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AXIOMS FOR MODELLING CUBICAL TYPE THEORY IN A TOPOS

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Abstract. The homotopical approach to intensional type theory views proofs of equality as paths. We explore what is required of an object $\mathtt{I}$ in a topos to give such a path-based model of type theory in which paths are just functions with domain $\mathtt{I}$. Cohen, Coquand, Huber and Mörtberg give such a model using a particular category of presheaves. We investigate the extent to which their model construction can be expressed in the internal type theory of any topos and identify a collection of quite weak axioms for this purpose. This clarifies the definition and properties of the notion of uniform Kan filling that lies at the heart of their constructive interpretation of Voevodsky's univalence axiom.

1. Introduction

Cubical type theory [CCHM18] provides a constructive justification of Voevodsky's univalence axiom, an axiom that has important consequences for the formalisation of mathematics within Martin-Löf type theory [Uni13]. Working informally in constructive set theory, Cohen *et al* [CCHM18] give a model of their type theory using the category $\hat{\mathcal{C}}$ of set-valued contravariant functors on a small category $\mathcal{C}$ that is the Lawvere theory for de Morgan algebra [BD75, Chapter XI]; see [Spi15]. The representable functor on the generic de Morgan algebra in $\mathcal{C}$ is used as an interval object $\mathtt{I}$ in $\hat{\mathcal{C}}$, with proofs of equality modelled by the corresponding notion of path, that is, by morphisms with domain $\mathtt{I}$. Cohen *et al* call the objects of $\hat{\mathcal{C}}$ *cubical sets*. They have a richer structure compared with previous, synonymous notions [BCH14, Hub15]. For one thing they allow path types to be modelled simply by exponentials $X^{\mathtt{I}}$, rather than by name abstractions [Pit13, Chapter 4]. More importantly, the de Morgan algebra operations endow the interval $\mathtt{I}$ with structure that considerably simplifies the definition and properties of the constructive notion of Kan filling that lies at the heart of [CCHM18]. In particular, the filling operation is obtained from a simple special case that *composes* a filling at one end of the interval to a filling at the other end. Coquand [Coq15] has suggested that this distinctive composition operation can be understood in terms of the properties of partial elements and their extension to total elements, within the internal higher-order logic of toposes [LS86]. In this paper we show that that is indeed the case and usefully so. In particular, the *uniformity* condition on composition operations [CCHM18, Definition 13], which allows one to avoid the non-constructive aspects of the classical notion

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of Kan filling [BC15], becomes automatic when the operations are formulated internally. Our approach has the usual benefit of axiomatics – helping to clarify exactly which properties of a topos are sufficient to carry out each of the various constructions used to model cubical type theory [CCHM18], clearing the way for simplifications, generalisations and new examples.

To accomplish this, we find it helpful to work not in the higher-order predicate logic of toposes, but in an extensional type theory equipped with an impredicative universe of propositions Ω, standing for the subobject classifier of the topos [Mai05]. Working in such a language, our axiomatisation concerns two structures that a topos $\mathcal{E}$ may possess: an object $\mathtt{I}$ that is endowed with some elementary characteristics of the unit interval; and a subobject of propositions $\mathtt{Cof} \rightarrowtail \Omega$ whose elements we call *cofibrant propositions* and which determine the subobjects that are relevant for a Kan-like notion of filling. (In the case of [CCHM18], $\mathtt{Cof}$ classifies subobjects generated by unions of faces of hypercubes.) Working internally with cofibrant propositions rather than externally with a class of cofibrant monomorphisms leads to an appealingly simple notion of *fibration* (Section 5), with that of Cohen *et al* as an instance when the topos is $\hat{\mathcal{C}}$. These fibrations are type-families equipped with extra structure (*composition* operations) which are supposed to model intensional Martin-Löf type theory, when organised as a Category with Families (CwF) [Dyb96], say. In order that they do so, we make a series of postulates about the interval and cofibrant subobjects that are true of the presheaf model in [CCHM18]. An overview of these postulates is given in Section 3, followed in subsequent sections by constructions in the CwF of fibrations that show it to be a model of Martin-Löf type theory with Σ, Π, data types (we just consider the natural numbers and disjoint unions) and intensional identity types. In section 6 we give an internal treatment of *glueing*, needed for constructions relating to univalence [Uni13]. Our approach to glueing is a bit different from that in [CCHM18] and enables us to isolate a strictness property of cofibrant propositions (axiom $\mathtt{ax}_9$ in Figure 1) independently of glueing, as well as separating out the use of the fact that cofibrant predicates are closed under universal quantification over the interval (axiom $\mathtt{ax}_8$ in Figure 1). In section 7 we give a result about univalence that is provable from our axioms. However, for reasons discussed at the end of that section, one cannot give an internal account of the univalent universe construction from [CCHM18]. (This problem is circumvented in [LOPS18] by extending the internal type theory with a suitable modality.)

In Section 8 we indicate why the model in [CCHM18] satisfies our axioms and more generally which other presheaf toposes satisfy them. There is some freedom in choosing the subobject of cofibrant propositions; and the connection algebra structure we assume for the interval $\mathtt{I}$ (axioms $\mathtt{ax}_3$ and $\mathtt{ax}_4$ in Figure 1) is weaker than being a de Morgan algebra, since we can avoid the use of a de Morgan involution operation. In Section 9 we conclude by considering other related work.

Agda formalisation. The definitions and constructions we carry out in the internal type theory of toposes are sufficiently involved to warrant machine-assisted formalisation. Our tool of choice is Agda [Agd]. We persuaded it to provide an impredicative universe of mere propositions [Uni13, Section 3.3] using a method due to Escardo [Esc15]. This gives an intensional, proof-relevant version of the subobject classifier Ω and of the type theory described in Section 2. To this we add postulates corresponding to the axioms in Figure 1. We also made modest use of the facility for user-defined rewriting in recent versions of Agda [CA16], in order to make the connection algebra axioms $\mathtt{ax}_3$ and $\mathtt{ax}_4$ definitional, rather than just propositional equalities, thereby eliminating a few proofs in favour of computation.

Using Agda required us to construct and pass around proof terms that are left implicit in the paper version; we found this to be quite bearable and also invaluable for getting the details right. Our development can be found at `https://doi.org/10.17863/CAM.21675`.

2. Internal Type Theory of a Topos

We rely on the categorical semantics of dependent type theory in terms of *categories with families* (CwF) [Dyb96]. For each topos $\mathcal{E}$ (with subobject classifier $\top : 1 \to \Omega$) one can find a CwF with the same objects, such that the category of families at each object X is equivalent to the slice category $\mathcal{E}/X$. This can be done in a number of different ways; for example [Pit00, Example 6.14], or the more recent references [KL16, Section 1.3], [LW15] and [Awo16], which cater for categories more general than a topos (and for contextual/comprehension categories rather than CwFs in the first two cases). Using the objects, families and elements of this CwF as a signature, we get an internal type theory along the lines of those discussed in [Mai05], canonically interpreted in the CwF in the standard fashion [Hof97]. We make definitions and postulates in this internal language for $\mathcal{E}$ using a concrete syntax inspired by Agda [Agd]. Dependent function types are written as $(x : A) \to B$; their canonical terms are function abstractions, written as $\lambda(x : A) \to t$. Dependent product types are written as $(x : A) \times B$; their canonical terms are pairs, written as (s, t). In the text we use this language informally, similar to the way that Homotopy Type Theory is presented in [Uni13]. For example, the typing contexts of the judgements in the formal version, such as $[x_0 : A_0, x_1 : A_1(x_0), x_2 : A_2(x_0, x_1)]$, become part of the running text in phrases like "given $x_0 : A_0$, $x_1 : A_1(x_0)$ and $x_2 : A_2(x_0, x_1)$, then…"

In the internal type theory the subobject classifier Ω of the topos becomes an impredicative universe of propositions, with logical connectives $(\top, \bot, \neg, \wedge, \vee, \Rightarrow)$, quantifiers $(\forall(x : A), \exists(x : A))$ and equality $(=)$ satisfying function and proposition extensionality properties. The universal property of the subobject classifier gives rise to comprehension subtypes: given $\Gamma, x : A \vdash \varphi(x) : \Omega$, then $\Gamma \vdash \{x : A \mid \varphi(x)\}$ is a type whose terms are those $t : A$ for which $\varphi(t)$ is provable, with the proof being treated irrelevantly.[1] Taking $A = 1$ to be terminal, for each $\varphi : \Omega$ we have a type whose inhabitation corresponds to provability of φ:

$$[\varphi] \triangleq \{\_ : 1 \mid \varphi\} \tag{2.1}$$

We will make extensive use of these types in connection with the partial elements of a type; see Section 5.1.

Instead of quantifying externally over the objects, families and elements of the CwF associated with $\mathcal{E}$, we will assume $\mathcal{E}$ comes with an internal full subtopos $\mathcal{U}$. In the internal language we use $\mathcal{U}$ as a Russell-style universe (that is, if $A : \mathcal{U}$, then A itself denotes a type) containing Ω and closed under forming products, exponentials and comprehension subtypes.

3. The axioms

In this section we present the axioms that we require to hold in the internal type theory of a topos $\mathcal{E}$. We provide an overview of each axiom, giving some intuition as to its purpose and we explain where it is used in the construction of a model of cubical type theory. This allows us to see that certain axioms are only required for modelling specific parts of cubical type

[1] Our Agda development is proof relevant, so that terms of comprehension types contain a proof of membership as a component.

The interval $\mathtt{I} : \mathcal{U}$ is connected

$$\mathtt{ax}_1 : [\forall(\varphi : \mathtt{I} \to \Omega).\ (\forall(i : \mathtt{I}).\ \varphi\, i \vee \neg\varphi\, i) \Rightarrow (\forall(i : \mathtt{I}).\ \varphi\, i) \vee (\forall(i : \mathtt{I}).\ \neg\varphi\, i)]$$

has distinct end-points $\mathtt{0}, \mathtt{1} : \mathtt{I}$

$$\mathtt{ax}_2 : [\neg(\mathtt{0} = \mathtt{1})]$$

and a connection algebra structure $\_ \sqcap \_, \_ \sqcup \_ : \mathtt{I} \to \mathtt{I} \to \mathtt{I}$

$$\mathtt{ax}_3 : [\forall(i : \mathtt{I}).\ \mathtt{0} \sqcap x = \mathtt{0} = x \sqcap \mathtt{0}\ \wedge\ \mathtt{1} \sqcap x = x = x \sqcap \mathtt{1}]$$

$$\mathtt{ax}_4 : [\forall(i : \mathtt{I}).\ \mathtt{0} \sqcup x = x = x \sqcup \mathtt{0}\ \wedge\ \mathtt{1} \sqcup x = \mathtt{1} = x \sqcup \mathtt{1}].$$

Cofibrant propositions $\mathtt{Cof} = \{\varphi : \Omega \mid \mathtt{cof}\,\varphi\}$ (where $\mathtt{cof} : \Omega \to \Omega$)
include end-point-equality

$$\mathtt{ax}_5 : [\forall(i : \mathtt{I}).\ \mathtt{cof}(i = \mathtt{0})\ \wedge\ \mathtt{cof}(i = \mathtt{1})]$$

and are closed under binary disjunction

$$\mathtt{ax}_6 : [\forall(\varphi\ \psi : \Omega).\ \mathtt{cof}\,\varphi \Rightarrow \mathtt{cof}\,\psi \Rightarrow \mathtt{cof}(\varphi \vee \psi)]$$

dependent conjunction

$$\mathtt{ax}_7 : [\forall(\varphi\ \psi : \Omega).\ \mathtt{cof}\,\varphi \Rightarrow (\varphi \Rightarrow \mathtt{cof}\,\psi) \Rightarrow \mathtt{cof}(\varphi \wedge \psi)]$$

and universal quantification over $\mathtt{I}$

$$\mathtt{ax}_8 : [\forall(\varphi : \mathtt{I} \to \Omega).\ (\forall(i : \mathtt{I}).\ \mathtt{cof}(\varphi\ i)) \Rightarrow \mathtt{cof}(\forall(i : \mathtt{I}).\ \varphi\ i)].$$

Strictness axiom: any cofibrant-partial type A
that is isomorphic to a total type B everywhere that A is defined,
can be extended to a total type B' that is isomorphic to B:

$$\begin{aligned}\mathtt{ax}_9 : \{\varphi : \mathtt{Cof}\}(A : [\varphi] \to \mathcal{U})(B : \mathcal{U})(s : (u : [\varphi]) \to (A\,u \cong B)) \to \\ (B' : \mathcal{U}) \times \{s' : B' \cong B \mid \forall(u : [\varphi]).\ A\,u = B' \wedge s\,u = s'\}\end{aligned}$$

Figure 1: The axioms

theory, for example definitional identity types (Section 5.3). These axioms can therefore be dropped when, for example, looking for models of cubical type theory with only propositional identity types (Section 4). For ease of reference the axioms are collected together in Figure 1, written in the language described in Section 2.

Notation 3.1 (**Infix and implicit arguments**)**.** In the figure and elsewhere we adopt a couple of useful notational conventions from Agda [Agd]. First, function arguments that are written with infix notation are indicated by the placeholder notation "_"; for example $\_ \sqcap \_ : \mathtt{I} \to \mathtt{I} \to \mathtt{I}$ applied to $i, j : \mathtt{I}$ is written $i \sqcap j$. Secondly, we use the convention that braces $\{\}$ indicate implicit arguments; for example, the application of $\mathtt{ax}_9$ in Figure 1 to $\varphi : \mathtt{Cof}$, $A : [\varphi] \to \mathcal{U}$, $B : \mathcal{U}$ and $s : (u : [\varphi]) \to (A\,u \cong B)$ is written $\mathtt{ax}_9\,A\,B\,s$, or $\mathtt{ax}_9\,\{\varphi\}\,A\,B\,s$ if φ cannot be deduced from the context.

The homotopical approach to type theory [Uni13] views elements of identity types as paths between the two elements being equated. We take this literally, using paths in the topos $\mathcal{E}$ that are morphisms out of a distinguished object $\mathtt{I}$, called the *interval*. Recall from Section 2 that we assume the given topos $\mathcal{E}$ comes with a Russell-style universe $\mathcal{U}$. We assume that the interval $\mathtt{I}$ is an element of $\mathcal{U}$. We also assume that $\mathtt{I}$ is equipped with morphisms $\mathtt{0}, \mathtt{1} : 1 \to \mathtt{I}$ and $\_ \sqcap \_, \_ \sqcup \_ : \mathtt{I} \to \mathtt{I} \to \mathtt{I}$ satisfying axioms $\mathtt{ax}_1$–$\mathtt{ax}_4$ in Figure 1. The other axioms ($\mathtt{ax}_5$–$\mathtt{ax}_9$) concern *cofibrant propositions*, which are used in Section 5 to define fibrations, the (indexed families of) types in the model of cubical type theory.

Axiom $\mathtt{ax}_1$ expresses that the interval $\mathtt{I}$ is internally connected, in the sense that any decidable subset of its elements is either empty or the whole of $\mathtt{I}$. This implies that if a path in an inductive datatype has a certain constructor form at one point of the path, it has the same form at any other point. This is used at the end of Section 5.2 to show that the natural number object in the topos is fibrant (that is, denotes a type) and that fibrations are closed under binary coproducts. It also gets used in proving properties of the glueing construct in Section 6. Together with axiom $\mathtt{ax}_2$, connectedness of $\mathtt{I}$ implies that there is no path from $\mathtt{inl} *$ to $\mathtt{inr} *$ in $1 + 1$ and hence that the path-based model of Martin-Löf type theory determined by the axioms is logically non-degenerate.

Axioms $\mathtt{ax}_3$ and $\mathtt{ax}_4$ endow $\mathtt{I}$ with a form of *connection* algebra structure [BM99]. They capture some very simple properties of the minimum and maximum operations on the unit interval $[0, 1]$ of real numbers that suffice to ensure contractibility of singleton types (Section 4) and, in combination with $\mathtt{ax}_2$, $\mathtt{ax}_5$ and $\mathtt{ax}_6$, to define path lifting from composition for fibrations (see Section 5.2). In the model of [CCHM18] the connection algebra structure is given by the lattice structure of the interval, taking $\_ \sqcap \_$ to be binary meet, $\_ \sqcup \_$ to be binary join and using the fact that $\mathtt{0}$ and $\mathtt{1}$ are respectively least and greatest elements.

Remark 3.2 (De Morgan involution). In the model of [CCHM18] $\mathtt{I}$ is not just a lattice, but also has an involution operation $1 - (\_) : \mathtt{I} \to \mathtt{I}$ (so that $(1 - (1 - i) = i)$ making $\sqcup$ the de Morgan dual of $\sqcap$, in the sense that $i \sqcup j = 1 - ((1 - i) \sqcap (1 - j))$. Although this involution structure is convenient, it is not really necessary for the constructions that follow. Instead we just give a $\mathtt{0}$-version and a $\mathtt{1}$-version of certain concepts; for example, "composing from $\mathtt{1}$ to $\mathtt{0}$" as well as "composing from $\mathtt{0}$ to $\mathtt{1}$" in Section 5.2.

Axioms $\mathtt{ax}_2$–$\mathtt{ax}_6$ allow us to show that fibrations provide a model of Π- and Σ-types; and furthermore to show that the path types determined by the interval object $\mathtt{I}$ (Section 4) satisfy the rules for identity types propositionally [CD13, van16]. Axiom $\mathtt{ax}_7$ is used to get from these propositional identity types to the proper, definitional identity types of Martin-Löf type theory, via a version of Swan's construction [Swa16]; see Section 5.3.

In Section 7 we consider univalence [Uni13, Section 2.10] – the correspondence between type-valued paths in a universe and functions that are equivalences modulo path-based equality. To do so, we first give in Section 6 a non-strict, "up-to-isomorphism" version of the *glueing* construct of Cohen *et al* in the internal type theory of the topos. Axiom $\mathtt{ax}_8$ is used in the definition of this weak form of glueing to ensure that the induced fibration structure extends the fibration structure on the family that we are "glueing". Axiom $\mathtt{ax}_9$ allows us to regain the strict form of glueing used by Cohen *et al* [CCHM18]. Its validity in presheaf models depends on a construction in the external meta-theory that cannot be replicated internally; see Theorem 8.4 for details.

4. Path Types

Given $A : \mathcal{U}$, we call elements of type $\mathtt{I} \to A$ *paths* in A. The *path type* associated with A is $\_ \sim \_ : A \to A \to \mathcal{U}$ where

$$a_0 \sim a_1 \triangleq \{p : \mathtt{I} \to A \mid p\,\mathtt{0} = a_0 \wedge p\,\mathtt{1} = a_1\} \tag{4.1}$$

Can these types be used to model the rules for Martin-Löf identity types? We can certainly interpret the identity introduction rule (reflexivity), since degenerate paths given by constant functions

$$\mathtt{k}\,a\,i \triangleq a \tag{4.2}$$

satisfy $\mathtt{k} : \{A : \mathcal{U}\}(a : A) \to a \sim a$. However, we need further assumptions to interpret the identity elimination rule, otherwise known as path induction [Uni13, Section 1.12.1]. Coquand has given an alternative (propositionally equivalent) formulation of identity elimination in terms of substitution functions $a_0 \sim a_1 \to P\,a_0 \to P\,a_1$ and contractibility of singleton types $(a_1 : A) \times (a_0 \sim a_1)$; see [BCH14, Figure 2]. The connection algebra structure gives the latter, since using $\mathtt{ax_3}$ and $\mathtt{ax_4}$ we have

$$\begin{aligned} &\mathtt{ctr} : \{A : \mathcal{U}\}\{a_0\ a_1 : A\}(p : a_0 \sim a_1) \to (a_0, \mathtt{k}\,a_0) \sim (a_1, p) \\ &\mathtt{ctr}\,p\,i \triangleq (p\,i, \lambda j \to p(i \sqcap j)) \end{aligned} \tag{4.3}$$

However, to get suitably behaved substitution functions we have to consider families of types endowed with some extra structure; and that structure has to lift through the type-forming operations (products, functions, identity types, etc). This is what the definitions in the next section achieve.

