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RECURRENCE ANALYSIS OF FORCED SYNCHRONIZATION IN A SELF-EXCITED THERMOACOUSTIC SYSTEM

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We use recurrence analysis to investigate the forced synchronization of a self-excited thermoacoustic system. The system consists of a swirl-stabilized turbulent premixed flame in an open-ended duct. We apply periodic acoustic forcing to this system at different amplitudes and frequencies around its natural self-excited frequency, and examine its response via unsteady pressure measurements. On increasing the forcing amplitude, we observe two bifurcations: from a periodic limit cycle (unforced) to quasiperiodicity (weak forcing) and then to lock-in (strong forcing). To analyse these nonlinear dynamics, we use cross-recurrence plots (CRPs) of the unsteady pressure and acoustic forcing. We find that the different time scales characterizing the quasiperiodicity and the transition to lock-in appear as distinct structures in the CRPs. Using cross recurrence quantification analysis (CRQA), we examine those structures and find that their recurrence quantities change even before the system transitions to lock-in. This shows that CRPs and CRQA can be used as alternative nonlinear tools to study forced synchronization in thermoacoustic systems, complementing existing linear tools such as spectral analysis.

Keywords: Combustion instability, thermoacoustics, synchronization, recurrence analysis, nonlinear dynamics, bifurcations

1. Introduction

Thermoacoustic instability continues to be a challenging problem in the development of gas turbines [1]. The driving mechanism for instability is the in-phase coupling between the unsteady heat release rate (HRR) and the chamber acoustics [2]. To analyse this interaction, it is common practice to force the flame acoustically over a range of frequencies and to examine its HRR response at those frequencies [3]. The overall flame response is then derived as the sum of the HRR response at each of those frequencies [4]. However, studies have shown that such a linear superposition of the flame response is often inadequate to account for the energy transfer between different frequencies, e.g. between a forced mode and a self-excited mode [5, 6].

Balusamy *et al.* [7] adopted a nonlinear dynamical systems approach to investigate the forced synchronization of a self-excited thermoacoustic system over a range of frequencies, while forcing the system at one frequency at a time. Using spectral analysis and Poincaré maps, they found rich dynamical behaviour arising from the interaction between the external forcing and the self-excited mode, including (i) a torus-birth bifurcation from periodicity to two-frequency quasiperiodicity for weak forcing and (ii) lock-in of the self-excited mode to the forced mode for strong forcing.

In this paper, we examine those bifurcations and nonlinear dynamics using the cross-recurrence plot (CRP) [8, 9]. The CRP is a bivariate extension of the classical univariate recurrence plot (RP), which was introduced by Eckmann *et al.* [10] to visualize the time evolution of high-dimensional phase-space trajectories. In thermoacoustics, RPs have proven to be useful in characterizing the nonlinear dynamics of both laminar and turbulent systems. For example, their structural patterns have helped to identify limit cycles, quasiperiodicity, chaos and intermittency [11, 12]. Their recurrence quantities have been used to analyse the bifurcations between different system states [13]. However, owing to their univariate formulation, RPs are incapable of isolating the interdependencies between two interacting systems, making them unsuitable for studying synchronization, a process necessarily involving the interaction between two (or more) systems. Instead, we use CRPs because they are specifically designed to examine the nonlinear cross-correlations between two interacting systems [8, 9]. We also make use of cross recurrence quantification analysis (CRQA) to identify and quantify the dynamical changes occurring in the lead up to lock-in.

The rest of this paper is organised as follows. The experimental setup is described in Section 2. The background and application of CRPs and CRQA are explained in Section 3. The key results are presented and discussed in Section 4, and the conclusions are summarized in Section 5.

