Hertz Theory and Carlson Elliptic Integrals **14 June 2018**

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Abstract.

Legendre's well-known elliptic integrals are not the only version of elliptic integrals. Carlson's form, developed in the late 1970s, have many advantages, and are particularly well suited for Hertzian contact analysis. They fit immediately into the basic formulation: they make no distinction between the major and minor axes of the ellipse (reducing the number of equations needed): and the extension to the study of the deformation outside the contact area is barely noticeable: nothing like the switch from complete to incomplete integrals needed when using Legendre's integrals is required. And finally, their computation is rapid and straightforward.

In addition, equations as Carlson integrals are given for the displacements due to tangential loading (Cattaneo-Mindlin theory), and notes given on the elliptic integrals needed in the evaluation of the internal stresses in a Hertzian contact.

§1 Elliptic integrals.

The term "Elliptic Integrals", if it does not just produce a shudder, immediately brings up some recollection of $K(k)=\int_0^{\pi/2}\frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}$; $E(k)=\int_0^{\pi/2}\sqrt{1-k^2\sin^2\phi}\,d\phi$; or perhaps of even sadder recollections of

$$F(k,\alpha)=\int_0^{\alpha}\frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}\ :\qquad E(k,\alpha)=\int_0^{\alpha}\sqrt{1-k^2\sin^2\phi}\,d\phi:$$

the "incomplete elliptic integrals".

It does not bring up the qualification "in Legendre's form", for there is no need: these ***are*** the elliptic integrals, are they not? In fact already in 1931 Emde explained that in electromagnetic theory the integrals

$$B(k)=\int_0^{\pi/2}\frac{\cos^2\phi}{\sqrt{1-k^2\sin^2\phi}}\,d\phi;\quad D(k)=\int_0^{\pi/2}\frac{\sin^2\phi}{\sqrt{1-k^2\sin^2\phi}}\,d\phi \tag{1}$$

are more use. This is certainly the case in Hertz theory, where for example the equations (Johnson, Contact Mechanics (§4.26a,b))

$$\frac{1}{2R_1}=\frac{p_0}{E'}\frac{b}{k^2a^2}[K(k)-E(k)];\qquad \frac{1}{2R_2}=\frac{p_0}{E'}\frac{b}{k^2a^2}[\,(a^2/b^2)E(k)-K(k)\,]. \tag{2}$$

where $k=\sqrt{1-b^2/a^2}$. (we need not here discuss their meaning: see below)

could (should?) have been just $\frac{1}{2R_1}=\frac{p_0}{E'}\frac{b}{a^2}D(k);\qquad \frac{1}{2R_2}=\frac{p_0}{E'}\frac{1}{b}B(k).$ (3)

Note the difference between the equations for the major and minor axes…less striking with Emde's functions: but still firmly distinguishing between major and minor axes. For of course we must have $b \le a$, or how can we calculate the eccentricity?

The answer to the question above is no: there is an alternative, and in Engineering, and particularly in the study of Hertzian contact, these are more convenient.

Carlson elliptic integrals are defined as

$$R_F(p,q,r) = \frac{1}{2}\int_0^\infty \frac{dt}{(t+p)^{1/2}(t+q)^{1/2}(t+r)^{1/2}} \tag{4}$$

and $$R_D^\otimes(p,q;r) = \frac{1}{2}\int_0^\infty \frac{dt}{(t+p)^{1/2}(t+q)^{1/2}(t+r)^{3/2}} \tag{5}$$

[Actually Carlson defines not $R_D^\otimes$ but R_D, three times $R_D^\otimes$ in order that $R_D(x,x,x) = x^{-3/2}$, matching $R_F(x,x,x) = x^{-1/2}$: as a result, it seems that all his relations involving R_D are preceded by a factor $1/3$.]

These integrals will replace both the complete[1] and incomplete Legendre integrals, so we have only increased the number of parameters from 2 to 3. But we can return to 2, for we have the scaling rules

$$(c^{1/2})\, R_F(cp,cq,cr) = R_F(p,q,r), \qquad (c^{3/2})\, R_D^\otimes(cp,cq;cr) = R_D^\otimes(p,q;r), \tag{6a,b}$$

(Substitute $s = ct$ in the integral); these can be used to make one parameter equal to unity.

Thus for example, (6a) gives . $$R_F(p,q,r) = p^{-1/2} R_F(1, q/p, r/p) \tag{7}$$

Note that all three parameters in R_F may be interchanged, but only the first two in $R_D^\otimes$: the difference is noted by the semi-colon before the third parameter.

§2 Hertz Theory

Johnson, Contact Mechanics §4.2a,b (p98), explains that from potential theory, the displacements due to a Hertzian pressure distribution $p = p_0[1 - x^2/a^2 - y^2/b^2]^{1/2}$ are

$$w = \frac{ab\,p_0}{2E'}\int_0^\infty \left(1 - \frac{x^2}{a^2+s} - \frac{y^2}{b^2+s}\right)\frac{ds}{[s(a^2+s)(b^2+s)]^{1/2}} \tag{8}$$

[A factor π in Johnson's equation has been removed, on the evidence of CM §3.5 p63/65].

Thus $$w = \frac{ab\,p_0}{E'}[I - Ax^2 - By^2] \quad \text{where } I \equiv \frac{1}{2}\int_0^\infty \frac{ds}{[s(a^2+s)(b^2+s)]^{1/2}}, \tag{9}$$

$$A \equiv \frac{1}{2}\int_0^\infty \frac{ds}{s^{1/2}(a^2+s)^{3/2}(b^2+s)^{1/2}} \quad \text{and } B \equiv \frac{1}{2}\int_0^\infty \frac{ds}{s^{1/2}(a^2+s)^{1/2}(b^2+s)^{3/2}} \tag{10}$$

Note that no distinction between major and minor axes has been needed.