5. Cohen-Coquand-Huber-Mörtberg (CCHM) Fibrations

In this section we show how to generalise the notion of fibration introduced in [CCHM18, Definition 13] from the particular presheaf model considered there to any topos with an interval object as in the previous sections. To do so we use the notion of *cofibrant proposition* from Figure 1 to internalise the composition and filling operations described in [CCHM18].

5.1. **Cofibrant propositions.** Kan filling and other cofibrancy conditions on collections of subspaces have to do with extending maps defined on a subspace to maps defined on the whole space. Here we take "subspaces of spaces" to mean subobjects of objects in toposes. Since subobjects are classified by morphisms to Ω, it is possible to consider collections of subobjects that are specified generically by certain propositions. More specifically, given a property of propositions, $\mathtt{cof} : \Omega \to \Omega$, we get a corresponding collection of propositions

$$\mathtt{Cof} \triangleq \{\varphi : \Omega \mid \mathtt{cof}\,\varphi\} \tag{5.1}$$

Consider the class of monomorphisms $m : A \rightarrowtail B$ whose classifying morphism

$$\lambda(y : B) \to \exists(x : A).\ m\,x = y : B \to \Omega$$

factors through $\mathtt{Cof} \rightarrowtail \Omega$. We call such monomorphisms *cofibrations*. Kan-like filling properties have to do with when a morphism $A \to X$ can be extended along a cofibration $m : A \rightarrowtail B$. Instead, working in the internal language of $\mathcal{E}$, we will consider when partial elements whose domains of definition are in $\mathtt{Cof}$ can be extended to totally defined elements. Recall that in intuitionistic logic, partial elements of a type A are often represented by sub-singletons, that is, by functions $s : A \to \Omega$ satisfying

$$\forall(x\ x' : A).\ s\,x \wedge s\,x' \Rightarrow x = x'$$

However, it will be more convenient to work with an extensionally equivalent representation as dependent pairs $\varphi : \Omega$ and $f : [\varphi] \to A$, as in the next definition. The proposition φ is the extent of the partial element; in terms of sub-singletons it is equal to $\exists(x : A).\ s\,x$.

Definition 5.1 (**Cofibrant partial elements, $\Box A$**)**.** We assume we are given a subobject $\mathtt{Cof} \rightarrowtail \Omega$ satisfying axioms $\mathtt{ax}_5$–$\mathtt{ax}_8$ in Figure 1. We call elements of type $\mathtt{Cof}$ *cofibrant propositions*. Given a type $A : \mathcal{U}$, we define the type of *cofibrant partial elements* of A to be

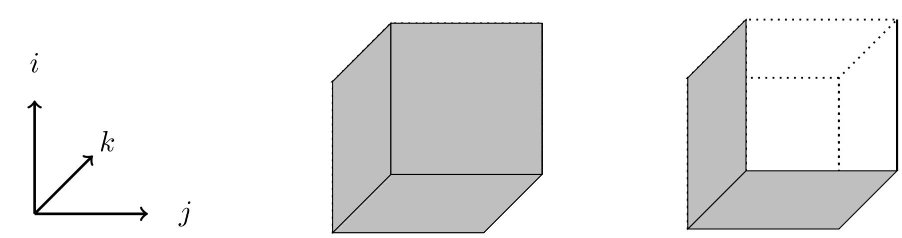
$$\Box A \triangleq (\varphi : \mathtt{Cof}) \times ([\varphi] \to A) \tag{5.2}$$

An *extension* of such a partial element $(\varphi, f) : \Box A$ is an element $a : A$ together with a proof of the following relation:

$$(\varphi, f) \nearrow a \triangleq \forall(u : [\varphi]).\ f\,u = a \tag{5.3}$$

Note that by taking $i = \mathtt{0}$ in axiom $\mathtt{ax}_5$ we have $\mathtt{cof}(\mathtt{0} = \mathtt{0})$ (that is, $\mathtt{cof}\,\top$) and $\mathtt{cof}(\mathtt{0} = \mathtt{1})$; and combining the latter with axiom $\mathtt{ax}_2$ we deduce also that $\mathtt{cof}\,\bot$ holds. So $A \rightarrowtail A$ and $\emptyset \rightarrowtail A$ are always cofibrations, where $\emptyset$ is the initial object. Since $\mathtt{cof}\,\top$ holds, for every $a : A$ there is a *total* cofibrant partial element $(\top, \lambda\_ \to a) : \Box A$ with a the unique element that extends $(\top, \lambda\_ \to a)$. Since $\mathtt{cof}\,\bot$ holds, every object A has an empty cofibrant partial element given by $(\bot, \mathtt{elim}_\emptyset) : \Box A$ such that every $a : A$ is an extension of $(\bot, \mathtt{elim}_\emptyset)$. (For any $B : \mathcal{U}$, $\mathtt{elim}_\emptyset : [\bot] \to B$ denotes the unique function given by initiality of $[\bot]$.)

Example 5.2. It is helpful to think of variables of type $\mathtt{I}$ as names of dimensions in space, so that working in a context $i_1, ..., i_n : \mathtt{I}$ corresponds to working in n dimensions. Assume that we are working in a context with $i, j, k : \mathtt{I}$; this therefore corresponds to working in three dimensions. We think of an element $i, j, k : \mathtt{I} \vdash a : A$ as a cube in the space A, as shown below. Let $\varphi \triangleq (i = \mathtt{0}) \vee (j = \mathtt{0}) \vee (j = \mathtt{1} \wedge k = \mathtt{1})$. From $\mathtt{ax}_5$–$\mathtt{ax}_7$ we have $i, j, k : \mathtt{I} \vdash \varphi : \mathtt{Cof}$. We think of φ as specifying certain faces and edges of a cube, in this case the bottom face ($i = \mathtt{0}$), the left face ($j = \mathtt{0}$) and the front-right edge ($j = \mathtt{1} \wedge k = \mathtt{1}$), as in the right-hand picture below. Then a cofibrant partial element $f : [\varphi] \to A$ can be thought of as a partial cube, only defined on the region specified by φ.



$i, j, k : \mathtt{I}$ $\qquad$ $a : A$ $\qquad$ $f : [\varphi] \to A$

Definition 5.3 (**Join of compatible partial elements**)**.** Say that two partial elements $f : [\varphi] \to A$ and $g : [\psi] \to A$ are *compatible* if they agree wherever they are both defined:

$$(\varphi, f) \smile (\psi, g) \triangleq \forall(u : [\varphi])(v : [\psi]).\ f\,u = g\,v \tag{5.4}$$

In that case we can form their join $f \cup g : [\varphi \vee \psi] \to A$, such that

$$\forall(u : [\varphi]).\ (f \cup g)\,u = f\,u \qquad\qquad \forall(v : [\psi]).\ (f \cup g)\,v = g\,v$$

To see why, consider the following pushout square in the topos:

$$\begin{tikzcd}
{[\varphi \wedge \psi]} \arrow[r, tail] \arrow[d, tail] & {[\psi]} \arrow[d, tail] \arrow[ddr, bend left, "g"] & \\
{[\varphi]} \arrow[r, tail] \arrow[drr, bend right, "f"'] & {[\varphi \vee \psi]} \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dotted, "f \cup g"] & \\
& & A
\end{tikzcd}$$

The outer square commutes because $(\varphi, f) \smile (\psi, g)$ holds and then $f \cup g$ is the unique induced morphism out of the pushout. Note that axiom $\mathtt{ax}_6$ in Figure 1 implies that the collection of cofibrant partial elements is closed under taking binary joins of compatible partial elements.

The following lemma gives an alternative characterization of axioms $\mathtt{ax}_7$ and $\mathtt{ax}_8$. Since we noted above that $\mathtt{cof}\,\top$ holds, part (i) of the lemma tells us that cofibrations form a *dominance* in the sense of synthetic domain theory [Ros86]; we only use this property of $\mathtt{Cof}$ in order to construct definitional identity types from propositional identity types (see Section 5.3).

Lemma 5.4. *(i) Axiom $\mathtt{ax}_7$ is equivalent to requiring the class of cofibrations to be closed under composition.*
(ii) Axiom $\mathtt{ax}_8$ is equivalent to requiring the class of cofibrations to be closed under exponentiation by $\mathtt{I}$.

Proof. For part (i), first suppose that $\mathtt{ax}_7$ holds and that $f : A \rightarrowtail B$ and $g : B \rightarrowtail C$ are cofibrations. So both $\forall(b : B).\ \mathtt{cof}(\exists(a : A).\ f\,a = b)$ and $\forall(c : C).\ \mathtt{cof}(\exists(b : B).\ g\,b = c)$ hold and we wish to prove $\forall(c : C).\ \mathtt{cof}(\exists(a : A).\ g(f\,a) = c)$. Note that for $b : B$ and $c : C$

$$\begin{aligned}
g\,b = c \;&\Rightarrow\; (\exists(a : A).\ g(f\,a) = c) = (\exists(a : A).\ f\,a = b) \qquad \text{since } g \text{ is a monomorphism}\\
&\Rightarrow\; \mathtt{cof}(\exists(a : A).\ g(f\,a) = c) = \mathtt{cof}(\exists(a : A).\ f\,a = b) = \top
\end{aligned}$$

So for $\varphi \triangleq \exists(b : B).\ g\,b = c$ and $\psi \triangleq \exists(a : A).\ g(f\,a) = c$, we have $\mathtt{cof}\,\varphi$ and $\varphi \Rightarrow \mathtt{cof}\,\psi$. Therefore by $\mathtt{ax}_7$ we get $\mathtt{cof}(\phi \wedge \psi)$, which is equal to $\mathtt{cof}(\psi)$ since $\psi \Rightarrow \varphi$. So we do indeed have $\forall(c : C).\ \mathtt{cof}(\exists(a : A).\ g(f\,a) = c)$.

Conversely, suppose cofibrations are closed under composition and that $\varphi, \psi : \Omega$ satisfy $\mathtt{cof}\,\varphi$ and $\varphi \Rightarrow \mathtt{cof}\,\psi$. That $\mathtt{cof}\,\varphi$ holds is equivalent to the monomorphism $[\varphi] \rightarrowtail 1$ being a cofibration; and since

$$\varphi \;\Rightarrow\; (\psi = \varphi \wedge \psi) \;\Rightarrow\; (\mathtt{cof}\,\psi = \mathtt{cof}(\varphi \wedge \psi))$$

from $\varphi \Rightarrow \mathtt{cof}\,\psi$ we get $\varphi \Rightarrow \mathtt{cof}(\varphi \wedge \psi)$ and hence the monomorphism $[\varphi \wedge \psi] \rightarrowtail [\varphi]$ is a cofibration. Composing these monomorphisms, we have that $[\varphi \wedge \psi] \rightarrowtail 1$ is a cofibration, that is, $\mathtt{cof}(\varphi \wedge \psi)$ holds.

For part (ii), first suppose that $\mathtt{ax}_8$ holds and that $f : A \rightarrowtail B$ is a cofibration. We have to show that $\mathtt{I} \to f : (\mathtt{I} \to A) \rightarrowtail (\mathtt{I} \to B)$ is also a cofibration. Given $\beta : \mathtt{I} \to B$ we have

$$\begin{aligned}
&(\forall i : \mathtt{I})(\exists a : A).\ f\,a = \beta\,i\\
&\Rightarrow (\forall i : \mathtt{I})(\exists! a : A).\ f\,a = \beta\,i && \text{(since } f \text{ is a monomorphism)}\\
&\Rightarrow (\exists \alpha : \mathtt{I} \to A)(\forall i : \mathtt{I}).\ f(\alpha\,i) = \beta\,i && \text{(by unique choice in the topos)}\\
&\Rightarrow (\forall i : \mathtt{I})(\exists a : A).\ f\,a = \beta\,i
\end{aligned}$$

so that $(\forall i:\mathtt{I})(\exists a:A).\ f\,a=\beta\,i$ is equal to $(\exists\alpha:\mathtt{I}\to A)(\forall i:\mathtt{I}).\ f(\alpha\,i)=\beta\,i$; and the latter is equal to $(\exists\alpha:\mathtt{I}\to A).\ (\mathtt{I}\to f)\,\alpha=\beta$ by function extensionality in the topos. Since f is a cofibration, for each $i:\mathtt{I}$ we have $\mathtt{cof}(\exists(a:A).\ f\,a=\beta\,i)$. Hence by axiom $\mathtt{ax}_8$ we also have $\mathtt{cof}((\forall i:\mathtt{I})(\exists a:A).\ f\,a=\beta\,i)$, that is, $\mathtt{cof}(\exists\alpha:\mathtt{I}\to A).\ (\mathtt{I}\to f)\,\alpha=\beta)$, as required for $\mathtt{I}\to f$ to be a cofibration.

Conversely, suppose cofibrations are closed under $\mathtt{I}\to(\_)$ and that $\varphi:\mathtt{I}\to\Omega$ satisfies $(\forall i:\mathtt{I}).\ \mathtt{cof}(\varphi\,i)$. The latter implies that $\{i:\mathtt{I}\mid\varphi\,i\}\rightarrowtail\mathtt{I}$ is a cofibration. Hence so is the monomorphism $(\mathtt{I}\to\{i:\mathtt{I}\mid\varphi\,i\})\rightarrowtail(\mathtt{I}\to\mathtt{I})$. Since $id:\mathtt{I}\to\mathtt{I}$ is in the image of this monomorphism iff $(\forall i:\mathtt{I}).\ \varphi\,i$ holds, we have $\mathtt{cof}((\forall i:\mathtt{I}).\ \varphi\,i)$, as required for axiom $\mathtt{ax}_8$. □

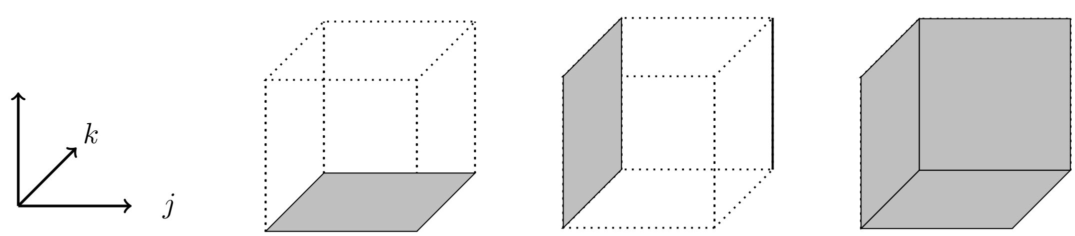
5.2. **Composition and filling structures.** Axioms $\mathtt{ax}_5$–$\mathtt{ax}_7$ in Figure 1 give the simple properties of cofibrant propositions we use to define an internal notion of fibration generalising Definition 13 of [CCHM18] and to show that it is closed under forming Σ-, Π- and $\mathtt{Id}$-types, as well as basic datatypes.

Given an interval-indexed family of types $A:\mathtt{I}\to\mathcal{U}$, we think of elements of the dependent function type $\Pi_{\mathtt{I}}A\triangleq(i:\mathtt{I})\to A\,i$ as dependently typed paths. We call elements of type $\square(\Pi_{\mathtt{I}}A)$ *cofibrant-partial paths*. Given $(\varphi,f):\square(\Pi_{\mathtt{I}}A)$, we can evaluate it at a point $i:\mathtt{I}$ of the interval to get a cofibrant partial element $(\varphi,f)\ @\ i:\square(A\,i)$:

$$(\varphi,f)\ @\ i\triangleq(\varphi,\lambda(u:[\varphi])\to f\,u\,i)\tag{5.5}$$

An operation for *filling from* $\mathtt{0}$ in $A:\mathtt{I}\to\mathcal{U}$ takes any $(\varphi,f):\square(\Pi_{\mathtt{I}}A)$ and any $a_0:A\,\mathtt{0}$ with $(\varphi,f)\ @\ \mathtt{0}\nearrow a_0$ and extends (φ,f) to a dependently typed path $g:\Pi_{\mathtt{I}}A$ with $g\,\mathtt{0}=a_0$. This is a form of uniform *Homotopy Extension and Lifting Property* (HELP) [May99, Chapter 10, Section 3] stated internally in terms of cofibrant propositions rather than externally in terms of cofibrations. A feature of our internal approach compared with Cohen *et al* is that their *uniformity* condition on composition/filling operations [CCHM18, Definition 13], which allows one to avoid the non-constructive aspects of the classical notion of Kan filling [BC15], becomes automatic when the operations are formulated in terms of the internal collection $\mathtt{Cof}$ of cofibrant propositions.

Example 5.5. For some intuition as to why such an operation is referred to as filling, consider the following example. For simplicity, assume that $A:\mathtt{I}\to\mathcal{U}$ is a constant family $A\triangleq\lambda(\_:\mathtt{I})\to A'$. Recall that we think of variables of type $\mathtt{I}$ as dimensions in space; so that, given an element $a:A$ in an ambient context $j,k:\mathtt{I}$, we think of a as a square in the space A. We are interested in extending this two dimensional square to a three dimensional cube as indicated below.



$j,k:\mathtt{I}$ $\quad$ $a:A$ $\quad$ $f:[\varphi]\to\Pi_{\mathtt{I}}A$ $\quad$ $g:\Pi_{\mathtt{I}}A$

However, let us imagine that we already know how to extend a on certain faces and edges of the cube, for example, on the faces/edges specified by $\varphi\triangleq(j=\mathtt{0})\vee(j=\mathtt{1}\wedge k=\mathtt{1})$. This means that we have a cofibrant partial path $f : [\varphi] \to \Pi_{\mathtt{I}} A$ which agrees with a where they are both defined, that is $(\varphi, f) \mathbin{@} \mathtt{0} \nearrow a$. Note that f is a partial path rather than partial element because, on the faces/edges where it is defined, it must be defined at all points along the new dimension by which we are extending a, i.e. φ cannot depend on this new dimension. A filling for this data is a cube $g : \Pi_{\mathtt{I}} A$ which agrees with the faces/edges that we started with. That is, it extends f and agrees with a at the base of the cube: $(\varphi, f) \nearrow g$ and $g\,\mathtt{0} = a$.

Since we are not assuming any structure on the interval for reversing paths (see Remark 3.2), we also need to consider the symmetric notion of filling from $\mathtt{1}$. Let

$$\{\mathtt{0},\mathtt{1}\} \triangleq \{i : \mathtt{I} \mid i = \mathtt{0} \vee i = \mathtt{1}\} \tag{5.6}$$

Note that because of axiom $\mathtt{ax}_2$, this is isomorphic to the object of Booleans, $1 + 1$ and hence there is a function

$$\overline{} : \{\mathtt{0},\mathtt{1}\} \to \{\mathtt{0},\mathtt{1}\} \tag{5.7}$$

satisfying $\overline{\mathtt{0}} = \mathtt{1}$ and $\overline{\mathtt{1}} = \mathtt{0}$. In what follows, instead of using path reversal we parameterise definitions with $e : \{\mathtt{0},\mathtt{1}\}$ and use (5.7) to interchange $\mathtt{0}$ and $\mathtt{1}$.