2. Experimental Setup

The experimental setup used in this study (Fig. 1) is identical to that of Balusamy *et al.* [7]. It has two main components: a burner and a combustor. The burner consists of a mixing plenum with two round concentric tubes (diameters of 15.05 and 27.75 mm) and a central shaft (diameter of 6.35 mm) that acts as an axisymmetric bluff body. A premixed supply of fuel (methane) and air is sent through the mixing plenum via a bypass valve and a siren. The siren is used to force the system with acoustic velocity perturbations (u'). Its rotational speed, which determines the forcing frequency (f_f), is controlled by a motor. The forcing amplitude ($A \equiv u'/u$) is controlled via the bypass valve. In the mixing plenum, two swirlers are mounted in each concentric tube to stabilize the flame. The combustor consists of a round fused-silica tube (diameter of 94 mm and length of 700 mm) mounted downstream of the burner exit. The unsteady pressure (p') is measured at 8192 Hz for 4 seconds using a condenser microphone (Model 40BP by GRAS) mounted 70 mm upstream of the burner exit (PT1 in Fig. 1). Further details on the experimental setup can be found in Balusamy *et al.* [7].

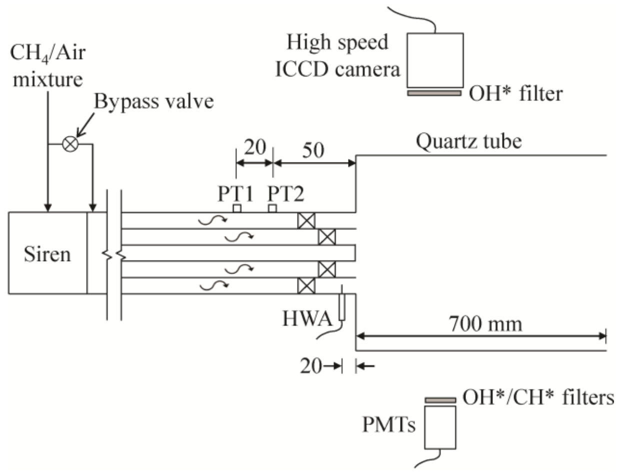

Figure 1: Diagram of the thermoacoustic system.

3. Recurrence Analysis

3.1 Cross-Recurrence Plots (CRPs)

Zbilut, Giuliani, and Webber [8] proposed the CRP as a tool to visualize cross-recurrence structures in bivariate data, improving on the classical RP that could only show auto-recurrence patterns in univariate data. We use a Matlab® toolbox (http://tocsy.agnld.uni-potsdam.de) to generate CRPs from two temporal signals: the unsteady pressure $p'(t)$ and the acoustic velocity forcing $u'(t)$. These signals are first normalised by their respective means (p and u) and used to generate phase-space trajectories (p_i and u_j) via Takens' time-delay embedding theorem [14]. The distance between those trajectories is then computed to form the cross-recurrence matrix ($CR_{i,j}$):

$$CR_{i,j} \equiv \Theta\left(\varepsilon - \left\|p_i - u_j\right\|\right), \qquad\qquad i,j = 1,2,\ldots N$$

where Θ is the Heaviside step function and ϵ is a threshold distance. A CRP displays a point at coordinates (i,j) if $CR_{i,j} = 1$ but displays nothing there if $CR_{i,j} = 0$. Unlike a RP, the main diagonal of a CRP is not necessarily populated by points because the trajectories of p_i and u_j represent two different dynamical processes (i.e. auto-recurrence is not guaranteed). Short diagonal lines in a CRP, called lines of synchronization (LOS), indicate that the corresponding trajectories run parallel to each other for the associated time interval. The length and frequency of these LOS are measures of the degree of synchronicity between the two temporal signals. These measures arise from nonlinear interrelations and cannot be determined using traditional linear cross-correlation methods [7,12].

When making CRPs, it is essential to use a suitable value of the threshold distance ϵ. If ϵ is too small, there may be only a few cross-recurring points, making it difficult to investigate the various regimes of synchronization. If ϵ is too large, the evolution of the two dynamical systems always appears to be synchronized. There are several criteria for selecting ϵ [15]. The one that we use is 5% of the recurrence density, often known as the method of a fixed number of nearest neighbours.

3.2 Cross-Recurrence Quantification Analysis (CRQA)

Cross-recurrence quantification analysis (CRQA) offers several measures to quantify the distribution of LOS in CRPs. Examples include determinism (DET), the average diagonal line length (ADL) and entropy (ENT); these are defined below. In CRQA, the main diagonal is treated as a reference for assessing the degree of synchronicity between two input signals. The statistical distribution of diagonal lines of length l that are parallel to the main diagonal is represented by $P(l)$.