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[1] $K(k^2) \equiv R_F(0,\ (1-k^2),\ 1)$; $\qquad D(k^2) \equiv R_D^\otimes(0,(1-k^2);1)$

The desired deformation is $w = \delta - x^2/2R_1 - y^2/2R_2$, so from (9) we require

$$\delta = \frac{ab\,p_0}{2E'}\int_0^\infty \frac{ds}{[s(a^2+s)(b^2+s)]^{1/2}} \tag{11}$$

How do we express this as an elliptic integral? Not readily: but fortunately Johnson §4.26c gives the answer: we decide which is the minor axis and calculate $k = \sqrt{1-b^2/a^2}$: then, *we are told*,

$$\delta = \frac{p_0}{E'}b\,K(k)\,. \tag{12a}$$

But for the Carlson version, by inspection,

$$\delta = \frac{ab\,p_0}{2E'}\int_0^\infty \frac{ds}{[s(a^2+s)(b^2+s)]^{1/2}} = \frac{ab\,p_0}{E'}R_F(0, a^2, b^2\ )\,. \tag{12b}$$

If we want a non-dimensional form, then the scaling rule (6a) gives at once $\delta = \dfrac{\sqrt{ab}\,p_0}{E'}R_F(0, a/b, b/a\ )$.

Similarly $\dfrac{1}{2R_1} = \dfrac{ab\,p_0}{2E'}\displaystyle\int_0^\infty \frac{ds}{s^{1/2}(a^2+s)^{3/2}(b^2+s)^{1/2}}$: how to proceed? Rely on Johnson[2], and believe

$$\frac{1}{2R_1} = \frac{p_0}{E'}\frac{b}{k^2a^2}[K(k)-E(k)]\ ? \tag{13a}$$

Or by inspection, recognise that

$$\frac{1}{2R_1} = \frac{ab\,p_0}{E'}R_D^{\otimes}(0, b^2; a^2)\,, \text{ and its partner } \frac{1}{2R_2} = \frac{ab\,p_0}{E'}R_D^{\otimes}(0, a^2; b^2)\,. \tag{14a,b}$$

Note that this does not require a separate analysis as is needed to obtain

$$\frac{1}{2R_2} = \frac{p_0}{E'}\frac{b}{k^2a^2}[\ (a^2/b^2)\,E(k) - K(k)\ ]. \tag{13b}$$

Once again the scaling rule (6b) can usefully be employed to get

$$\frac{1}{2R_1} = \frac{p_0}{E'\sqrt{ab}}R_D^{\otimes}(0, b/a; a/b) \quad \text{and} \quad \frac{1}{2R_2} = \frac{p_0}{E'\sqrt{ab}}R_D^{\otimes}(0, a/b; b/a) \tag{15a,b}$$

The first step in analysing an elliptical Hertz contact is to find the relation between the macroscopic geometry, characterised by R_1 and R_2, and the ellipticity[3] $e = b/a$. Dividing the last two equations gives

$$\frac{R_1}{R_2} = \frac{R_D^{\otimes}(0, a/b; b/a)}{R_D^{\otimes}(0, b/a; a/b)} \tag{16}$$

However a different scaling gives a more useful form

[2] Barber, Contact Mechanics, derives these from first principles, using the Boussinesq point-load solution

[3] e is frequently used as the symbol for the eccentricity: but k is so standard as the argument of Legendre elliptic integrals that we retain it, which frees e. Of course, when $e < 1$, we have $k = \sqrt{1-e^2}$ .and $k' = e$.

$$\frac{R_1}{R_2} = \frac{a^{3/2} R_D(0, (a/b)^{3/2}; (b/a)^{1/2})}{b^{3/2} R_D(0, (b/a)^{3/2}; (a/b)^{1/2})}. \tag{17}$$

The ratio of the two Carlson integrals in (17) is insensitive to the value of b/a, so we get immediately the well-known approximation $\frac{a^{3/2}}{b^{3/2}} \approx \frac{R_1}{R_2}$.

Indeed, rewriting equation (17) as $(a/b)^{3/2} = \frac{R_1}{R_2} \cdot \frac{R_D(0, (b/a)^{3/2}; (a/b)^{1/2})}{R_D(0, (a/b)^{3/2}; (b/a)^{1/2})}$ (18)

the same property makes this a neat iterative rule for finding the a/b exactly.

[For example, for $R_1/R_2 = 5$, starting from $a/b \approx (R_1/R_2)^{2/3} = 2.924$, we get successively a / b = 2.8891, 2.890215, 2.890180, 2.890181…]

The load will be $P = (2/3)\pi ab\, p_0$: so substituting in equations (14) gives

$$\frac{3P}{2\pi E'} R_D^{\otimes}(0, b^2; a^2) = 1/(2R_1): \quad \frac{3P}{2\pi E'} R_D^{\otimes}(0, a^2; b^2) = 1/(2R_2) \tag{19}$$

which, scaling the first by $1/a^2$ and the second by $1/b^2$ can be written

$$\frac{3PR_1}{\pi E'} R_D^{\otimes}(0, (b/a)^2; 1) = a^3 : \quad \frac{3PR_2}{\pi E'} R_D^{\otimes}(0, (a/b)^2; 1) = b^3 \tag{20a,b}$$

corresponding to the equation $a^3 = \frac{3}{4}\frac{PR}{E'}$ for a circular contact. Setting $b = a$ and noting that $R_D^{\otimes}(0, 1; 1) = \pi/4$, we recover the circular result.