Definition 5.6 (Filling structures). Given $e : \{\mathtt{0},\mathtt{1}\}$, the type $\mathtt{Fill}\, e\, A : \mathcal{U}$ of *filling structures* for an $\mathtt{I}$-indexed families of types $A : \mathtt{I} \to \mathcal{U}$, is defined by:

$$\begin{aligned} \mathtt{Fill}\, e\, A \triangleq (\varphi : \mathtt{Cof})(f : [\varphi] \to \Pi_{\mathtt{I}} A)(a : \{a' : A\, e \mid (\varphi, f) \mathbin{@} e \nearrow a'\}) \to \\ \{g : \Pi_{\mathtt{I}} A \mid (\varphi, f) \nearrow g \ \wedge \ g\, e = a\} \end{aligned} \tag{5.8}$$

A notable feature of [CCHM18] compared with preceding work [BCH14] is that such filling structure can be constructed from a simpler *composition* structure that just produces an extension at one end of a cofibrant-partial path from an extension at the other end. We will deduce this using axioms $\mathtt{ax}_3$–$\mathtt{ax}_6$ from the following, which is the main notion of this paper.

Definition 5.7 (CCHM fibrations). A *CCHM fibration* (A, α) over a type $\Gamma : \mathcal{U}$ is a family $A : \Gamma \to \mathcal{U}$ equipped with a fibration structure $\alpha : \mathtt{isFib}\, A$, where $\mathtt{isFib} : \{\Gamma : \mathcal{U}\}(A : \Gamma \to \mathcal{U}) \to \mathcal{U}$ is defined by

$$\mathtt{isFib}\,\{\Gamma\}\, A \triangleq (e : \{\mathtt{0},\mathtt{1}\})(p : \mathtt{I} \to \Gamma) \to \mathtt{Comp}\, e\, (A \circ p) \tag{5.9}$$

Here $\mathtt{Comp} : (e : \{\mathtt{0},\mathtt{1}\})(A : \mathtt{I} \to \mathcal{U}) \to \mathcal{U}$ is the type of *composition structures* for $\mathtt{I}$-indexed families:

$$\begin{aligned} \mathtt{Comp}\, e\, A \triangleq (\varphi : \mathtt{Cof})(f : [\varphi] \to \Pi_{\mathtt{I}} A) \to \\ \{a_0 : A\, e \mid (\varphi, f) \mathbin{@} e \nearrow a_0\} \to \{a_1 : A\, \overline{e} \mid (\varphi, f) \mathbin{@} \overline{e} \nearrow a_1\} \end{aligned} \tag{5.10}$$

Unwinding the definition, if $\alpha : \mathtt{isFib}\, A$ then $\alpha\, \mathtt{0}$ satisfies that for each cofibrant partial path $f : [\varphi] \to \Pi_{\mathtt{I}}(A \circ p)$ over a path $p : \mathtt{I} \to \Gamma$, if $a_0 : A\, \mathtt{0}$ extends the partial element $(\varphi, f) \mathbin{@} \mathtt{0}$, *i.e.* $\forall (u : [\varphi]).\ f\, u\, \mathtt{0} = a_0$, then $\alpha\, \mathtt{0}\, p\, \varphi\, f\, a_0 : A\, \mathtt{1}$ extends $(\varphi, f) \mathbin{@} \mathtt{1}$, *i.e.* $\forall (u : [\varphi]).\ f\, u\, \mathtt{1} = \alpha\, \mathtt{0}\, p\, \varphi\, f\, a_0$; and similarly for $\alpha\, \mathtt{1}$.

Definition 5.8 (The CwF of CCHM fibrations). Let $\mathtt{Fib}\, \Gamma$ be the type of CCHM fibrations over an object Γ, defined by

$$\mathtt{Fib}\ \Gamma \triangleq (A : \Gamma \to \mathcal{U}) \times \mathtt{isFib}\, A \tag{5.11}$$

CCHM fibrations are closed under re-indexing: given $\gamma : \Delta \to \Gamma$ and $A : \Gamma \to \mathcal{U}$, we get a function $\_[\gamma] : \mathtt{isFib}\, A \to \mathtt{isFib}(A \circ \gamma)$ defined by $\alpha[\gamma]\, e\, p \triangleq \alpha\, e\, (\gamma \circ p)$. Therefore we get a function $\_[\_] : (\Delta \to \Gamma) \to \mathtt{Fib}\, \Gamma \to \mathtt{Fib}\, \Delta$ given by

$$(A, \alpha)[\gamma] \triangleq (A \circ \gamma, \alpha[\gamma]) \tag{5.12}$$

which is functorial: $((A, \alpha)[id] = (A, \alpha)$ and $(A, \alpha)[g \circ f] = (A, \alpha)[g][f]$. It follows that $\mathtt{Fib}$ has the structure of a Category with Families by taking families to be CCHM fibrations $(A, \alpha) : \mathtt{Fib}\, \Gamma$ over each $\Gamma : \mathcal{U}$ and elements of such a family to be dependent functions in $(x : \Gamma) \to A\, x$.

Remark 5.9 (**Fibrant objects**)**.** We say $A : \mathcal{U}$ is a *fibrant object* if we have a fibration structure for the constant family $\lambda(\_ : 1) \to A$ over the terminal object 1. Note that if $(A, \alpha) : \mathtt{Fib}\, \Gamma$ is a fibration, then for each $x : \Gamma$ the type $A\, x : \mathcal{U}$ is fibrant, with the fibration structure given by reindexing α by the map $\lambda(\_ : 1) \to x : 1 \to \Gamma$. However the converse is not true: having a family of fibration structures, that is, an element of $(x : \Gamma) \to \mathtt{isFib}(\lambda(\_ : 1) \to A\, x)$, is weaker than having a fibration structure for $A : \Gamma \to \mathcal{U}$. To see why, consider the family, $P : \mathtt{I} \to \mathcal{U}$ defined by

$$P\ i \triangleq [i = \mathtt{0}] \tag{5.13}$$

For each $i : \mathtt{I}$ the fibre $P\, i : \mathcal{U}$ is a fibrant object, with a fibration structure, $\rho_i : \mathtt{isFib}(\lambda(\_ : 1) \to P\, i)$, given by $\rho_i\ e\ p\ \phi\ f\ x \triangleq x$. However, it is not possible to construct a $\rho : \mathtt{isFib}\, P$. For if it were, then we could define $\rho\ \mathtt{0}\ id\ \bot\ \mathtt{elim}_{\emptyset}\ * : [1 = 0]$; combined with $\mathtt{ax}_2$, this would lead to contradiction.

If $\alpha : \mathtt{Fill}\, e\, A$, then $\lambda \varphi\, f\, a \to \alpha\, \varphi\, f\, a\, \overline{e} : \mathtt{Comp}\, e\, A$ and so every filling structure gives rise to a composition structure. Conversely, the composition structure of a CCHM fibration gives rise to filling structure:

Lemma 5.10 (**Filling structure from composition structure**)**.** *Given $\Gamma : \mathcal{U}$, $A : \Gamma \to \mathcal{U}$, $e : \{\mathtt{0},\mathtt{1}\}$, $\alpha : \mathtt{isFib}\, A$ and $p : \mathtt{I} \to \Gamma$, there is a filling structure $\mathtt{fill}\, e\, \alpha\, p : \mathtt{Fill}\, e\, (A \circ p)$ that agrees with α at $\overline{e}$, that is:*

$$\forall (\varphi : \mathtt{Cof})(f : [\varphi] \to \Pi_{\mathtt{I}} A)(a : A(p\, e)).$$
$$(\varphi, f) \mathbin{@} e \nearrow a \;\Rightarrow\; \mathtt{fill}\, e\, \alpha\, p\, \varphi\, f\, a\, \overline{e} = \alpha\, e\, p\, \varphi\, f\, a \tag{5.14}$$

Furthermore, $\mathtt{fill}$ is stable under re-indexing in the sense that for all $\gamma : \Delta \to \Gamma$ and $p : \mathtt{I} \to \Delta$

$$\mathtt{fill}\, e\, \alpha\, (\gamma \circ p) = \mathtt{fill}\, e\, (\alpha[\gamma])\, p \tag{5.15}$$

Proof. The construction of filling from composition follows [CCHM18, Section 4.4], but just using the connection algebra structure on $\mathtt{I}$ (axioms $\mathtt{ax}_3$ and $\mathtt{ax}_4$), rather than a de Morgan algebra structure. Suppose $\Gamma : \mathcal{U}$, $A : \Gamma \to \mathcal{U}$, $e : \{\mathtt{0},\mathtt{1}\}$, $\alpha : \mathtt{isFib}\, A$, $p : \mathtt{I} \to \Gamma$, $\varphi : \mathtt{Cof}$, $f : [\varphi] \to \Pi_{\mathtt{I}}(A \circ p)$, $a : A(p\, e)$ with $(\varphi, f) \mathbin{@} e \nearrow a$, and $i : \mathtt{I}$. Then using Definition 5.3 we can define

$$\mathtt{fill}\, e\, \alpha\, p\, \varphi\, f\, a\, i \triangleq \alpha\, e\, (p'\, i)\, (\varphi \vee i = e)\, (f'\, i \cup g\, i)\, a \tag{5.16}$$

where

$p' : \mathtt{I} \to \mathtt{I} \to \Gamma$ is defined by $p'\, i\, j \triangleq p(i \sqcap_e j)$

$f' : (i : \mathtt{I}) \to [\varphi] \to \Pi_{\mathtt{I}}(A \circ (p'\, i))$ is defined by $f'\, i\, u\, j \triangleq f\, u\, (i \sqcap_e j)$

$g : (i : \mathtt{I}) \to \{g' : [i = e] \to \Pi_{\mathtt{I}}(A \circ (p' i)) \mid (\varphi, f'\, i) \smile (i = e, g')\}$ is defined by $g\, i\, v\, j \triangleq a$

and where $\sqcap_e$ is given by $\sqcap_0 \triangleq \sqcap$ and $\sqcap_1 \triangleq \sqcup$. Finally, property (5.15) is immediate from definitions (5.12) and (5.16). $\square$

Compared with [BCH14], the fact that filling can be defined from composition considerably simplifies the process of lifting fibration structure through the usual type-forming constructs, as the following two theorems demonstrate. Their proofs are internalisations of those in [CCHM18, Section 4.5], except that we avoid the use Cohen *et al* make of de Morgan involution.

Theorem 5.11 (**Fibrant Σ-types**). *There is a function*

$$\begin{aligned}\mathtt{isFib}_\Sigma : \{\Gamma : \mathcal{U}\}\{A_1 : \Gamma \to \mathcal{U}\}\{A_2 : (x : \Gamma) \times A_1\, x \to \mathcal{U}\} \to \\ \mathtt{isFib}\, A_1 \to \mathtt{isFib}\, A_2 \to \mathtt{isFib}(\Sigma\, A_1\, A_2)\end{aligned} \tag{5.17}$$

where $\Sigma\, A_1\, A_2\, x \triangleq (a_1 : A_1\, x) \times A_2(x, a_1)$*. The function is stable under re-indexing, in the sense that for all* $\gamma : \Delta \to \Gamma$

$$(\mathtt{isFib}_\Sigma\, \alpha_1\, \alpha_2)[\gamma] = \mathtt{isFib}_\Sigma(\alpha_1[\gamma])(\alpha_2[\gamma \times id]) \tag{5.18}$$

Hence the category with families given by CCHM fibrations supports the interpretation of Σ*-types* [Hof97, Definition 3.18]*.*

Proof. The construction of $\mathtt{isFib}_\Sigma$ makes use of the filling operation from Lemma 5.10. Given $\Gamma : \mathcal{U}$, $A_1 : \Gamma \to \mathcal{U}$, $A_2 : (x : \Gamma) \times A_1\, x \to \mathcal{U}$, $\alpha_1 : \mathtt{isFib}\, A_1$, $\alpha_2 : \mathtt{isFib}\, A_2$, $e : \{\mathtt{0},\mathtt{1}\}$, $p : \mathtt{I} \to \Gamma$, $\varphi : \mathtt{Cof}$, $f : [\varphi] \to \Pi_{\mathtt{I}}((\Sigma\, A_1\, A_2) \circ p)$ and $(a_1, a_2) : (\Sigma\, A_1\, A_2)(p\, e)$ with $(\varphi, f) @ e \nearrow (a_1, a_2)$, define

$$\mathtt{isFib}_\Sigma\ \alpha_1\, \alpha_2\, e\, p\, \varphi\, f\, (a_1, a_2) \triangleq (\alpha_1\, e\, p\, \varphi\, f_1\, a_1\ ,\ \alpha_2\, e\, q\, \varphi\, f_2\, a_2) \tag{5.19}$$

where

$$\begin{aligned}&f_1 : [\varphi] \to \Pi_{\mathtt{I}}(A_1 \circ p)\\ &f_1\, u\, i \triangleq \mathtt{fst}(f\, u\, i)\\ &q : \mathtt{I} \to (x : \Gamma) \times A_1\, x\\ &q \triangleq \langle p\ ,\ \mathtt{fill}\, e\, \alpha_1\, p\, \varphi\, f_1\, a_1\rangle\\ &f_2 : [\varphi] \to \Pi_{\mathtt{I}}(A_2 \circ q)\\ &f_2\, u\, i \triangleq \mathtt{snd}(f\, u\, i)\end{aligned}$$

Thus $\mathtt{isFib}_\Sigma\ \alpha_1\, \alpha_2\, e\, p\, \varphi\, f\, (a_1, a_2) : (\Sigma\, A_1\, A_2)(p\, \overline{e})$; and since

$$\begin{aligned}&\forall (u : [\varphi]).\ f_1\, u\, \overline{e} = \alpha_1\, e\, p\, \varphi\, f_1\, a_1 = \mathtt{fill}\, e\, \alpha_1\, p\, \varphi\, f_1\, a_1\, \overline{e}\\ &\forall (u : [\varphi]).\ f_2\, u\, \overline{e} = \alpha_2\, e\, q\, \varphi\, f_2\, a_2\end{aligned}$$

hold, it follows that

$$(\varphi, f) @ \overline{e} \nearrow \mathtt{isFib}_\Sigma\ \alpha_1\, \alpha_2\, e\, p\, \varphi\, f\, (a_1, a_2).$$

Hence $\mathtt{isFib}_\Sigma\ \alpha_1\, \alpha_2 : \mathtt{isFib}(\Sigma\, A_1\, A_2)$. Finally, property (5.18) follows from (5.15) and (5.19). $\square$

Theorem 5.12 (**Fibrant Π-types**). *There is a function*

$$\begin{aligned}\mathtt{isFib}_\Pi : \{\Gamma : \mathcal{U}\}\{A_1 : \Gamma \to \mathcal{U}\}\{A_2 : (x : \Gamma) \times A_1\, x \to \mathcal{U}\} \to \\ \mathtt{isFib}\, A_1 \to \mathtt{isFib}\, A_2 \to \mathtt{isFib}(\Pi\, A_1\, A_2)\end{aligned} \tag{5.20}$$

where $\Pi\, A_1\, A_2\, x \triangleq (a_1 : A_1\, x) \to A_2(x, a_1)$. *This function is stable under re-indexing (cf. 5.18) and hence the category with families given by CCHM fibrations supports the interpretation of* Π*-types* [Hof97, Definition 3.15].

Proof. Given $\Gamma : \mathcal{U}$, $A_1 : \Gamma \to \mathcal{U}$, $A_2 : (x : \Gamma) \times A_1\, x \to \mathcal{U}$, $\alpha_1 : \mathtt{isFib}\, A_1$, $\alpha_2 : \mathtt{isFib}\, A_2$, $e : \{\mathtt{0},\mathtt{1}\}$, $p : \mathtt{I} \to \Gamma$, $\varphi : \mathtt{Cof}$, $f : [\varphi] \to \Pi_{\mathtt{I}}((\Pi\, A_1\, A_2) \circ p)$, $g : (\Pi\, A_1\, A_2)(p\, e)$ with $(\varphi, f) @ e \nearrow g$ and $a_1 : A_1(p\, \overline{e})$, using Lemma 5.10 we define

$$\mathtt{isFib}_\Pi\ \alpha_1\, \alpha_2\, e\, p\, \varphi\, f\, g\, a_1 \triangleq \alpha_2\, e\, q\, \varphi\, f_2\, a_2 \tag{5.21}$$

where

$$\begin{aligned}
&f_1 : \Pi_{\mathtt{I}}(A_1 \circ p)\\
&f_1 \triangleq \mathtt{fill}\, \overline{e}\, \alpha_1\, p \perp \mathtt{elim}_\emptyset\ a_1\\
&q : \mathtt{I} \to (x : \Gamma) \times A_1\, x\\
&q \triangleq \langle p\ ,\ f_1 \rangle\\
&f_2 : [\varphi] \to \Pi_{\mathtt{I}}(A_2 \circ q)\\
&f_2\, u\, i \triangleq f\, u\, i\, (f_1\, i)\\
&a_2 : \{a_2' : A_2(q\, e) \mid (\varphi, f_2) @ e \nearrow a_2'\}\\
&a_2 \triangleq g(f_1\, e)
\end{aligned}$$

Since we know that $\mathtt{fill}\, \overline{e}\, \alpha_1\, p \perp \mathtt{elim}_\emptyset\ a_1\, \overline{e} = a_1$, therefore we have

$$\mathtt{isFib}_\Pi\ \alpha_1\, \alpha_2\, e\, p\, \varphi\, f\, g\, a_1 : A_2(q\, \overline{e}) = A_2(p\, \overline{e}, f_1\, \overline{e}) = A_2(p\, \overline{e}, a_1) \tag{5.22}$$

Furthermore, since $(\varphi, f_2) @ \overline{e} \nearrow \alpha_2\, e\, q\, \varphi\, f_2\, a_2$, for any $u : [\varphi]$ we have

$$f\, u\, \overline{e}\, a_1 = f\, u\, \overline{e}\, (f_1\, \overline{e}) = f_2\, u\, \overline{e} = \alpha_2\, e\, q\, \varphi\, f_2\, a_2 = \mathtt{isFib}_\Pi\ \alpha_1\, \alpha_2\, e\, p\, \varphi\, f\, g\, a_1 \tag{5.23}$$

Since (5.22) and (5.23) hold for all $a_1 : A_1(p\, \overline{e})$, from the first if follows that

$$\mathtt{isFib}_\Pi\ \alpha_1\, \alpha_2\, e\, p\, \varphi\, f\, g : (\Pi\, A_1\, A_2)(p\, \overline{e})$$

and from the second that $(\varphi, f) @ \overline{e} \nearrow \mathtt{isFib}_\Pi\ \alpha_1\, \alpha_2\, e\, p\, \varphi\, f\, g$. Therefore we have that (5.21) does give an element of $\mathtt{isFib}(\Pi\, A_1\, A_2)$. Finally, stability of $\mathtt{isFib}_\Pi\ \alpha_1\, \alpha_2$ under re-indexing follows from (5.15). ☐

These theorems allow us to construct fibration structures for Σ- and Π-types, given fibration structures for their constituent types. But are there any fibration structures to begin with? We answer this question by showing that the natural number object $\mathtt{N}$ in the topos is always fibrant. This is proved for the topos of cubical sets $\hat{\mathcal{C}}$ in [BCH14, Section 4.5] by defining a composition structure by primitive recursion. We give a more elementary proof using the fact that the interval object in $\hat{\mathcal{C}}$ satisfies axiom $\mathtt{ax_1}$ (see Theorem 8.2).