Determinism is defined as the ratio of the recurring points that form diagonal lines to the total number of recurring points:

$$DET \equiv \frac{\sum_{l=l_{min}}^{N} lP(l)}{\sum_{l=1}^{N} lP(l)}$$

The average diagonal line length is related to the divergence rate of the phase-space trajectories of the two systems:

$$ADL \equiv \frac{\sum_{l=l_{min}}^{N} lP(l)}{\sum_{l=l_{min}}^{N} P(l)}$$

Entropy is calculated as the Shannon information entropy of the probability distribution of diagonal line lengths:

$$ENT \equiv -\sum_{l=l_{min}}^{N} p(l)\ln p(l)$$

where $p(l)$ is the probability ($p(l) = P(l)/N_l$) of the occurrence of a diagonal line with length l.

4. Results and Discussion

4.1 Overview of the Nonlinear Dynamics

We start by reviewing the nonlinear dynamics of the thermoacoustic system as reported in our previous study [7]. When unforced ($A = 0$), the system is naturally self-excited at an equivalence ratio of 0.8 and a Reynolds number of 8000. This can be seen in the pressure spectrum (Fig. 2a), which shows a sharp peak at $f_s = 195 \pm 3$ Hz, indicating a periodic limit cycle at the fundamental longitudinal mode of the combustor. When forced at $f_f = 300$ Hz, the system exhibits a series of bifurcations as the forcing amplitude increases, reaches a maximum, and then decreases in turn.

For weak forcing (Fig. 2b,c: $A = 0.080$ and 0.097), the system undergoes a torus-birth (Neimark-Sacker) bifurcation from the initial periodic limit cycle to quasiperiodic oscillations at two incommensurate frequencies: one due to the self-excited mode ($f_s = 209$ Hz) and one due to the forced mode ($f_f = 300$ Hz). The fact that the self-excited mode has shifted towards the forced mode and is now at a slightly higher frequency than it was at when unforced (209 Hz now vs 195 Hz before) is evidence of frequency pulling, which was also seen in our previous study [7] and is a characteristic nonlinear feature of forced self-excited oscillators [16].

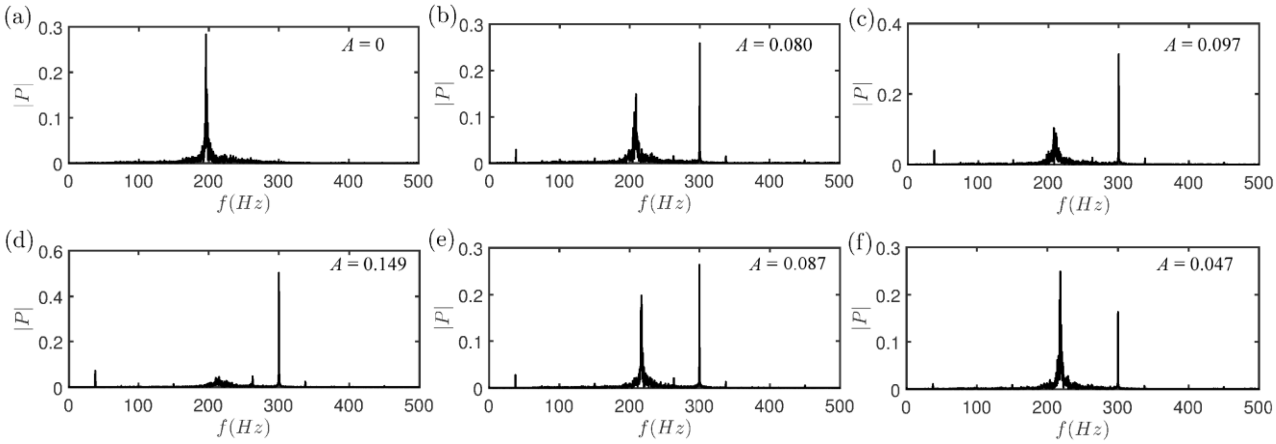

Figure 2: Spectra of the normalized unsteady pressure measured in a self-excited thermoacoustic system subjected to external acoustic forcing at a frequency of $f_f = 300$ Hz and at multiple amplitudes ($A \equiv u'/u$).

