, having the smallest possible order. Since p and q are nonadjacent vertices of $\Gamma_v(G)$, Theorem B yields $\mathbf{F}(G) \neq 1$. As a consequence, there exists an abelian minimal normal subgroup A of G . We will proceed through a number of steps.

(i) *A is the unique abelian minimal normal subgroup of G . Moreover, we can assume $p \in V(G/A)$ and $q \notin V(G/A)$.*

The factor group G/A clearly has a nonabelian minimal normal subgroup. If p and q are vertices of $\Gamma(G/A)$, then they cannot be adjacent in $\Gamma_v(G/A)$, as otherwise they would be adjacent in $\Gamma_v(G)$ as well; therefore, by our minimality assumption, G/A is $\{p, q\}$ -solvable. But then G is so, and this is a contradiction. On the other hand, if both p and q are not in $V(G/A)$, then G/A is both p -nilpotent and q -nilpotent and G is not a counterexample. Hence $\{p, q\} \cap V(G/A)$ cannot be empty, and we will assume $p \in V(G/A)$, $q \notin V(G/A)$ (thus G is q -solvable). Now, let $B \neq A$ be an abelian minimal normal subgroup of G . The above discussion applies to B as well, hence $\{p, q\} \cap V(G/B)$ contains precisely one element; if this element is q , then G/B is p -solvable and so is G , a contradiction. But if $q \notin V(G/B)$, then both G/A and G/B have a central Sylow q -subgroup, and the same holds for G , which embeds into $(G/A) \times (G/B)$. This would imply $q \in V(G)$, which is not the case. Therefore, assuming the existence of a minimal normal subgroup of G other than A we get a contradiction, and the claim is proved.

(ii) *The Frattini subgroup $\Phi(G)$ of G is trivial, and $\mathbf{F}(G) = A$.*

Assume $\Phi(G) \neq 1$, so $A \leq \Phi(G)$. Let Q be a Sylow q -subgroup of G . Then, as we are assuming $q \notin V(G/A)$, we have $QA \trianglelefteq G$ and hence $Q \trianglelefteq G$ by the Frattini argument. As A is the unique abelian minimal normal subgroup of G , this clearly implies that A is a q -group. Moreover, G/A has a normal q -complement K/A and, again by the Frattini argument, a Hall q' -subgroup K_0 of K is actually a normal q -complement of G , thus $G = K_0 \times Q$. Observe that Q cannot be abelian, as otherwise it would lie in $\mathbf{Z}(G)$, thus we can choose $x \in Q \setminus \mathbf{Z}(Q)$; note that $|x^G|$ is divisible by q , and that x is a vanishing element of Q (see [16, Theorem B]), whence clearly a vanishing element of G as well. Consider now $y \in K_0$ such that $|y^G|$ is

divisible by p . Then xy is a vanishing element of G whose conjugacy class in G has size divisible by pq , and G is not a counterexample. We conclude that $\Phi(G) = 1$; in particular, $\mathbf{F}(G)$ is a direct product of abelian minimal normal subgroups of G (see [13, III.4.5]), whence $\mathbf{F}(G) = A$.

(iii) *A is a q-group*

For a proof by contradiction, we assume that $q \nmid |A|$. Hence, if Q is a Sylow q -subgroup of G , we get $A = [A, Q] \times \mathbf{C}_A(Q)$. Note that $[A, Q]$ cannot be trivial, as otherwise the abelian Sylow subgroup Q of G would be normal in $QA \trianglelefteq G$ and hence central in G . Moreover, every G -conjugate of Q is of the form Q^a for a suitable $a \in A$, whence $[A, Q]$ is normal in G . As a consequence, we get $[A, Q] = A$ and $\mathbf{C}_A(Q) = 1$; in particular, as this holds for any Sylow q -subgroup of G , for every $x \in A \setminus \{1\}$ we have that q is a divisor of $|x^G|$.

Now, let M be a nonabelian minimal normal subgroup of G , and set $C = \mathbf{C}_G(M)$. If p divides $|G/C|$ then, by Lemma 3.2, there exists $y \in M$ which is a vanishing element of G such that $p \mid |y^G|$. But then, for any $x \in A \setminus \{1\}$, xy is a vanishing element of G with $pq \mid |(xy)^G|$, a contradiction. We conclude that C contains a Sylow p -subgroup P of G , and clearly $P \not\leq \mathbf{Z}(C)$ (as otherwise P would be normal in G , so G would be p -solvable). Moreover, any Sylow q -subgroup Q of G lies in C because $[Q, G] \leq A$, and therefore $[Q, M] \leq A \cap M = 1$; but Q is not centralized by A , so $Q \not\leq \mathbf{Z}(C)$ as well. As a consequence, p and q are vertices of $\Gamma(C)$. Observe that $\{p, q\}$ must be an edge of $\Gamma(C)$, as otherwise C would be $\{p, q\}$ -solvable by [5, Theorem B(i)], and so would be G , because G/C is a $\{p, q\}'$ -group. Hence, there exists an element $c \in C$ such that $|c^G|$ is divisible by pq . Take any $y \in M$ which is vanishing in G : we get that xy is vanishing in G and $|(xy)^G|$ is divisible by pq .

(iv) *Final contradiction.*

By the previous step, we deduce that A is a Sylow q -subgroup of G ; in fact, if $Q \in \text{Syl}_q(G)$, then Q/A is normal in G/A , and so $Q \leq \mathbf{F}(G) = A$. In particular, for every $x \in G \setminus \mathbf{C}_G(A)$, we have $q \mid |x^G|$. Observe also that $\overline{G} = G/\mathbf{C}_G(A)$ does not have any vanishing element whose \overline{G} -class size is divisible by p (otherwise, if \overline{x} is such an element, then x would be a vanishing element of G such that $pq \mid |x^G|$); whence, by Theorem A in [6], \overline{G} is p -nilpotent.