Theorem 5.13 ($\mathtt{N}$ **is fibrant**). *If* $\mathtt{N}$ *is an object with decidable equality, then there is a function* $\mathtt{isFib}_{\mathtt{N}} : \{\Gamma : \mathcal{U}\} \to \mathtt{isFib}(\lambda(\_ : \Gamma) \to \mathtt{N})$. *In particular, if the topos* $\mathcal{E}$ *has a natural number object* $1 \xrightarrow{\mathtt{Z}} \mathtt{N} \xrightarrow{\mathtt{S}} \mathtt{N}$, *then the category with families given by CCHM fibrations has a natural number object.*

Proof. Suppose $\Gamma : \mathcal{U}$, $e : \{\mathtt{0},\mathtt{1}\}$, $p : \mathtt{I} \to \Gamma$, $\varphi : \mathtt{Cof}$, $f : [\varphi] \to \Pi_{\mathtt{I}}(\lambda\_ \to \mathtt{N})$ and $n : \mathtt{N}$ with $(\varphi, f) @ e \nearrow n$. By assumption on $\mathtt{N}$, for each $u : [\varphi]$ the property $\lambda(i : \mathtt{I}) \to (f\, u\, i = n) : \mathtt{I} \to \Omega$ is decidable; hence by axiom $\mathtt{ax_1}$ and the fact that $f\, u\, e = n$, we also have $f\, u\, \overline{e} = n$. Therefore we can get $\mathtt{isFib}_{\mathtt{N}}\, e\, p\, \varphi\, f\, n : \{n' : \mathtt{N} \mid (\varphi, f) @ \overline{e} \nearrow n'\}$ just by defining: $\mathtt{isFib}_{\mathtt{N}}\, e\, p\, \varphi\, f\, n \triangleq n$.

For the last part of the theorem we use the fact that in a topos with natural number object, equality of numbers is decidable. □

A similar use of axiom $\mathtt{ax}_1$ suffices to prove:

Theorem 5.14 (**Fibrant coproducts**). *Writing* $A_1 \xrightarrow{\mathtt{inl}} A_1 + A_2 \xleftarrow{\mathtt{inr}} A_2$ *for the coproduct of* A_1 *and* A_2 *in* $\mathcal{E}$*, we lift this to families of types,* $\_ \uplus \_ : \{\Gamma : \mathcal{U}\}(A_1\ A_2 : \Gamma \to \mathcal{U}) \to \Gamma \to \mathcal{U}$*, by defining* $(A_1 \uplus A_2)\, x \triangleq A_1\, x + A_2\, x$*. Then there is a function*

$$\mathtt{isFib}_{\uplus} : \{\Gamma : \mathcal{U}\}\{A_1\ A_2 : \Gamma \to \mathcal{U}\} \to \mathtt{isFib}\, A_1 \to \mathtt{isFib}\, A_2 \to \mathtt{isFib}(A_1 \uplus A_2) \tag{5.24}$$

and this fibration structure on coproducts is stable under re-indexing. Hence the category with families given by CCHM fibrations has binary coproducts.

Proof. The proof makes use of the principle of unique choice, which holds in the internal type theory of a topos:

$$\mathtt{uc} : (A : \mathcal{U})(\varphi : A \to \Omega) \to [\exists!(a : A).\ \varphi\, a] \to \{a : A \mid \varphi\, a\} \tag{5.25}$$

where $\exists!(a : A).\ \varphi\, a \triangleq \exists(a : A).\ \varphi\, a \wedge \forall(a' : A).\ \varphi\, a' \Rightarrow a = a'$.

Suppose we have $\Gamma : \mathcal{U}$, $A_1\ A_2 : \Gamma \to \mathcal{U}$, $\alpha_1 : \mathtt{isFib}\, A_1$, $\alpha_2 : \mathtt{isFib}\, A_2$, $e : \{\mathtt{0},\mathtt{1}\}$, $p : \mathtt{I} \to \Gamma$, $\varphi : \mathtt{Cof}$, $g : [\varphi] \to \Pi_{\mathtt{I}}((A_1 \uplus A_2) \circ p)$ and $c : A_1(p\, e) + A_2(p\, e)$ with $(\varphi, g) \mathbin{@} e \nearrow c$. Note that for all $u : [\varphi]$ and $i : \mathtt{I}$

$$\begin{aligned} &P_1, P_2 : [\varphi] \to \mathtt{I} \to \Omega \\ &P_1\, u\, i \triangleq \exists!(a_1 : A_1(p\, i)).\ g\, u\, i = \mathtt{inl}\, a_1 \\ &P_2\, u\, i \triangleq \exists!(a_2 : A_2(p\, i)).\ g\, u\, i = \mathtt{inr}\, a_2 \end{aligned}$$

are complementary propositions ($P_1\, u\, i \ \wedge\ P_2\, u\, i = \bot$ and $P_1\, u\, i \ \vee\ P_2\, u\, i = \top$); and hence by $\mathtt{ax}_1$ we have that $(\forall(i : \mathtt{I}).\ P_1\, u\, i) \vee (\forall(i : \mathtt{I}).\ P_2\, u\, i)$. Either $c = \mathtt{inl}\, a_1$ for some $a_1 : A_1(p\, e)$, or $c = \mathtt{inr}\, a_2$ for some $a_2 : A_2(p\, e)$. In the first case, since $\forall(u : [\varphi]).\ g\, u\, e = c$, it follows that $\forall(u : [\varphi])(i : \mathtt{I}).\ P_1\, u\, i$; then using $\mathtt{uc}$ we get some $f_1 : [\varphi] \to \Pi_{\mathtt{I}}(A_1 \circ p)$ with $\forall(u : [\varphi])(i : \mathtt{I}).\ g\, u\, i = \mathtt{inl}(f_1\, u\, i)$ and we can define

$$\mathtt{isFib}_{\uplus}\, \alpha_1\, \alpha_2\, e\, p\, \varphi\, g\, (\mathtt{inl}\, a_1) \triangleq \mathtt{inl}(\alpha_1 e\, p\, \varphi\, f_1\, a_1)$$

Similarly if $c = \mathtt{inr}\, a_2$, then there is some $f_2 : [\varphi] \to \Pi_{\mathtt{I}}(A_2 \circ p)$ with $\forall(u : [\varphi])(i : \mathtt{I}).\ g\, u\, i = \mathtt{inr}(f_2\, u\, i)$ and we can define

$$\mathtt{isFib}_{\uplus}\, \alpha_1\, \alpha_2\, e\, p\, \varphi\, g\, (\mathtt{inr}\, a_2) \triangleq \mathtt{inr}(\alpha_2 e\, p\, \varphi\, f_2\, a_2).$$

□

5.3. Identity types.

Theorem 5.15 (**Fibrant path types**). *There is a function*

$$\mathtt{isFib}_{\mathtt{Path}} : \{\Gamma : \mathcal{U}\}\{A : \Gamma \to \mathcal{U}\} \to \mathtt{isFib}\, A \to \mathtt{isFib}(\mathtt{Path}\, A) \tag{5.26}$$

where $\mathtt{Path}\, A : (x : \Gamma) \times (A\, x \times A\, x) \to \mathcal{U}$ *is given by*

$$\mathtt{Path}\, A\, (x, (a_0, a_1)) \triangleq a_0 \sim a_1 \tag{5.27}$$

and where $\sim$ *is as in* (4.1)*. This fibration structure on path types is stable under re-indexing, in the sense that for all* $\gamma : \Delta \to \Gamma$

$$(\mathtt{isFib}_{\mathtt{Path}}\, \alpha)[\gamma \times (id \times id)] = \mathtt{isFib}_{\mathtt{Path}}(\alpha[\gamma]) \tag{5.28}$$

Proof. Given $\Gamma : \mathcal{U}$, $A : \Gamma \to \mathcal{U}$, $\alpha : \mathtt{isFib}\, A$, $e : \{\mathtt{0},\mathtt{1}\}$, $p : \mathtt{I} \to (x : \Gamma) \times (A\, x \times A\, x)$, $\varphi : \mathtt{Cof}$, $f : [\varphi] \to \Pi_{\mathtt{I}}((\mathtt{Path}\, A) \circ p)$, $q : \mathtt{Path}\, A\, (p\, e)$ with $(\varphi, f) \mathbin{@} e \nearrow q$ and $i : \mathtt{I}$, suppose $p = \langle p' , \langle q_0 , q_1 \rangle\rangle$ where $p' : \mathtt{I} \to \Gamma$ and $q_0, q_1 : \Pi_{\mathtt{I}}(A \circ p)$, and define

$$\mathtt{isFib}_{\mathtt{Path}}\, \alpha\, e\, p\, \varphi\, f\, q\, i \triangleq \alpha\, e\, p'\, (\varphi \vee i = \mathtt{0} \vee i = \mathtt{1})\, (f' \cup f_0 \cup f_1)\, (q\, i) \tag{5.29}$$

where

$$\begin{aligned}
& f' : [\varphi] \to \Pi_{\mathtt{I}}(A \circ p') \\
& f'\, u\, j \triangleq f\, u\, j\, i \\
& f_0 : \{g : [i = \mathtt{0}] \to \Pi_{\mathtt{I}}(A \circ p') \mid (\varphi, f') \smile (i = \mathtt{0}, g)\} \\
& f_0\, _ \triangleq q_0 \\
& f_1 : \{g : [i = \mathtt{1}] \to \Pi_{\mathtt{I}}(A \circ p') \mid (\varphi \vee i = \mathtt{0}, f' \cup f_0) \smile (i = \mathtt{1}, g)\} \\
& f_1\, _ \triangleq q_1
\end{aligned}$$

Thus for each $i : \mathtt{I}$ we have $\mathtt{isFib}_{\mathtt{Path}}\, \alpha\, e\, p\, \varphi\, f\, q\, i : A(p' \overline{e})$, so that $\mathtt{isFib}_{\mathtt{Path}}\, \alpha\, e\, p\, \varphi\, f\, q : \mathtt{I} \to A(p' \overline{e})$. Since $\alpha\, e\, p' : \mathtt{Comp}\, e\, (A \circ p')$, we have

$$\begin{aligned}
\forall (u : [\varphi \vee i = \mathtt{0} \vee i = \mathtt{1}]).\ & (f' \cup f_0 \cup f_1)\, u\, \overline{e} = \\
& \alpha\, e\, p'\, (\varphi \vee i = \mathtt{0} \vee i = \mathtt{1})\, (f' \cup f_0 \cup f_1)\, (q\, i) = \mathtt{isFib}_{\mathtt{Path}}\, \alpha\, e\, p\, \varphi\, f\, q\, i
\end{aligned}$$

Hence $\mathtt{isFib}_{\mathtt{Path}}\, \alpha\, e\, p\, \varphi\, f\, q\, \mathtt{0} = q_0\, \overline{e}$ and $\mathtt{isFib}_{\mathtt{Path}}\, \alpha\, e\, p\, \varphi\, f\, q\, \mathtt{1} = q_1\, \overline{e}$, so that

$$\mathtt{isFib}_{\mathtt{Path}}\, \alpha\, e\, p\, \varphi\, f\, q : \mathtt{Path}\, A\, (p'\, \overline{e}, (q_0\, \overline{e}, q_1\, \overline{e})) = \mathtt{Path}\, A\, (p\, \overline{e})$$

and furthermore, $(\varphi, f) \mathbin{@} \overline{e} \nearrow \mathtt{isFib}_{\mathtt{Path}}\, \alpha\, e\, p\, \varphi\, f\, q$. Thus $\mathtt{isFib}_{\mathtt{Path}}\, \alpha : \mathtt{isFib}(\mathtt{Path}\, A)$. Finally, property (5.28) is immediate from (5.29) and the definition of $\_[\_]$ (5.12). □

These path types in the CwF of CCHM fibrations (Definition 5.8) satisfy the Coquand formulation of identity types with propositional computation properties [BCH14, Figure 2]. Thus in addition to the contractibility of singleton types (4.3), we get *substitution functions* for transporting elements of a fibration along a path

$$\begin{aligned}
& \mathtt{subst} : \{\Gamma : \mathcal{U}\}\{A : \Gamma \to \mathcal{U}\}\{\alpha : \mathtt{isFib}\, A\}\{x_0\ x_1 : \Gamma\} \to (x_0 \sim x_1) \to A\, x_0 \to A\, x_1 \\
& \mathtt{subst}\, p\, a \triangleq \alpha\, \mathtt{0}\, p\, \bot\ \mathtt{elim}_{\emptyset}\ a
\end{aligned} \tag{5.30}$$

using the cofibrant partial elements $(\bot, \mathtt{elim}_{\emptyset})$ mentioned after Definition 5.1. By Lemma 5.10 we have that these substitution functions satisfy a propositional computation rule for constant paths (4.2):

$$\begin{aligned}
& \mathtt{H} : \{\Gamma : \mathcal{U}\}\{A : \Gamma \to \mathcal{U}\}\{\alpha : \mathtt{isFib}\, A\}\{x : \Gamma\}(a : A\, x) \to (a \sim \mathtt{subst}(\mathtt{k}\, x)\, a) \\
& \mathtt{H}\, a \triangleq \mathtt{fill}\, \mathtt{0}\, \alpha\, (\mathtt{k}\, x)\, \bot\ \mathtt{elim}_{\emptyset}\ a
\end{aligned} \tag{5.31}$$

Remark 5.16 (**Function extensionality**)**.** As one might expect from [Uni13, Lemma 6.3.2], the path types of Theorem 5.15 satisfy function extensionality. Given $A : \mathcal{U}$, $B : A \to \mathcal{U}$, $f, g : (x : A) \to B\, x$ and $p : (x : A) \to (f\, x \sim g\, x)$, we get a path $\mathtt{funext}\, p : f \sim g$ in $(x : A) \to B\, x$ given by

$$\mathtt{funext}\, p\, i \triangleq \lambda(x : A) \to p\, x\, i$$

for all $i : \mathtt{I}$.

To get Martin-Löf identity types with standard definitional, rather than propositional computation properties from these path types, we use a version of Swan's construction [Swa16] like the one in Section 9.1 of [CCHM18], but only using the connection algebra structure on $\mathtt{I}$, rather than a de Morgan algebra structure. This is the only place that axiom $\mathtt{ax}_7$ is used; we need the fact that the universe given by $\mathtt{Cof}$ and $[\_] : \mathtt{Cof} \to \mathcal{U}$ is closed under dependent products:

Lemma 5.17. *The following element of type Ω is provable: $\forall(\varphi : \Omega)(f : [\varphi] \to \Omega).\ \mathtt{cof}\,\varphi \Rightarrow (\forall(u : [\varphi]).\ \mathtt{cof}(f\,u)) \Rightarrow \mathtt{cof}(\exists(u : [\varphi]).\ f\,u)$.*

Proof. Note that if $u : [\varphi]$ then $(\exists(v : [\varphi]).\ f\,v) = f\,u$ and hence $\mathtt{cof}(\exists(v : [\varphi]).\ f\,v) = \mathtt{cof}(f\,u)$. So $\forall(u : [\varphi]).\ \mathtt{cof}(f\,u)$ equals $\varphi \Rightarrow \mathtt{cof}(\exists(v : [\varphi]).\ f\,v)$. Therefore from $\mathtt{cof}\,\varphi$ and $\forall(u : [\varphi]).\ \mathtt{cof}(f\,u)$ by axiom $\mathtt{ax}_7$ we get $\mathtt{cof}(\varphi \wedge \exists(v : [\varphi]).\ f\,v)$ and hence $\mathtt{cof}(\exists(v : [\varphi]).\ f\,v)$, since $(\exists(v : [\varphi]).\ f\,v) \Rightarrow \varphi$. □

Theorem 5.18 (**Fibrant identity types**). *Define identity types by:*

$$\mathtt{Id} : \{\Gamma : \mathcal{U}\}(A : \Gamma \to \mathcal{U}) \to (x : \Gamma) \times (A\,x \times A\,x) \to \mathcal{U} \tag{5.32}$$

$$\mathtt{Id}\,A\,(x, (a_0, a_1)) \triangleq (p : \mathtt{Path}\,A\,(x, (a_0, a_1))) \times \{\varphi : \mathtt{Cof} \mid \varphi \Rightarrow \forall(i : \mathtt{I}).\ p\,i = a_0\}$$

Then there is a function $\mathtt{isFib}_{\mathtt{Id}} : \{\Gamma : \mathcal{U}\}\{A : \Gamma \to \mathcal{U}\} \to \mathtt{isFib}\,A \to \mathtt{isFib}(\mathtt{Id}\,A)$ and the fibrations $(\mathtt{Id}\,A, \mathtt{isFib}_{\mathtt{Id}}\,A)$ can be given the structure of Martin-Löf identity types in the CwF of CCHM fibrations [Hof97, Defintion 3.19].