As the forcing amplitude increases (Fig. 2a → 2d: $A = 0 \rightarrow 0.149$), the forced and self-excited modes compete against each other, with the former eventually dominating the latter. This can be seen most clearly in Fig. 2d, where the forcing is at its maximum amplitude ($A = 0.149$) and has suppressed the self-excited mode. This state, known as lock-in (or phase locking in the synchronization literature [16]), is characterized by the system becoming completely synchronized in amplitude and phase with the forcing signal. The suppression of the self-excited mode is gradual (rather than abrupt) suggesting that the transition to lock-in occurs via a torus-death (inverse Neimark-Sacker) bifurcation, rather than a saddle-node bifurcation [16]. On the return path (Fig. 2d → 2f), as the forcing amplitude decreases from its maximum (Fig. 2d: $A = 0.149$) to its minimum (Fig. 2f: $A = 0.047$), the self-excited mode re-emerges to become dominant again.

All of these nonlinear dynamics were discussed in our previous study [7] using spectral analysis and Poincaré maps, and were identified as universal features of forced self-excited oscillators. Our aim now is to see whether further insight into these dynamics can be gained with CRPs and CRQA.

4.2 Cross-Recurrence Plots (CRPs)

Before presenting the results, it is perhaps helpful to review how CRPs are read. A visual inspection of a CRP can reveal many similarities but also important differences between two dynamical systems. Diagonal lines indicate that the phase-space trajectories of both systems evolve similarly on a time scale proportional to the line length. The vertical spacing between diagonal lines is a measure of the oscillation frequency. When two systems are locked into each other (i.e. synchronized), the main diagonal becomes continuous, making the CRP appear identical to a RP. Any temporal compression or dilatation of the parallel trajectories appears as distortion in the main diagonal.