Let now $\overline{K} = K/\mathbf{C}_G(A)$ be a minimal normal subgroup of \overline{G} . Clearly K does not have a central Sylow q -subgroup, because $K > \mathbf{C}_G(A)$; but K does not have a central Sylow p -subgroup as well, since otherwise K would be p -solvable and so would be G (recall that \overline{G} is p -nilpotent). As a consequence, p and q are vertices of $\Gamma(K)$. Moreover, K must have conjugacy classes of size divisible by pq ; otherwise K would be p -solvable by [5, Theorem B(i)] and, as above, G would be p -solvable. Note that such conjugacy classes of K are contained in $K \setminus \mathbf{C}_G(A)$. But an application of Corollary 4.4 yields that every element in $K \setminus \mathbf{C}_G(A)$ is actually a vanishing element of G , and this is the final contradiction that completes the proof. \blacksquare

As for the Corollary stated in the Introduction, observe that if G is a group having a nonabelian minimal normal subgroup, and p is a prime that is not a complete vertex of $\Gamma_v(G)$, then either $p \notin V_v(G) = V(G)$ (and so G is p -nilpotent) or $p \in V_v(G)$ is not adjacent in $\Gamma_v(G)$ to some other vertex q . In the latter case, Theorem A yields the p -solvability of G .

ACKNOWLEDGEMENTS

The fourth author wishes to thank Silvio Dolfi for a couple of very pleasant discussions on the subject of this paper.

REFERENCES

- [1] M. Bianchi, D. Chillag, M.L. Lewis, E. Pacifici, *Character degree graphs that are complete graphs*, Proc. Amer. Math. Soc. 135 (2007), 671–676.
- [2] J.S. Brodkey, *A note on finite groups with an abelian Sylow group*, Proc. Amer. Math. Soc. 14 (1963), 132–133.
- [3] J. Brough, *On vanishing criteria that control finite group structure*, J. Algebra 458 (2016), 207–215.
- [4] J. Brough, *Non-vanishing elements in finite groups*, J. Algebra 460 (2016), 387–391.
- [5] C. Casolo, S. Dolfi, *Products of primes in conjugacy class sizes and irreducible character degrees*, Israel J. Math. 174 (2009), 403–418.
- [6] S. Dolfi, E. Pacifici, L. Sanus, *Groups whose vanishing class sizes are not divisible by a given prime*, Arch. Math. 94 (2010), 311–317.
- [7] S. Dolfi, E. Pacifici, L. Sanus, P. Spiga, *On the vanishing prime graph of finite groups*, J. London Math. Soc. 82 (2010), 167–183.
- [8] P. Fleischmann, *Finite Groups with Regular Orbits on Vector Spaces*, J. Algebra 103 (1986), 211–215.
- [9] D.P.M. Goodwin, *Regular Orbits of Linear Groups with an Application to the $k(GV)$ -Problem, 1*, J. Algebra 227 (2000), 395–432.
- [10] D. Gorenstein, *Finite Groups*, Harper & Row, New York, 1967.
- [11] D. Gorenstein, R. Lyons, R. Solomon, *The classification of the finite simple groups. Number 3. Part I. Chapter A. Almost simple K -groups*, Mathematical Surveys and Monographs, 40.3. AMS, Providence, RI, 1998.
- [12] A. Granville, K. Ono, *Defect zero p -blocks for finite simple groups*, Trans. Amer. Math. Soc. 348 (1996), 331–347.
- [13] B. Huppert, *Endliche Gruppen I*, Springer, Berlin, 1979.
- [14] B. Huppert, N. Blackburn, *Finite Groups II*, Springer, Berlin, 1982.
- [15] I.M. Isaacs, *Character theory of finite groups*, Dover Publications, New York, 1994.
- [16] I.M. Isaacs, G. Navarro, T.R. Wolf, *Finite Group Elements where No Irreducible Character Vanishes*, J. Algebra 222 (1999), 413–423.
- [17] C. Köhler, H. Pahlings, *Regular orbits and the $k(GV)$ -problem*, in: Groups and Computation III: Proceedings of the International Conference at the Ohio State University, June 15–19, (1999, 2001), 209–228.
- [18] B. Sagan, *The Symmetric Group*, Springer, Graduate Texts in Mathematics 203, 1991.

MARIAGRAZIA BIANCHI, DIPARTIMENTO DI MATEMATICA F. ENRIQUES,
UNIVERSITÀ DEGLI STUDI DI MILANO, VIA SALDINI 50, 20133 MILANO, ITALY.

E-mail address: `mariagrazia.bianchi@unimi.it`

JULIAN M.A. BROUGH, TECHNISCHE UNIVERSITÄT KAISERSLAUTERN,
67663, KAISERSLAUTERN, GERMANY.

E-mail address: `brough@mathematik.uni-kl.de`

RACHEL D. CAMINA, FITZWILLIAM COLLEGE, CAMBRIDGE, CB3 0DG, UK.

E-mail address: `rdc26@dpms.cam.ac.uk`

EMANUELE PACIFICI, DIPARTIMENTO DI MATEMATICA F. ENRIQUES,
UNIVERSITÀ DEGLI STUDI DI MILANO, VIA SALDINI 50, 20133 MILANO, ITALY.

E-mail address: `emanuele.pacifici@unimi.it`