Proof. Given $\Gamma : \mathcal{U}$, $A : \Gamma \to \mathcal{U}$ and $\alpha : \mathtt{isFib}\,A$, using Theorems 5.11 and 5.15 we define $\mathtt{isFib}_{\mathtt{Id}}\,\alpha \triangleq \mathtt{isFib}_{\Sigma}(\mathtt{isFib}_{\mathtt{Path}}\,\alpha)\,\beta$, where $\beta : \mathtt{isFib}\,\Phi$ with

$$\Phi : (y : (x : \Gamma) \times (A\,x \times A\,x)) \times \mathtt{Path}\,A\,y \to \mathcal{U}$$

$$\Phi((x, (a_0, a_1)), p) \triangleq \{\varphi : \mathtt{Cof} \mid \varphi \Rightarrow \forall(i : \mathtt{I}).\ p\,i = a_0\}$$

and the fibration structure β mapping $e : \{\mathtt{0},\mathtt{1}\}$, $p : \mathtt{I} \to (y : (x : \Gamma) \times (A\,x \times A\,x)) \times \mathtt{Path}\,A\,y$, $\varphi : \mathtt{Cof}$, $f : [\varphi] \to \Pi_{\mathtt{I}}(\Phi \circ p)$ and $\varphi' : \Phi(p\,e)$ with $(\varphi, f) \mathbin{@} e \nearrow \varphi'$ to the element

$$\beta\,e\,p\,\varphi\,f\,\varphi' \triangleq \exists(u : [\varphi]).\ f\,u\,\overline{e}$$

(using Lemma 5.17 to see that this is well defined). We get the usual introduction, elimination and computation rules for these identity types as follows. Since $\top : \mathtt{Cof}$ holds by axiom $\mathtt{ax}_5$, identity introduction

$$\mathtt{refl} : \{\Gamma : \mathcal{U}\}\{A : \Gamma \to \mathcal{U}\}\{x : \Gamma\}(a : A\,x) \to \mathtt{Id}\,A\,(x, (a, a)) \tag{5.33}$$

can be defined by $\mathtt{refl}\,a \triangleq (\lambda a\, i \to a, \top)$. Identity elimination

$$\begin{aligned}\mathtt{J} : \{\Gamma : \mathcal{U}\}(A : \Gamma \to \mathcal{U})(x : \Gamma)(a_0 : A\,x)(B : (a : A\,x) \times \mathtt{Id}\,A\,(x, (a_0, a)) \to \mathcal{U})\\ (\beta : \mathtt{isFib}\,B)(a_1 : A\,x)(e : \mathtt{Id}\,A\,(x, (a_0, a_1))) \to B(a_0, \mathtt{refl}\,a_0) \to B(a_1, e)\end{aligned} \tag{5.34}$$

is given by

$$\mathtt{J}\,A\,x\,a_0\,B\,\beta\,a_1(p, \varphi)\,b \triangleq \beta\,\mathtt{0}\,\langle p\ ,\ q\rangle\,\varphi\,f\,b$$

where $q : (i : \mathtt{I}) \to \mathtt{Id}\,A\,x\,(a_0, p\,i)$ is $q\,i\,j \triangleq (p(i \sqcap j), \varphi \vee i = \mathtt{0})$ and $f : [\varphi] \to \Pi_{\mathtt{I}}(B \circ \langle p\ ,\ q\rangle)$ is $f\,u\,i \triangleq b$. (In the above element, since $(p, \varphi) : \mathtt{Id}\,A\,(x, (a_0, a_1))$ we have $p\,\mathtt{0} = a_0$, $p\,\mathtt{1} = a_1$ and $\varphi \Rightarrow \forall(i : \mathtt{I}).\ p\,i = a_0$; hence $\varphi \Rightarrow \forall(i : \mathtt{I}).\ q\,i = \mathtt{refl}\,a_0$, so that f is well-defined.) Note that by axioms $\mathtt{ax}_3$ and $\mathtt{ax}_4$ we have $q\,\mathtt{0} = \mathtt{refl}\,a_0$ and $q\,\mathtt{1} = (p, \varphi)$, so that $\mathtt{J}\,A\,x\,a_0\,B\,\beta\,a_1(p,\varphi)\,b = \beta\,\mathtt{0}\,\langle p\ ,\ q\rangle\,\varphi\,f\,b : B(p\,\mathtt{1}, q\,\mathtt{1}) = B(a_1,(p,\varphi))$, as required. Furthermore, since $(\varphi, f)\ @\ \mathtt{1} \nearrow \beta\,\mathtt{0}\,\langle p\ ,\ q\rangle\,\varphi\,f\,b$, we have

$$\forall (u : [\varphi]).\ b = f\,u\,\mathtt{1} = \mathtt{J}\,A\,x\,a_0\,B\,\beta\,a_1(p,\varphi)\,b$$

So when $(p,\varphi) = \mathtt{refl}\,a_0 = (\mathtt{k}\,a_0, \top)$ and hence $a_1 = a_0$, we have

$$\mathtt{J}\,A\,x\,a_0\,B\,\beta\,a_o\,(\mathtt{refl}\,a_0)\,b = b \tag{5.35}$$

In other words the computation property for identity elimination holds as a judgemental equality and not just a propositional one. Finally, to correctly support the interpretation of intensional identity types, one needs stability of $(\mathtt{Id}\,A, \mathtt{isFib}_{\mathtt{Id}}\,A)$, $\mathtt{refl}$ and $\mathtt{J}\,A$ under re-indexing; but this follows from the stability properties of $\mathtt{isFib}_\Sigma$ and $\mathtt{isFib}_{\mathtt{Path}}$. □

6. Glueing

In this section we give an internal presentation of the *glueing* construction given by Cohen *et al* [CCHM18]. Glueing is similar to a composition structure (Definition 5.7) for type-families, except that instead of partial paths of types it involves partial equivalences between types. Glueing is crucial for the constructions relating to univalence [Uni13] given in Section 7.

We begin by defining the glueing construction for cofibrant-partial types, that is, for functions $A : [\varphi] \to \mathcal{U}$ where $\varphi : \mathtt{Cof}$:

Definition 6.1 (**Glueing**). Given $\varphi : \mathtt{Cof}$, $A : [\varphi] \to \mathcal{U}$, $B : \mathcal{U}$ and $f : (u : [\varphi]) \to A\,u \to B$, the type $\mathtt{Glue}\,\varphi\,A\,B\,f : \mathcal{U}$ is defined to be

$$\mathtt{Glue}\,\varphi\,A\,B\,f \triangleq (a : (u : [\varphi]) \to A\,u) \times \{b : B \mid \forall (u : [\varphi]).\ f\,u\,(a\,u) = b\} \tag{6.1}$$

Elements of this type consist of pairs (a, b) where a is a partial element of the partial type A and b is an element of type B, such that f applied to a gives a partial element of B that extends to b. When $\varphi = \top$ then A and f are both total and so $\mathtt{Glue}\,\varphi\,A\,B\,f$ essentially consists of pairs $(a, f\,a)$ for every $a : A$ and hence is clearly isomorphic to A. When $\varphi = \bot$ then A and f are both uniquely determined and $\mathtt{Glue}\,\varphi\,A\,B\,f$ will consist of pairs $(\mathtt{elim}_\emptyset, b)$ for $b : B$ and hence is clearly isomorphic to B.

We now extend this glueing operation from cofibrant-partial types to *cofibrant-partial type-families*:

Definition 6.2 (**Cofibrant-partial families**). Given a object $\Gamma : \mathcal{U}$ and a cofibrant property $\Phi : \Gamma \to \mathtt{Cof}$ define the *restriction of* Γ *by* Φ to be

$$\Gamma|\Phi \triangleq (x : \Gamma) \times [\Phi\,x] \tag{6.2}$$

Thus $\Gamma|\Phi : \mathcal{U}$ and there is a monomorphism $\iota : \Gamma|\Phi \rightarrowtail \Gamma$ given by first projection. (Note that $\Gamma|\Phi$ is isomorphic to the comprehension subtype $\{x : \Gamma \mid \Phi\,x\}$, but we use the above representation to make proofs of $\Phi\,x$ more explicit in various constructions.) Then given an object $\Gamma : \mathcal{U}$ and a cofibrant property $\Phi : \Gamma \to \mathtt{Cof}$, a *cofibrant-partial type-family* over Γ is a family A of types over the restriction $\Gamma|\Phi$, that is $A : (\Gamma|\Phi) \to \mathcal{U}$.

Definition 6.3 (**Glueing for families**). We lift the glueing operation from types to type-families as follows. Given $\Gamma : \mathcal{U}$, $\Phi : \Gamma \to \mathtt{Cof}$, $A : \Gamma|\Phi \to \mathcal{U}$, $B : \Gamma \to \mathcal{U}$ and $f : (x : \Gamma)(v : [\Phi\,x]) \to A(x, v) \to B\,x$, define the family $\mathtt{Glue}\,\Phi\,A\,B\,f : \Gamma \to \mathcal{U}$ by

$$\mathtt{Glue}\,\Phi\,A\,B\,f\,x \triangleq \mathtt{Glue}\,(\Phi\,x)\,(A(x, \_))\,(B\,x)\,(f(x, \_)) \tag{6.3}$$

The glueing construction works for any map $f : (x : \Gamma)(v : [\Phi\, x]) \to A(x, v) \to B\, x$. However, we want to see that this construction lifts to the CwF of CCHM fibrations. This means that $\texttt{Glue}\,\Phi\, A\, B\, f$ should have a fibration structure whenever A and B do and this puts some requirements on f. We begin by introducing the notion of an *extension structure*:

Definition 6.4 (**Extension structures**)**.** The type of extension structures, $\texttt{Ext} : \mathcal{U} \to \mathcal{U}$, is given by

$$\texttt{Ext}\, A \triangleq (\tilde{a} : \Box\, A) \to \{a : A \mid \tilde{a} \nearrow a\}$$

Having an extension structure for a type $A : \mathcal{U}$ allows us to extend any partial element of A to a total element. We say that a family $A : \Gamma \to \mathcal{U}$ has an extension structure if each of its fibres do, and we abusively write

$$\texttt{Ext}\, A \triangleq (x : \Gamma) \to \texttt{Ext}(A\, x)$$

An extension structure for $A : \Gamma \to \mathcal{U}$ is similar to having a composition structure for A in the sense that both allow us to extend partial elements; and in fact every family with an extension structure is a fibration. However, an extension structure does not require a total element from which we extend/compose and so is in fact a stronger notion than a composition structure. First note that having an extension structure for $A : \mathcal{U}$ implies that A is inhabited, because we can always extend the empty partial element. Further, given any element $a : A$ we can use the extension structure to show that it is path equal to the extension of the empty partial element. Together these facts tell us that A is *contractible*:

Definition 6.5 (**Contractibility** [Uni13, Section 3.11])**.** A type A is said to be *contractible* if it has a centre of contraction $a_0 : A$ and every element $a : A$ is propositionally equal to a_0, that is, there exists a path $a_0 \sim a$. Therefore a type is contractible if $\texttt{Contr}\, A$ is inhabited, where $\texttt{Contr} : \mathcal{U} \to \mathcal{U}$ is defined by

$$\texttt{Contr}\, A \triangleq (a_0 : A) \times ((a : A) \to a_0 \sim a)$$

As with extension structures, we say that a family $A : \Gamma \to \mathcal{U}$ is contractible if each of its fibres is and write

$$\texttt{Contr}\, A \triangleq (x : \Gamma) \to \texttt{Contr}(A\, x)$$

As mentioned above, having an extension structure for a family $A : \Gamma \to \mathcal{U}$ implies that A is both fibrant and contractible. In fact the converse is true as well (*cf.* [CCHM18, Lemma 5]):

Lemma 6.6. *There are functions*

$$\texttt{fromExt} : \{\Gamma : \mathcal{U}\}\{A : \Gamma \to \mathcal{U}\} \to \texttt{Ext}\, A \to \texttt{isFib}\, A \times \texttt{Contr}\, A \tag{6.4}$$

$$\texttt{toExt} : \{\Gamma : \mathcal{U}\}\{A : \Gamma \to \mathcal{U}\} \to \texttt{isFib}\, A \to \texttt{Contr}\, A \to \texttt{Ext}\, A \tag{6.5}$$

Proof. Given $\Gamma : \mathcal{U}, A : \Gamma \to \mathcal{U}$ and $\varepsilon : \texttt{Ext}\, A$ we define $\alpha : \texttt{isFib}\, A$

$$\alpha\ e\ p\ \varphi\ f\ a_0 \triangleq \varepsilon\ (p\ \overline{e})\ ((\varphi, f) \mathbin{@} \overline{e})$$

For every $x : \Gamma$ we use the totally undefined cofibrant-partial element $(\bot, \texttt{elim}_\emptyset) : \Box\, A$ to define $a_0 : A$

$$a_0 \triangleq \varepsilon\, x\, (\bot, \texttt{elim}_\emptyset)$$

For each $a : A\, x$ and $i : \texttt{I}$, we have $\lambda\_ \to a : [i = \texttt{1}] \to A\, x$; so we get a path $p_a : \texttt{I} \to A\, x$

$$p_a\, i \triangleq \varepsilon\, x\, (i = \texttt{1}, \lambda\_ \to a)$$

By the definition of $\mathtt{Ext}\,A$ we have $p_a\,\mathtt{1} = \varepsilon\,x\,(\top, \lambda_{\_} \to a) = a$, and by $\mathtt{ax_2}$ we have $p_a\,\mathtt{0} = \varepsilon\,x\,(\bot, \mathtt{elim}_\emptyset) = a_0$. Therefore $p_a : a_0 \sim a$. Hence $A\,x$ is contractible. Together this shows that, given $\varepsilon : \mathtt{Ext}\,A$, we can define elements of type $\mathtt{isFib}\,A$ and $\mathtt{Contr}\,A$. Therefore there is a function $\mathtt{fromExt} : \{\Gamma : \mathcal{U}\}\{A : \Gamma \to \mathcal{U}\} \to \mathtt{Ext}\,A \to \mathtt{isFib}\,A \times \mathtt{Contr}\,A$.

Conversely, given $\Gamma : \mathcal{U}$, $A : \Gamma \to \mathcal{U}$, $\alpha : \mathtt{isFib}\,A$, $\langle a_0, p\rangle : \mathtt{Contr}\,A$, note that for any $x : \Gamma$, $\varphi : \mathtt{Cof}$ and $f : [\varphi] \to A\,x$ we have $(p\,x) \circ f : [\varphi] \to (\mathtt{I} \to A\,x)$ such that $\forall (u : [\varphi]).\ ((p\,x) \circ f)\,u : a_0\,x \sim f\,u$; therefore $(\varphi, (p\,x) \circ f) \mathbin{@} \mathtt{0} \nearrow (a_0\,x)$ and so defining

$$\varepsilon\,x\,(\varphi, f) \triangleq \alpha\,\mathtt{0}\,(\lambda_{\_} \to x)\,\varphi\,((p\,x) \circ f)\,(a_0\,x)$$

we get $\varepsilon\,x\,(\varphi, f) : A\,x$. Furthermore, since $(\varphi, (p\,x) \circ f) \mathbin{@} \mathtt{1} = (\varphi, f)$ by the type of α we get $(\varphi, f) \nearrow \varepsilon\,x\,(\varphi, f)$. Thus $\varepsilon\,x\,(\varphi, f) : \mathtt{Ext}(A\,x)$, as required. □

We now come to the main result of this section: showing that fibrations are closed under glueing. Proving this requires that the function f is an *equivalence*:

Definition 6.7 (**Equivalences** [Uni13, Section 4.4])**.** Given types $A, B : \mathcal{U}$, a function $f : A \to B$ is an *equivalence* if the type $\mathtt{Equiv}\ f$ is inhabited, where

$$\mathtt{Equiv}\,f \triangleq (b : B) \to \mathtt{Contr}((a : A) \times f\,a \sim b)$$

Again, this lifts to families in the obvious way: given $\Gamma : \mathcal{U}$, $A, B : \Gamma \to \mathcal{U}$ and $f : (x : \Gamma) \to A\,x \to B\,x$, define

$$\mathtt{Equiv}\,f \triangleq (x : \Gamma) \to \mathtt{Equiv}\,(f\,x)$$

Theorem 6.8 (**Composition for glueing**)**.** *Let Φ, A, B and f be as in Definition 6.3. Then $\mathtt{Glue}\,\Phi\,A\,B\,f$ has a fibration structure if A and B both have one and f has the structure of an equivalence. In other words there is a function*

$$\begin{aligned} \mathtt{isFib}_{\mathtt{Glue}} : {} & \{\Gamma : \mathcal{U}\}\{\Phi : \Gamma \to \mathtt{Cof}\}\{A : \Gamma|\Phi \to \mathcal{U}\}\{B : \Gamma \to \mathcal{U}\} \\ & (f : (x : \Gamma)(u : [\Phi\,x]) \to A(x, u) \to B\,x) \to \\ & ((x : \Gamma)(v : [\Phi\,x]) \to \mathtt{Equiv}(f\,x\,v)) \to \\ & \mathtt{isFib}\,A \to \mathtt{isFib}\,B \to \mathtt{isFib}(\mathtt{Glue}\,\Phi\,A\,B\,f) \end{aligned} \tag{6.6}$$

Proof. Given $\Gamma : \mathcal{U}$, $\Phi : \Gamma \to \mathtt{Cof}$, $A : \Gamma|\Phi \to \mathcal{U}$, $B : \Gamma \to \mathcal{U}$, $f : (x : \Gamma)(u : [\Phi\,x]) \to A(x, u) \to B\,x$, $eq : (x : \Gamma)(u : [\Phi\,x]) \to \mathtt{Equiv}(f\,x\,v)$, $\alpha : \mathtt{isFib}\,A$, $\beta : \mathtt{isFib}\,B$, we wish to define an element of type $\mathtt{isFib}(\mathtt{Glue}\,\Phi\,A\,B\,f)$. Therefore, taking

$$e : \{\mathtt{0},\mathtt{1}\},\ p : \mathtt{I} \to \Gamma,\ \psi : \mathtt{Cof},\ q : [\psi] \to \Pi_{\mathtt{I}}(\mathtt{Glue}\,\Phi\,A\,B\,f)$$

$$(a_0, b_0) : \{(a_0, b_0) : (\mathtt{Glue}\,\Phi\,A\,B\,f)(p\,e) \mid (\psi, q) \mathbin{@} e \nearrow (a_0, b_0)\}$$

our goal is to define $(a_1, b_1) : (\mathtt{Glue}\,\Phi\,A\,B\,f)\,x$ such that $(\psi, \tilde{a_1}) \nearrow a_1$ and $(\psi, \tilde{b_1}) \nearrow b_1$, where $x : \Gamma$, $\tilde{a_1} : [\psi] \to ((u : [\Phi\,x]) \to A(x, u))$ and $\tilde{b_1} : [\psi] \to B\ x$ are defined by

$$x \triangleq p\,\overline{e}$$

$$\tilde{a_1}\,v \triangleq \mathtt{fst}(q\ v\ \overline{e})$$

$$\tilde{b_1}\,v \triangleq \mathtt{snd}(q\ v\ \overline{e})$$

and satisfy $\forall (v : [\psi])(u : [\Phi\,x]).\ f\,x\,u\,(\tilde{a_1}\,v\,u) = \tilde{b}\,v$ by the definition of $\mathtt{Glue}\,\Phi\,A\,B\,f$.

We start by composing over p in B to get $b_1' : B\ x$ which can be thought of as a first approximation to b_1:

$$b_1' \triangleq \beta\ e\ p\ \psi\ (\lambda(v : [\psi])(i : \mathtt{I}) \to \mathtt{snd}(q\,v\,i))\ b_0$$

Recall that a_1 will have type $(u : [\Phi\ x]) \to A(x,u)$ and so we assume $u : [\Phi\, x]$ in order to define an element of type $A(x,u)$. Note that we cannot simply compose over p in A because we do not know that $\Phi(p\, i)$ holds for all $i : \mathtt{I}$. Instead we will use the equivalence structure to define a_1.