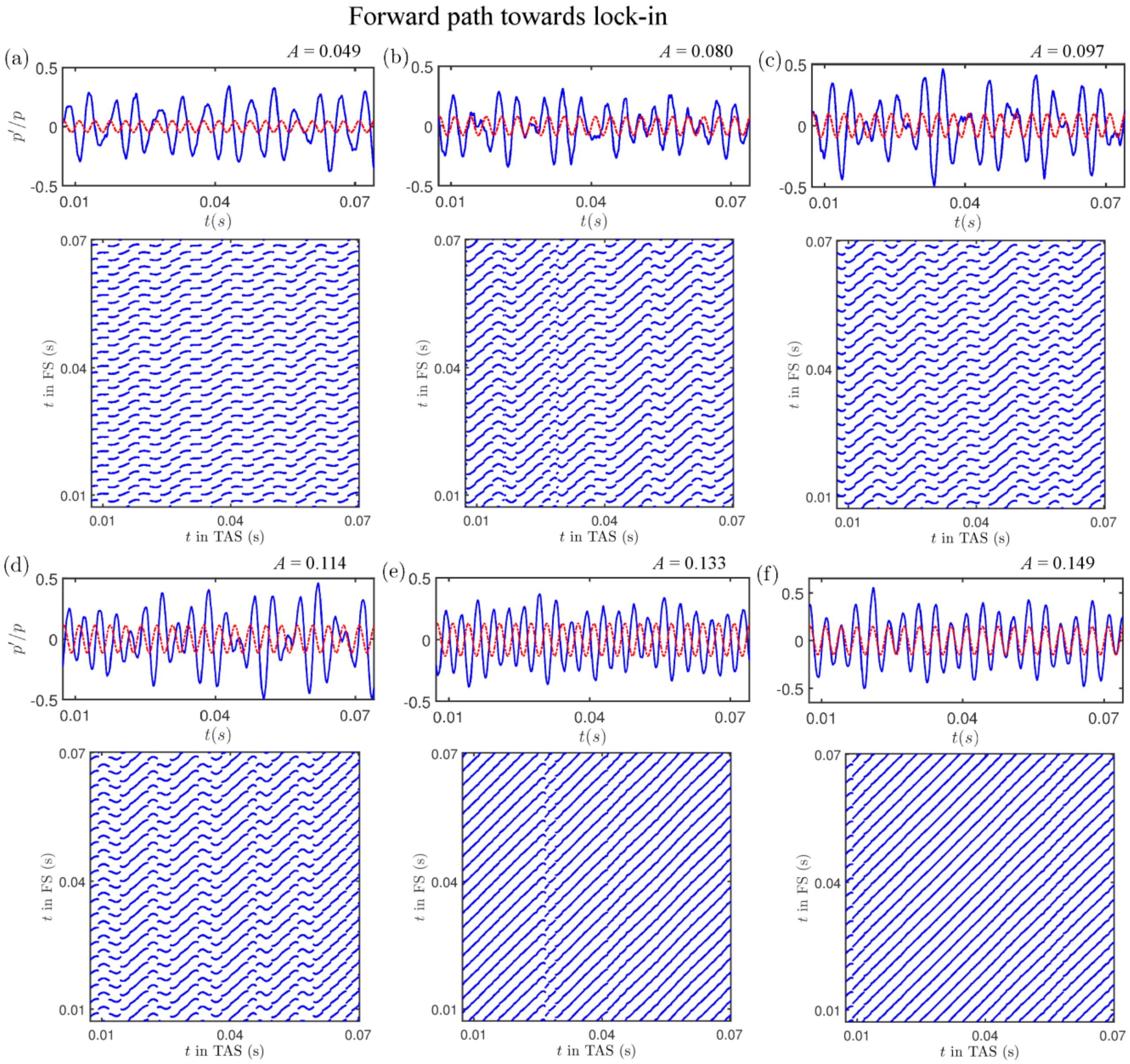

Figure 3: [Top frame] Time traces of the unsteady pressure (blue line) and external forcing (red line), and [Bottom frame] their corresponding CRPs for six different forcing amplitudes leading up to the onset of lock-in at $A = 0.149$. The forcing frequency is $f_f = 300$ Hz and the self-excited frequency is $f_s = 195 \pm 3$ Hz (when unforced). In the CRPs, the forcing signal (FS) is plotted on the vertical axis, while the thermo-acoustic signal (TAS) is plotted on the horizontal axis.

Figure 3 shows the CRPs of the system when it is forced at a frequency of $f_f = 300$ Hz and at six different amplitudes leading up to lock-in. The synchronization dynamics between the system and the forcing can be readily examined. For weak forcing ($A = 0.049$), a pattern of short broken curved lines appears amidst white space, indicating transient cross-correlations. This is characteristic of the initial stages of quasiperiodicity in which the system slips in and out of synchronicity as it oscillates. As the forcing amplitude increases (Fig. 3a → 3f), those lines of synchronization (LOS) become longer and straighter, indicating that the system is spending increasingly more time oscillating at the forcing frequency and that its phase-space trajectory is visiting increasingly more of the same regions as that of the forcing. For strong forcing ($A = 0.149$), the LOS are perfectly straight diagonal lines spaced equally from each other, which implies lock-in. On the return path (Fig. 4a

→ 4f), as the forcing amplitude decreases from its maximum (Fig. 4a: $A = 0.149$) to its minimum (Fig. 4f: $A = 0.047$), the long straight LOS give way to short curved lines (similar to those in Fig. 3a), indicating a reverse transition from lock-in to a partially synchronous state of quasiperiodicity.