Symmetric forms are often desirable, so taking the square root of the product of (20a) and (20b) gives

$$(ab)^{3/2} = \frac{3P\sqrt{R_1R_2}}{\pi E'} [\, R_D^{\otimes}(0, (b/a)^2; 1) \cdot R_D^{\otimes}(0, (a/b)^2; 1)\,]^{1/2}.$$

Alternatively, in terms of the maximum pressure,

$$(ab)^{1/2} = \frac{2p_0\sqrt{R_1R_2}}{E'} [\, R_D^{\otimes}(0, (b/a)^2; 1) \cdot R_D^{\otimes}(0, (a/b)^2; 1)\,]^{1/2}$$

which we may rewrite as $\frac{p_0}{\sqrt{ab}} = \frac{E'}{2\sqrt{R_1R_2}} [\, R_D^{\otimes}(0, (b/a)^2; 1) \cdot R_D^{\otimes}(0, (a/b)^2; 1)\,]^{-1/2}$ (21a)

or, with a different scaling, as $\frac{p_0}{\sqrt{ab}} = \frac{E'}{2\sqrt{R_1R_2}} [R_D^{\otimes}(0, b/a; a/b) \cdot R_D^{\otimes}(0, a/b; b/a)]^{-1/2}$ (21b)

corresponding to (and agreeing with) $\frac{p_0}{a} = \frac{2E'}{\pi R}$ since $R_D^{\otimes}(0,1;1)\,] = \pi/4$.

§3 Evaluation of Carlson Integrals

The ease with which the Hertz equations may be written as Carlson integrals would be pointless if the integrals were not easy to evaluate. But the possibility of readily evaluating them by duplication (corresponding to the evaluation of Legendre integrals using Landen's transformation) removes this worry..and indeed, it would seem that this possibility is what led Carlson to introduce them.

The basic algorithm is that for $R_F(p,q,r) = \frac{1}{2}\int_0^\infty \frac{dt}{(t+p)^{1/2}(t+q)^{1/2}(t+r)^{1/2}}$ we calculate

$\lambda = \sqrt{pq} + \sqrt{qr} + \sqrt{rp}$. Then let $p_1 = (p+\lambda)/4;\quad q_1 = (q+\lambda)/4;\quad r_1 = (r+\lambda)/4;$

Then $$R_F(p_1, q_1, r_1) = R_F(p, q, r). \tag{22}$$

… and the maximum difference between p, q and r has been reduced by a factor 4. Clearly, repeating the process ultimately results in almost equal values of p_n, q_n, r_n. Now let $\mu_n = (p_n + q_n + r_n)/3$. Then a series expansion in terms of $\varepsilon_p = (p-\mu)/\mu$ etc. and some algebra (noting that $\varepsilon_p + \varepsilon_q + \varepsilon_r = 0$) shows that

$$R_F(p, q, r) = \frac{1}{\sqrt{\mu_n}}\left[1 + O(\varepsilon^2)\right], \text{ where } \varepsilon = \max[|\varepsilon_p|, |\varepsilon_q|, |\varepsilon_r|) \tag{23}$$

Carlson recommends the use of further terms, which can reduce the error to $O(\varepsilon^6)$, but for engineering use this seems unnecessary.

The algorithm for $R_D(p, q; r)$ is more complex: Carlson shows that

$$R_D^{\otimes}(p_{n+1}, q_{n+1}, r_{n+1}) = \tfrac{1}{4} R_D^{\otimes}(p_n, q_n, r_n) + \frac{1}{r_n^{1/2}(r_n + \lambda_n)} \tag{24}$$

where the successive sets of (p,q,r) are found as above. Thus its use involves the collection of partial results found during the set of duplications...reminiscent of the procedure for $E(k)$ given in the NBS tables. Ultimately $R_D^{\otimes}(p_n, q_n, r_n)$ tends to a limit $\frac{1}{3}\tilde{\mu}^{-3/2}$ where now $\tilde{\mu}_n = (p_n + q_n + 3r_n)/5$. Once again Carlson recommends that for efficiency, a series expansion of the differences $(p_n - \tilde{\mu}_n)$ etc should be used. This is detailed in Carlson (1977), but seems unnecessary for 6-decimal accuracy. A simple MATLAB program without the series, but believed to give both R_F and $R_D^{\otimes}$ to an accuracy of 10^{-6} is given in appendix 1.

§4 Displacements outside the contact area.

The basic Hertz analysis giving the shape and size of the contact area uses only "complete" Legendre elliptic integrals: these correspond to Carlson integrals where one of the arguments is zero (see footnote (1) above). The deformation outside the contact area requires "incomplete" elliptic integrals, and the path to these from the fundamental equations requires some skill. Once again, the path to the Carlson form is elementary.

From Johnson Contact Mechanics §4.2b (p98), the displacements outside the contact area due to the Hertzian pressure distribution are

$$w = \frac{ab\, p_0}{2E'} \int_{\lambda_1}^{\infty} \left(1 - \frac{x^2}{a^2+s} - \frac{y^2}{b^2+s}\right) \frac{ds}{[s(a^2+s)(b^2+s)]^{1/2}}$$ (again omitting the false π), where

now the lower limit of the integral in (8) has become λ_1, the positive root of the quadratic equation

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1 . \qquad (25)$$

A simple change of variable shifts the lower limit back to zero, so that

$$I \equiv \frac{1}{2}\int_{\lambda_1}^{\infty} \frac{ds}{[s(a^2+s)(b^2+s)]^{1/2}} = \frac{1}{2}\int_{0}^{\infty} \frac{ds}{[(\lambda_1+s)(a^2+\lambda_1+s)(b^2+\lambda_1+s)]^{1/2}} = R_F(\lambda_1, a^2+\lambda_1, b^2+\lambda_1),$$

and

$$A = \frac{1}{2}\int_{\lambda_1}^{\infty} \frac{ds}{[s(b^2+s)(a^2+s)^3]^{1/2}} = R_D^{\otimes}(\lambda_1, b^2+\lambda_1; a^2+\lambda_1);$$

$$B = \frac{1}{2}\int_{\lambda_1}^{\infty} \frac{ds}{[s(a^2+s)(b^2+s)^3]^{1/2}} = R_D^{\otimes}(\lambda_1, a^2+\lambda_1; b^2+\lambda_1)$$

so that $w(x, y) = \dfrac{ab\, p_0}{E'}\left[I - A\, x^2 - B\, y^2\right]$.