Let $C \triangleq (a : A(x,u)) \times f\, x\, u\, a \sim b_1'$ be the fibre of $f\, x\, u$ at b_1'. Using Theorems 5.11 and 5.15 and the fact that both A and B are fibrations (as witnessed by α and β respectively) we can deduce that C is a fibrant object. Combined with the fact that $eq\ x\ u\ b_1' : \mathtt{Contr}\, C$ we can use Lemma 6.6 to define $\varepsilon : \mathtt{Ext}\, C$. We can then define

$$(\psi, (\lambda(v : [\psi]) \to \tilde{a_1}\, v\, u,\ \mathtt{refl} \circ \tilde{b_1})) : \Box\, C$$

This is well-defined because, as mentioned above, we have $\forall (v : [\psi]).\ f\ x\ u\ (\tilde{a_1}\, v\, u) = \tilde{b_1}\ v$ and, by the type of composition structures, we have $(\psi, \tilde{b_1}) \nearrow b_1'$ and so given $v : [\psi]$ we have $\mathtt{refl}(\tilde{b_1}\, v) : f\ x\ u\ (\tilde{a_1}\, v\, u) \sim b_1'$. Now we can define

$$\varepsilon\, (\psi, (\lambda(v : [\psi]) \to \tilde{a_1}\, v\, u,\ \mathtt{refl} \circ \tilde{b_1})) : C$$

Discharging our assumption $u : [\Phi\, x]$ and taking first and second projections of the pair defined above we get:

$$a_1 : (u : [\Phi\, x]) \to A(x,u) \qquad p_b : (u : [\Phi\, x]) \to f\, x\, u\, (a_1\, u) \sim b_1'$$

We now have a_1 and b_1' such that $\tilde{a_1} \nearrow a_1$ and $\tilde{b_1} \nearrow b_1'$. However, we cannot simply take b_1 to be b_1' because we do not know that $\forall (u : \Phi\ x).\ f\ x\ u\ (a_1\ u) = b_1'$ and therefore cannot conclude that $(a_1, b_1') : (\mathtt{Glue}\, \Phi\, A\, B\, f)\, x$. In order to solve this problem we perform one final composition in $B\ x$ in order to "correct" b_1' to achieve this property. Consider the following join

$$p_b \cup (\mathtt{refl} \circ \tilde{b_1}) : [\Phi\, x \vee \psi] \to \Pi_{\mathtt{I}}(B\, x)$$

This is well defined because p_b is defined by extending $\mathtt{refl} \circ \tilde{b_1}$ and so they must be equal where they are both defined. We use this to perform one final composition in $B\, x$:

$$b_1 \triangleq \beta\ \mathtt{1}\ (\lambda(\_ : \mathtt{I}) \to x)\ (\Phi\, x \vee \psi)\ (p_b \cup (\mathtt{refl} \circ \tilde{b_1}))\ b_1'$$

We now have $(a_1, b_1) : (\mathtt{Glue}\, \Phi\, A\, B\, f)\, x$ such that $(\psi, \tilde{a_1}) \nearrow a_1$ and $(\psi, \tilde{b_1}) \nearrow b_1$, as required. $\square$

We now have a way to interpret the glueing operation from [CCHM18] that meets some of the necessary requirements; see [CCHM18, Figure 4]. However, the current construction fails the requirement that $\mathtt{Glue}\, \Phi\, A\, B\, f$ should be equal to A when reindexing along the inclusion $\iota : \Gamma|\Phi \rightarrowtail \Gamma$. In fact, this equality should hold in the CwF of CCHM fibrations. This means that not only should $A = (\mathtt{Glue}\, \Phi\, A\, B\, f) \circ \iota : \Gamma|\Phi \to \mathcal{U}$, but also that reindexing the fibration structure derived in Theorem 6.8 should result in the same fibration structure with which we started. To be precise, what we require is:

$$(A, \alpha) = (\mathtt{Glue}\, \Phi\, A\, B\, f, \mathtt{isFib}_{\mathtt{Glue}}\ f\, eq\, \alpha\, \beta)[\iota] \tag{6.7}$$

What we have at present is that the families A and $(\mathtt{Glue}\, \Phi\, A\, B\, f) \circ \iota$ are isomorphic in the following sense:

Definition 6.9 (**Isomorphisms**)**.** Given objects $A, B : \mathcal{U}$, an isomorphism between A and B is a function $f : A \to B$ that has a 2-sided inverse. Let $A \cong B$ be the type of isomorphisms between A and B, defined by

$$A \cong B \triangleq \{f : A \to B \mid (\exists g : B \to A)\, (g \circ f = id) \wedge (f \circ g = id)\}$$

We say that A and B are isomorphic if there exists an isomorphism $f : A \cong B$. This notion lifts to families in the obvious, point-wise way. Isomorphisms have inverses up to the extensional equality of the internal type theory, in contrast to equivalences which only have inverses up to path equality.

We get to (6.7) in two steps. First we use Axiom $\mathtt{ax_9}$ in order to *strictify* the glueing construction to get a new, strict form of glueing, $\mathtt{SGlue}$, such that $\mathtt{Glue}\,\Phi\,A\,B\,f \cong \mathtt{SGlue}\,\Phi\,A\,B\,f$ but where $A = (\mathtt{SGlue}\,\Phi\,A\,B\,f) \circ \iota$. We then use Axiom $\mathtt{ax_8}$ to adapt the fibration structure on $\mathtt{SGlue}$ so that under reindexing along ι it is equal the the fibration structure on A. The order of these steps does not seem to be important; we could equally have first adapted the fibration structure on $\mathtt{Glue}$ and then strictified $\mathtt{Glue}$ with this new fibration structure.

Remark 6.10. Apart from the use of the internal language of a topos, our approach to getting a glueing operation with good properties diverges from that taken by Cohen *et al* [CCHM18], where glueing is defined directly with all the required properties. However it is possible to see where our final two steps occur in the original work. The strictification can be seen in the use of the case split on $\varphi\rho = 1_{\mathbb{F}}$ in [CCHM18, Definition 15]; see Section 8.2 for more details. Rather than defining an initial composition structure for glueing and then modifying it to get the required reindexing property, Cohen *et al* define the composition structure directly. Removing all uses of the $\forall$ operator from [CCHM18, Section 6.2] would yield our initial composition structure, and we then use the closure of $\mathtt{Cof}$ under $\forall(i : \mathtt{I})$ (axiom $\mathtt{ax_8}$) in a separate step to modify this composition. We prefer this approach because it simplifies the core composition structure for glueing and makes more explicit what role $\mathtt{ax_8}$ plays in the construction of a model of cubical type theory.

We now recall axiom $\mathtt{ax_9}$ from Figure 1:

$$\begin{aligned}\mathtt{ax_9} : \{\varphi : \mathtt{Cof}\}(A : [\varphi] \to \mathcal{U})(B : \mathcal{U})(s : (u : [\varphi]) \to (A\,u \cong B)) \to \\ (B' : \mathcal{U}) \times \{s' : B' \cong B \mid \forall(u : [\varphi]).\ A\,u = B' \wedge s\,u = s'\}\end{aligned}$$

This states that any partial type A, which is isomorphic to a total type B everywhere that it is defined, can be extended to a total type B' that is isomorphic to B. We investigate why the cubical presheaf topos [CCHM18] satisfies this axiom in section 8.2. Given $\mathtt{ax_8}$, it is straightforward to define a strict form of glueing.

Definition 6.11 (**Strict glueing**). Given $\Gamma : \mathcal{U}$, $\Phi : \Gamma \to \mathtt{Cof}$, $A : \Gamma|\Phi \to \mathcal{U}$, $B : \Gamma \to \mathcal{U}$ and $f : (x : \Gamma)(u : [\Phi\,x]) \to A(x,u) \to B\,x$, define $\mathtt{SGlue}\,\Phi\,A\,B\,f : \Gamma \to \mathcal{U}$ by

$$\begin{aligned}&\mathtt{SGlue}\,\Phi\,A\,B\,f\,x \triangleq \\ &\quad \mathtt{fst}(\mathtt{ax_9}\,(\lambda u : [\Phi\,x] \to A(x,u))\,(\mathtt{Glue}\ \Phi\,A\,B\,f\,x)\,(\lambda u : [\Phi\,x] \to \mathtt{glue}\,(x,u)) \qquad (6.8)\end{aligned}$$

where $\mathtt{glue}\,(x,u) : A(x,u) \cong \mathtt{Glue}\ \Phi\,A\,B\,f\,x$ is the isomorphism alluded to in Definition 6.1 given by

$$\mathtt{glue}\ (x,u)\ a \triangleq (\lambda\_ \to a, f\,x\,u\,a)$$

Note that $\mathtt{SGlue}$ has the desired strictness property: given any $(x,u) : \Gamma|\Phi$, by $\mathtt{ax_9}$ we have $A(x,u) = \mathtt{fst}(\mathtt{ax_9}\,(\lambda u : [\Phi\,x] \to A(x,u))\,(\mathtt{Glue}\ \Phi\,A\,B\,f\,x)\,(\lambda u : [\Phi\,x] \to \mathtt{glue}\,(x,u)))$ and hence

$$\forall(x : \Gamma)(u : [\Phi\,x]).\ \mathtt{SGlue}\,\Phi\,A\,B\,f\,x = A(x,u) \qquad (6.9)$$

Theorem 6.12 (**Composition structure for strict glueing**). *Given Γ, Φ, A, B, and f as in Definition 6.11, $\mathtt{SGlue}\,\Phi\,A\,B\,f : \Gamma \to \mathcal{U}$ has a fibration structure if A and B have one and f has the structure of an equivalence.*

Proof. It is easy to show that (fibrewise) isomorphisms preserve fibration structures. Hence we obtain a fibration structure on $\texttt{SGlue}$ by transporting the structure obtained from $\texttt{isFib}_{\texttt{Glue}}$ (Theorem 6.8) along the isomorphism from $\texttt{ax}_9$. $\square$

The final step in this section in this section is to use axiom $\texttt{ax}_8$ to adapt the composition structure for $\texttt{SGlue}$ so that we recover the original composition structure on A.

Theorem 6.13 (**Adapting composition structures**). *Given $\Gamma : \mathcal{U}$ and $\Phi : \Gamma \to \texttt{Cof}$, let $\iota : \Gamma|\Phi \rightarrowtail \Gamma$ be as in Definition 6.2. For any $G : \Gamma \to \mathcal{U}$, $\alpha : \texttt{isFib}(G \circ \iota)$ and $\gamma : \texttt{isFib}\, G$, there exists a composition structure $\gamma' : \texttt{isFib}\, G$ such that $\alpha = \gamma'[\iota]$.*

Proof. Given $\Gamma, \Phi, A, \alpha, \gamma$ as above, using axiom $\texttt{ax}_8$ (and $\texttt{ax}_6$) we can define γ' by

$$\gamma'\, e\, p\, \psi\, f\, g \triangleq \gamma\, e\, p\, (\psi \vee (\forall (i : \texttt{I}).\, \Phi\, (p\, i)))\, (f \cup f')\, g \tag{6.10}$$

where $f' : [\forall (i : \texttt{I}).\, \Phi\, (p\, i)] \to \Pi_{\texttt{I}} G$ is given by $f'\, u \triangleq \texttt{fill}\; e\, \alpha\, (\lambda i \to (p, u\, i))\, \psi\, f\, g$. The fact that f and f' are compatible follows from the fact that we use ψ and f in the definition of f' and so by the properties of filling f' must agree with f wherever they are mutually defined. $\square$

Corollary 6.14. *Given $\Gamma : \mathcal{U}$, $\Phi : \Gamma \to \texttt{Cof}$, $(A, \alpha) : \texttt{Fib}(\Gamma|\Phi)$, $(B, \beta) : \texttt{Fib}\,\Gamma$ and $f : (x : \Gamma)(v : [\Phi\, x]) \to A(x, v) \to B\, x$, there exists $(G, \gamma) : \texttt{Fib}\,\Gamma$ such that $(A, \alpha) = (G, \gamma)[\iota]$.*

Proof. Simply take $G = \texttt{SGlue}\, \Phi\, A\, B\, f$; by Theorem 6.12 we get a composition structure for G, which we then adapt using Theorem 6.13 to get a new composition structure γ satisfying the required equality. $\square$

7. Univalence

Voevodsky's *univalence axiom* [Uni13, Section 2.10] for a universe $\mathcal{V}$ in a CwF (with at least Σ-, Π- and $\texttt{Id}$-types) states that for every $A, B : \mathcal{V}$ the canonical function from $\texttt{Id}\,\mathcal{V}\, A\, B$ to $(f : A \to B) \times \texttt{Equiv}\, f$ is an equivalence. Cohen *et al* construct a universe in the (CwF associated to the) presheaf topos of cubical sets whose family of types is generic for CCHM fibrations with small fibres (for a suitable notion of smallness in the meta-theory) and prove that it satisfies the univalence axiom. They do so by adapting the Hofmann-Streicher universe construction for presheaf categories [HS99]. It is not possible to express their universe construction just using the internal type theory of a general topos, for reasons that we discuss below in Remark 7.5. In recent work the authors, along with Licata and Spitters [LOPS18], have shown how to axiomatize the CCHM universe construction in a modal extension of the internal type theory. Here we just prove a version of univalence without reference to a universe of fibrations.

To understand what this might mean, consider the following: were there to be a universe $\mathcal{V}$ whose elements are codes for CCHM fibrations, then given fibrations $(A, \alpha), (B, \beta) : \texttt{Fib}\,\Gamma$ named by functions $a, b : \Gamma \to \mathcal{V}$ into the universe, a path-equality between a and b gives (by Currying) a function $p : \Gamma \times \texttt{I} \to \mathcal{V}$ such that $p(x, \texttt{0}) = a\, x$ and $p(x, \texttt{1}) = b\, x$ for all $x : \Gamma$. Then p names a fibration $(P, \rho) : \texttt{Fib}(\Gamma \times \texttt{I})$ such that $(P, \rho)[\langle id, \texttt{0}\rangle] = (A, \alpha)$ and $(P, \rho)[\langle id, \texttt{1}\rangle] = (B, \beta)$. The latter gives a notion of path-equality between type-families whether or not there is such a $\mathcal{V}$, which we study in this section in relation to equivalences between fibrations.

To give the definitions more formally, we expand our running assumption (see section 2) that the ambient topos $\mathcal{E}$ comes with a universe (internal full subtopos) $\mathcal{U}$ to the case where there is a second universe $\mathcal{U}_1$ with $\mathcal{U} : \mathcal{U}_1$. We sometimes refer to objects of type $\mathcal{U}$ as *small* types and objects of type $\mathcal{U}_1$ as *large* types.

Definition 7.1 (**Path equality between fibrations**). Define the type of paths between CCHM fibrations $\_ \sim_{\mathcal{U}} \_ : \{\Gamma : \mathcal{U}\} \to \mathtt{Fib}\,\Gamma \to \mathtt{Fib}\,\Gamma \to \mathcal{U}_1$ by

$$A \sim_{\mathcal{U}} B \triangleq \{P : \mathtt{Fib}(\Gamma \times \mathtt{I}) \mid P[\langle id, \mathtt{0}\rangle] = A \wedge P[\langle id, \mathtt{1}\rangle] = B\}$$

We show that given such a path $(P, \rho) : (A, \alpha) \sim_{\mathcal{U}} (B, \beta)$ it is always possible to construct an equivalence $f : (x : \Gamma) \to A\,x \to B\,x$. Conversely, given an equivalence $f : (x : \Gamma) \to A\,x \to B\,x$ between fibrations (A, α) and (B, β), it is always possible to construct such a (P, ρ).

Theorem 7.2 (**Converting paths to equivalences**). *There is a function*

$$\mathtt{pathToEquiv} : \{\Gamma : \mathcal{U}\}\{A\ B : \mathtt{Fib}\,\Gamma\}(P : A \sim_{\mathcal{U}} B) \to (f : (x : \Gamma) \to A\,x \to B\,x) \times \mathtt{Equiv}\,f \quad (7.1)$$

Proof. Given $\Gamma : \mathcal{U}, (A, \alpha), (B, \beta) : \mathtt{Fib}\,\Gamma$ and $(P, \rho) : A \sim_{\mathcal{U}} B$ we define maps $f : (x : \Gamma) \to A\,x \to B\,x$ and $g : (x : \Gamma) \to B\,x \to A\,x$. First, given $x : \Gamma$ write $\langle x, id\rangle : (x, \mathtt{0}) \sim (x, \mathtt{1})$ for the path given by $\langle x, id\rangle\, i \triangleq (x, i)$. Now define f and g as follows:

$$f\,x\,a \triangleq \rho\,\mathtt{0}\,\langle x, id\rangle \perp \mathtt{elim}_\emptyset\ a \qquad\qquad g\,x\,b \triangleq \rho\,\mathtt{1}\,\langle x, id\rangle \perp \mathtt{elim}_\emptyset\ b$$

This definition is well-typed since $P(x, \mathtt{0}) = A\,x$ and $P(x, \mathtt{1}) = B\,x$. Since both functions are defined using composition structure we can use filling (Lemma 5.10) to find dependently typed paths:

$$p : (x : \Gamma)(a : A\,x) \to \Pi_{\mathtt{I}} P \text{ defined by } p\,x\,a \triangleq \mathtt{fill}\,\mathtt{0}\,\rho\,\langle x, id\rangle \perp \mathtt{elim}_\emptyset\ a$$

$$q : (x : \Gamma)(b : B\,x) \to \Pi_{\mathtt{I}} P \text{ defined by } q\,x\,b \triangleq \mathtt{fill}\,\mathtt{1}\,\rho\,\langle x, id\rangle \perp \mathtt{elim}_\emptyset\ b$$

Note that for all $x : \Gamma$ and $a : A\,x$ we have $p\,x\,a\,\mathtt{0} = a$ and $p\,x\,a\,\mathtt{1} = f\,x\,a$. Similarly, for all $b : B\,x$ we have $q\,x\,b\,\mathtt{0} = g\,x\,b$ and $q\,x\,b\,\mathtt{1} = b$. Now we define:

$$r : (x : \Gamma)(a : A\,x) \to a \sim g\,x\,(f\,x\,a)$$

$$r\,x\,a\,i \triangleq \rho\,\mathtt{1}\,\langle x, id\rangle\,(i = \mathtt{0} \vee i = \mathtt{1})\,((\lambda\,\_ \to p\,x\,a) \cup (\lambda\,\_ \to q\,x\,(f\,x\,a)))\,(f\,x\,a)$$

$$s : (x : \Gamma)(b : B\,x) \to b \sim f\,x\,(g\,x\,b)$$

$$s\,x\,b\,i \triangleq \rho\,\mathtt{0}\,\langle x, id\rangle\,(i = \mathtt{0} \vee i = \mathtt{1})\,((\lambda\,\_ \to q\,x\,b) \cup (\lambda\,\_ \to p\,x\,(g\,x\,b)))\,(g\,x\,b)$$

Hence f and g are quasi-inverses; from which we can construct an equivalence structure [Uni13, Chapter 4]. □

We now wish to show that, conversely, one can convert equivalences to paths between fibrations. To do so we use the glueing construction given in Section 6.

Theorem 7.3 (**Converting equivalences to paths**). *There is a function*

$$\begin{aligned}\mathtt{equivToPath} : &\{\Gamma : \mathcal{U}\}\{A\ B : \mathtt{Fib}\,\Gamma\}\\ &(f : (x : \Gamma) \to \mathtt{fst}\ A\,x \to \mathtt{fst}\ B\,x) \to (\mathtt{Equiv}\,f) \to A \sim_{\mathcal{U}} B\end{aligned} \quad (7.2)$$

Proof. Define the following:

$$\Phi : \Gamma \times \mathtt{I} \to \mathtt{Cof}$$
$$\Phi\,(x,i) \triangleq (i = 0) \vee (i = 1)$$
$$C : (\Gamma \times \mathtt{I})|\Phi \to \mathcal{U}$$
$$C\,((x,i),u) \triangleq ((\lambda\_ : [i = 0] \to A\,x) \cup (\lambda\_ : [i = 1] \to B\,x))\,u$$
$$f' : ((x,i) : \Gamma \times \mathtt{I})(u : [\Phi(x,i)]) \to C((x,i),u) \to B\,x$$
$$f'\,(x,i) \triangleq (\lambda\_ : [i = 0] \to f\,x) \cup (\lambda\_ : [i = 1] \to id)$$

Now let $P \triangleq \mathtt{SGlue}\ \Phi\,C\,(\lambda(x,\_) \to B\,x)\,f'$ and observe that $P(x,\mathtt{0}) = A\,x$ and $P(x,\mathtt{1}) = B\,x$ by the strictness property of $\mathtt{SGlue}$.