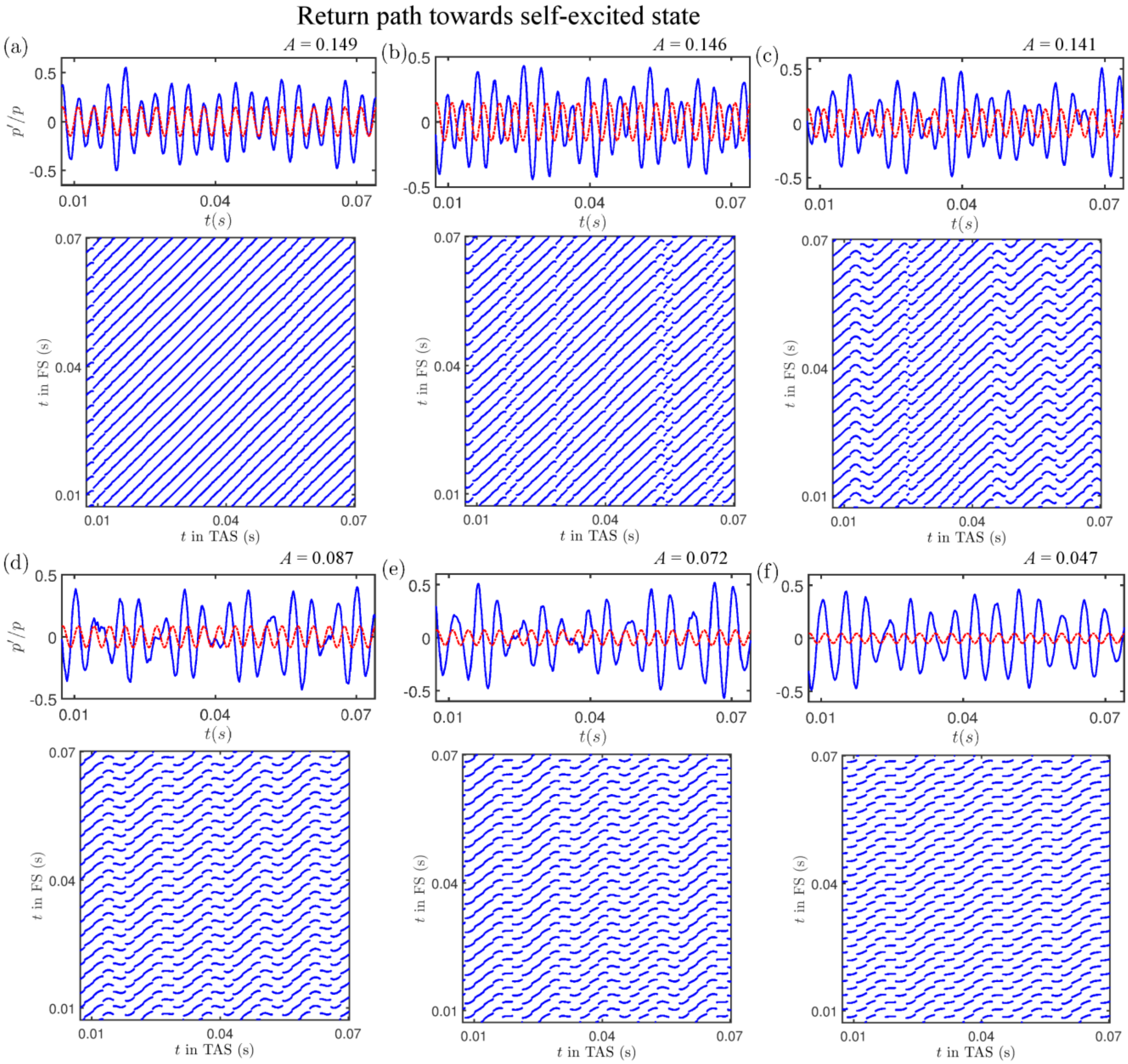

Figure 4: The same as for Fig. 3 but with decreasing forcing amplitudes.

4.3 Cross-Recurrence Quantification Analysis (CRQA)

In this section, the synchronization dynamics shown in Figs. 3 and 4 are quantified with three CRQA measures: determinism (DET), the average diagonal line length (ADL), and the Shannon entropy (ENT). These measures are found by computing the probability of occurrence of similar states in the thermoacoustic system and the external forcing. They are plotted in Fig. 5 for a range of forcing amplitudes, both in the forward direction (towards lock-in) and in the return direction (away from lock-in). As Figs. 3 and 4 showed, when the phase-space trajectories of the system and the forcing evolve similarly to each other (e.g. near or at lock-in), the diagonals in the CRPs appear longer and straighter, indicating increased determinism. This causes the value of DET to increase with the forcing amplitude (Fig. 5a). When the forcing amplitude decreases from its maximum, the value of DET retraces its forward path in the return direction, indicating a smooth transition from a fully synchronous state of lock-in to a partially synchronous state of quasiperiodicity.