Inside the ellipse, where $\lambda_1 = 0$, the two coefficients A, B are constants, and this gives the expected parabolic variation, but outside the ellipse, because of the variation of λ_1 with x and y, A and B are no longer constants and the variation is no longer parabolic.

Note how easily the change from inside the contact area (equations (12b, 14a,b)) to outside it is made: and the algorithms discussed above already provide the values needed. But remember that λ_1 depends on (x, y) and must be recalculated from (25) for each point considered.

For points along the axes, this is simple: for example, if $y = 0$, eqn. (25) reduces to $\dfrac{x^2}{a^2+\lambda_1} = 1$ and

$\lambda_1 = x^2 - a^2$ giving the simpler form

$$w(x, 0) = \frac{ab\, p_0}{E'}\left[R_{\mathbf{F}}(x^2-a^2,\ (x^2-a^2+b^2),\ x^2) - (x^2)\, R_{\mathbf{D}}^{\otimes}(x^2-a^2,\ (x^2-a^2+b^2);\ x^2)\right]. \qquad (26a)$$

Note once again the ease with which we can switch from major to minor axes: there is no requirement that $b < a$, so we have at once

$$w(0, y) = \frac{ab\, p_0}{E'}\left[R_{\mathbf{F}}(y^2-b^2,\ (y^2-b^2+a^2),\ y^2) - (y^2)\, R_{\mathbf{D}}^{\otimes}(y^2-b^2,\ (y^2-b^2+a^2);\ y^2)\right] \qquad (26b)$$

§5 Tangential loading over an ellipse.

Mindlin's classic analysis of the initiation of slip (Mindlin (1949) used a tangential load

$$q_x(\xi,\eta) = q_0 / \sqrt{1-(\xi/a)^2-(\eta/b)^2} \quad \text{over the ellipse } \frac{\xi^2}{a^2}+\frac{\eta^2}{b^2}<1 \; . \tag{27}$$

This leads by well-known methods (see for example Barber 2017) to an equation for the displacement in the direction of the traction $u_x(x,y) = \dfrac{q_0}{G}\displaystyle\int_0^{\pi/2} \frac{1-\nu\sin^2\phi}{\sqrt{\cos^2\phi/a^2+\sin^2\phi/b^2}}d\phi$ (28)

We need to convert this to standard elliptic integrals.

For the Legendre form, provided $b \le a$, we proceed

$$u_x(x,y) = \frac{bq_0}{G}\int_0^{\pi/2} \frac{1-\nu\sin^2\phi}{\sqrt{(b^2/a^2)\cos^2\phi+1-\cos^2\phi}}d\phi = \frac{bq_0}{G}\int_0^{\pi/2} \frac{1-\nu\sin^2\phi}{\sqrt{1-m\cos^2\phi}}d\phi \tag{29a}$$

where the parameter is, as usual, $m = 1-b^2/a^2$. [or modulus $k \equiv \sqrt{m} = \sqrt{1-b^2/a^2}$]

Then setting $\theta = \pi/2-\varphi$ we get

$$u_x(x,y) = \frac{bq_0}{G}\int_0^{\pi/2} \frac{1-\nu\cos^2\theta}{\sqrt{1-m\sin^2\theta}}d\phi = \frac{bq_0}{G}\left[K(m)-\nu B(m)\right] \tag{30a}$$

But if $a<b$, we write

$$u_x(x,y) = \frac{aq_0}{G}\int_0^{\pi/2} \frac{1-\nu\sin^2\phi}{\sqrt{\cos^2\phi+(a^2/b^2)\}\sin^2\phi}}d\phi = \frac{aq_0}{G}\int_0^{\pi/2} \frac{1-\nu\sin^2\phi}{\sqrt{1-\tilde{m}\sin^2\phi}}d\phi \tag{29b}$$

where $\tilde{m} = 1-a^2/b^2$: so that $u_x(x,y) = \dfrac{aq_0}{G}\left[K(\tilde{m})-\nu D(\tilde{m})\right]$ (30b)

where K and D are complete elliptic integrals with parameter $\tilde{m}$: [m would be negative].

For the Carlson form, without needing to pick the smaller of (a,b), we set $t = \cot^2\phi$. Then

$$\sin^2\phi = \frac{1}{1+t}; \quad \cos^2\phi = \frac{t}{1+t} \quad \text{and} \quad d\phi = -dt/(2(1+t)\sqrt{t})$$

$$u_x(x,y) = \frac{q_0}{G}\int_0^{\pi/2} \frac{1-\nu\sin^2\phi}{\sqrt{\cos^2\phi/a^2+\sin^2\phi/b^2}}d\phi = \frac{q_0}{2G}\int_0^{\infty} \frac{[1-\nu/(1+t)]\sqrt{(1+t)}}{\sqrt{(t/a^2+1/b^2)}}\frac{dt}{(1+t)\sqrt{t}}$$

$$= \frac{aq_0}{2G}\left[\int_0^{\infty}\frac{1}{\sqrt{t\,(t+1)(t+a^2/b^2)}}dt - \int_0^{\infty}\frac{\nu}{\sqrt{t\,(t+1)^3(t+a^2/b^2)}}dt\right] \tag{31}$$

$$\therefore \quad u_x(x,y) = \frac{aq_0}{G}\left[R_K(0,1,(a/b)^2)-\nu\, R_D(0,(a/b)^2\,;1)\right] \tag{32}$$

It is difficult to claim that this was easier to obtain than the Legendre forms. But we have arrived at a **single** equation (32), instead of the pair (30 a,b): the result for a traction along the “b-axis” is implicit.