Now we show that P has a fibration structure. First, we observe that C has a fibration structure, using axiom $\mathtt{ax_1}$. In order to define $\gamma : \mathtt{isFib}\ C$ we take:

$$e : \{\mathtt{0},\mathtt{1}\}, \quad p : \mathtt{I} \to (\Gamma \times \mathtt{I})|\Phi, \quad \varphi : \mathtt{Cof}, \quad f : [\varphi] \to \Pi(C \circ p), \quad c : C\,(p\,e)$$

and aim to define $\gamma\,e\,p\,\varphi\,f\,c : C(p\,\overline{e})$. First, define the predicate $\mathtt{pZero} : \mathtt{I} \to \Omega$ by:

$$\mathtt{pZero}\,i \triangleq (\mathtt{snd}(\mathtt{fst}(p\,i)) = \mathtt{0})$$

Observe that since $p : \mathtt{I} \to (\Gamma \times \mathtt{I})|\Phi$ we know, from $\mathtt{snd} \circ p$, that:

$$(\forall i : \mathtt{I})\ \mathtt{snd}(\mathtt{fst}(p\,i)) = \mathtt{0} \vee \mathtt{snd}(\mathtt{fst}(p\,i)) = \mathtt{1} \tag{7.3}$$

and so, using $\mathtt{ax_2}$, we have $(\forall i : \mathtt{I})\,(\mathtt{pZero}\,i \vee \neg(\mathtt{pZero}\,i))$ and using $\mathtt{ax_1}$ we get

$$((\forall i : \mathtt{I})\,\mathtt{pZero}\,i) \vee ((\forall i : \mathtt{I})\,\neg(\mathtt{pZero}\,i))$$

In case $(\forall i : \mathtt{I})\,\mathtt{pZero}\,i$, we deduce that $C \circ p = A \circ \mathtt{fst} \circ \mathtt{fst} \circ p$ and define:

$$\gamma\,e\,p\,\varphi\,f\,c \triangleq \alpha\,e\,(\mathtt{fst} \circ \mathtt{fst} \circ p)\,\varphi\,f\,c$$

Otherwise, in case $(\forall i : \mathtt{I})\,\neg(\mathtt{pZero}\,i)$, we use $\mathtt{ax_2}$ to deduce $(\forall i : \mathtt{I})\ \mathtt{snd}(\mathtt{fst}(p\,i)) = \mathtt{1}$, hence $C \circ p = B \circ \mathtt{fst} \circ \mathtt{fst} \circ p$, and so define:

$$\gamma\,e\,p\,\varphi\,f\,c \triangleq \beta\,e\,(\mathtt{fst} \circ \mathtt{fst} \circ p)\,\varphi\,f\,c$$

Therefore C has a fibration structure. Next we show that f' is an equivalence for every $x : \Gamma$, $i : \mathtt{I}$ and $u : [\Phi\,i]$. First note that the identity function $id : B\,x \to B\,x$ is always an equivalence; let $idEq : \mathtt{Equiv}\ id$ be a proof of this fact. Define $eq' : (x : \Gamma)(i : \mathtt{I})(u : [\Phi\,i]) \to \mathtt{Equiv}\,(f'\,x\,i\,u)$ by:

$$eq'x\ i \triangleq (\lambda u : [i = \mathtt{0}] \to eq) \cup (\lambda u : [i = \mathtt{1}] \to idEq)$$

Hence, by Corollary 6.14, we get a fibration structure, $\rho : \mathtt{isFib}\,P$, such that

$$(P,\rho)[\iota : (\Gamma \times \mathtt{I})|\Phi \rightarrowtail \Gamma \times \mathtt{I}] = (C,\gamma)$$

and hence $(P,\rho)[\langle id,\mathtt{0}\rangle] = (P,\rho)[\iota \circ \langle id,\mathtt{0},*\rangle] = (C,\gamma)[\langle id,\mathtt{0},*\rangle] = (A,\alpha)$ and similarly $(P,\rho)[\langle id,\mathtt{1}\rangle] = (B,\beta)$. Therefore we define $\mathtt{equivToPath}\ f\,e \triangleq (P,\rho)$. □

The univalence axiom [Uni13, Axiom 2.10.3] becomes here the property that the map `pathToEquiv` is itself an equivalence. A result by Licata [unpublished] tells us that this is actually the same as having a map from equivalences to paths, which we have in `equivToPath`, such that coercion along $\mathtt{equivToPath}\ f\,e$ is path equal to f. This property does indeed hold:

Theorem 7.4 (**Univalence for** $\sim_{\mathcal{U}}$)**.** *Define*

$$\texttt{coerce} : \{\Gamma : \mathcal{U}\}\{A\ B : \texttt{Fib}\,\Gamma\}(P : A \sim_{\mathcal{U}} B)(x : \Gamma) \to \texttt{fst}\ A\,x \to \texttt{fst}\ B\,x$$

$$\texttt{coerce}(P, \rho) \triangleq \texttt{fst}(\texttt{pathToEquiv}(P, \rho))$$

Given $\Gamma : \mathcal{U}$*,* $(A, \alpha), (B, \beta) : \texttt{Fib}\,\Gamma$*,* $f : (x : \Gamma) \to A\,x \to B\,x$ *and* $eq : \texttt{Equiv}\ f$*, there exists a path* $f \sim \texttt{coerce}(\texttt{equivToPath}\ f\,e)$*.*

Proof. Let $(P, \rho) \triangleq \texttt{equivToPath}\ f\,e$. Unfolding the definition of $\texttt{coerce}$ we have

$$\texttt{coerce}\,(P, \rho)\,x\,a = \rho\,\texttt{0}\,\langle x, id\rangle\perp \texttt{elim}_{\emptyset}\ a$$

Recalling that ρ is the composition structure for $\texttt{SGlue}$ we can calculate $\rho\,\texttt{0}\,\langle x, id\rangle \perp \texttt{elim}_{\emptyset}\ a$. We first move across the isomorphism with $\texttt{Glue}$ so that a becomes $(\lambda\_ \to a, f\,x\,a)$. We now trace the algorithm for composition in $\texttt{Glue}$: we begin by composing in $B\,x$ to get $b'_1 \triangleq \beta\,\texttt{0}\,(\lambda\_ \to x) \perp \texttt{elim}_{\emptyset}\,(f\,x\,a)$. Next, we use the equivalence structure to derive $a_1 : (u : [\texttt{1} = \texttt{0} \vee \texttt{1} = \texttt{1}]) \to B\,x$ and $p_b : (u : [\texttt{1} = \texttt{0} \vee \texttt{1} = \texttt{1}]) \to id\,(a_1\,u) \sim b'_1$, where in particular, a_1 is given by $a_1 u \triangleq \beta\,\texttt{1}\,(\lambda\_ \to x) \perp \texttt{elim}_{\emptyset}\ b'_1$. We then perform the final step of the composition, which is to compose from b'_1 in $B\,x$ to get b_1. This leaves us with the result of composing in $\texttt{Glue}$ as (a_1, b_1). When transferring back across the isomorphism we simply take the first component of the pair, namely a_1, but now regarded as a total element to get the final result of composing in $\texttt{SGlue}$ as $\beta\,\texttt{1}\,(\lambda\_ \to x) \perp \texttt{elim}_{\emptyset}\ b'_1$. So, in summary we have:

$$\begin{aligned}\texttt{coerce}\,(P, \rho)\,x\,a &= \rho\,\texttt{0}\,\langle x, id\rangle \perp \texttt{elim}_{\emptyset}\ a\\ &= \beta\,\texttt{1}\,(\lambda\_ \to x) \perp \texttt{elim}_{\emptyset}\ b'_1\\ &= \beta\,\texttt{1}\,(\lambda\_ \to x) \perp \texttt{elim}_{\emptyset}\,(\beta\,\texttt{0}\,(\lambda\_ \to x) \perp \texttt{elim}_{\emptyset}\,(f\,x\,a))\end{aligned}$$

Since this is simply two trivial compositions applied to $f\,x\,a$ we can use use filling to derive a path $f\,x\,a \sim \texttt{coerce}\,(P, \rho)\,x\,a$. Now, two applications of function extensionality (Remark 5.16) yields a path $f \sim \texttt{coerce}\,(P, \rho)$ as required. □

Remark 7.5. Suppose we have a "Tarski-style" universe in the CwF of CCHM fibrations (Definition 5.8), that is, a (large) fibrant object $\mathcal{V} : \mathcal{U}_1$ and a fibration $(\mathcal{E}\ell, \eta) : \texttt{Fib}\,\mathcal{V}$ (with small fibres, $\mathcal{E}\ell : \mathcal{V} \to \mathcal{U}$). We can regard the elements $a : \mathcal{V}$ as *codes* for small types $\mathcal{E}\ell\,a : \mathcal{U}$ and ask that the universe have functions on codes for type-forming constructs of interest, such as Σ, Π and $\texttt{Id}$; *cf.* [Hof97, Section 2.1.6]). Re-indexing $(\mathcal{E}\ell, \eta)$ along a function $a : \Gamma \to \mathcal{V}$ gives a fibration $(\mathcal{E}\ell \circ a, \eta[a]) : \texttt{Fib}\,\Gamma$; and re-indexing turns paths in $\mathcal{V}$ into path equalities between fibrations in the sense of Definition 7.1. In order to use Theorem 7.4 to deduce that such a universe is univalent, one would at least need to construct functions

$$(\mathcal{E}\ell \circ a, \eta[a]) \sim_{\mathcal{U}} (\mathcal{E}\ell \circ b, \eta[b]) \to (a \sim b) \qquad (\text{for } a, b : \Gamma \to \mathcal{V})$$

converting path equalities between fibrations arising from codes back into paths between the codes themselves. For the universe in the presheaf topos of cubical sets constructed by Cohen *el al* [CCHM18, Definition 18], the existence of such functions with good properties follows from the fact that the universe is generic, in the sense that for every object Γ and every CCHM fibration (A, α) over Γ, there is a morphism $a = \ulcorner A, \alpha \urcorner : \Gamma \to \mathcal{V}$ with $(A, \alpha) = (\mathcal{E}\ell \circ a, \eta[a])$. One difficulty for our approach of using the internal type-theoretic language of a topos is that this genericity is an external, or global, property: postulating an internal version of it, i.e. the existence of a function

$$\ulcorner\_, \_\urcorner : \{\Gamma : \mathcal{U}_1\}(A : \Gamma \to \mathcal{U}_1)(\alpha : \texttt{isFib}\,A) \to \Gamma \to \mathcal{V}$$

such that

$$(\mathcal{E}\ell \circ \ulcorner A, \alpha \urcorner, \eta[\ulcorner A, \alpha \urcorner]) = (A, \alpha)$$

leads to contradiction. To see why, recall Remark 5.9 which points out that a family of fibrant objects is not necessarily fibrant. Were we to assume the existence of such a map $\ulcorner \_, \_ \urcorner$, then we could derive a fibration structure for a family of fibrant objects like so: given an object $\Gamma : \mathcal{U}$, and family $A : \Gamma \to \mathcal{U}$ write $A_x : 1 \to \mathcal{U}$, where $x : \Gamma$, for the object $A\,x$ as a family over 1, i.e. $A_x \triangleq \lambda(\_ : 1) \to A\,x$. If A has fibrant fibres then we have a map $\alpha_{\_} : (x : \Gamma) \to \mathtt{Fib}\, A_x$, so that $\alpha_x : \mathtt{Fib}\, A_x$ is a fibration structure for the object $A\,x$ regarded as a family over 1 (for $x : \Gamma$). Then we can define a function $a : \Gamma \to \mathcal{V}$ as follows:

$$a\,x \triangleq \ulcorner A_x, \alpha_x \urcorner(*)$$

(where $*$ is the unique element of the terminal type 1). It can be checked that $El \circ a = A$ and therefore $\eta[a] : \mathtt{Fib}\, A$. Hence we have a fibration structure for A whenever A has fibrant fibres; but this gives a contradiction as in Remark 5.9. This issue can be resolved by moving to a different internal type theory where one has access to a modal operator which is able to express properties of global, rather than local, elements [LOPS18].

8. Satisfying the Axioms

Working informally in a constructive set theory, the authors of [CCHM18] give a model of their type theory using the topos $\hat{\mathcal{C}} = \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ of contravariant set-valued functors on a particular small category $\mathcal{C}$ that they call the *category of cubes*. In this section we present sufficient conditions on an arbitrary small category $\mathbf{C}$ for the topos $\hat{\mathbf{C}} = \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ of set-valued presheaves (within Intuitionistic ZF set theory [AR10, Section 3.2], say) to have an interval object and subobject of cofibrant propositions satisfying the axioms in Figure 1. We show that the category of cubes is an instance of such a $\mathbf{C}$ (in more than one way); and we also show that in the presence of the Law of Excluded Middle, so is the simplex category $\mathbf{\Delta}$.

We begin by briefly recalling the definition of $\mathcal{C}$ from [CCHM18, Section 8]. Recall that a *De Morgan algebra* is a distributive lattice equipped with a function $d \mapsto \mathtt{1}$ - d which is involutive $\mathtt{1}$ - $(\mathtt{1}$ - $d) = d$ and satisfies De Morgan's Law $\mathtt{1}$ - $(d_1 \vee d_2) = (\mathtt{1}$ - $d_1) \wedge (\mathtt{1}$ - $d_2)$; see [BD75, Chapter XI]. A homomorphism of De Morgan algebras is a function preserving finite meets and joins and the involution function. Let $\mathbf{DM}$ denote the category of De Morgan algebras and homomorphisms and $\mathsf{dM}(I)$ the free De Morgan algebra on a set I.

Definition 8.1 (**The category of cubes,** $\mathcal{C}$)**.** Fix a countably infinite set $\mathbb{D}$ whose elements we call *directions* and write as $\mathsf{x}, \mathsf{y}, \mathsf{z}, \ldots$ The objects of $\mathcal{C}$ are the finite subsets of $\mathbb{D}$, which we write as $I, J, K, \ldots$ The morphisms $\mathcal{C}(I, J)$ are all functions $J \to \mathsf{dM}(I)$. Such functions are in bijection with the De Morgan algebra homomorphisms $\mathsf{dM}(J) \to \mathsf{dM}(I)$ and the composition in $\mathcal{C}$ of $f \in \mathcal{C}(I, J)$ with $g \in \mathcal{C}(J, K)$ is the composition in $\mathbf{Set}$ of $g : K \to \mathsf{dM}(J)$ with the homomorphism $\mathsf{dM}(J) \to \mathsf{dM}(I)$ corresponding to f.

Thus $\mathcal{C}$ is equivalent to the opposite of the full subcategory of $\mathbf{DM}$ whose objects are the free, finitely generated De Morgan algebras. Hence it is the algebraic theory of De Morgan algebra as a Lawvere theory [Law63, ARV10]: it is a category with finite products equipped with an internal De Morgan algebra (whose underlying object is $\{\mathsf{x}\}$ for some chosen $\mathsf{x} \in \mathbb{D}$) and is universal among such categories.

8.1. **The interval object ($\mathtt{ax}_1$–$\mathtt{ax}_4$).** Returning to the general case of a small category $\mathbf{C}$ with its associated presheaf topos $\hat{\mathbf{C}}$, if we take the interval object $\mathtt{I} \in \hat{\mathbf{C}}$ to be a representable functor $\mathrm{y}_{\mathtt{i}} \triangleq \mathbf{C}(\_\,, \mathtt{i})$ for some object $\mathtt{i} \in \mathbf{C}$, then the following theorem gives a useful criterion for such an interval object to satisfy axiom $\mathtt{ax}_1$.

Theorem 8.2. *In a presheaf topos $\hat{\mathbf{C}}$, a representable functor $\mathtt{I} = \mathrm{y}_{\mathtt{i}}$ satisfies axiom $\mathtt{ax}_1$ if $\mathbf{C}$ is a* cosifted *category, that is, if finite products in* **Set** *commute with colimits over $\mathbf{C}^{\mathrm{op}}$* [GU71].

Proof. $\mathbf{C}$ is cosifted if the colimit functor $\mathrm{colim}_{\mathbf{C}^{\mathrm{op}}} : \hat{\mathbf{C}} \to \mathbf{Set}$ preserves finite products. Recall that $\mathrm{colim}_{\mathbf{C}^{\mathrm{op}}} : \hat{\mathbf{C}} \to \mathbf{Set}$ is left adjoint to the constant presheaf functor $\Delta : \mathbf{Set} \to \hat{\mathbf{C}}$ and (hence) that for any $c \in \mathbf{C}$ it is the case that $\mathrm{colim}_{\mathbf{C}^{\mathrm{op}}}\, \mathrm{y}_c \cong 1$. So when $\mathbf{C}$ is cosifted we have for any $c \in \mathbf{C}$

$$\begin{aligned}\hat{\mathbf{C}}(\mathrm{y}_c \times \mathrm{y}_{\mathtt{i}}, \Delta\{0,1\}) &\cong \mathbf{Set}(\mathrm{colim}_{\mathbf{C}^{\mathrm{op}}}(\mathrm{y}_c \times \mathrm{y}_{\mathtt{i}}), \{0,1\}) \cong \\ &\qquad \mathbf{Set}(\mathrm{colim}_{\mathbf{C}^{\mathrm{op}}}\, \mathrm{y}_c \times \mathrm{colim}_{\mathbf{C}^{\mathrm{op}}}\, \mathrm{y}_{\mathtt{i}}, \{0,1\}) \cong \mathbf{Set}(1 \times 1, \{0,1\}) \cong \{0,1\}\end{aligned}$$

Since decidable subobjects in $\hat{\mathbf{C}}$ are classified by $1 + 1 = \Delta\{0,1\}$, this means that the only two decidable subobjects of $\mathrm{y}_c \times \mathrm{y}_{\mathtt{i}}$ are the smallest and the greatest subobjects. Since this is so for all $c \in \mathbf{C}$, it follows that $\mathtt{I} = \mathrm{y}_{\mathtt{i}}$ satisfies $\mathtt{ax}_1$. □

A more elementary characterisation of cosiftedness is that $\mathbf{C}$ is inhabited and for every pair of objects $c, c' \in \mathbf{C}$ the category of spans $c \leftarrow \cdot \rightarrow c'$ is a connected category [ARV10, Theorem 2.15]. Any category with finite products trivially has this property. This is the case for the category $\mathcal{C}$ of cubes defined above and thus the interval in the model of [CCHM18] (where $\mathbf{C} = \mathcal{C}$ and $\mathtt{i}$ is the generic De Morgan algebra) satisfies $\mathtt{ax}_1$. A relevant example of a category that does not have finite products, but which is nevertheless cosifted is $\boldsymbol{\Delta}$, the category of inhabited finite linearly ordered sets $[0 < 1 < \cdots < n]$, for which $\hat{\mathbf{C}}$ is the category of *simplicial sets*, widely used in homotopy theory [GJ09]. Thus the natural candidate for an interval in $\hat{\boldsymbol{\Delta}}$, namely $\mathrm{y}_{\mathtt{i}}$ when $\mathtt{i}$ is the 1-simplex $[0 < 1]$, satisfies $\mathtt{ax}_1$.

In addition to $\mathtt{ax}_1$, the other axioms in Figure 1 concerning the interval say that $\mathtt{I}$ is a non-trivial ($\mathtt{ax}_2$) model of the algebraic theory given by $\mathtt{ax}_3$ and $\mathtt{ax}_4$, which we call *connection algebra*. (See also Definition 1.7 of [GS17], which considers a similar notion in a more abstract setting.) For cubical sets, the Yoneda embedding $\mathrm{y} : \mathcal{C} \to \hat{\mathcal{C}}$ sends the generic De Morgan algebra in $\mathcal{C}$ to a De Morgan algebra in $\hat{\mathcal{C}}$. This is a non-trivial connection algebra: the constants are the least and greatest elements and the binary operations are meet and join. An obvious variation on the theme of [CCHM18] would be to replace $\mathcal{C}$ by the Lawvere theory for connection algebras. Note also that the 1-simplex in $\hat{\boldsymbol{\Delta}}$ is a non-trivial connection algebra, the constants being its two end points and the binary operations being induced by the order-preserving binary operations of minimum and maximum on $[0 < 1]$.