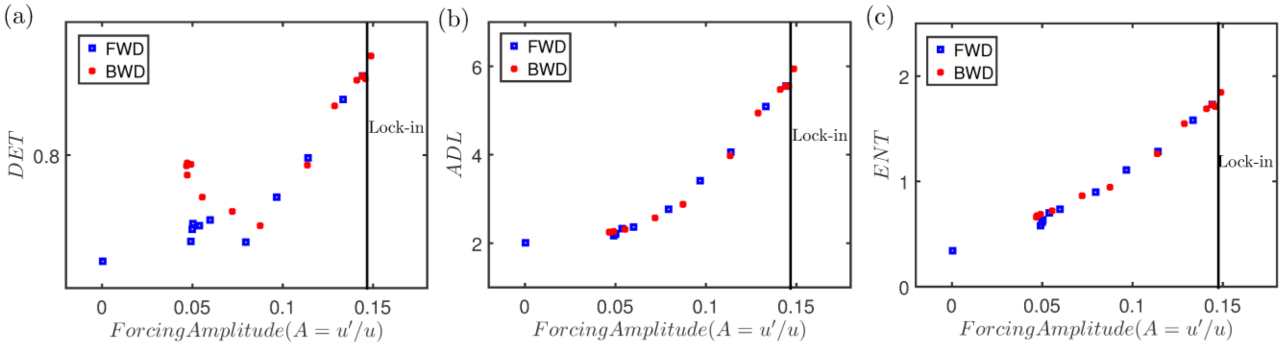

Figure 5: Variation of CRQA measures as the forcing amplitude (blue squares) increases towards lock-in and then (red circles) decreases away from lock-in: (a) determinism, (b) the average diagonal line length and (c) the Shannon entropy. The forcing frequency is $f_f = 300$ Hz and the self-excited frequency is $f_s = 195 \pm 3$ Hz (when unforced). The onset of lock-in occurs at $A = 0.149$.

If a diagonal line of length l appears in a CRP, it implies that the phase-space trajectories of the two input signals run parallel to each other during that time interval l. Thus, the average diagonal line length (ADL) quantifies the degree of parallelism in an interacting system. As the forcing amplitude increases in our thermoacoustic system, its phase-space trajectories evolve to become more parallel to those of the forcing, increasing ADL (Fig. 5b). Like DET, ADL varies smoothly as the forcing amplitude varies, increasing gradually towards lock-in and then retracing its forward path in the return direction as it moves away from lock-in. Figure 5c shows the Shannon entropy. Like the two other CRQA measures, this too increases and decreases smoothly as the system moves towards and away from lock-in. In summary, this demonstrates that CRPs and CRQA are robust nonlinear tools with which one can investigate forced synchronization in thermoacoustic systems.

5. Conclusions

Recent research has shown that an acoustically forced thermoacoustically self-excited system (a swirl-stabilized turbulent premixed flame in a duct driven by a siren) undergoes multiple bifurcations as the forcing amplitude increases, producing a wide range of nonlinear dynamics including quasiperiodicity and lock-in [7]. In this study, we have examined those dynamics using cross-recurrence plots (CRPs) of the unsteady pressure and acoustic forcing. Our findings show that the different time scales characterizing the quasiperiodicity and the transition to lock-in appear as distinct structures in the CRPs. Furthermore, the approach to lock-in can be identified as a gradual increase in the length and parallelism of the lines of synchronization (LOS). We quantified the structural features of these LOS using cross recurrence quantification analysis (CRQA) and found that various statistical measures of their cross-recurrence (e.g. determinism, the average diagonal line length, and the Shannon entropy) increase smoothly as the system approaches lock-in and then decrease along the same path as the system returns to its self-excited state. The fact that all of these cross-recurrence measures vary smoothly in the lead up to lock-in suggests that CRPs and CRQA are sufficiently sensitive to reveal even subtle changes in the degree of synchronicity. This study shows that CRPs and CRQA can be used to analyse forced synchronization in thermoacoustic systems, offering an alternative to existing linear methods such as spectral analysis.

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