We are not, of course, suggesting that a and b are interchangeable: the traction is in the “x-direction”, and a is the semi-axis in that direction: we are merely not specifying whether this is the major or the minor semi-axis.

Ellipsoidal traction

For an ellipsoidal traction $q_x(\xi.\eta)=q_1\sqrt{1-(\xi/a)^2-(\eta/b)^2}$ the same procedure of setting up a polar co-ordinate system centred on the point of interest leads to a more complicated equation:

$$u_x(x,y)=\frac{q_1}{4G}\int_0^{\pi}[1-\nu\sin^2\phi]\frac{M\sin\phi\cos\phi+A\sin^2\phi+B\cos^2\phi}{[\cos^2\phi/a^2+\sin^2\phi/b^2]^{3/2}}\quad d\phi \tag{33a}$$

where $M=2xy/a^2b^2;\qquad A=(1-x^2/a^2)/b^2;\qquad B=(1-y^2/b^2)/a^2$ (33b)

From symmetry, the integral is twice the integral from 0 to $\pi/2$, except that the M-term is dropped.

To express this as elliptic integrals, we again[4] set $t=\cot^2\phi$. Then

$$\sin^2\phi=\frac{1}{1+t};\quad \cos^2\phi=\frac{t}{1+t}\quad\text{and}\quad d\phi=-dt/(2\sqrt{t}\,(1+t))$$

$$\therefore u_x(x,y)=\frac{q_1}{8G}\int_0^{\infty}(1-\nu/(1+t))\,\left[t/a^2+1/b^2\right]^{-3/2}(t+1)^{3/2}\left[\frac{A}{t+1}+\frac{Bt}{t+1}\right]\frac{dt}{\sqrt{t}\,(1+t)}$$

$$=\frac{q_1}{8G}\int_0^{\infty}Ba^3\,\frac{[t+1-\nu][t+A/B]}{\sqrt{t}\,(t+1)^{3/2}\left(t+e^2\right)^{3/2}}dt\qquad\text{where } e=a/b. \tag{34}$$

The repeated cubic term is removed by partial fractions: we find

$$\frac{(t+1-\nu)(t+A/B)}{(t+1)(t+e^2)}=1+\frac{1}{e^2-1}\left[\frac{(\nu)\,(1-A/B)}{t+1}-\frac{(e^2-\{1-\nu\})(e^2-A/B)}{t+e^2}\right]$$

or writing $\nu\equiv\mu(e^2-1)$ this becomes $1+\left[\frac{(\mu)\,(1-A/B)}{t+1}-\frac{(1+\mu\})(e^2-A/B)}{t+e^2}\right]$ (35)

$$u_x(x,y)=\frac{q_1a^3}{8G}\left[\int_0^{\infty}\frac{B\,dt}{\sqrt{t\,(t+1)(t+e^2)}}+\mu(B-A)\int_0^{\infty}\frac{dt}{\sqrt{t\,(t+1)^3(t+e^2)}}-(1+\mu)(Be^2-A)\int_0^{\infty}\frac{dt}{\sqrt{t\,(t+1)(t+e^2)^3}}\right]$$

Thus $u_x(x,y)=\frac{q_1a^3}{4G}\left[B\,R_F(0,1,e^2)+\mu(B-A)R_D^{\otimes}(0,e^2;1)-(1+\mu)(Be^2-A)R_D^{\otimes}(0,1;e^2)\right]$ (36)

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[4] Why not $t=\tan^2\phi$? No good reason here: but for the incomplete integral $\int_0^{\alpha}(..)d\phi$ the limits must be interchanged, so the substitution is then $t=\cot^2\phi-\cot^2\alpha$. This is needed for points outside the contact area: see Barber (2017).

Since $A=(1-x^2/a^2)/b^2$ and $B=(1-y^2/b^2)/a^2$ it is convenient to write

$X=1-x^2/a^2$; $Y=1-y^2/b^2$ so that $a^2A=e^2X$; $a^2B=Y$:

$$u_x(x,y)=\frac{q_1 a}{4G}\left[\begin{array}{l} Y\{R_F(0,1,e^2)-e^2 R_D^{\otimes}(0,1;e^2)\}+e^2X\,R_D^{\otimes}(0,1;e^2) \\ +\mu\left[Y\{R_D^{\otimes}(0,e^2;1)-e^2R_D^{\otimes}(0,1;e^2)\}-e^2X\{R_D^{\otimes}(0,e^2;1)-R_D^{\otimes}(0,1;e^2)\}\right]\end{array}\right] \tag{37}$$

The constant term in the displacement is found by setting $X=Y=1$ and adding:

$$u_x(0,0)=\frac{q_1 a}{4G}\left[R_F(0,e^2;1)-\nu\,R_D^{\otimes}(0,e^2;1)\right]$$

Using the addition theorem (A5) in the form $R_F(0,e^2,1)=R_D^{\otimes}(0,e^2;1)+e^2\,R_D^{\otimes}(0,1;e^2)$, the first term of (37) can be simplified to give

$$u_x(x,y)=\frac{q_1 a}{4G}\left[\begin{array}{l} Y\{R_D^{\otimes}(0,e^2;1)\}+e^2X\,R_D^{\otimes}(0,1;e^2) \\ +\mu\left[Y\{R_D^{\otimes}(0,e^2;1)-e^2R_D^{\otimes}(0,1;e^2)\}-e^2X\{R_D^{\otimes}(0,e^2;1)-R_D^{\otimes}(0,1;e^2)\}\right]\end{array}\right] \tag{38}$$

Transverse displacements.