8.2. **Cofibrant propositions ($\mathtt{ax}_5$–$\mathtt{ax}_8$) and the strictness axiom ($\mathtt{ax}_9$).** In a topos with an interval object, there are many candidates for a subobject $\mathtt{Cof} \rightarrowtail \Omega$ satisfying axioms $\mathtt{ax}_5$–$\mathtt{ax}_8$ in Figure 1. At one extreme, one could just take $\mathtt{Cof}$ to be the whole of Ω. At the opposite extreme, one could take the subobject (internally) inductively defined by the requirements that it contains the propositions $i = \mathtt{0}$ and $i = \mathtt{1}$ (for all $i : \mathtt{I}$) and is closed under binary disjunction, dependent and $\mathtt{I}$-indexed conjunction, thereby obtaining the smallest $\mathtt{Cof}$ satisfying $\mathtt{ax}_5$–$\mathtt{ax}_8$. However, cofibrant propositions also have to satisfy the strictness axiom $\mathtt{ax}_9$ and we consider that next.

Given a presheaf topos $\hat{\mathbf{C}}$, we work in the CwF associated with $\hat{\mathbf{C}}$ as in [Hof97, Section 4]. In particular, families over a presheaf $\Gamma \in \hat{\mathbf{C}}$ are given by functors $(\int \Gamma)^{\mathrm{op}} \to \mathbf{Set}$, where $\int \Gamma$ is the usual category of elements of Γ, with $\mathrm{obj}(\int \Gamma) = (c \in \mathrm{obj}\,\mathbf{C}) \times \Gamma\, c$ and $(\int \Gamma)((c,x),(d,y)) = \{f \in \mathbf{C}(c,d) \mid \Gamma\, f\, y = x\}$. If $\mathcal{S}$ is a Grothendieck universe in the ambient set theory, then its Hofmann-Streicher lifting [HS99] to a universe $\mathcal{U}$ in that CwF satisfies that the morphisms $\Gamma \to \mathcal{U}$ in $\hat{\mathbf{C}}$ name the families $(\int \Gamma)^{\mathrm{op}} \to \mathcal{S}$ taking values in $\mathcal{S} \subseteq \mathbf{Set}$.

Definition 8.3 (Ω_{dec})**.** The subobject classifier Ω in a presheaf topos $\hat{\mathbf{C}}$ maps each $c \in \mathbf{C}$ to the set $\Omega(c)$ of *sieves* on c, that is, pre-composition closed subsets $S \subseteq \mathrm{obj}(\mathbf{C}/c)$. Let $\Omega_{\mathrm{dec}} \rightarrowtail \Omega$ be the subpresheaf whose value at each $c \in \mathrm{obj}\,\mathbf{C}$ is the subset of $\Omega(c)$ consisting of those sieves S that are decidable subsets of $\mathrm{obj}(\mathbf{C}/c)$.

Of course if the ambient set theory satisfies the Law of Excluded Middle, then $\Omega_{\mathrm{dec}} = \Omega$. In general Ω_{dec} classifies monomorphisms $\alpha : F \rightarrowtail G$ in $\hat{\mathbf{C}}$ such that for all $c \in \mathrm{obj}\,\mathbf{C}$ the (injective) function $\alpha_c : F\,c \to G\,c$ has decidable image.

Theorem 8.4. *Interpreting the universe $\mathcal{U}$ as the Hofmann-Streicher lifting* [HS99] *of a Grothendieck universe in* **Set***, a subobject* `Cof` $\rightarrowtail \Omega$ *in a presheaf topos $\hat{\mathbf{C}}$ satisfies the strictness axiom* $\mathtt{ax}_9$ *if it is contained in $\Omega_{\mathrm{dec}} \rightarrowtail \Omega$.*

Proof. For each $c \in \mathrm{obj}\,\mathbf{C}$, suppose we are given $S \in \Omega_{\mathrm{dec}}(c)$. Thus S is a sieve on c and for each $c' \in \mathrm{obj}\,\mathbf{C}$ and $\mathbf{C}$-morphism $f : c' \to c$, it is decidable whether or not $f \in S$. We can also regard S as a subpresheaf $S \hookrightarrow \mathrm{y}_c$.

Suppose that we have families $A : (\int S)^{\mathrm{op}} \to \mathcal{S}$, $B : (\int \mathrm{y}_c)^{\mathrm{op}} \to \mathcal{S}$ and a natural isomorphism s between A and the restriction of B along $S \hookrightarrow \mathrm{y}_c$. For each $\mathbf{C}$-morphism $\cdot \xrightarrow{f} c$, using the decidability of S, we can define bijections $s'(f) : B'(f) \cong B(f)$ given by

$$B'(f) \triangleq \begin{cases} A(f) & \text{if } f \in S \\ B(f) & \text{if } \neg(f \in S) \end{cases} \qquad \text{and} \qquad s'(f) \triangleq \begin{cases} s(f) & \text{if } f \in S \\ f & \text{if } \neg(f \in S) \end{cases}$$

(compare this with Definition 15 in [CCHM18]). We make B' into a functor $(\int \mathrm{y}_c)^{\mathrm{op}} \to \mathcal{S}$ by transferring the functorial action of B across these bijections. Having done that, s' becomes a natural isomorphism $B' \cong B$; and by definition its restriction along $S \hookrightarrow \mathrm{y}_c$ is s. □

Corollary 8.5. *Let* $\mathbf{C}$ *be a small category with finite products containing an object* `i` *with the structure of a non-trivial connection algebra* $0, 1 : 1 \to \mathtt{i}$, $\sqcap, \sqcup : \mathtt{i} \times \mathtt{i} \to \mathtt{i}$ *(cf. Figure 1). Suppose that for each object $c \in \mathbf{C}$, the set $\mathbf{C}(c, \mathtt{i})$ has decidable equality. Then the topos of presheaves $\hat{\mathbf{C}}$ satisfies all the axioms in Figure 1 if we take the interval* `I` *to be* $\mathrm{y}_{\mathtt{i}}$ *and* `Cof` *to be* Ω_{dec}*.*

Proof. We already noted in section 8.1 that axioms $\mathtt{ax}_1$–$\mathtt{ax}_4$ are satisfied by this choice of `I`. The decidability of each set $\mathbf{C}(c, \mathtt{i})$ implies that the subobjects $\{0\} \rightarrowtail \mathtt{I}$ and $\{1\} \rightarrowtail \mathtt{I}$ in $\hat{\mathbf{C}}$ factor through $\Omega_{\mathrm{dec}} \rightarrowtail \Omega$ and hence that axiom $\mathtt{ax}_5$ is satisfied when $\mathtt{Cof} = \Omega_{\mathrm{dec}}$. Note that this choice of `Cof` automatically satisfies $\mathtt{ax}_6$ and $\mathtt{ax}_7$; and it satisfies axiom $\mathtt{ax}_9$ by Theorem 8.4. So it just remains to check that axiom $\mathtt{ax}_8$ is satisfied. We saw in Lemma 5.4(ii) that this axiom is equivalent to requiring cofibrations to be closed under exponentiation by `I`. In this case cofibrations are the monomorphisms $\alpha : F \rightarrowtail G$ classified by Ω_{dec} and we noted after Definition 8.3 that they are characterized by the fact that each function $\alpha_c \in \mathbf{Set}(F\,c, G\,c)$ has decidable image. Closure of these monomorphisms under exponentiating by $\mathtt{I}$ follows from the fact that $\mathbf{C}$ has finite products and that $\mathtt{I} = \mathrm{y}_{\mathtt{i}}$ is representable; for then $\mathtt{I} \to (\_)$ is isomorphic to the functor $\hat{\mathbf{C}} \to \hat{\mathbf{C}}$ induced by precomposition with $(\_) \times \mathtt{i} : \mathbf{C} \to \mathbf{C}$, which clearly preserves the componentwise decidable image property. □

The above argument for axiom $\mathtt{ax_8}$ does not apply to $\mathbf{\Delta}$, since it does not have finite products; we do not know whether that axiom is satisfied by constructive simplicial sets. However, in the presence of the Law of Excluded Middle (LEM), $\Omega_{\mathrm{dec}} = \Omega$ and we have:

Corollary 8.6 (Classical simplicial sets). *Assuming LEM holds in the set-theoretic metatheory, then the presheaf topos of simplicial sets $\hat{\mathbf{\Delta}}$ satisfies the axioms in Figure 1 if we take* $\mathtt{I}$ *to be the representable presheaf on the 1-simplex and* $\mathtt{Cof}$ *to be the whole of* Ω.

Proof. We already noted in section 8.1 that axioms $\mathtt{ax_1}$–$\mathtt{ax_4}$ are satisfied by this choice of $\mathtt{I}$. If $\mathtt{Cof} = \Omega$ (that is, $\mathtt{cof} = \lambda\_ \to \top$), then axioms $\mathtt{ax_5}$–$\mathtt{ax_8}$ hold trivially. Furthermore, if LEM holds, then $\Omega_{\mathrm{dec}} = \Omega$ and so axiom $\mathtt{ax_9}$ holds by Theorem 8.4. □

Remark 8.7. As a partial converse of Theorem 8.4, we have that if $\mathtt{ax_9}$ is satisfied by the Hofmann-Streicher universe in the CwF associated with $\hat{\mathbf{C}}$, then each cofibrant mono $\alpha : F \rightarrowtail G$ has component functions $\alpha_c \in \mathbf{Set}(F\,c, G\,c)$ $(c \in \operatorname{obj}\mathbf{C})$ whose images are $\neg\neg$-closed subsets of $G\,c$. To see this we can apply an argument due to Andrew Swan [private communication] that relies upon the fact that in the ambient set theory one has

$$(X = \emptyset) \;=\; \forall x \in X.\, \bot \;=\; \neg\neg(\forall x \in X.\, \bot) \;=\; \neg\neg(X = \emptyset) \tag{8.1}$$

For suppose given $c \in \operatorname{obj}\mathbf{C}$ and $S \in \mathtt{Cof}(c)$. We have to use axiom $\mathtt{ax_9}$ to show that S is a $\neg\neg$-closed subset of $\operatorname{obj}(\mathbf{C}/c)$. Let $A : (\int S)^{\mathrm{op}} \to \mathcal{S}$ be the constant functor mapping each (c', f) to $\{\emptyset\}$; and let $B : (\int \mathrm{y}_c)^{\mathrm{op}} \to \mathcal{S}$ map each (c', f) to $\{\{\emptyset\}, \{\emptyset \mid f \in S\}\}$ (which does extend to a functor, because S is a sieve). The restriction of B along $S \hookrightarrow \mathrm{y}_c$ is isomorphic to A and so by $\mathtt{ax_9}$ there is some $B' : (\int \mathrm{y}_c)^{\mathrm{op}} \to \mathcal{S}$ whose restriction along $S \hookrightarrow \mathrm{y}_c$ is equal to A and some isomorphism $s' : B' \cong B$. For any $(c', f) \in \operatorname{obj}(\int \mathrm{y}_c)$, suppose $X \in B'(c', f)$; then $f \in S \Rightarrow X = \emptyset$, hence $\neg\neg(f \in S) \Rightarrow \neg\neg(X = \emptyset)$ and therefore by (8.1), $\neg\neg(f \in S) \Rightarrow (X = \emptyset)$. Therefore $\neg\neg(f \in S) \Rightarrow B'(c', f) = \{\emptyset\} \Rightarrow B(c', f) \cong \{\emptyset\} \Rightarrow f \in S$. So S is indeed a $\neg\neg$-closed subset of $\operatorname{obj}(\mathbf{C}/c)$.

Note that this result implies that it is not possible to take $\mathtt{Cof}$ to be the whole of Ω and satisfy $\mathtt{ax_9}$ unless the ambient set theory satisfies LEM.

Since in a constructive setting equality in free De Morgan algebras is decidable, it follows that Corollary 8.5 gives a model of our axioms when $\mathbf{C} = \mathcal{C}$, the category of cubes (Definition 8.1). However, this uses a different choice of cofibrancy from the one in [CCHM18, Section 4.1]. In the remainder of this section we check that the CCHM notion of fibrancy satisfies our axioms.

Definition 8.8 (**Cofibrant propositions in** [CCHM18])**.** For each object of the category of cubes $I \in \operatorname{obj}\mathcal{C}$, Cohen *et al* define the *face lattice* $\mathbb{F}(I)$ to be the distributive lattice generated by symbols $(\mathsf{x} = \mathtt{0})$ and $(\mathsf{x} = \mathtt{1})$ for each $\mathsf{x} \in I$, subject to the equations $(\mathsf{x} = \mathtt{0}) \wedge (\mathsf{x} = \mathtt{1}) = \bot$.

Since the free De Morgan algebra $\mathsf{dM}(I)$ is freely generated as a distributive lattice by symbols x and $\mathtt{1}\text{-}\mathsf{x}$ (as x ranges over the finite set I), we can regard $\mathbb{F}(I)$ as a quotient lattice of $\mathsf{dM}(I)$ via the function mapping x to $(\mathsf{x} = \mathtt{1})$ and $\mathtt{1}\text{-}\mathsf{x}$ to $(\mathsf{x} = \mathtt{0})$; we write $q_I : \mathsf{dM}(I) \to \mathbb{F}(I)$ for the quotient function.[2] It is not hard to see that for each $f \in \mathbf{C}(J, I)$,

[2] If $r \in \mathsf{dM}(I)$, then [CCHM18] uses the notation $(r = \mathtt{1})$ for $q_I(r)$.

the corresponding De Morgan algebra homomorphism $\mathsf{dM}(I) \to \mathsf{dM}(J)$ (which we also write as f) induces a lattice homomorphism between the face lattices:

$$\begin{tikzcd} \mathsf{dM}(I) \arrow[r, "f"] \arrow[d, "q_I"'] & \mathsf{dM}(J) \arrow[d, "q_J"] \\ \mathbb{F}(I) \arrow[r, "\mathbb{F}\,f"'] & \mathbb{F}(J) \end{tikzcd}$$

This makes $\mathbb{F}$ into an object of $\hat{\mathcal{C}}$ and there is a monomorphism $m : \mathbb{F} \rightarrowtail \Omega$ whose component at $I \in \operatorname{obj}\mathcal{C}$ sends each $\varphi \in \mathbb{F}(I)$ to the sieve

$$m_I(\varphi) = \{\cdot \xrightarrow{f} I \mid \mathbb{F}\,f\,\varphi = \top\} \tag{8.2}$$

We now take $\mathtt{Cof} \rightarrowtail \Omega$ in $\hat{\mathcal{C}}$ to be the subobject given by this monomorphism $m : \mathbb{F} \rightarrowtail \Omega$. Axioms $\mathtt{ax_1}$–$\mathtt{ax_4}$ hold without change from Corollary 8.5; and axioms $\mathtt{ax_5}$–$\mathtt{ax_7}$ follow from the definition of the face lattices $\mathbb{F}(I)$. So it just remains to check axioms $\mathtt{ax_8}$ and $\mathtt{ax_9}$.

Recall that the interval object in $\hat{\mathcal{C}}$ is the representable presheaf $\mathtt{I} = \mathrm{y}_{\{\mathsf{x}\}}$ on a one-element subset $\{\mathsf{x}\} \in \operatorname{obj}\mathcal{C}$. For an arbitrary object $I \in \operatorname{obj}\mathcal{C}$, with n distinct elements $\mathsf{x}_1, \dots \mathsf{x}_n \in \mathbb{D}$ say, the representable y_I is isomorphic to an n-fold product $\mathtt{I}^n$ in $\hat{\mathcal{C}}$. Thus the sieve (8.2) corresponds to a subobject of the n-cube $\mathtt{I}^n$. Indeed, each φ in the face lattice $\mathbb{F}(I)$ is a finite join of irreducible elements; and each of those irreducibles is a finite conjunction of atomic conditions of the form $\mathsf{x} = \mathtt{0}$ or $\mathsf{y} = \mathtt{1}$. The corresponding subobject of $\mathtt{I}^n$ is a finite union of *faces*, that is, subobjects of $\mathtt{I}^n$ obtained by setting some co-ordinates to either $\mathtt{0}$ or $\mathtt{1}$. This disjunctive normal form for elements of $\mathbb{F}(I)$ entails that its equality is decidable and hence this $\mathtt{Cof}$ is contained in Ω_{dec} and axiom $\mathtt{ax_9}$ is satisfied (Theorem 8.4). Finally, axiom $\mathtt{ax_8}$ follows from the fact that pullback of cofibrations along a projection $\pi_1 : \mathtt{I}^n \times \mathtt{I} \to \mathtt{I}^n$ has a (stable) right adjoint, or equivalently that the lattice morphisms $\mathbb{F}\,\pi_1 : \mathbb{F}(I) \to \mathbb{F}(I \cup \{\mathsf{x}\})$ (for any $I \in \operatorname{obj}\mathcal{C}$ and $\mathsf{x} \in \mathbb{D} - I$) have (stable) right adjoints. This is the quantifier elimination result for face lattices; see [CCHM18, Lemma 2].

9. Related Work

The work reported here was inspired by the suggestion of Coquand [Coq15] that some of the constructions developed in [CCHM18] paper might be better understood using the internal logic of a topos. We have shown how to express Cohen, Coquand, Huber and Mörtberg's notion of fibration in the internal type theory of a topos. The use of internal language permits an appealingly simple description (Definition 5.7) compared, for example, with the more abstract category-theoretic methods of weak factorization systems and model categories, which have used for the same purpose by Gambino and Sattler [GS17, Section 3]. Birkedal *et al* [BBC$^+$16] develop guarded cubical type theory with a semantics based on an axiomatic version of [CCHM18] within the internal logic of a presheaf topos.

Within the framework of a topos equipped with an interval-like object, we found that quite a simple collection of axioms (Figure 1) suffices for this to model Martin-Löf type theory with intensional identity types satisfying a weak form of univalence. In particular, only a simple connection algebra, rather than a de Morgan algebra structure, is needed on the interval. Furthermore, the collection of propositions suitable for uniform Kan filling is not tightly constrained and can be chosen in various ways. In Section 8 we only considered how presheaf categories can satisfy our axioms. It might be interesting to consider models in general Grothendieck toposes (where presheaves are restricted to be sheaves for a given notion of covering), particularly *gros toposes* such as Johnstone's topological topos [Joh79]; this allows the interval object to be (a representable sheaf corresponding to) the usual topological interval and hence for the model of type theory to have a rather direct connection with classical homotopy types of spaces. However, although the Hofmann-Streicher [HS99] universe construction (the basis for the construction in Section 8.2 of [CCHM18] of a fibrant universe satisfying the full univalence axiom) can be extended from presheaf to sheaf toposes via the use of sheafification [Str05, Section 3], it seems that sheafification does not interact well with the CCHM notion of fibration. In another direction, recent work of Frumin and Van Den Berg [Fv18] makes use of our elementary, axiomatic approach using a non-Grothendieck topos, namely the effective topos [Hyl82].

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