The transverse displacement u_y no longer vanishes. We have

$$u_y(x,y)=\frac{q_1}{4G}\int_0^{\pi}[\sin\phi\cos\phi]\frac{M\sin\phi\cos\phi+A\sin^2\phi+B\cos^2\phi}{[\cos^2\phi/a^2+\sin^2\phi/b^2]^{3/2}}\quad d\phi \tag{39}$$

Now the symmetry knocks out the A and B terms, leaving

$$u_y(x,y)=\frac{\nu q_1}{2G}\int_0^{\pi/2}(\cos^2\phi/a^2+\sin^2\phi/b^2)^{-3/2}[M\sin^2\phi\cos^2\phi]\,d\phi \quad \text{where } M=2xy/a^2b^2$$

Then $u_y(x,y)=\dfrac{\nu q_1}{2G}\displaystyle\int_0^{\infty}(t/a^2+1/b^2)^{-3/2}(t+1)^{3/2}\frac{M\,t}{(t+1)^2}\,\frac{dt}{2(t+1)\sqrt{t}}$

$$u_y(x,y)=\frac{\nu q_1 a^3}{4G}\int_0^{\infty}\frac{M\,t\,dt}{(t+e^2)^{3/2}(t+1)^{3/2}\sqrt{t}} \tag{40}$$

Partial fractions now give $\dfrac{t}{(t+e^2)(t+1)}=\dfrac{1}{e^2-1}\left[\dfrac{e^2}{t+e^2}-\dfrac{1}{t+1}\right]$

so, again writing μ for $\nu/(e^2-1)$,

$$u_y(x,y)=\frac{\mu q_1 a^3 M}{4G}\left[\int_0^{\infty}\frac{e^2\,dt}{(t+e^2)^{3/2}(t+1)^{3/2}\sqrt{t}}-\int_0^{\infty}\frac{dt}{(t+e^2)^{1/2}(t+1)^{3/2}\sqrt{t}}\right]$$

$$u_y(x,y)=\frac{\mu q_1 a}{G}\,\frac{x\,y}{b^2}\left[e^2R_D^{\otimes}(0,1;e^2)-R_D^{\otimes}(0,e^2;1)\right] \tag{41}$$

The expressions found above all agree with the results given by Vermeulen & Johnson (1964).

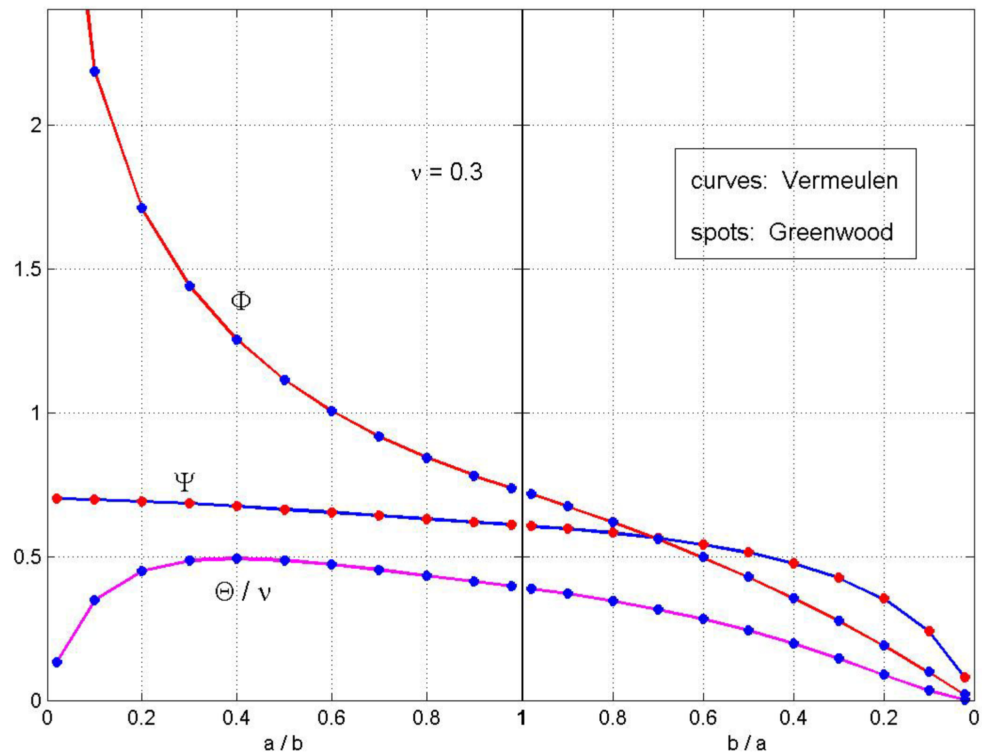

Figure 1: Equivalence of the Legendre and Carlson expressions.
Vermeulen's symbols in the plot above signify $u_x(x,y)=(q_1 a/2G)[\Gamma-\Psi\cdot(y^2/b^2)-\Phi\cdot(x^2/a^2)]$;

$u_y(x,y)=(q_1 a/2G)[\Theta\cdot(xy/ab)]$

§6 Internal stresses for an elliptical Hertzian contact.

Sackfield & Hills (1983) show that the complete set of the six internal stresses can all be expressed as the sum of a very complicated but algebraic expression $L(x,y,z)$ [Fessler & Ollerton (1957) study the reduced forms of this: $Q(x,z)$ when $y=0$, and $R(y,z)$ when $x=0$], and "three elliptic integrals"

$$I_1 \equiv \int_s^\infty \frac{dw}{(1+w^2)^{3/2}(k^2+w^2)^{1/2}}, \quad I_2 \equiv \int_s^\infty \frac{dw}{(1+w^2)^{1/2}(k^2+w^2)^{3/2}} \text{ and}$$

$$I_3 \equiv \int_s^\infty \frac{dw}{w^2(1+w^2)^{1/2}(k^2+w^2)^{1/2}}. \tag{41}$$

Sackville & Hills comment that these can be transformed into a standard form or evaluated directly using, for example, Simpson's rule. Readers of this paper will already have recognised that these are just our Carlson integrals and need no numerical integration. By substituting $w=\sqrt{u}$ to make the factors linear, followed by $u=t+s^2$ to bring the lower limit to zero, we have simply

$$I_1 = R_D^{\otimes}(s^2, k^2+s^2\,; 1+s^2), \quad I_2 = R_D^{\otimes}(s^2, 1+s^2\,; k^2+s^2), \quad I_3 = R_D^{\otimes}(1+s^2, k^2+s^2\,; s^2). \tag{42}$$

We note that from the addition theorem (A3) we have $I_1+I_2+I_3 = \dfrac{1}{\sqrt{(s^2)(1+s^2)(k^2+s^2)}}$.

For the full equations for the six stresses, and the details of the notation, see Sackfield & Hills (1983)

§7 Discussion.

Nothing written above is intended to disparage Legendre's magnificent transformation of the elliptic integrals into his standard forms $K(k)=\int_0^{\pi/2}\frac{1}{\sqrt{1-k^2\sin^2\phi}}\,d\phi;\quad E(k)=\int_0^{\pi/2}\sqrt{1-k^2\sin^2\phi}\,d\phi$; and of course tables of these are widely available. But are they now much used? More usually, Fortran or Matlab routines will be used to evaluate the integrals as needed. Routines for evaluating the Carlson elliptic integrals are also widely available (via the internet for Matlab). Carlson's duplication procedure for evaluating his integrals is no more complex than (and as quick as) the corresponding processes for evaluating $K(m)$,[5] and $E(m)$, and distinctly simpler than the procedure for evaluating the incomplete Legendre integrals $F(m,\alpha),\quad E(m,\alpha)$. Carlson would clearly have despised (ridiculed?) the program given below, believing the duplication process to be only a preliminary to the power series expansion. For that, consult the arxiv reference. There he also gives 14 figure reference values, for real and complex arguments, from which $R_F(2, 3, 4) = 0.58408\ 28416\ 7715$ and $R_D(2, 3; 4) = 0.16510\ 52729\ 4261$ ($\equiv 3R_D^{\otimes}$) have been abstracted. The simple program below reproduces the first 8 decimals.

The algebraic integrand makes them easier to manipulate than Legendre's trigonometric form. To demonstrate useful techniques, examples of differentiation and simple addition theorems are given in appendix 2. The reader may like the challenge of proving that $\int_0^{\pi/2}\frac{d\theta}{(1-m\sin^2\theta)^{3/2}}=\frac{E(m)}{1-m}$; and comparing it with the Carlson equivalent!

§8 Conclusion.

Carlson elliptic integrals are particularly well suited to the analysis of Hertzian contacts, both for determining the area of contact and for studying the deformation outside the contact. They eliminate the inconvenient distinction between major and minor axes. Their calculation is fast and straightforward, and there is no distinction between "complete" and "incomplete" integrals. It is time for Legendre's elliptic integrals to be pensioned off.

[5] Indeed, $K(m)=R_K(0\,,1\,,\,1-m)$

Appendix 1 *A simple MATLAB program for Carlson Integrals*

```
%x = [p  q  r];

function [Rk,Rd]=carlson(x);

%Calculates one-third of Carlson's R_D
% for 6 decimal accuracy

if x(3)<1E-15,  display('r ~ 0:  RD suspect');  end

eps=1E-3;
quam=1; sigma=0; e=1;
while e>eps,
  xr=sqrt(x);
  lambda=xr(1)*(xr(2)+xr(3))+xr(2)*xr(3);
  delta=quam/((x(3)+lambda)*xr(3));
  igma=sigma+delta;
  x=(x+lambda)/4;
  mu=(x(1)+x(2)+x(3))/3;
  X=x/mu-1;
  quam=quam/4;
  e=max(abs(X));
end
Rk=1/sqrt(mu);
mud=(x(1)+x(2)+3*x(3))/5;
Rd=sigma+quam/(3*mud^(3/2));
XQ=sprintf('Rk %9.7f  Rd* %9.7f',Rk,Rd); disp(XQ);
%---------------------------------
```

Notes: the series expansion of R_K is $R_K \approx (1/\sqrt{\mu})[1+\frac{1}{20}(X_1^2+X_2^2+X_3^2)+..]$ so the relative error in using $R_K \approx 1/\sqrt{\mu}$ is less than $eps^2/20$. The error in the series for $R_D^{\otimes}$ is comparable.

Appendix 2 Additional properties

Throughout a,b,c are merely any three (real, positive) quantities: no ordering is implied or necessary.

Define $\Delta \equiv \sqrt{(t+a)(t+b)(t+c)}$

Then $\frac{1}{2}\int_0^\infty \frac{dt}{\Delta} \equiv R_K(a,b,c)$ and $\frac{1}{2}\int_0^\infty \frac{dt}{(t+c)\Delta} \equiv R_D^{\otimes}(a,b\,;c)$ (A1)

Two cubic factors

Since $\frac{1}{(t+b)(t+c)} = \frac{1}{c-b}\left[\frac{1}{t+b} - \frac{1}{t+c}\right]$ we have $\frac{1}{2}\int_0^\infty \frac{dt}{\sqrt{(t+a)(t+b)^3(t+c)^3}} =$

$$=\frac{1}{2(c-b)}\int_0^{\infty}\left[\frac{1}{\sqrt{(t+a)(t+b)^3(t+c)}}-\frac{1}{\sqrt{(t+a)(t+b)(t+c)^3}}\right]dt=$$

$$=\frac{1}{c-b}\left[R_D^{\otimes}(a,c\,;b)-R_D^{\otimes}(a,b\,;c)\right] \qquad \text{(A2)}$$

For a repeated cubic factor, see (A9) below.

Addition theorems

(I) $$\frac{d}{dt}\left(\frac{1}{\Delta}\right)=-\frac{1}{2\Delta}\left[\frac{1}{t+a}+\frac{1}{t+b}+\frac{1}{t+c}\right]$$

Integrating wrt t: $\left.\frac{1}{\Delta}\right|_0^{\infty}=-\frac{1}{2}\int_0^{\infty}\left[\frac{1}{\Delta(t+a)}+\frac{1}{\Delta(t+b)}+\frac{1}{\Delta(t+c)}\right]dt$

so $$\frac{1}{\sqrt{abc}}=R_D^{\otimes}(b,c\,;a)+R_D^{\otimes}(c,a\,;b)+R_D^{\otimes}(a,b\,;c) \qquad \text{(A3)}$$

(II) A second addition theorem, particularly useful when one of the three parameters is zero ("complete elliptic integral") is found as follows:

$$\frac{d}{dt}\left(\frac{t}{\Delta}\right)=\frac{1}{\Delta}-\frac{1}{2\Delta}\left[\frac{t}{t+a}+\frac{t}{t+b}+\frac{t}{t+c}\right]=\frac{1}{\Delta}-\frac{1}{2\Delta}\left[\frac{(t+a)-a}{t+a}+\frac{(t+b)-b}{t+b}+\frac{(t+c)-c}{t+c}\right]$$

Integrating: $\left.\frac{t}{\Delta}\right|_0^{\infty}=\int_0^{\infty}\frac{1}{2\Delta}\left[-1+\frac{a}{(t+a)}+\frac{b}{(t+b)}+\frac{c}{(t+c)}\right]dt$ and $\left.\frac{t}{\Delta}\right|_0^{\infty}=0$

so $$R_F(a,b,c)=aR_D^{\otimes}(b,c\,;a)+bR_D^{\otimes}(c,a\,;b)+cR_D^{\otimes}(a,b\,;c) \qquad \text{(A4)}$$

Thus, if $c=0$, we have

$$R_F(a,b,0)=a\,R_D^{\otimes}(b,0\,;a)+b\,R_D^{\otimes}(0,a\,;b) \qquad \text{(A5)}$$

Differentiation

$$\frac{1}{2}\int_0^{\infty}\frac{\partial}{\partial a}\left(\frac{1}{\Delta}\right)dt=\frac{1}{2}\int_0^{\infty}\left(-\frac{1}{2}\frac{1}{(t+a)\Delta}\right)dt \quad \text{or} \quad \frac{\partial}{\partial a}R_F(b,c\,,a)=-\frac{1}{2}R_D^{\otimes}(b,c\,;a)$$

Then $\frac{\partial}{\partial a}R_D^{\otimes}(a,b\,;c\,)=\frac{1}{2}\int_0^{\infty}\frac{\partial}{\partial a}\left(\frac{1}{(t+c)\Delta}\right)dt=\frac{1}{2}\int_0^{\infty}-\frac{1}{2}\frac{dt}{(t+a)(t+c)\Delta}$

and by (A2) this is $-\frac{1}{2(c-a)}\left[R_D^{\otimes}(b,c\,;a)-R_D^{\otimes}(a,b\,;c)\right]$ (A6)

Thus $$\frac{\partial^2}{\partial a\partial c}R_F(a,b,c)=+\frac{1}{4(c-a)}\left[R_D^{\otimes}(b,c\,;a)-R_D^{\otimes}(a,b\,;c)\right] \qquad \text{(A7)}$$

For $\frac{\partial^2}{\partial a^2}R_F(a,b,c)$ we need to be more elaborate, for $\frac{\partial}{\partial a}R_D^{\otimes}(b,c\,;a\,)=\frac{1}{2}\int_0^{\infty}-\frac{1}{2}\frac{dt}{(t+a)^2\Delta}$

But differentiating (A3): $\frac{\partial R_D^{\otimes}(b,c\,;a)}{\partial a}+\frac{\partial R_D^{\otimes}(c,a\,;b)}{\partial a}+\frac{\partial R_D^{\otimes}(a,b\,;c)}{\partial a}=-\frac{1}{2}\frac{1}{\sqrt{a^3\,bc}}$

so $$\frac{\partial}{\partial a}R_D^{\otimes}(b,c;a)=-\left[\frac{1}{2\sqrt{a^3bc}}+\frac{\partial}{\partial a}R_D^{\otimes}(a,b;c)+\frac{\partial}{\partial a}R_D^{\otimes}(c,a;b)\right]$$

and all the terms on the RHS are known:

$$\frac{\partial}{\partial a}R_D^{\otimes}(b,c;a)=$$

$$=\frac{1}{2}\left[\frac{1}{c-a}[R_D^{\otimes}(b,c;a)-R_D^{\otimes}(a,b;c)]+\frac{1}{b-a}[R_D^{\otimes}(b,c;a)-R_D^{\otimes}(a,c;b)]-\frac{1}{\sqrt{a^3bc}}\right] \quad \text{(A8)}$$

Thus, we have inadvertently proved that

$$\frac{1}{2}\int_0^{\infty}\frac{dt}{\Delta(t+a)^2}\equiv\frac{1}{2}\int\frac{dt}{\sqrt{(t+b)(t+c)(t+a)}^{5/2}}=$$

$$=-\frac{2}{3}\left[\frac{1}{c-a}[R_D^{\otimes}(b,c;a)-R_D^{\otimes}(a,b;c)]+\frac{1}{b-a}[R_D^{\otimes}(b,c;a)-R_D^{\otimes}(a,c;b)]-\frac{1}{\sqrt{a^3bc}}\right] \quad \text{(A9)}$$

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